Introduction to Integrability

Problem Sets

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Contents

Introduction to Integrability — Problem Set 1

ETH Zurich, HS16 — Prof. N. Beisert, A. Garus

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1.1. Classical Mechanics Redux

Recall that a transformation $Q(q,p,t)$, $P(q,p,t)$ is canonical if the Poisson brackets in the new variables take the canonical form. The new Hamiltonian $K(Q,P,t)$ is related to the old one $H(q,p,t)$ by

$$H = K + \frac{\partial F}{\partial t}\,, \tag{1.1}$$

where F is a generating function of the transformation. Specifically, for a time-independent transformation the canonical transformation will yield $H(q,p) = K(Q(q,p), P(q,p))$.

a) Show that for a one-dimensional system the transformation $Q = q + \mathring{\imath}p$, $P = q - \mathring{\imath}p$ is *not* canonical. Can you rescale Q and P by constants to make it canonical?

b) Write dQ/dt using Hamiltonian equations of motion and conclude that the transformation is canonical if

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i}\,, \qquad \frac{\partial Q_i}{\partial p_j} = -\frac{\partial q_j}{\partial P_i}\,. \tag{1.2}$$

c) Using the results of the previous part verify that the following transformation is canonical

$$Q = \log\left(\frac{\sin p}{q}\right), \qquad P = q\cot p. \tag{1.3}$$

d) For the transformation given above we want to find a generating function. It is easy to express $q = q(p,P)$ and $Q = Q(p,P)$, hence we will be interested in the generating function $F = F(p,P)$. Determine $F(p,P)$ by solving the following PDE's

$$\frac{\partial F}{\partial P} = Q, \qquad \frac{\partial F}{\partial p} = -q. \tag{1.4}$$

e) Canonical transformations need to preserve the Poisson structure, namely if $\{q,p\} = 1$, then we also demand $\{Q,P\} = 1$. Consider a transformation given by

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p). \tag{1.5}$$

For what values of α and β is the Poisson bracket preserved (and the transformation canonical)?

f) Find a canonical transformation that brings the Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{1.6}$$

to the one of the harmonic oscillator.

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1.2. Euler top

Consider a rotating solid body attached to a fixed point. Without external forces, its Hamiltonian (in the co-moving frame) takes the form

$$H = \sum_{i=1}^{3} \frac{J_i^2}{2I_i}, \tag{1.7}$$

where J_i are the components of the angular momentum and I_i the principal moments of inertia.

One introduces the Poisson bracket by

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k. \tag{1.8}$$

a) Show that the above defined Poisson bracket is degenerate (i.e. that there exists a non-zero quantity Poisson-commuting with everything).

In order to circumvent the issue one fixes the value of J^2. Then the phase space is of dimension two.

b) Argue that the system is integrable by exhibiting one integral of motion. Why is it not super-integrable?

c) Use the fixed value of J^2 and the integral of motion to express $J_{2,3}$ in terms of J_1.

d) Derive the equation of motion for J_1 and argue that it can be brought to a form

$$\dot{J}_1 = \sqrt{\alpha + \beta J_1^2 + \gamma J_1^4}\,. \tag{1.9}$$

1.3. Not every system is integrable (advanced)

Consider a system given by the Hamiltonian

$$H = \tfrac{1}{2}\big(p_x^2 + p_y^2 + x^2 + y^2 + \epsilon x^2 y^2\big). \tag{1.10}$$

Clearly for $\epsilon = 0$ it describes two uncoupled harmonic oscillators.

a) Exhibit two independent constants of motion thus proving that the system is integrable

b) Show that if two solutions start close to each other, they will remain close at every time.

Now let us turn on the perturbation, i.e. consider $\epsilon \neq 0$.

c) How many constants of motion can you find now? What does it imply for the system?

d) Demonstrate that now the distance between two different solutions will grow exponentially even if they start close to each other. Do it analytically by perturbing around a known solution as well as numerically, e.g. using *Mathematica*'s `NDSolve`.

2.1. The Kepler problem and action-angle variables

Consider a motion in an arbitrary centrally symmetric potential $V(r)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In the Kepler problem $V(r) = C/r$, but we can keep the discussion general without any additional difficulties. The Hamiltonian of the system is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(r), \tag{2.1}$$

where p_i is a momentum conjugate to x_i.

a) The phase space is 6-dimensional, hence we need three conserved quantities for the system to be integrable. Show that the components of the angular momentum vector $\vec{J} = (J_1, J_2, J_3)$ are conserved. Recall that $J_k = \varepsilon_{kij} x_i p_j$.

b) Compute the Poisson bracket $\{J_i, J_j\}$ between the components of $\vec{J}$ to demonstrate that they are *not* in involution. Do you recognise the algebra that they span?

c) Another conserved quantity we have at our disposal is the Hamiltonian H itself. Show that the combination $J^2 := J_1^2 + J_2^2 + J_3^2$ commutes with both the Hamiltonian and an arbitrary J_i. From now on we take $P := (H, J^2, J_3)$ as our three integrals of motion.

d) Spherical symmetry encourages us to rewrite the problem in spherical coordinates

$$x_1 = r \sin\vartheta \cos\varphi, \qquad x_2 = r \sin\vartheta \sin\varphi, \qquad x_3 = r\cos\vartheta. \tag{2.2}$$

What are the momenta $p := (p_r, p_\vartheta, p_\varphi)$ conjugate to the positions $q := (r, \vartheta, \varphi)$? Rewrite the conserved quantities P in terms of spherical coordinates.

e) Invert the above relations to express the momenta in terms of conserved quantities.

f) The generating function of a canonical transformation is given by

$$S\big(q, P\big) = \int^r dr\, p_r + \int^\vartheta d\vartheta\, p_\vartheta + \int^\varphi d\varphi\, p_\varphi. \tag{2.3}$$

We now would like to find the positions $Q_i := \partial S / \partial P_i$ conjugate to our momenta P. Show that $\dot{Q}_H = 1$ and $\dot{Q}_{J^2} = \dot{Q}_{J_3} = 0$. Using the first of these equations, derive the standard solution of the Kepler problem

$$t - t_0 = \int^r \frac{dr}{\sqrt{2m\big(H - V(r)\big) - J^2/r^2}}\,. \tag{2.4}$$

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2.2. Euler top and Lax pair

In problem 1.2 we have solved the Euler top by quadratures. Let us now find its formulation in terms of a Lax pair which will allow us to claim its integrability without calculating integrals.

a) Using the fact that $J_k = I_k \omega_k$, where $\vec{\omega}$ is the rotation vector of the co-moving frame, show that the equation of motion for $\vec{J}$ is

$$\frac{dJ_k}{dt} = \varepsilon_{kij}\omega_i J_j. \tag{2.5}$$

Hint: Use the Poisson structure and the Hamiltonian equation of motion $dF/dt = -\{H, F\}$.

b) Introduce the 3×3 matrices $L_{ij} = \varepsilon_{ijk} J_k$ and $M_{ij} = -\varepsilon_{ijk}\omega_k$. Show that the equation of motion can be written as

$$\frac{dL}{dt} = [M, L]. \tag{2.6}$$

c) Calculate the few first conserved quantities $\operatorname{Tr} L^n$. Why is the Hamiltonian not among them?

We see that our naive approach fails. In order to find a Lax pair that would give us all the necessary integrals of motion, let us take a step back.

d) In problem 1.2 you showed that the equations of motion for the Euler top take the form

$$\frac{dJ_i}{dt} = \frac{I_k - I_l}{I_k I_l} J_k J_l, \tag{2.7}$$

with (i, k, l) a cyclic permutation of $(1, 2, 3)$. Show that you can introduce rescaled $\mathcal{J}_i := \alpha_i J_i$ so that the equations can be alternatively written as

$$\frac{d\mathcal{J}_i}{dt} = 2\mathcal{J}_k \mathcal{J}_l. \tag{2.8}$$

Under which assumptions is this possible?

e) Show that the following pair of 2×2 matrices $L(z)$ and $M(z)$ is a Lax pair for the system, that is that the equation of motion can be written as $dL(z)/dt = [M(z), L(z)]$

$$\begin{aligned} L(z) &= \begin{pmatrix} -2z\mathcal{J}_3 & (1-z^2)\mathcal{J}_1 + (1+z^2)\mathcal{J}_2 \\ (1-z^2)\mathcal{J}_1 - (1+z^2)\mathcal{J}_2 & 2z\mathcal{J}_3 \end{pmatrix}, \\ M(z) &= \begin{pmatrix} \mathcal{J}_3 & z\mathcal{J}_1 - z\mathcal{J}_2 \\ z\mathcal{J}_1 + z\mathcal{J}_2 & -\mathcal{J}_3 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Additionally, show that you can expand $L(z)$ and $M(z)$ in terms of Pauli matrices. Why is it *not* surprising?

f) Compute all the independent integrals of motion $\operatorname{Tr} L(z)^k$. By explicit calculation show that they are indeed conserved, and write them in terms of H and J^2.

g) Compute the eigenvalues and eigenvectors of $L(z)$. Argue that the spectral curve is a Riemann surface of genus $g = 1$.

Introduction to Integrability — Problem Set 3

ETH Zurich, HS16 — Prof. N. Beisert, A. Garus

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3.1. Conservation laws for the KdV equation

We first show that there are infinitely many conserved local charges for the KdV equation

$$\dot{h} = 6hh' - h'''. \tag{3.1}$$

a) Start by the following change of field variable $h \to w$

$$h = w + \mathring{\imath}\epsilon w' - \epsilon^2 w^2. \tag{3.2}$$

Demonstrate that h satisfies (3.1) if w satisfies the following conservation law

$$\dot{w} = \frac{\partial}{\partial x}\big(3w^2 - w'' - 2\epsilon^2 w^3\big). \tag{3.3}$$

b) Write w as a formal power series in ϵ

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n. \tag{3.4}$$

Obtain the recursion relations satisfied by w_n's (you may stop at the order ϵ^3). Show that only even coefficients of ϵ will give you new information.

c) Finally show that the conservation law (3.3) implies

$$F_n = \int_{-\infty}^{+\infty} \mathrm{d}x\, w_n, \qquad \dot{F}_n = 0. \tag{3.5}$$

Write down F_0 and F_2 in terms of h and its derivatives.

3.2. Solitons and soliton trains

Let us consider solutions to the KdV equation $\dot{h} = 6hh' - h'''$ with constant velocity v by assuming $h(t,x) = f(x - vt)$. Show that f satisfies the following equation

$$\tfrac{1}{2}f'^2 = f^3 + \tfrac{1}{2}vf^2 + \alpha f + \beta, \tag{3.6}$$

with α and β arbitrary constants. Assuming that the function vanishes asymptotically, how do you fix α and β? Can you find a solution?

Hint: The equation can be solved most easily for the particular values of α and β.

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3.3. KdV equation and the Inverse Scattering Method

In the problem we will now discuss the Inverse Scattering Method on the example of the KdV equation. Recall that the GLM equation takes the form

$$K(x,y) + \hat{r}(x+y) + \int_x^\infty \mathrm{d}z\, K(x,z)\, \hat{r}(z+y) = 0, \qquad h(x) = -2\frac{\partial}{\partial x} K(x,x). \tag{3.7}$$

For reflectionless potentials $\hat{r}(x)$ is a sum over poles and residues only

$$\hat{r}(x) = \sum_{j=1}^{N} \lambda_j\, \mathrm{e}^{-\kappa_j x}, \quad \text{where } \lambda_j = \lambda_j(t) = \lambda_j(0)\, \mathrm{e}^{8\kappa_j^3 t} \tag{3.8}$$

First, consider the case $N = 1$ in (3.8) with $\kappa_1 = \kappa$ and $\lambda_1 = \lambda$.

a) Solve the GLM equation (3.7) by means of the separation ansatz $K(x,y) = K(x)\,\mathrm{e}^{-\kappa y}$, and obtain the solution $h(x,t)$ from $K(x,y)$.

b) Compute the minimum x_0 of the function $h(x)$ as a function of κ, λ, x and t.

Now consider the case $N = 2$ in (3.8).

c) Assume again that $K(x,y)$ separates completely

$$K(x,y) = K_1(x)\,\mathrm{e}^{-\kappa_1 y} + K_2(x)\,\mathrm{e}^{-\kappa_2 y}\,. \tag{3.9}$$

Show that you can write the GLM equation in the matrix form $AK + L = 0$ with

$$A_{n,m} = \delta_{n,m} + \lambda_n \frac{\mathrm{e}^{-(\kappa_n+\kappa_m)x}}{\kappa_n + \kappa_m}, \qquad K_m = K_m(x), \qquad L_m = \lambda_m\, \mathrm{e}^{-\kappa_m x}\,. \tag{3.10}$$

d) Show that

$$K(x,x) = K_1(x)\,\mathrm{e}^{-\kappa_1 x} + K_2(x)\,\mathrm{e}^{-\kappa_2 x} = \frac{\partial}{\partial x} \log\det A(x). \tag{3.11}$$

Show that, as expected, $h(x,t) \to 0$ for $x, t \to -\infty$.

e) Compare the combinations 1, $\lambda_1(t)\,\mathrm{e}^{-2\kappa_1 x}$ and $\lambda_2(t)\,\mathrm{e}^{-2\kappa_2 x}$ in the limit $t \to -\infty$ and $t \to +\infty$ (assuming $\kappa_1 < \kappa_2$).

f) Now assume that $\lambda_2(t)\,\mathrm{e}^{-2\kappa_2 x}$ is either extremely small or extremely large compared to the other relevant quantities. Find the leading behaviour for $\det A = \det A^\pm$ in both of these limits. You may drop all the terms that become irrelevant in $(\partial/\partial x)^2 \log\det A$.

g) Show that both resulting expressions for $h^\pm = -2(\partial/\partial_x)^2 \log\det A$ have the same form as in part a). Compute the parameters (κ, λ) for the solution in part a) in terms of $(\kappa_1^\pm, \lambda_1^\pm)$ in both limits.

h) Compare the minima $x_0 = x_0^\pm$ of $h(x)$ between both limits. What can you say about the difference of $x_0^\pm$? Interpret the result. Is the interaction between the solitons effectively attractive or repulsive?

4.1. Circle solutions for the Heisenberg magnet

The Heisenberg magnet has two fields $\vartheta(t,x)$, $\varphi(t,x)$ with the equations of motion

$$\dot{\vartheta} = 2\cos\vartheta\,\vartheta'\varphi' + \sin\vartheta\,\varphi'', \qquad \dot{\varphi} = \cos\vartheta\,\varphi'^2 - \frac{\vartheta''}{\sin\vartheta}\,. \tag{4.1}$$

The momentum P, energy E and angular momentum Q are given by the expressions

$$P = \int \mathrm{d}x\,\bigl(1-\cos\vartheta\bigr)\,\varphi', \qquad E = \int \mathrm{d}x\,\bigl[\tfrac{1}{2}\vartheta'^2 + \tfrac{1}{2}\sin^2\vartheta\,\varphi'^2\bigr], \qquad Q = \int \mathrm{d}x\cos\vartheta. \tag{4.2}$$

a) The model has a simple class of solutions whose shape is a circle at constant latitude $\vartheta(t,x) = \vartheta_0$. Find the most general solution for $\varphi(t,x)$.

b) Impose periodic boundary conditions with period L. Note that the boundary condition needs to be satisfied only modulo the equivalence $\varphi \equiv \varphi + 2\pi n$.

c) Determine the momentum P, energy E and angular momentum Q for these solutions.

4.2. Spectral curve for the Heisenberg magnet

Here, we are going to construct the simplest spectral curve for the Heisenberg magnet. An appropriate finite-gap ansatz for the quasi-momentum $q(u)$ with a single cut is

$$q'(u) = \frac{au+b}{u^2\sqrt{u^2+cu+d}}\,. \tag{4.3}$$

An admissible quasi-momentum has the following additional properties:
Let A be a cycle around the branch cut of q' and B a path going from $u=\infty_2$ through the cut back to $u=\infty_1$. Then $q'(u)$ should satisfy the following conditions

$$\oint_A \mathrm{d}u\,q'(u) = 0, \qquad \frac{1}{2\pi}\int_B \mathrm{d}u\,q'(u) = n \in \mathbb{Z}, \qquad I = \frac{1}{2\pi i}\oint_A \mathrm{d}u\,u q'(u), \tag{4.4}$$

where I is the filling of the cut. Furthermore, the length L, momentum P, energy E and angular momentum Q are related to the expansions at $u=0$ and $u=\infty$ as follows

$$q(u) = \frac{L}{u} - \tfrac{1}{2}P + \tfrac{1}{4}uE + \mathcal{O}(u^2), \qquad q(u) = \frac{Q}{u} + \mathcal{O}(u^{-2}). \tag{4.5}$$

a) Determine the coefficients a,b,c,d in terms of I,n,L by using the above relations. *Hint:* The A-cycle integral can be computed as the difference of residues at $u=\infty$ and $u=0$. Note also that $q'(u)$ changes sign upon passing the cut.

b) Integrate $q'(u)$ to get the quasi-momentum $q(u)$. Fix the integration constant using one of the conditions of (4.5).

c) Match the expansions of $q(u)$ at $u=0$ and $u=\infty$ to (4.5) to derive the charges P,E,Q. Compare to the results of problem 4.1c).

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4.3. Explicit diagonalisation for the Heisenberg spin chain

Consider the Hamiltonian of the Heisenberg spin chain with periodic boundary conditions

$$\mathcal{H} = \sum_{j=1}^{L} \big(\mathcal{I}_{j,j+1} - \mathcal{P}_{j,j+1}\big). \tag{4.6}$$

Compute the spectrum of $\mathcal{H}$ in the cases specified below, where K denotes the number of up spins and $L-K$ the number of down spins.

a) Compute the spectrum for the states $L = 3$ and arbitrary number of spin flips K. How do these fit into multiplets of SU(2)?

From now on restrict to cyclic states, i.e. identify all states which are equivalent under cyclic permutations.

b) Compute the spectrum for the states $L = 4$, $K = 2$ and $L = 6$, $K = 2, 3$.

c) Compute the spectrum for the states with $K = 2$ and arbitrary length L.

4.4. Spectrum by means of the Bethe equations

The Bethe equations for the Heisenberg chain are

$$\left(\frac{u_k + \frac{\mathring{\imath}}{2}}{u_k - \frac{\mathring{\imath}}{2}}\right)^{L} = \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k - u_j + \mathring{\imath}}{u_k - u_j - \mathring{\imath}}, \qquad \text{for } k = 1, \ldots, K. \tag{4.7}$$

For each solution of these equations (with distinct u_k) there exist an eigenstate of the Heisenberg Hamiltonian with energy and (exponentiated) momentum

$$E = \sum_{k=1}^{K} \left(\frac{\mathring{\imath}}{u_k + \frac{\mathring{\imath}}{2}} - \frac{\mathring{\imath}}{u_k - \frac{\mathring{\imath}}{2}}\right) \quad \text{and} \quad e^{\mathring{\imath}P} = \prod_{k=1}^{K} \frac{u_k + \frac{\mathring{\imath}}{2}}{u_k - \frac{\mathring{\imath}}{2}}. \tag{4.8}$$

Use the Bethe equations to rederive the energies of states considered in problem 4.3.

a) Compute the spectrum for the states $L = 3$ and arbitrary number of spin flips $K \leq 3$. Compare to the results of problem problem 4.3a) How can the multiplets of SU(2) be realised? How do you interpret additional solutions?

From now on restrict to cyclic states, i.e. demand $e^{\mathring{\imath}P} = 1$.

b) Compute the spectrum for the states $L = 4$, $K = 2$ and $L = 6$, $K = 2, 3$. Compare to the results of problem 4.3b). *Hint:* the state $L = 6$, $K = 3$ is singular, can you find it?

c) Compute the spectrum for the states with $K = 2$ and arbitrary length L. Compare to the results of problem 4.3c).

Introduction to Integrability — Problem Set 5

ETH Zurich, HS16 — Prof. N. Beisert, A. Garus

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5.1. Finite-size corrections to the magnon spectrum

In this problem, we would like to compute the leading finite-size corrections to the magnon spectrum around the ferromagnetic vacuum for long chains.

Recall the Bethe equations for the Heisenberg XXX spin chain in the logarithmic form

$$\mathring{\imath}L\log\frac{u_k+\frac{\mathring{\imath}}{2}}{u_k-\frac{\mathring{\imath}}{2}}-\mathring{\imath}\sum_{\substack{j=1\\j\neq k}}^{M}\log\frac{u_k-u_j+\mathring{\imath}}{u_k-u_j-\mathring{\imath}}+2\pi n_k=0. \tag{5.1}$$

a) First, we would like to determine the correction in the case of distinct mode numbers $\{n_k\}$. We write $u_k = L/2\pi n_k + \delta u_k$. Show that expanding in $1/L$ the leading correction to the Bethe equations (5.1) reads

$$4\pi^2 n_k^2\delta u_k+\sum_{\substack{j=1\\j\neq k}}^{M}\frac{4\pi n_j n_k}{n_j-n_k}=0. \tag{5.2}$$

Show that the solution for arbitrary M is

$$\delta u_k=\frac{1}{\pi n_k}\sum_{\substack{j=1\\j\neq k}}^{M}\frac{n_j}{n_k-n_j}\,, \tag{5.3}$$

and then verify that the energy becomes

$$E=\sum_{k=1}^{M}\frac{4\pi^2n_k^2}{L^2}-\sum_{k=1}^{M}\sum_{j=k+1}^{M}\frac{16\pi^2n_kn_j}{L^3}+\mathcal{O}\big(1/L^4\big). \tag{5.4}$$

b) In the lecture you showed that for two magnons sharing the same n_k, the corrections δu to u's are determined by the following set of equations

$$\frac{4\pi^2n^2}{L}\,\delta u_k+\sum_{\substack{j=1\\j\neq k}}^{M}\frac{2}{\delta u_k-\delta u_j}=0. \tag{5.5}$$

Show that the solutions δu_k correspond to the roots of Hermite polynomials $H_n(a\,u)$, that is for given M determine order of the polynomial n as well as the coefficient a.

c) *optional:* In the situation of part b) the leading correction to the energy occurs at $\mathcal{O}(1/L^3)$, hence one needs second order perturbation theory to compute it. Compute the leading energy correction for $M = 2, 3, \ldots$.

Can you predict or compute the behaviour for arbitrary M? Can you guess the finite-size corrections to the energy spectrum for arbitrary excitation numbers $\{M_n\}$ of several modes n.

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5.2. Fluctuations of the one-cut solution

In problem 4.2 you considered a one-cut solution for the Heisenberg magnet which corresponds to a state of the ferromagnetic XXX spin chain in the limit of long chains. We now want to add a second, infinitesimally short branch cut to this solution corresponding to one additional magnon excitation.

a) A cut of length $\sqrt{\epsilon}$ ending in square root singularities can be modelled by the function

$$\sqrt{\left(u - u_* - \tfrac{i}{2}\sqrt{\epsilon}\right)\left(u - u_* + \tfrac{i}{2}\sqrt{\epsilon}\right)}. \tag{5.6}$$

Show that for small ϵ it is approximated by a pole at u_* with residue $\sim \epsilon$.

b) Justify the following ansatz for the quasi-momentum $q(u)$

$$q(u) = \frac{a(u-b)}{u-u_*} \frac{\sqrt{u^2 + cu + d}}{u} + e \tag{5.7}$$

with the unknown coefficients a, b, c, d, e. Explain why the vanishing of the A-periods is automatic for this ansatz.

The leading order $q_0(u)$ of $q(u)$ is given by the solution of problem 4.2

$$q_0(u) = \pi n \sqrt{1 + \frac{2(L-I)}{\pi n u} + \frac{L^2}{\pi^2 n^2 u^2}} - \pi n. \tag{5.8}$$

We will be interested in the leading correction in ϵ, i.e. we need a, b, c, d, e to order ϵ.

c) Show that the position u_* of the new branch cut is given the equation

$$q_0(u_*) = -\pi m \in \pi\mathbb{Z}. \tag{5.9}$$

Solve it for the position u_* which you will need only at leading order.

d) Let the filling of the short cut be ϵ,

$$\epsilon = \frac{1}{2\pi i} \oint_{u^*} \mathrm{d}u\, u q'(u). \tag{5.10}$$

Express this as a relationship for the coefficients.

e) The remaining conditions we discussed in the context of the spectral curve were

$$\frac{1}{2\pi} \int_B \mathrm{d}u\, q'(u) = n, \qquad I = \frac{1}{2\pi i} \oint_A \mathrm{d}u\, u q'(u), \qquad \frac{1}{2\pi i} \oint_0 \mathrm{d}u\, q(u) = L. \tag{5.11}$$

Use them to fix the remaining coefficients. *Hint:* Follow the hints in problem 4.2 and pay attention to all the possible poles you need to take into account!

f) Recall that the energy E, momentum P and angular momentum Q of the solution can be read off from the expansions

$$q(u) = \frac{L}{u} - \tfrac{1}{2}P + \tfrac{1}{4}uE + \mathcal{O}(u^2), \qquad q(u) = \frac{Q}{u} + \mathcal{O}(u^{-2}). \tag{5.12}$$

Compute the leading corrections and write them in terms of I, n, m, L.

g) *optional:* Determine ϵ such that the short branch cut corresponds to one magnon.