Symmetries in Physics

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Contents

1 Finite Groups

1.1 Definition and Basic Examples

Let us begin by reminding ourselves of the definition of a (finite) group. Consider a set G of finitely (or infinitely) many elements, for which we define a law of combination that assigns to every ordered pair $a, b \in G$ a unique element $a \cdot b \in G$. Here the products $a \cdot b$ and $b \cdot a$ need not be identical. We call G a group if

(i) G contains an element $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$. (e is then called the unit element or identity element.)

(ii) To every $a \in G$, there exists an inverse $a^{-1} \in G$ so that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

(iii) The composition is associative, i.e.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \ . \tag{1.1.1}$$

Simple examples of groups are the set of all integers $\mathbb{Z}$, with the group operation being addition; or the set of all non-zero rationals $\mathbb{Q}^*$ or reals $\mathbb{R}^*$ with the group operation being multiplication.

A more general class of examples is provided by the *transformation groups*. Let W be some set, and let G be the set of transformations

$$f : W \to W \tag{1.1.2}$$

that are one-to-one, i.e. $x \neq y$ implies $f(x) \neq f(y)$, and onto, i.e. for every $y \in W$, there exists an $x \in W$ such that $y = f(x)$. Then G defines a group, where the group operation is composition, i.e.

$$(f \cdot g)(x) = f(g(x)) \ . \tag{1.1.3}$$

The identity element of G is the trivial transformation $e(x) = x$. Inverses exist since the transformations are one-to-one and onto, and the composition of maps is always associative. A prominent example of this kind are the **symmetric groups** S_n that are the transformation groups associated to $W_n = \{1, \ldots, n\}$. The elements of S_n are the permutations, i.e. the invertible maps from $\{1, \ldots, n\}$ to itself. We may label them compactly as (here $n = 7$)

$$(1324)(5)(67) \longleftrightarrow \left\{ \begin{array}{ccc} 1 & \mapsto & 3 \\ 3 & \mapsto & 2 \\ 2 & \mapsto & 4 \\ 4 & \mapsto & 1 \\ 5 & \mapsto & 5 \\ 6 & \mapsto & 7 \\ 7 & \mapsto & 6 \end{array} \right. \tag{1.1.4}$$

A group G is called *abelian* if $a \cdot b = b \cdot a$ for all $a, b \in G$. For example, $(\mathbb{Z}, +)$ and $(\mathbb{R}^*, \cdot)$ are abelian, while the permutation groups S_n with $n \geq 3$ are not. A group is cyclic if it consists exactly of the powers of some element a, i.e. the group elements are

$$e \equiv a^0 \equiv a^p \ , \ a \ , \ a^2 \ , \ldots \ , \ a^{p-1} \tag{1.1.5}$$

with the group operation

$$a^l \cdot a^m = a^{l+m} \ , \tag{1.1.6}$$

where $l+m$ is evaluated mod p. This **cyclic group** will be denoted by C_p in the following, and it also defines an abelian group.

Another important finite group is the **dihedral group** D_n, which is the symmetry group of a regular n-gon in the plane. Its elements are of the form

$$e \equiv d^n \ , \ d \ , \ d^2 \ , \ldots \ , \ d^{n-1} \ , \qquad s \ , \ sd \ , \ sd^2 \ , \ldots \ , \ sd^{n-1} \ , \tag{1.1.7}$$

with the relations

$$s^2 = d^n = e \ , \qquad d^{-k} s = s d^k \ . \tag{1.1.8}$$

(Here d denotes the clockwise rotation of the n-gon, while s is the reflection around an axis.) Using (1.1.8) it is easy to see that the group elements of the form (1.1.7) close under the group operation.

For a finite group G, the order $|G|$ of G is the number of elements in G. For example, for the cyclic group defined by (1.1.5), the order is $|C_p| = p$, while for the dihedral group we have $|D_n| = 2n$ and for the symmetric group the order is

$$|S_n| = n! \ . \tag{1.1.9}$$

1.2 Representations

A representation of a finite group G on a finite-dimensional complex vector space V is a homomorphism

$$\rho : G \to \mathrm{Aut}(V) \tag{1.2.1}$$

of G into the group of automorphisms of V. This is to say, to every $a \in G$, $\rho(a)$ is an endomorphism of V that is invertible. Furthermore, the structure of G is respected by this map, i.e.

$$\rho(a \cdot b) = \rho(a) \circ \rho(b) \qquad \forall a, b \in G \ , \tag{1.2.2}$$

where '$\circ$' stands for composition of maps in $\mathrm{Aut}(V)$. In particular, it follows from this property that

$$\rho(e) = \mathbf{1} \ , \tag{1.2.3}$$

the identity map from $V \to V$, and that

$$\rho(a^{-1}) = (\rho(a))^{-1} \ . \tag{1.2.4}$$

We say that such a map ρ gives V the structure of a G-module; when there is little ambiguity about the map ρ (and we are afraid, even sometimes when there is) we sometimes call V itself the representation of G. In this vein we will often suppress the symbol ρ and write $a \cdot v$ or av for $(\rho(a))(v)$. The dimension of V will sometimes be called the dimension of the representation ρ.

If V has finite dimension, then upon introducing a basis in V, the linear transformations can be described by non-singular $n \times n$ matrices. Thus a finite-dimensional representation ρ is an assignment of matrices $\rho(a)$ for each group element $a \in G$ such that (1.2.2) holds, where '$\circ$' on the right-hand-side stands for matrix multiplication. In this lecture we will always only consider finite-dimensional representations of groups.

As an example consider the symmetric group S_3, that consists of the 6 permutations of the three symbols $\{1, 2, 3\}$. It is generated by the two transpositions

$$\sigma_1 = (12)(3) \ , \qquad \sigma_2 = (1)(23) \tag{1.2.5}$$

subject to the relations that

$$\sigma_1^2 = \sigma_2^2 = e \ , \qquad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \ ; \tag{1.2.6}$$

the entire group then consists of the six group elements

$$e\,, \quad \sigma_1\,, \quad \sigma_2\,, \quad \sigma_1\sigma_2\,, \quad \sigma_2\sigma_1\,, \quad \sigma_1\sigma_2\sigma_1 \ . \tag{1.2.7}$$

It posseses a natural 3-dimensional representation, for which the group elements act as permutation matrices; thus the two generators σ_1 and σ_2 are represented by

$$\rho(\sigma_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \qquad \rho(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \tag{1.2.8}$$

etc. It is easy to see that this defines a representation of S_3, i.e. that (1.2.2) is satisfied for all elements of S_3. (This is just a consequence of the fact that the relations (1.2.6) are satisfied for $\rho(\sigma_1)$ and $\rho(\sigma_2)$.) Another representation of S_3 is the 1-dimensional representation, the so-called *alternating representation*, defined by the determinant of these 3-dimensional matrices; it satisfies

$$\rho_d(\sigma_1) = \rho_d(\sigma_2) = \rho_d(\sigma_1\sigma_2\sigma_1) = -1 \ , \tag{1.2.9}$$

as well as

$$\rho_d(\sigma_1\sigma_2) = \rho_d(\sigma_2\sigma_1) = \rho_d(e) = +1 \ . \tag{1.2.10}$$

Again, it is straightforward to check that ρ_d satisfies (1.2.2), now with '$\circ$' standing for regular multiplication.

A **subrepresentation** W is a vector subspace $W \subseteq V$ wich is invariant under G, i.e. which has the property that

$$\rho(a)\, w \in W \qquad \forall w \in W \ , \ a \in G \ . \tag{1.2.11}$$

A representation V is called **irreducible** if V does not contain any subrepresentation other than $W = V$ or $W = \{0\}$. Otherwise the representation is called reducible.

Note that the above 3-dimensional representation (1.2.8) of S_3 is *not* irreducible (i.e. it is reducible). In particular, it contains the 1-dimensional subrepresentation

$$W = \langle e_1 + e_2 + e_3 \rangle \ . \tag{1.2.12}$$

Indeed, it is easy to see that $\rho(a)(e_1 + e_2 + e_3) = (e_1 + e_2 + e_3)$, i.e. that $\rho_W(a) = 1$ for all $a \in S_3$. This representation is therefore called the *trivial representation*; obviously the trivial representation with $\rho(a) \equiv 1$ for all $a \in G$ exists for every group G.

If V and W are representations of G, the **direct sum** $V \oplus W$ is also a representation of G, where the action of G is defined by

$$\rho = \rho_V \oplus \rho_W \ . \tag{1.2.13}$$

Similarly, if V and W are representations of G, their **tensor product** $V \otimes W$ is also a representation of G. Recall that if e_i, $i = 1, \ldots, n$ and f_j, $j = 1, \ldots, m$ are basis vectors for V and W, respectively, then a basis for $V \otimes W$ is described by the pairs

$$e_i \otimes f_j \ , \qquad i \in \{1, \ldots, n\} \, , \ j \in \{1, \ldots, m\} \ . \tag{1.2.14}$$

(Thus the tensor product $V \otimes W$ has dimension $\dim(V) \dim(W)$.) The tensor product representation is then defined by

$$a(v \otimes w) = (a \cdot v) \otimes (a \cdot w) \qquad \forall a \in G \, , \ v \in V \, , \ w \in W \ . \tag{1.2.15}$$

It is immediate that this defines a representation of G.

Finally, if V is a representation of G, then also the dual vector space V^* is a G-representation. In order to define the G-action on V^* we use as a guiding principle that the natural pairing $\langle \cdot, \cdot \rangle$ between V^* and V should be invariant under G, i.e. that

$$\langle \rho^*(a)(v^*), \rho(a)(v) \rangle = \langle v^*, v \rangle \ , \tag{1.2.16}$$

where $a \in G$, $v \in V$ and $v^* \in V^*$ are arbitrary. By rewriting this condition, replacing v by $\rho(a^{-1})v$, we obtain the defining relation for $\rho^*(a)$

$$\langle \rho^*(a)(v^*), v \rangle = \langle v^*, \rho(a^{-1})v \rangle \ . \tag{1.2.17}$$

Note that combining the definition of the dual and the tensor product then also allows us to make

$$\mathrm{Hom}(V, W) \cong V^* \otimes W \tag{1.2.18}$$

into a G-representation, provided that V and W are G-representations. (Here $\mathrm{Hom}(V, W)$ is the space of vector space homomorphisms from V to W.) More explicitly, on the vector space homomorphisms $\varphi : V \to W$ the G-action is defined by

$$(a\varphi)\,(v) = a\, \varphi(a^{-1}v) \ . \tag{1.2.19}$$

1.3 Complete Reducibility

As we have seen above, there are many different representations of a finite group. Before we begin our attempt to classify them we should try to simplify life by restricting our search somewhat. Specifically, we have seen that representations of G can be built up out of other representations by linear algebraic operations, most simply by taking the direct sum. We should focus then on representations that are 'atomic' with respect to this operation, i.e. that cannot be expressed as a direct sum of others; the usual term for such a representation is *indecomposable*. Happily, the situation is as nice as it could possibly be: a representation is atomic in this sense if and only if it is irreducible, i.e. contains no proper subpresentation; and every representation is equivalent to the direct sum of irreducibles, in a suitable sense uniquely so. The key to all of this is the following:

Proposition: If W is a subrepresentation of a representation V of a finite group G, then there is a complementary invariant subspace W' of V so that $V = W \oplus W'$.

Proof: One simple way to prove this is the following. We define a positive definite hermitian inner product $\langle \cdot, \cdot \rangle$ on V which is preserved by each $a \in G$ via

$$\langle v, w \rangle \equiv \sum_{a \in G} \langle av, aw \rangle_0 \ , \tag{1.3.1}$$

where $\langle \cdot, \cdot \rangle_0$ is any hermitian inner product on V. Recall that a hermitian inner product on V is an inner product satisfying

$$\langle v, w \rangle = \langle w, v \rangle^* \tag{1.3.2}$$

as well as

$$\langle v, v \rangle \geq 0 \ , \tag{1.3.3}$$

with equality if and only if $v = 0$ in V. On any n-dimensional complex vector space such an inner product exists. If $\langle \cdot, \cdot \rangle_0$ is an hermitian inner product on V, then so is $\langle \cdot, \cdot \rangle$ defined by (1.3.1). Furthermore, $\langle \cdot, \cdot \rangle$ is then invariant under the action of G, i.e. we have

$$\langle v, w \rangle = \langle bv, bw \rangle \qquad \text{for any } b \in G, \ v, w \in V. \tag{1.3.4}$$

(This last statement simply follows from the fact that

$$\langle bv, bw \rangle = \sum_{a \in G} \langle abv, abw \rangle_0 = \sum_{a' \in G} \langle a'v, a'w \rangle_0 = \langle v, w \rangle \ , \tag{1.3.5}$$

where in the middle step we have relabelled the sum over a by a sum over $a' = ab$ — as a runs over the whole group G, so does a', since the map from $a \mapsto a'$ is one-to-one with inverse $a = a'b^{-1}$.)

Now with respect to the invariant hermitian inner product $\langle \cdot, \cdot \rangle$ we define the orthogonal complement of W by

$$W^{\perp} := \{ u \in V : \langle u, w \rangle = 0 \, , \ \forall w \in W \} \ . \tag{1.3.6}$$

Since the hermitian inner product is non-degenerate, we have

$$V = W \oplus W^{\perp} \ , \tag{1.3.7}$$

i.e. the intersection of W and $W^{\perp}$ is just the zero vector. Thus $W^{\perp}$ plays the role of W' in the proposition.

Finally, we observe that $W^{\perp}$ also defines a subrepresentation of V; to see this we only need to show that $au \in W^{\perp}$ for any $a \in G$ and $u \in W^{\perp}$. But since

$$\langle w, au \rangle = \langle a^{-1}w, u \rangle = 0 \tag{1.3.8}$$

for any $w \in W$, this is the case, and we have shown that $W^{\perp}$ is a suberpresentation of V.

Note that by induction, this result implies that any (finite-dimensional) representation of a finite group G is completely reducible, i.e. that it can be written as a direct sum of irreducible representations. As we shall see also compact continuous groups have this property — in fact the proof is fairly analogous since we can simply replace the sum over the group elements in (1.3.1) by an integral, using the invariant Haar measure of the compact group. However, there are also (infinite) groups for which complete reducibility does not hold. For example, for the group $(\mathbb{Z}, +)$ the 2−dimensional representation

$$n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \tag{1.3.9}$$

leaves the 1-dimensional subspace spanned by e_1 invariant. However, there is no complementary subspace that would also be invariant under the action of the group, and hence the above representation is reducible but indecomposable, i.e. it cannot be written as a direct sum of two 1-dimensional representations. However, for the remainder of these lectures we shall always consider groups whose representations are completely reducible.

For the example of the 3-dimensional representation ρ from (1.2.8), we have seen above, see eq. (1.2.12), there it contains a one-dimensional subrepresentation W. With respect to the standard inner product, the orthogonal complement of W is then generated by

$$V \equiv W^{\perp} = \langle f_1 = (1, -1, 0) \, , \ f_2 = (0, 1, -1) \rangle \ . \tag{1.3.10}$$

One easily confirms that the two generators in (1.2.8) act as

$$\rho(\sigma_1) f_1 = -f_1 \ , \qquad \rho(\sigma_1) f_2 = f_1 + f_2 \ , \tag{1.3.11}$$

as well as

$$\rho(\sigma_2) f_1 = f_1 + f_2 \ , \qquad \rho(\sigma_2) f_2 = -f_2 \ , \tag{1.3.12}$$

i.e. the two generators are represented by the 2×2 matrices

$$\rho(\sigma_1)_V = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \ , \qquad \rho(\sigma_2)_V = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \ . \tag{1.3.13}$$

Again, one easily confirms that these matrices satisfy (1.2.6), as they must. Furthermore, it is easy to see that $W^\perp$ does not contain any invariant 1-dimensional subspace, and hence that it must be irreducible. Thus the 3-dimensional representation (1.2.8) is in fact a direct sum of the 1-dimensional representation W as well as the 2-dimensional representation $V \equiv W^\perp$. V is called the 'standard' representation.

1.4 Equivalence of Representations and Schur's Lemma

For our example of S_3 we have now found three irreducible representations: the trivial representation W (1.2.12), the representation by signs ρ_d defined by (1.2.9) and (1.2.10), as well as the representation $W^\perp$ defined by (1.3.13). As we shall see later, these are in fact the *only* irreducible representations of S_3. However, before we can explain this, we first need to understand when we should regard two representations as 'being the same'.

We shall say that two representations $\rho_1 : G \to V_1$ and $\rho_2 : G \to V_2$ are equivalent (or 'the same') provided that there exists an invertible linear map (in particular, this therefore implies that V_1 and V_2 have the same vector space dimension, $n = \dim(V_1) = \dim(V_2)$)

$$T : V_1 \to V_2 \tag{1.4.1}$$

such that

$$T \circ \rho_1(a) = \rho_2(a) \circ T \ , \qquad \forall a \in G \ , \tag{1.4.2}$$

i.e. that $\rho_1 = T^{-1} \circ \rho_2 \circ T$. In terms of $n \times n$ matrices, this condition therefore means that the $n \times n$ matrices $\rho_1(a)$ and $\rho_2(a)$ are conjugate to one another for all $a \in G$, where the conjugating matrix T is independent of a. Note that for 1-dimensional representations, two representations ρ_1 and ρ_2 are equivalent if and only if $\rho_1(a) = \rho_2(a)$ for all $a \in G$.

Now that we have understood when two representations are the same, we can explain in which sense the decomposition of a representation into irreducibles is unique. This is a consequence of

Schur's Lemma: Let V and W be irreducible representations of G and $T : V \to W$ a G-module homomorphism, i.e. a linear map satisfying

$$T \circ \rho_V = \rho_W \circ T \ . \tag{1.4.3}$$

Then

(i) Either T is an isomorphism or $T = 0$.

(ii) If $V = W$, then $T = \lambda \cdot \mathbf{1}$ for some $\lambda \in \mathbb{C}$. (Here $\mathbf{1}$ is the identity map.)

Proof: First we note that the kernel of T is an invariant subspace of V, since if $v \in \ker(T)$ then (1.4.3) implies that

$$T\big(\,\rho_V(a)\,(v)\,\big) = \rho_W(a)\big(\,T\,(v)\,\big) = 0 \ , \tag{1.4.4}$$

i.e. $\rho_V(a)\,(v)$ is also in the kernel of T. Similarly, the image of T is an invariant subspace of W: if $w \in W$ is in the image of T, i.e. $w = T(v)$, then

$$\rho_W(a)\,\big(T(v)\big) = T\big(\rho_V(a)(v)\big) \tag{1.4.5}$$

is also in the image of T. Since V is irreducible, it follows that either $\ker(T) = V$ — in which case $T \equiv 0$ — or $\ker(T) = \{0\}$, in which case T is one-to-one. Similarly, since W is irreducible, either $\mathrm{im}(T) = \{0\}$ — in which case $T \equiv 0$ — or $\mathrm{im}(T) = W$, i.e. T is onto. Combining these two statements either $T \equiv 0$, or T is both one-to-one and onto, i.e. an isomorphism. This proves (i).

In order to prove (ii), we use that T must have at least one eigenvalue $\lambda \in \mathbb{C}$. But then $T - \lambda \cdot \mathbf{1}$ has a non-zero kernel, and hence, since it is also a G-module homomorphism, by the argument of case (i), it follows that $T - \lambda \cdot \mathbf{1} \equiv 0$. This proves that $T = \lambda \cdot \mathbf{1}$, i.e. the statement (ii).

As we have seen above, any representation V can be completely decomposed into irreducibles, i.e. we can write

$$V = V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{\oplus n_k} \ , \tag{1.4.6}$$

where the V_i are distinct irreducible representations, and the n_i denote the multiplicities with which these representations appear in the decomposition. Schur's Lemma now implies that this decomposition is unique in the sense that the same factors and the same multiplicities always appear. In order to see this, let us assume that we can also write V as

$$V = W_1^{\oplus m_1} \oplus \cdots \oplus W_l^{\oplus m_l} \ , \tag{1.4.7}$$

where W_j are distinct irreducible representations, and m_j denote the multiplicities of these representations. Then since the identity map $\mathbf{1} : V \to V$ is a G-module homomorphism, i.e. satisfies (1.4.3), Schur's Lemma implies that $\mathbf{1}$ must map the summand $V_i^{\oplus n_i}$ into the summand $W_j^{\oplus m_j}$ for which $W_j \cong V_i$, and furthermore that the multiplicities must agree. This proves the uniqueness of the decomposition (4.2.1).

As we will see below, each finite group G only has finitely many irreducible representations. Once these are known, we therefore know in effect, because of (4.2.1), the most general representation of the group G. Thus in the following we shall concentrate on classifying the irreducible representations. One very powerful statement that we will derive is that the dimensions of these irreducible representations satisfies the identity

$$|G| = \sum_{R \text{ irrep}} \dim(R)^2 \ . \tag{1.4.8}$$

For example, for the case of S_3, this is the identity

$$3! = 6 = 1 + 1 + 2^2 \ , \tag{1.4.9}$$

showing that the three irreducible representations from above, two 1-dimensional representations and the two-dimensional standard representation V, are in fact the only irreducible representations of S_3.

1.5 Characters

There is a remarkably effective tool for understanding the representations of a finite group G, called character theory. It is in effect a way of keeping track of the eigenvalues of the action of the group elements. Of course, specifying all eigenvalues of the action of each element of G would be unwieldy, but fortunately, this would also be redundant. For example, if we know the eigenvalues $\{\lambda_i\}$ of an element $a \in G$, we also know the eigenvalues of a^k for any k — they are just $\{\lambda_i^k\}$. The key observation here is that it is enough to give, for example, the sum of the eigenvalues of each element $a \in G$ since knowing the sums $\sum_i \lambda_i^k$ is equivalent to knowing the eigenvalues $\{\lambda_i\}$ themselves. This then motivates the definition of the character of a representation as follows.

If V is a representation of G, its character χ_V is the complex-valued function on the group defined by

$$\chi_V(a) = \mathrm{Tr}(a|_V) \ , \tag{1.5.1}$$

i.e. the trace of a on V. Note that in particular, we have

$$\chi_V(bab^{-1}) = \chi_V(a) \ , \tag{1.5.2}$$

i.e. the character χ_V is constant on the conjugacy classes of G. (Recall that the conjugacy class $[a]$ of $a \in G$ consists of all group elements that are conjugate to a, i.e. that are of the form bab^{-1} for some $b \in G$. This relation defines an equivalence relation, and hence the group can be split up into disjoint conjugacy classes. Note that the conjugacy class of the identity element just consists of the identity element itself.) A function that does not depend on the representative of each conjugacy class is called a *class function*. Note that for the conjugacy class of the identity we simply have $\chi_V(e) = \dim(V)$.

Suppose V and W are representations of G. Then we have

$$\chi_{V\oplus W} = \chi_V + \chi_W \ , \qquad \chi_{V\otimes W} = \chi_V \cdot \chi_W \qquad \chi_{V^*} = \overline{\chi_V} \ . \tag{1.5.3}$$

In order to understand these identities, we consider a fixed group element $a \in G$. Then for the action of a, V has the eigenvalues $\{\lambda_i\}$ while W has the eigenvalues $\{\mu_j\}$. Then the eigenvalues on $V \oplus W$ and $V \otimes W$ are $\{\lambda_i, \mu_j\}$ and $\{\lambda_i \cdot \mu_j\}$, respectively, from which the first two formulae follow. Similarly $\{\lambda_i^{-1} = \overline{\lambda_i}\}$ are the eigenvalues for g on V^*, where we have used that all eigenvalues are n'th roots of unity, with n the order of the group element a; this proves the last identity.

As we have said before, the character of a representation of a group G is really a function on the set of conjugacy classes in G. This suggests expressing the basic information about the irreducible representations of a group G in the form of a *character table*. This is a table with the conjugacy classes $[a]$ of G listed across the top, usually given by a representative a, with the number of elements in each conjugacy class over it; the irreducible representations V of G are listed on the left, and in the appropriate box the value of the character χ_V on the conjugacy class $[a]$ is given. For example, for the group S_3 the character table takes the form given in Table 1.

S_3	1 e	3 (12)	2 (123)
trivial	1	1	1
alternating	1	-1	1
standard V	2	0	-1

Table 1: Character Table of S_3

Note that the conjugacy class (12) contains the three generators (12), (23) and (13), while the conjugacy class (123) contains the two generators (123) and (132). The traces in the trivial and alternating — the former is the representation W defined in (1.2.12), while the latter is the representation defined by ρ_d in (1.2.9) and (1.2.10) — are immediate from the definition; the trace for (12) in V follows directly from (1.3.13), while that of $(123) = (23)(12) = \sigma_2\sigma_1$ or $(132) = (12)(23) = \sigma_1\sigma_2$ follows from

$$\rho_V(\sigma_2\sigma_1) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \rho_V(\sigma_1\sigma_2) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} . \tag{1.5.4}$$

Note that, by (1.5.3), the character in the 3-dimensional representation ρ in (1.2.8) (that can be written as $W \oplus V$) is then

$$\chi_\rho(e) = 3 , \qquad \chi_\rho(12) = 1 , \qquad \chi_\rho(123) = 0 \tag{1.5.5}$$

as one also directly verifies. In fact, we could have determined the decomposition of ρ into $W \oplus V$ from this character identity since the three character functions are independent. This is to say, if we make the ansatz

$$V_\rho = n_1\,(\text{trivial}) + n_2(\text{alternating}) + n_3\,V \tag{1.5.6}$$

then the right-hand-side implies that

$$\chi_V(e) = n_1 + n_2 + 2n_3 , \qquad \chi_V(12) = n_1 - n_2 , \qquad \chi_V(123) = n_1 + n_2 - n_3 . \tag{1.5.7}$$

So for the case at hand, we get the identities

$$n_1 + n_2 + 2n_3 = 3 , \qquad n_1 - n_2 = 1 , \qquad n_1 + n_2 - n_3 = 0 , \tag{1.5.8}$$

from which $n_1 = 1$, $n_2 = 0$ and $n_3 = 1$ is the only solution. This basic idea will play an important role in the following.

1.6 The Projection Formula

We now want to make this more systematic, and in particular, determine the trivial factor in the decomposition of an arbitrary representation. Suppose V is a representation of G. Then we define the invariant subspace of V as

$$V^G = \{v \in V : av = v \ \ \forall a \in G\} . \tag{1.6.1}$$

We define the map

$$\varphi = \frac{1}{|G|} \sum_{a \in G} a \in \mathrm{End}(V) \ . \tag{1.6.2}$$

Note that φ defines a G-module homomorphism, i.e. $b \circ \varphi = \varphi \circ b$ for any $b \in G$; this follows simply from the fact

$$\sum_{a \in G} a = \sum_{a \in G} b^{-1} a b = \sum_{a' \in G} a' \ , \tag{1.6.3}$$

where we have defined $a' = b^{-1}ab$ in the last step. Next we claim that φ defines the projection of V onto V^G. First, we show that $\varphi(V) \subset V^G$: let $v = \varphi(w)$. Then we have

$$b\,\varphi(w) = \frac{1}{|G|} \sum_{a \in G} baw = \frac{1}{|G|} \sum_{a' \in G} a'w = \varphi(w) \ , \tag{1.6.4}$$

where $a' = ba$, thus showing $\varphi(V) \subset V^G$. Conversely, suppose $v \in V^G$. Then

$$\varphi(v) = \frac{1}{|G|} \sum_{a \in G} av = \frac{1}{|G|} \sum_{a \in G} v = v \ , \tag{1.6.5}$$

thus showing $V^G \subset \varphi(V)$. Hence we conclude that $\varphi(V) = V^G$. Note that the last identity also implies that $\varphi \circ \varphi = \varphi$.

We can use this method to determine explicitly the multiplicity with which the trivial representation appears in V; this number is just

$$n_1 = \dim V^G = \mathrm{Tr}_V(\varphi) = \frac{1}{|G|} \sum_{a \in G} \mathrm{Tr}_V(a) = \frac{1}{|G|} \sum_{a \in G} \chi_V(a) \ . \tag{1.6.6}$$

For example, for the case of the 3-dimensional representation ρ in (1.2.8) (that can be written as $W \oplus W^\perp$) the multiplicity with which the trivial representation (that we called W) appears in ρ equals

$$n_1 = \frac{1}{6}\Big(\chi_\rho(e) + 3\chi_\rho\big((12)\big) + 2\chi_\rho\big((123)\big)\Big) = \frac{1}{6}\big(3 + 3 \cdot 1 + 2 \cdot 0\big) = 1 \ . \tag{1.6.7}$$

However, we can actually do much more with this idea. First we note that

$$\mathrm{Hom}(V, W)^G = \{G\text{-module homomorphism from } V \text{ to } W\} \ , \tag{1.6.8}$$

since the condition that $\varphi : V \to W$ is G-invariant means (see eq. (1.2.19)) that for all $a \in G$

$$a\,\varphi(a^{-1}v) = \varphi(v) \quad \Longrightarrow \quad \varphi \circ a' = a' \circ \varphi \quad (a' = a^{-1}) \ , \tag{1.6.9}$$

compare (1.4.3). Next we observe that if V is irreducible, then by the proof of Schur's Lemma, $\ker(\varphi)$ is either 0 or V itself, and

$$\dim\big(\mathrm{Hom}(V, W)^G\big) = m_V \equiv \text{multiplicity with which } V \text{ appears in } W \ . \tag{1.6.10}$$

On the other hand, we know that

$$\mathrm{Hom}(V,W) \cong V^* \otimes W \ , \tag{1.6.11}$$

and thus we can apply the above trick to this tensor product. Using the character formula for the tensor product and the dual representation, see eq. (1.5.3), we therefore conclude that

$$m_V = \frac{1}{|G|}\sum_{a\in G} \overline{\chi_V(a)} \cdot \chi_W(a) \ . \tag{1.6.12}$$

Thus we can determine the multiplicity with which every irreducible representation appears in a given representation using these character techniques. For example, for the above 3-dimensional representation ρ of S_3, see (1.2.8), we deduce from this that

$$\begin{aligned} n_2 &= \frac{1}{6}\Big(\overline{\chi}_a(e)\chi_\rho(e) + 3\cdot\overline{\chi}_a\big((12)\big)\chi_\rho\big((12)\big) + 2\cdot\overline{\chi}_a\big((123)\big)\chi_\rho\big((123)\big)\Big) \\ &= \frac{1}{6}\Big((1)(3) + 3\cdot(-1)\cdot 1 + 2\cdot(1)(0)\Big) = 0 \end{aligned} \tag{1.6.13}$$

and

$$\begin{aligned} n_3 &= \frac{1}{6}\Big(\overline{\chi}_{W^\perp}(e)\chi_\rho(e) + 3\cdot\overline{\chi}_{W^\perp}\big((12)\big)\chi_\rho\big((12)\big) + 2\cdot\overline{\chi}_{W^\perp}\big((123)\big)\chi_\rho\big((123)\big)\Big) \\ &= \frac{1}{6}\Big((2)(3) + 3\cdot(0)\cdot 1 + 2\cdot(-1)(0)\Big) = 1 \ , \end{aligned} \tag{1.6.14}$$

in agreement with what we saw above.

Note that if W is also irreducible, we learn from this that the characters of the irreducible representations are orthonormal with respect to the hermitian inner product on the space of class functions defined by

$$(\alpha,\beta) = \frac{1}{|G|}\sum_{a\in G} \overline{\alpha(a)}\,\beta(a) \ . \tag{1.6.15}$$

For example, for the case of S_3 above, this property can be directly read off from the character table, see Tab. 1, where the numbers over each conjugacy class tell us how many times to count entries in that column.

As a consequence it immediately follows that the number of irreducible representations of G is less than or equal to the number of conjugacy classes. In fact, we shall soon see that the number is always exactly equal, i.e. that there are no non-zero class functions that are orthogonal to all characters.

It is very instructive to apply these ideas to the regular representation of G, i.e. to the left-action of G on itself. For the regular representation, the underlying vector space R has a basis $\{e_a\}$ labelled by the group elements $a \in G$, and the action of $b \in G$ on e_a equals

$$b\,(e_a) = e_{ba} \ . \tag{1.6.16}$$

This clearly defines a representation R of G of dimension $\dim(R) = |G|$. It is clear by construction that the character of R equals

$$\chi_R(a) = \begin{cases} 0 & \text{if } a \neq e \\ |G| & \text{if } a = e. \end{cases} \tag{1.6.17}$$

If we decompose R into irreducibles as $R = \bigoplus_i V_i^{\oplus n_i}$, then

$$n_i = \frac{1}{|G|} \sum_{a \in G} \overline{\chi_{V_i}(a)} \chi_R(a) = \overline{\chi_{V_i}(e)} = \dim(V_i) \ . \tag{1.6.18}$$

Thus we derive the desired relation between the size of the group and the dimensions of the irreducible representations,

$$|G| = \sum_i \dim(V_i)^2 \ . \tag{1.6.19}$$

It remains to show that there are no non-zero class functions that are orthogonal to all characters, i.e. that the number of irreducible representations equals the number of conjugacy classes of G. To this end suppose that $\alpha : G \to \mathbb{C}$ is any function on the group G. Given any representation V of G we define the endomorphism of V

$$\varphi_{\alpha,V} = \sum_{a \in G} \alpha(a) \cdot a : V \to V \ . \tag{1.6.20}$$

We now claim that $\varphi_{\alpha,V}$ is a homomorphism of G-modules, i.e. satisfies

$$b \circ \varphi_{\alpha,V} = \varphi_{\alpha,V} \circ b \qquad \forall\ b \in G \tag{1.6.21}$$

for all representations V if and only if α is a class function, i.e. if

$$\alpha(bab^{-1}) = \alpha(a) \qquad \forall\ a, b \in G \ . \tag{1.6.22}$$

In order to see this let us write

$$\begin{aligned} (\varphi_{\alpha,V} \circ b)(v) &= \sum_{a \in G} \alpha(a) \cdot a(bv) \\ &= \sum_{a \in G} \alpha(bab^{-1}) \cdot bab^{-1}(bv) \qquad \qquad [\text{substitute } a \mapsto bab^{-1}] \\ &= \sum_{a \in G} \alpha(bab^{-1}) \cdot b(av) \ . \end{aligned} \tag{1.6.23}$$

Now if α is a class function, $\alpha(bab^{-1}) = \alpha(a)$, then the last equation can be written as

$$b\Big(\sum_{a \in G} \alpha(a) \cdot av\Big) = (b \circ \varphi_{\alpha,V})(v) \ . \tag{1.6.24}$$

Conversely, suppose α is not a class function, i.e. there exist $a_0, b_0 \in G$ such that $\alpha(b_0 a_0 b_0^{-1}) \neq \alpha(a_0)$. We consider the regular representation of G, i.e. the representation of G on itself by left-action, see eq. (1.6.16). For $v = e_e$ we then have

$$(\varphi_{\alpha,G} \circ b_0)(e_e) = \sum_{a\in G} \alpha(b_0 a b_0^{-1}) \cdot e_{b_0 a} \neq \sum_{a\in G} \alpha(a) \cdot e_{b_0 a} = (b_0 \circ \varphi_{\alpha,G})(e_e) \ , \tag{1.6.25}$$

where the inequality is a consequence of the fact that the coefficient of the basis vector $e_{b_0 a_0}$ is different between the two sides. This proves the converse direction.

This result now implies that the characters of the irreducible representations form an orthonormal basis for the space of class functions. To see this suppose that α is a class function that is orthonormal to all the characters of the irreducible representations, i.e.

$$\sum_{a\in G} \alpha(a)\, \chi_V(a) = 0 \tag{1.6.26}$$

for all (irreducible) V. (Note that since this is true for all representations, we may use $\chi_{V^*} = \overline{\chi}_V$ to remove the complex conjugate from the second factor.) Let V be one of these irreducible representations, and consider again the endomorphism $\varphi_{\alpha,V}$ defined as above in eq. (1.6.20). Then by Schur's Lemma $\varphi_{\alpha,V} = \lambda \cdot \mathbf{1}$ for some $\lambda \in \mathbb{C}$, and with $n = \dim(V)$ we have

$$\lambda = \frac{1}{n}\mathrm{Tr}_V(\varphi_{\alpha,V}) = \frac{1}{n}\sum_{a\in G} \alpha(a)\chi_V(a) = 0 \ . \tag{1.6.27}$$

Thus $\varphi_{\alpha,V} = 0$ or $\sum_{a\in G} \alpha(a) \cdot a = 0$ on any irreducible representation V of G. But since any representation can be written as a direct sum of irreducibles, this statement is true on any representation, in particular the regular representation. But then, by a similar argument as above in eq. (1.6.25), it follows that $\alpha \equiv 0$. This completes the proof.

1.7 The group algebra

There is an important notion that we have already dealt with implicitly, but not explicitly: this is the group algebra $\mathbb{C}G$ associated to a finite group G. This is an object that, for all intents and purposes, can completely replace the group G itself. Any statement about the representations of G has an exact equivalent statement about the group algebra.

The underlying vector space of the group algebra of G is the vector space with basis $\{e_a\}$ corresponding to the different elements $a \in G$, i.e. the underlying vector space of the regular representation. We define the algebra structure on this vector space simply by

$$e_a \cdot e_b = e_{ab} \ . \tag{1.7.1}$$

By a representation of the group algebra $\mathbb{C}G$ on a vector space V we mean simply an algebra homomorphism

$$\mathbb{C}G \to \mathrm{End}(V) \ , \tag{1.7.2}$$

i.e. a representation V of $\mathbb{C}G$ is the same thing as a left $\mathbb{C}G$-module. Note that a representation $\rho : G \to \mathrm{End}(V)$ will extend, by linearity, to a map $\tilde{\rho} : \mathbb{C}G \to \mathrm{End}(V)$ so that representations of $\mathbb{C}G$ correspond exactly to representations of G. The left $\mathbb{C}G$-module given by $\mathbb{C}G$ itself corresponds to the regular representation.

If $\{W_i\}$ are the irreducible representations of G, then we have seen that the regular representation R decomposes as

$$R = \bigoplus_i (W_i)^{\oplus \dim(W_i)} \ . \tag{1.7.3}$$

We can now refine this in terms of the group algebra as the statement that, as algebras,

$$\mathbb{C}G \cong \bigoplus_i \mathrm{End}(W_i) \ . \tag{1.7.4}$$

To see this we recall that for any representation W of G, the map $\rho : G \to \mathrm{End}(W)$ extends by linearity to a map $\mathbb{C}G \to \mathrm{End}(W)$. Applying this to each of the irreducible representations W_i gives us a canonical map

$$\varphi : \mathbb{C}G \to \bigoplus_i \mathrm{End}(W_i) \ . \tag{1.7.5}$$

This map is injective since the representation on the regular representation is faithful, i.e. different group elements act differently. Since both have dimension $\sum(\dim W_i)^2$, the map is an isomorphism.

1.8 Crystal-field splitting

As an application of these techniques we consider the following physical problem which can be solved elegantly using group theoretic methods. (This analysis was pioneered by Bethe in the 1920s.)

When an atom or ion is located not in free space, but in a crystal, it is subjected to various inhomogeneous electric fields which destroy the isotropy of free space. In particular, the symmetry group is reduced from that of the full three-dimensional rotation group to some finite group of rotations through finite angles (and perhaps also reflections).

We shall discuss the representation theory of continuous groups (such as the three-dimensional rotation group) in later sections, but we recall for the moment that the energy spectrum of electrons in an atom (e.g. the hydrogen atom) are most conveniently described in terms of spherical harmonics. These are nothing but special families of functions that transform in irreducible representations of the rotation group. Indeed, as you learned in quantum mechanics, the set of spherical harmonics

$$Y^{l,m}(\theta,\varphi) \ , \qquad m = -l,\ldots,l \ , \tag{1.8.1}$$

where $l = 0, 1, \ldots$ is fixed, transform under rotations into one another and correspond to the irreducible representation D_l of the rotation group of dimension $2l + 1$. Since rotations change the value of m, this in particular implies that the energy spectrum of a rotationally symmetric problem such as the hydrogen atom can only depend on n (the additional quantum number characterising the radial behaviour) and l, but not on m. In fact, as you probably remember, for the case of the hydrogen atom, the energy spectrum is even independent of l, but this is a sign of an even bigger underlying symmetry that is associated to the Runge-Lenz vector.

Let us consider the case of an ion in a crystal at a site where it is surrounded by a regular octahedron of negative ions; this is a reasonable approximation of the real situation in a large number of instances. The discrete symmetry group that maps the adjacent ions into one another contains then the octahedral group O, i.e. the group of proper rotations that take a cube or a octahedron into itself. (The relation between the octahedron and the cube is that we can embed the octahedron into the cube so that the vertices of the octahedron sit at the face centers of the cube, while the face centers of the octahedron are in one-to-one correspondence with the vertices of the cube. Thus we may equivalently think of O as being the group of rotational symmetries of the cube.)

The rotational symmetries of the cube contain 24 elements: (i) for each of the 4 body diagonals, there are 2 non-trivial rotations, leading to $8 = 4 \cdot 2$ diagonal rotations; (ii) for each of the three axes through the face centers of opposite faces — we can naturally think of them as the x, y and z-axis — there are 3 non-trivial rotations, leading to $9 = 3 \cdot 3$ x, y, z rotations; (iii) finally for each of the 6 axes through the origin that are parallel to face diagonals there is one non-trivial 180-degree rotation, leading to $6 = 6 \cdot 1$ Altogether, and including the identity generator, we therefore have a group of $8 + 9 + 6 + 1 = 24$ elements, $|O| = 24$.

In terms of conjugacy classes, there is the conjugacy class of the identity E; the conjugacy class containing the order 3 rotations along the diagonal C_3 — it contains all 8 such elements; the conjugacy class C_4 of the order 4 rotations along x, y, z — it contains all 6 such elements. In addition there are two conjugacy classes of order 2 elements, one associated to the order 2 rotations along the x, y, z axis — we shall denote it by $3\,C_2$ since it contains 3 elements — and one containing all 6 order two rotations of (iii) — this will be denoted by $6\,C_2$. The character table then has the form given in table 2, where the different representations have been labelled (following the usual convention in molecular physics) A_1, A_2, E, T_1 and T_2.

	1	8	3	6	6
O	E	C_3	$3\,C_2$	$6\,C_2$	C_4
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	-1	1
T_2	3	0	-1	1	-1

Table 2: Character Table of O

Note that A_1 is the identity representation, and that the different characters are orthonormal with respect to the standard inner product on the space of class functions (1.6.15).

Now consider an electron with angular momentum quantum number L with respect to the usual 3-dimensional rotation group. The $(2L+1)$ spherical harmonics that describe the different values for M all have degenerate energy eigenvalues in isotropic space. However, with respect to the smaller octahedral symmetry group, this irreducible representation of the rotation group will now not be irreducible, but will rather be a direct sum of irreducible O-representations. Thus in the presence of the lattice, one should expect that the $(2L+1)$-fold degeneracy of the eigenstate will be lifted according to the decomposition into O-representations. (The states that sit in the same irreducible O-representation will continue to have degenerate energy eigenvalues, but for those that sit in different irreducible O-representations this will not generically be the case.) Thus without doing any real calculation, we can make a qualitative prediction for how the degeneracies will lift in the presence of the crystal!

In order to understand this more concretely, all we have to do is to understand how the irreducible D_L representation decomposes with respect to the O-action. In fact, given our results above, it is enough to know the character of the various conjugacy classes of O in the D_L representation. Recall that in the representation D_L, a rotation by an angle α leads to the trace

$$\begin{aligned}\chi_L(\alpha) &= \sum_{M=-L}^{L} e^{iM\alpha} = e^{-iL\alpha}\sum_{m=0}^{2L} e^{im\alpha} = e^{-iL\alpha}\frac{1-e^{i(2L+1)\alpha}}{1-e^{i\alpha}} && (1.8.2)\\ &= \frac{e^{i(L+\frac{1}{2})\alpha}-e^{-i(L+\frac{1}{2})\alpha}}{e^{i\frac{\alpha}{2}}-e^{-i\frac{\alpha}{2}}} && (1.8.3)\\ &= \frac{\sin\big((L+\frac{1}{2})\alpha\big)}{\sin(\frac{\alpha}{2})}\ . && (1.8.4)\end{aligned}$$

Since the different conjugacy classes of O all correspond to rotations we therefore find

$$\begin{aligned}\chi_L(E) &= 2L+1 && (1.8.5)\\ \chi_L(C_3) &= \chi_L(\tfrac{2\pi}{3}) = \frac{\sin\big((L+\frac{1}{2})\frac{2\pi}{3}\big)}{\sin(\frac{\pi}{3})} = \begin{cases} 1 & L=0,3,\ldots\\ 0 & L=1,4,\ldots\\ -1 & L=2,5,\ldots\end{cases} && (1.8.6)\\ \chi_L(C_2) &= \chi_L(\pi) = \frac{\sin\big((L+\frac{1}{2})\pi\big)}{\sin(\frac{\pi}{2})} = (-1)^L && (1.8.7)\\ \chi_L(C_4) &= \chi_L(\tfrac{\pi}{2}) = \frac{\sin\big((L+\frac{1}{2})\frac{\pi}{2}\big)}{\sin(\frac{\pi}{4})} = \begin{cases} 1 & L=0,1,4,5,\ldots\\ -1 & L=2,3,6,7,\ldots\ ,\end{cases} && (1.8.8)\end{aligned}$$

where the characters in the two C_2 conjugacy classes are the same — the corresponding rotations are in different conjugacy classes of O, but in the same conjugacy class of the full rotation group.

Now we have all the information to determine the decomposition of D_L into irreducible O-representations using (1.6.12). For the first few cases we find explicitly

$$\begin{array}{llll} D_0 & = A_1 & & \\ D_1 & = T_1 & 3 \to 3 & \\ D_2 & = E \oplus T_2 & 5 \to 2+3 & \quad (1.8.9) \\ D_3 & = A_2 \oplus T_1 \oplus T_2 & 7 \to 1+3+3 & \\ D_4 & = A_1 \oplus E \oplus T_1 \oplus T_2 & 9 \to 1+2+3+3\ , & \end{array}$$

as one easily verifies by comparing characters. So for example, this implies that the degeneracy of the 3 states with angular momentum number $L = 1$ is not lifted by the crystal, whereas the degeneracy of the 5 states with angular momentum number $L = 2$ is lifted into a two-fold and a three-fold degenerate level, etc.

In most crystals there are at least small departures from cubic symmetry at the lattice site of a magnetic ion. We can take these (smaller) effects into account by considering now the breaking of the O-representations into representations of the smaller symmetry group that are still respected by the deformed lattice. For example, let us assume that the octahedron of ions producing the crystal field is distorted by an elongation along one of the threefold axes. This reduces the rotational symmetry to the dihedral group D_3 that is a subgroup of O. The group D_3 consists of 6 elements that sit in three conjugacy classes: the identity element (that is a conjugacy class by itself); the conjugacy class of order 3 elements (that contains 2 elements, namely d and d^2 in the notation of (1.1.7)); as well as the conjugacy class of order 2 elements (that contains 3 elements, namely s, sd and sd^2, again in the notation of (1.1.7)). Its character table as well as the characters of the irreducible O representations is given in table 3

	1	2	3
D_3	E	C_3	C_2
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0
T_1	3	0	-1
T_2	3	0	1

Table 3: Character Table of D_3 (upper block). The representations below the double line refer to the O-representations.

We conclude from this that we have the branching rules from O to D_3 given by

$$\begin{aligned} A_1 &= A_1 \\ A_2 &= A_2 \\ E &= E \\ T_1 &= A_2 \oplus E \qquad\qquad 3 \to 1+2 \\ T_2 &= A_1 \oplus E \qquad\qquad 3 \to 1+2 \ , \end{aligned} \tag{1.8.10}$$

i.e. the O-representations A_1, A_2 and E are irreducible with respect to D_3 (where they are denoted by the same symbol), whereas the two 3-dimensional O-representations T_1 and T_2 decompose into a direct sum of a 1-dimensional and a 2-dimensional D_3 representation.

As a consequence, while the degeneracy of the 3 states with angular momentum number $L = 1$ is not lifted by the perfect crystal, a small elongation along one of the threefold axes will induce a small splitting of $3 \to 1+2$. Similarly, the 5 states at $L = 2$ are split by the perfect crystal into a two-fold and a three-fold degenerate level; the small elongation will not lift the degeneracy of the two-fold level, but will split the three-fold level further as $3 \to 1 + 2$, etc.

This example shows how simple group theoretic methods allow us to gain important structural insight into the qualitative features of a problem without doing actual detailed calculations. This is one of the central themes of this course.

1.9 Classification of crystal classes

In the above example we have considered two lattices, the perfect cubic lattice whose symmetry group (fixing a lattice point) contained the octahedral group O, as well as the deformed lattice whose symmetry group was the dihedral group D_3. One may ask what other groups may arise as symmetry groups of 3-dimensional lattices, and in fact one can show that there are only 32 possible symmetry groups. Accordingly, the lattices can be classified into 32 so-called crystal classes.

While we shall not attempt to prove this classification in detail, we want to understand at least schematically what sorts of groups can arise, and why there are only 32 possibilities. First of all, we are only interested in the point groups of the lattice, i.e. in the group of lattice symmetries that fix a specific lattice point which we may take to be the origin. The corresponding group elements can then be thought of as real 3×3 matrices. Furthermore, since the angles between the various lattice vectors are preserved, these 3×3 matrices must be orthogonal.

The first step of the classification consists of finding all finite subgroups of O(3), the group of orthogonal 3×3 matrices. The relevant subgroups can be classified; they are either pure rotation groups, i.e. only contain elements in SO(3) with determinant $+1$, or contain also some reflections. The possible pure rotation groups are

- the cyclic group C_g of order g, consisting of g proper rotations about a fixed axis;

- the dihedral group D_h of order $2h$ consisting of all proper rotations carrying a plane regular h-sided polygon in space onto itself: these include the h proper rotations about an axis perpendicular to the plane, together with h proper 180 degree rotations about the symmetry axes in the plane of the polygon;[1]
- the tetrahedral group T of order 12, consisting of the proper rotations carrying a regular tetahedron onto itself;
- the octahedral group O or order 24, consisting of the proper rotations carrying a cube into itself;
- the icosahedral group I of order 60, consisting of the proper rotations carrying a regular icosahedron into itself. (This group is isomorphic to the alternating group A_5.)

In all of these cases it is also possible to adjoin some reflection symmetries to these pure rotation groups, but in each case there are at most two ways of doing so. But since the order of the group elements for the first two classes of groups, C_g and D_h is not yet constrained, at this stage we still have an infinite list of possible symmetry groups.

To cut this list down to finite size, the following observation is crucial. So far we have studied the possible finite orthogonal symmetry groups of O(3), but we haven't yet used that this must map an actual lattice into itself. For example, in a suitable orthonormal basis the rotations in C_g or D_h are of the form

$$R(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} . \tag{1.9.11}$$

However, if this transformation maps the lattice to itself, it must map each basis vector of the lattice to a linear combination (with integer coefficients) of lattice basis vectors. Thus in the lattice basis (that is typically not orthonormal) the matrix $R(\varphi)$ must be described by a matrix with *integer entries*. The lattice basis is related, by a change of coordinates, to the above orthonormal basis, and hence, for example, the trace of the matrix is independent of which basis is being used. But this then implies that the trace of $R(\varphi)$ must be an integer since it equals an integer in the lattice basis. On the other hand, we have explicitly

$$\mathrm{Tr}\big(R(\varphi)\big) = 1 + 2\cos\varphi \ . \tag{1.9.12}$$

In order for this to be an integer, the possible angles are therefore

$$\varphi = \pm\frac{\pi}{3} \ , \ \pm\frac{\pi}{2} \ , \ \pm\frac{2\pi}{3} \ , \pm\pi \ . \tag{1.9.13}$$

[1]From the 2-dimensional viewpoint of the plane containing the polygon, this group also includes the 'reflection' s, see (1.1.7), but from the 3-dimensional viewpoint that is relevant here, this reflection is described by a 180 degree rotation about an axis in the plane of the polygon.

Thus we conclude that the possible orders of rotations are 1, 2, 3, 4 or 6, and thus the possible groups C_g and D_h that can appear have

$$h, g \in \{1, 2, 3, 4, 6\} . \tag{1.9.14}$$

It is then clear that the complete list of symmetry groups is finite, and a detailed analysis leads to the 32 cases mentioned before. (A more comprehensive discussion may be found in [FS, Chapter 8.2 & 8.3] or [T, Chapter 4.2].)

2 The symmetric group and Young diagrams

For some aspects of Lie theory (that will form the center of attention of the second half of this course) the symmetric group plays an important role. We therefore want to understand its representation theory in some detail.

2.1 Conjugacy classes and Young diagrams

Recall that S_n is the group of 1-to-1 transformations of the set $\{1, \ldots, n\}$. As we have seen before, each element of S_n may be described in terms of cycles, describing the orbits of the different elements of $\{1, \ldots, n\}$. It is not difficult to see that the conjugacy classes of S_n are then labelled by the cycle shapes, i.e. by the specification of how many cycles of which length the permutation has. For example, for the permutation described in (1.1.4), we have

$$(1324)(5)(67) \;\longleftrightarrow\; \text{cycle shape } 1^1\, 2^1\, 4^1 , \tag{2.1.1}$$

i.e. there is one cycle of length 1, 2 and 4 each. We shall denote the conjugacy classes by $C_{\mathbf{i}}$, where $\mathbf{i}$ is a multiindex

$$\mathbf{i} = (i_1, i_2 \ldots, i_n) \ , \tag{2.1.2}$$

with i_j denoting the number of cycles of length j. So for the above example we have $\mathbf{i} = (1, 1, 0, 1, 0, 0, 0)$. Note that we have the identity

$$n = \sum_{j=1}^{n} j\, i_j \ , \tag{2.1.3}$$

i.e. the cycle shapes define a *partition* of n into positive integers. The number of conjugacy classes of S_n is therefore equal to the number of partitions $p(n)$ of n. Its generating function equals

$$\begin{aligned} \sum_{n=0}^{\infty} p(n)\, t^n &= \prod_{m=1}^{\infty} \frac{1}{1-t^m} \\ &= (1+t+t^2+t^3+\cdots)(1+t^2+t^4+\cdots)(1+t^3+t^6+t^9+\cdots)\cdots \\ &= 1+t+2t^2+3t^3+5t^4+7t^5+\cdots \ , \end{aligned} \tag{2.1.4}$$

i.e. S_2 has 2 conjugacy classes, S_3 has 3 conjugacy classes, etc. (This last statement is obviously in agreement with what we saw explicitly above, see table 1.) The above generating function has interesting arithmetic properties and has been carefully studied; for example, the partition numbers grow asymptotically as

$$p(n) \sim e^{\pi\sqrt{\frac{2n}{3}}} \ . \tag{2.1.5}$$

We can describe partitions in terms of Young diagrams (sometimes also called Young frames). Suppose $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, i.e. $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Without loss of generality we may order the λ_i as $\lambda_1 \geq \cdots \geq \lambda_k$. For example, for $n = 10$ with $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 2$, $\lambda_4 = \lambda_5 = 1$, the corresponding Young frame has the form

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{2.1.6}$$

i.e. there are λ_1 boxes in the first row, λ_2 in the second, etc., with the rows of boxes lined up on the left. The conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_r)$ is defined by interchanging the rows and columns in the Young diagram, i.e. by reflecting the diagram along the 45 degree line. For example, for the diagram above, the conjugate partition is $\lambda' = (5, 3, 2)$ with Young diagram

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} & & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \tag{2.1.7}$$

If we want to define the conjugate partition without reference to the diagram, we can define λ'_i as the number of terms in the partition λ that are greater or equal than i.

The purpose of writing a Young diagram instead of just the partition, of course, is to put something in the boxes. Any way of putting a positive integer in each box of a Young diagram is called a filling. A *Young tableau* is a filling (by positive integers) such that the entries are

(1) weakly increasing (i.e. allowing for equalities as well) across each row from left to right

(2) strictly increasing (i.e. without equalities) down each column

We call a Young tableau *standard* if the entries are the numbers $\{1, \ldots, n\}$, each occuring once. For example, for the Young diagram (2.1.6), two standard Young tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 9 \\ \hline 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 10 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 4 & 10 \\ \hline 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} 9 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{2.1.8}$$

Young diagrams can be used to describe projection operators for the regular representation of S_n, which will then give rise to the irreducible representations of S_n. Given a standard tableau, say the one shown to the left above, define two subgroups of the symmetric group[2]

$$P = P_\lambda = \{\sigma \in S_n : \sigma \text{ preserves each row}\} \tag{2.1.9}$$

[2] If a tableau other than the canonical one were chosen, one would get different groups in place of P and Q and different elements in the group algebra, but the representations constructed this way will be isomorphic. We will come back to this point below.

and

$$Q = Q_\lambda = \{\sigma \in S_n : \sigma \text{ preserves each column}\} \ . \tag{2.1.10}$$

In the group algebra $\mathbb{C}G$ we introduce two elements corresponding to these subgroups by defining

$$a_\lambda = \sum_{\sigma \in P} e_\sigma \ , \qquad\qquad b_\lambda = \sum_{\sigma \in Q} \operatorname{sgn}(\sigma)\, e_\sigma \ , \tag{2.1.11}$$

where $\operatorname{sgn}(\sigma)$ is the parity of the permutation σ. To see what a_λ and b_λ do, suppose that V is any vector space. Then S_n acts on the n'th tensor power $V^{\otimes n}$ by permuting the factors, and the image of the element $a_\lambda \in \mathbb{C}S_n$ under the map $a_\lambda \in \mathbb{C}S_n \to \operatorname{End}(V^{\otimes n})$ is just the subspace

$$\operatorname{Im}(a_\lambda) = \big(\operatorname{Sym}^{\lambda_1} V\big) \otimes \big(\operatorname{Sym}^{\lambda_2} V\big) \otimes \cdots \otimes \big(\operatorname{Sym}^{\lambda_k} V\big) \subset V^{\otimes n} \ , \tag{2.1.12}$$

where the inclusion on the right is obtained by grouping the factors of $V^{\otimes n}$ according to the rows of the Young tableaux. Similarly, the image of b_λ on this tensor power is

$$\operatorname{Im}(b_\lambda) = \big(\wedge^{\mu_1} V\big) \otimes \big(\wedge^{\mu_2} V\big) \otimes \cdots \otimes \big(\wedge^{\mu_r} V\big) \subset V^{\otimes n} \ , \tag{2.1.13}$$

where μ is the conjugate partition to λ, and $\wedge^\mu V$ is the totally antisymmetric subspace of the tensor product $V^{\otimes \mu}$.

Finally we define the *Young symmetriser* associated to λ by

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_n \ . \tag{2.1.14}$$

For example, if λ is the Young diagram $\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}$, $b_\lambda = e_e$ and thus $c_\lambda = a_\lambda$. Then the image of c_λ on $V^{\otimes n}$ is $\operatorname{Sym}^n V$. Conversely, if λ is the Young diagram

$$\lambda = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \tag{2.1.15}$$

then $a_\lambda = e_e$ and thus $c_\lambda = b_\lambda$. Then the image of c_λ on $V^{\otimes n}$ is $\wedge^n V$. We will eventually see that the image of the symmetrisers c_λ on $V^{\otimes n}$ provide essentially all the finite-dimensional irreducible representations of $\mathrm{GL}(V)$.

What is important for us in the present context is that we can also obtain all irreducible representations of S_n in this manner. Concretely, one has the following Theorem (which we shall however not prove in the lecture, for a proof see e.g. [FH] chapter 4.2):

Theorem: Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, where $n_\lambda \in \mathbb{R}$, and the image of c_λ by right-multiplication on the group algebra $\mathbb{C}G$ is an irreducible representation V_λ of S_n. The representations corresponding to the different diagrams λ are inequivalent, and hence all irreducible representations can be obtained in this manner.

Note that this construction therefore establishes a direct correspondence between conjugacy classes in S_n and irreducible representations of S_n, something which is not available for general groups. Instead of proving the Theorem we shall illustrate it with a few examples. For example, for the Young diagram $\lambda = \ydiagram{5}$

$$V_{\ydiagram{5}} = \mathbb{C}S_5 \cdot \sum_{\sigma \in S_5} e_\sigma = \mathbb{C} \cdot \sum_{\sigma \in S_5} e_\sigma \tag{2.1.16}$$

is the trivial representation, since for any $\tau \in S_5$

$$\tau \cdot \sum_{\sigma \in S_5} e_\sigma = \sum_{\sigma \in S_5} e_{\tau\sigma} = \sum_{\sigma \in S_5} e_\sigma \ . \tag{2.1.17}$$

Thus each permutation acts trivially on the element $\sum_\sigma e_\sigma$. Conversely, (2.1.15) corresponds to

$$V_{\ydiagram{1,1,1,1,1}} = \mathbb{C}S_5 \cdot \sum_{\sigma \in S_5} \mathrm{sgn}(\sigma)\, e_\sigma = \mathbb{C} \cdot \sum_{\sigma \in S_5} \mathrm{sgn}(\sigma)\, e_\sigma \ , \tag{2.1.18}$$

which defines the alternating representation since now, for any $\tau \in S_5$,

$$\tau \cdot \sum_{\sigma \in S_5} \mathrm{sgn}(\sigma)\, e_\sigma = \sum_{\sigma \in S_5} \mathrm{sgn}(\sigma)\, e_{\tau\sigma} = \sum_{\sigma' \in S_5} \mathrm{sgn}(\tau^{-1}\sigma')\, e_{\sigma'} = \mathrm{sgn}(\tau) \sum_{\sigma' \in S_5} \mathrm{sgn}(\sigma')\, e'_\sigma \ , \tag{2.1.19}$$

where we have set $\sigma' = \tau\sigma$ in the middle step. For $\lambda = \ydiagram{2,1}$ with the canonical numbering, i.e., for

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \tag{2.1.20}$$

we have

$$c_{\ydiagram{2,1}} = (e_e + e_{(12)})(e_e - e_{(13)}) = e_e + e_{(12)} - e_{(13)} - e_{(132)} \in \mathbb{C}S_n \ . \tag{2.1.21}$$

Then $V_{\ydiagram{2,1}}$ is spanned by $\tilde{f}_1 \equiv c_{\ydiagram{2,1}}$ and $\tilde{f}_2 \equiv (13) \cdot c_{\ydiagram{2,1}}$, so $V_{\ydiagram{2,1}}$ is in fact the standard representation V of S_3. Indeed, using $(13)(12) = (123)$ and $(13)(132) = (23)$, we find

$$\tilde{f}_2 = e_{(13)} + e_{(123)} - e_e - e_{(23)} \ , \tag{2.1.22}$$

and then we have, with $(12)(13) = (132)$, $(12)(132) = (13)$, etc.

$$(12) \cdot \tilde{f}_1 = \tilde{f}_1 \ , \qquad (23) \cdot \tilde{f}_1 = e_{(23)} + e_{(132)} - e_{(123)} - e_{(12)} = -(\tilde{f}_1 + \tilde{f}_2) \tag{2.1.23}$$

and

$$(12) \cdot \tilde{f}_2 = e_{(23)} + e_{(132)} - e_{(123)} - e_{(12)} = -(\tilde{f}_1 + \tilde{f}_2) \ , \qquad (23) \cdot \tilde{f}_2 = \tilde{f}_2 \ , \tag{2.1.24}$$

thus reproducing, up to an overall sign, precisely (1.3.13). The overall sign can be undone by going to a different basis, i.e. by defining

$$f_1 = \tilde{f}_1 + 2\tilde{f}_2 \ , \qquad f_2 = -2\tilde{f}_1 - \tilde{f}_2 \ . \tag{2.1.25}$$

In terms of this basis we then have

$$\begin{array}{llll} (12)\, f_1 & = -f_1 \ , & (23)\, f_1 & = f_1 + f_2 \\ (12)\, f_2 & = f_1 + f_2 \ , & (23)\, f_2 & = -f_2 \ , \end{array} \tag{2.1.26}$$

in complete agreement with (1.3.13).

It is also instructive to see what happens if we use the other allowed standard filling for the Young diagram, i.e.

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{2.1.27}$$

Then the corresponding $\hat{c}_\lambda$ equals

$$\hat{c}_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} = (e_e + e_{(13)})(e_e - e_{(12)}) = e_e + e_{(13)} - e_{(12)} - e_{(123)} \in \mathbb{C}S_n \ , \tag{2.1.28}$$

and the corresponding basis vectors can be taken to be

$$\hat{f}_1 = \hat{c}_\lambda = e_e + e_{(13)} - e_{(12)} - e_{(123)} \qquad \hat{f}_2 = (23)\cdot\hat{c}_\lambda = e_{(23)} + e_{(123)} - e_{(132)} - e_{(13)} \ , \tag{2.1.29}$$

on which we have the actions

$$(12)\cdot\hat{f}_1 = -(\hat{f}_1 + \hat{f}_2) \ , \qquad (23)\cdot\hat{f}_1 = \hat{f}_2 \ , \tag{2.1.30}$$

as well as

$$(12)\cdot\hat{f}_2 = \hat{f}_2 \ , \qquad (23)\cdot\hat{f}_2 = \hat{f}_1 \ , \tag{2.1.31}$$

Then we get the standard form of the representation by defining

$$f_1 = 2\hat{f}_1 + \hat{f}_2 \ , \qquad f_2 = \hat{f}_2 - \hat{f}_1 \ . \tag{2.1.32}$$

Thus, in particular, c_λ and $\hat{c}_\lambda$ define equivalent representations, cf. footnote 2.

2.2 Frobenius formula

Next we want to describe an elegant formula for the character of the irreducible representation corresponding to the Young diagram λ. Let us introduce independent variables $x_1, \ldots, x_k$, with k at least as large as the number of rows of the Young diagram of λ. We want to evaluate the character on the conjugacy class labelled by $C_{\mathbf{i}}$ with $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, where i_j is the number of cycles of length j.

Let us define the power sums $P_j(x)$ for $1 \le j \le n$, and the discriminant $\Delta(x)$ by

$$P_j(\mathbf{x}) = x_1^j + x_2^j + \cdots + x_k^j \tag{2.2.1}$$

$$\Delta(\mathbf{x}) = \prod_{i<j} (x_i - x_j) \ . \tag{2.2.2}$$

If $f(\mathbf{x}) = f(x_1, \dots, x_k)$ is a formal power series, and $(l_1, \dots, l_k)$ is a k-tuple of non-negative integers, let us denote the corresponding Fourier coefficient by

$$[f(\mathbf{x})]_{(l_1,\dots,l_k)} = \text{coefficient of } x_1^{l_1} \cdots x_k^{l_k} \text{ in } f(\mathbf{x}) \ . \tag{2.2.3}$$

Given a partition λ of n with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, we define a strictly decreasing sequence of k non-negative integers by

$$l_j = \lambda_j + k - j \ , \tag{2.2.4}$$

i.e. $l_1 = \lambda_1 + k - 1 > l_2 = \lambda_2 + k - 2$, ..., finally leading to $l_k = \lambda_k \geq 0$. The character of the irreducible representation associated to λ evaluated on the conjugacy class $C_{\mathbf{i}}$ is now given by the remarkable formula

$$\chi_\lambda(C_{\mathbf{i}}) = \Big[\Delta(\mathbf{x}) \cdot \prod_{j=1}^{n} P_j(\mathbf{x})^{i_j}\Big]_{(l_1,\dots,l_k)} \ . \tag{2.2.5}$$

This is the famous **Frobenius formula**.

We shall not prove this formula in this lecture; a proof can, for example, be found in [FH], section 4.3. However, let us illustrate it with some simple examples. For example, for the case of S_3, the standard (2-dimensional) representation is described by the partition $\lambda_1 = 2$, $\lambda_2 = 1$ with $k = 2$. Then the corresponding l-values are $l_1 = 3$, $l_2 = 1$, and for the character of the conjugacy class containing the identity, i.e. $\mathbf{i} = (3, 0, 0)$, we get

$$\chi_{\yng(2,1)}(e) = \Big[(x_1 - x_2)(x_1 + x_2)^3\Big]_{(3,1)} = 2 \ , \tag{2.2.6}$$

in agreeement with the dimension of the standard representation. Similarly, for the conjugacy class containing the transpositions, say (12) — this corresponds to $\mathbf{i} = (1, 1, 0)$ — we have instead

$$\chi_{\yng(2,1)}\big((12)\big) = \Big[(x_1 - x_2)(x_1 + x_2)(x_1^2 + x_2^2)\Big]_{(3,1)} = \Big[x_1^4 - x_2^4\Big]_{(3,1)} = 0 \ , \tag{2.2.7}$$

and for the conjugacy class containing the cyclic permutation (123), i.e. $\mathbf{i} = (0, 0, 1)$ we have

$$\chi_{\yng(2,1)}\big((123)\big) = \Big[(x_1 - x_2)(x_1^3 + x_2^3)\Big]_{(3,1)} = -1 \ . \tag{2.2.8}$$

These results then reproduce exactly the entries of the character table, see Table 1.

Let us use the Frobenius formula to compute the **dimension** of the irreducible representation V_λ associated to λ. The conjugacy class of the identity always corresponds to $\mathbf{i} = (n)$, so that we have the formula

$$\dim(V_\lambda) = \chi_\lambda\big((n)\big) = \Big[\Delta(\mathbf{x}) \cdot (x_1 + \cdots + x_k)^n\Big]_{(l_1,\dots,l_k)} \ . \tag{2.2.9}$$

The discriminant equals the Vandermonde determinant

$$\Delta(\mathbf{x}) = \begin{vmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{vmatrix} = \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma)\, x_k^{\sigma(1)-1} \cdots x_1^{\sigma(k)-1} \ . \tag{2.2.10}$$

The other term equals

$$(x_1 + \cdots + x_k)^n = \sum_{r_1,\ldots,r_k} \frac{n!}{r_1! \cdots r_k!}\, x_1^{r_1}\, x_2^{r_2} \cdots x_k^{r_k} \ , \tag{2.2.11}$$

where the sum runs over all k-tuples $(r_1, \ldots, r_k)$ such that $\sum_j r_j = n$. To find the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ in the product, we pick, for each $\sigma \in S_k$, the relevant term from the second expression, i.e. we get

$$\dim(V_\lambda) = \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) \frac{n!}{(l_1 - \sigma(k) + 1)! \cdots (l_i - \sigma(k - i + 1) + 1)! \cdots (l_k - \sigma(1) + 1)!} \ , \tag{2.2.12}$$

where the sum only runs over those σ for which $l_{k-i+1} - \sigma(i) + 1 \geq 0$ for all $1 \leq i \leq k$. Note that both (2.2.10) and (2.2.11) are homogeneous polynomials of degree $\frac{k(k-1)}{2}$ and n, respectively, and their product therefore has degree

$$d = \frac{k(k-1)}{2} + n = \sum_{i=1}^{k} l_i \ . \tag{2.2.13}$$

Thus the only condition we have to worry about is whether the r_i are all non-negative, which is precisely the condition $l_{k-i+1} - \sigma(i) + 1 \geq 0$.
Next we observe that we can rewrite (2.2.12) as

$$\begin{aligned} \dim(V_\lambda) &= \frac{n!}{l_1! \cdots l_k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) \prod_{j=1}^{k} l_j\, (l_j - 1) \cdots (l_j - \sigma(k - j + 1) + 2) \\ &= \frac{n!}{l_1! \cdots l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k - 1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1 - 1) & \cdots \end{vmatrix} \ . \end{aligned} \tag{2.2.14}$$

By column reduction this determinant reduces to the Vandermonde determinant, so we get finally

$$\dim(V_\lambda) = \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j) \ , \tag{2.2.15}$$

where $l_i = \lambda_i + k - i$ for $i = 1, \ldots, k$.

For example, for the trivial representation we have $\lambda_1 = n$ with $k = 1$, and hence $l_1 = n$. Then (2.2.15) becomes

$$\dim(V_{(n)}) = \frac{n!}{n!} = 1 \ , \tag{2.2.16}$$

as expected. Similarly, for the alternating representation with have $\lambda_1 = \cdots = \lambda_n = 1$ with $k = n$. Then $l_i = n + 1 - i$, and hence (2.2.15) becomes

$$\dim(V_{(1\cdots 1)}) = n! \prod_{j=1}^{n} \frac{1}{j!} \ \prod_{i<j} (j - i) = 1 \ , \tag{2.2.17}$$

again as expected. Here the last product term equals

$$\prod_{i<j} (j - i) = (n-1)\,(n-2)^2 \,\cdots 2^{n-2}\, 1^{n-1} = \prod_{j=1}^{n-1} j! \ . \tag{2.2.18}$$

There is another useful way of expressing the dimension of the V_λ representation, the so-called **hook formula**. We define the hook length of a box in a Young diagram to be the number of squares directly below or directly to the right of the box, including the box once. So if we denote by r_i and c_j the row and column lengths, then the hook length of the box at position (i, j) equals

$$h(i,j) = r_i + c_j - (i + j - 1) \ . \tag{2.2.19}$$

If we label each box of the Young diagram by its hook length, we get for example

$$\begin{array}{|c|c|c|c|c|c|} \hline 9 & 8 & 7 & 5 & 4 & 1 \\ \hline 7 & 6 & 5 & 3 & 2 \\ \cline{1-5} 6 & 5 & 4 & 2 & 1 \\ \cline{1-5} 3 & 2 & 1 \\ \cline{1-3} \end{array} \tag{2.2.20}$$

Then the dimension of the representation associated to λ equals

$$\dim(V_\lambda) = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)} \ , \tag{2.2.21}$$

where the product runs over all boxes of the Young diagram λ. So for the example above (2.2.20) — this describes a representation of S_{19} — we get

$$\begin{aligned} \dim\big(V_{(6,5,5,3)}\big) &= \frac{19!}{9\cdot 8\cdot 7\cdot 5\cdot 4\cdot 1\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2\cdot 6\cdot 5\cdot 4\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1} \\ &= 19\cdot 17\cdot 16\cdot 13\cdot 11\cdot 9 = 6'651'216 \ . \end{aligned} \tag{2.2.22}$$

Note that for the trivial and alternating representations, the hook lengths are (for the case of S_5)

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 4 & 3 & 2 & 1 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{|c|}\hline 5 \\ \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \tag{2.2.23}$$

and thus the hook length formula gives indeed in each case again 1. On the other hand, for the regular representation of S_3 we get

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \tag{2.2.24}$$

and hence we produce

$$\dim(V_{\text{reg}}) = \frac{3!}{3 \cdot 1 \cdot 1} = 2 \ . \tag{2.2.25}$$

We remark that it is immediate from the hook length formula that the dimension of the representation associated to the Young diagram λ and the conjugate Young diagram, λ', is the same.

Finally, there is another way of describing the dimension of the representation corresponding to λ: it is the number of standard Young tableaux with shape λ. This is to say, it is the number of fillings of the Young diagram λ by the integers $\{1, \ldots, n\}$ such that the entries increase across each row and down each column. Again, for the trivial and alternating representation there is obvioulsy only one standard Young tableau, while for the regular representation of S_3 there are two — these are the two fillings

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \text{and} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \tag{2.2.26}$$

thus reflecting that the corresponding dimension is indeed $\dim(V_{\text{reg}}) = 2$.

2.3 Equivalent particles

As an application of these ideas let us consider a (quantum mechanical) system of n equivalent particles. [Here by the term 'equivalent' we mean that the Hamiltonian of the system is invariant under the interchange of the coordinates of all of these particles.] Suppose $\psi(x_1, \ldots, x_n)$ is an eigenfunction of the Hamiltonian, then any of the $n!$ permutations applied to $\psi(x_1, \ldots, x_n)$ will also define an eigenfunction (with the same energy eigenvalue).

If the particles are indistinguishable bosons or indistinguishable fermions, then we know that the relevant physical wave function is the totally symmetric or totally anti-symmetric wave-function, i.e. the wave-function we obtain from $\psi(x_1, \ldots, x_n)$ upon summing over all permutations as

$$\Psi^{\epsilon}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (\text{sgn}(\sigma))^{\epsilon} \, \psi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \ , \tag{2.3.27}$$

where $\epsilon = 0$ corresponds to the bosonic and $\epsilon = 1$ to the fermionic case.

Obviously, these two cases correspond precisely to the two 1-dimensional representations of the symmetric group, the trivial and the alternating representation. However, as we have seen above, the symmetric group also has other irreducible representations, and we may contemplate the idea that our particles are neither bosons nor fermions, but transform in one of these other irreducible representations of the symmetric group. In this case one says that the particles obey *parastatistics*. While it is now believed that, at least in $3+1$ spacetime dimensions, all fundamental particles are either bosons or fermions — for the lower-dimensional case, there are exceptions having to do with the braid group, see section 2.4 — parastatistics may very well occur for quasiparticles that appear in some effective description.

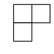
With the technology we have developed above, it is now easy to find a suitable basis for the wave-functions that transforms in a specific irreducible representation of the permutation group. To this end we simply apply the Young symmetriser c_λ to the wave-function — recall that the Young symmetriser, defined in eq. (2.1.14), is an element of the group algebra, and hence it makes perfect sense to apply it to the wave-function as above. In particular, we recover the totally symmetric or totally antisymmetric wave-function for the trivial and the alternating representation, respectively, see eqs. (2.1.16) and eqs. (2.1.18).

There is only one point that requires further comment: the way we introduced the Young symmetriser in eq. (2.1.14), it was associated to a Young diagram, but as is implicit from the construction there, it is actually associated to a standard Young tableau, i.e. to a Young diagram together with a choice of a standard filling. Now, as we have mentioned at the end of the previous section, the number of standard fillings is actually equal to the dimension of the corresponding representation of the symmetric group. And furthermore, since there are $n! = |S_n|$ different orderings for the variables $(x_1, \ldots, x_n)$, among the $n!$ functions, a given irreducible representation R of the symmetric group will appear $\dim(R)$ times — this is just the identity

$$n! = |S_n| = \sum_i \dim(R_i)^2 \ , \qquad (2.3.28)$$

where the sum runs over all irreducible representations of S_n.

So now it should be clear how to choose a suitable basis of functions: for each standard Young tableau we apply the corresponding Young symmetriser to the wave-function, and each of the resulting wave-functions will generate (upon the action of the symmetric group from the left) an irreducible representation of the symmetric group.

So for example, for the case of 3 particles, we have in addition to the totally symmetric and totally antisymmetric wave-function, the two wave-functions associated to the 2-dimensional regular representation with Young diagram ; if we use the 'canonical' standard filling we get, see eq. (2.1.21),

$$\Psi^{(1)} = \psi(x_1, x_2, x_3) + \psi(x_2, x_1, x_3) - \psi(x_3, x_2, x_1) - \psi(x_3, x_1, x_2) \qquad (2.3.29)$$

while for the other standard filling, see eq. (2.1.28), we get instead

$$\Psi^{(2)} = \psi(x_1, x_2, x_3) + \psi(x_3, x_2, x_1) - \psi(x_2, x_1, x_3) - \psi(x_2, x_3, x_1) \ . \tag{2.3.30}$$

Each of these functions generates a 2-dimensional representation of the symmetric group; for example, applying the (23) permutation to $\Psi^{(1)}$ we obtain

$$\Psi^{(1)'} = \psi(x_1, x_3, x_2) + \psi(x_3, x_1, x_2) - \psi(x_2, x_3, x_1) - \psi(x_2, x_1, x_3) \tag{2.3.31}$$

while applying the (23) permutation to $\Psi^{(2)}$ leads to

$$\Psi^{(2)'} = \psi(x_1, x_3, x_2) + \psi(x_2, x_3, x_1) - \psi(x_3, x_1, x_2) - \psi(x_3, x_2, x_1) \ . \tag{2.3.32}$$

The space of functions generated by $\Psi^{(1)}$ and $\Psi^{(1)'}$ forms an irreducible representation of the symmetric group, where the permutations act on the left via

$$(\sigma \cdot \Psi)\,(x_1, x_2, x_3) = \Psi(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \ . \tag{2.3.33}$$

For example,

$$\begin{aligned} ((13) \cdot \Psi^{(1)})\,(x_1, x_2, x_3) &= \psi(x_3, x_2, x_1) + \psi(x_2, x_3, x_1) - \psi(x_1, x_2, x_3) - \psi(x_1, x_3, x_2) \\ &= -\Psi^{(1)}(x_1, x_2, x_3) - \Psi^{(1)'}(x_1, x_2, x_3) \ , \end{aligned} \tag{2.3.34}$$

etc. Similarly, the space of functions generated by $\Psi^{(2)}$ and $\Psi^{(2)'}$ also forms an irreducible representation of the symmetric group (that is equivalent to the above). Furthermore, these 4 functions, together with the totally symmetric and totally antisymmetric wave-functions, span the full 6-dimensional space of all wave-functions. (We are assuming here that the initial wave-function is sufficiently generic — obviously, it may be that some of these linear combinations vanish identically, e.g. if the wave-function $\psi(x_1, \ldots x_n)$ is of a simple product form, say $\psi(x_1, \ldots, x_n) = \chi(x_1) \cdots \chi(x_n)$, all but the totally symmetric wave-function will vanish.)

2.4 Braid group statistics

As was alluded to above, the symmetric group is only expected to characterise the behaviour of identical particles in $3+1$ dimensions, but more general possibilities appear in lower dimensions. One way to motivate this is to consider the classical configuration space of identical particles. Suppose we have n identical particles moving in d space-dimensions, then the classical configuration space is

$$\mathcal{M}_{d,n} = \Big(\big\{ (\mathbf{x}_1, \ldots, \mathbf{x}_n) \, : \, \mathbf{x}_j \in \mathbb{R}^d \big\} - \big\{ (\mathbf{x}_1, \ldots, \mathbf{x}_n) \, : \, \mathbf{x}_i = \mathbf{x}_j \text{ for some } i \neq j \big\} \Big) / S_n \ , \tag{2.4.35}$$

where the symmetric group acts in the obvious manner, i.e. for $\sigma \in S_n$ the action is

$$\sigma(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(n)}) \ . \tag{2.4.36}$$

(Note that we have to remove the configurations where two points coincide in order to have a fixed-point free action of S_n.)

At each moment in time, the system of n identical particles is described by a single point in $\mathcal{M}_{d,n}$, and thus, as a function of time, the whole system is described by a curve in the configuration space $\mathcal{M}_{d,n}$. Now suppose that, after some time T, the system returns to the *same* point $\mathbf{X}$ in the configuration space, i.e. to a configuration where the n points are at the same positions (but may have interchanged places). If we can deform (within the configuration space) this curve to the curve where the system was at the position $\mathbf{X}$ all along, then we would expect the wave-function to be identically the same as before. However, if the two curves — the actual time evolution of the system and the constant curve — cannot be deformed into one another, the two wave-functions may differ, e.g. by a phase.

Obviously, a necessary condition for the curve to be 'trivial' in this sense is that the particles haven't changed position. For $d \geq 3$ this is also a sufficient condition, but this is not true in lower dimensions. More precisely, the question of which curves are deformable into one another is precisely what homotopy theory describes, and the relevant homotopy group is, for $d \geq 3$, simply described by the permutation group

$$\pi_1(\mathcal{M}_{d,n}) = S_n \ . \tag{2.4.37}$$

Thus, for $d \geq 3$, the 'statistics' of particles is characterised by a representation of the symmetric group.

However, for $d = 2$ something more interesting happens. Consider n points in the plane ($d = 2$), which we take to be the $x - y$ plane, and draw their time-evolution along the z-axis. Then the time evolution of the whole system is described by a configuration of threads, one for each of the n identical particles. Suppose that after time T, the first two particles have interchanged position (while the others have been far away and fixed). Now there are two ways in which this can happen, depending on which way the two particles have gone round each other. A more formal way of saying this is that the relevant homotopy group is, for $d = 2$,

$$\pi_1(\mathcal{M}_{2,n}) = B_n \tag{2.4.38}$$

the braid group. Thus 'particles' in $d = 2$ are characterised not by a representation of the symmetric group, but rather by a representation of the braid group.

The braid group has infinite order (and is hence much more complicated than the symmetric group), but it has a very simple description in terms of generators and relations. In order to motivate this, recall that we may describe the symmetric group to be generated by the elements $\sigma_i \equiv (i\, i+1)$, $i = 1, \ldots n-1$, subject to the relations

$$\sigma_i^2 = e \ , \qquad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \ , \tag{2.4.39}$$

as well as

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ , \quad i = 1, \ldots, n-2 \ . \tag{2.4.40}$$

Now the braid group has a very similar presentation: it is generated by the elements b_i, $i = 1, \ldots, n-1$, describing the clockwise interchange of threads i and $i+1$. The relevant relations are the same as those for the permutation group, except that the first relation is missing

$$b_i b_j = b_j b_i \quad \text{if } |i-j| \geq 2 \ , \qquad\qquad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \ , \quad i = 1, \ldots, n-2 \ . \tag{2.4.41}$$

It is not hard to see that these relations are true for geometric braids (and that b_i^2 isn't the identity.) As a consequence the braid group has infinite order. Another nice consequence of this is that there are not just two 1-dimensional representations of the braid group, but rather a whole continuum of them. Indeed the assignment

$$b_i = e^{i\theta} \tag{2.4.42}$$

defines a representation of the braid group for any value of θ. Note that the two values $\theta = 0$ and $\theta = \pi$ correspond to the trivial and alternating representation of the symmetric group — these are the only values for which $b_i^2 = 1$ — but that for the braid group any phase θ is allowed. Thus in 2 dimensions there are not just bosons and fermions, but also *anyons*. It is believed that anyonic statistics characterises certain quasiparticles in the quantum Hall effect (which is effectively a 2-dimensional system).

In the recent past also particles that have non-abelian braid group statistics, i.e. that transform in a higher dimensional irreducible representation of the braid group and hence exhibit the analogue of parastatistics, have attracted a lot of attention. In particular, particles with this property would allow one to construct what is called a 'topological quantum computer'. There is some chance that the $\nu = \frac{5}{2}$ state in the fractional quantum Hall effect may be described by such a quasiparticle.

3 Lie groups and Lie algebras

In the second half of this lecture course we shall explain the basics of Lie theory. Lie groups are groups that are at the same time differentiable manifolds such that the group operations (i.e. the group multiplication as well as the map that sends a group element to its inverse) are smooth — typical examples are the matrix groups $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$, etc. Thus Lie groups seem to stand at the opposite end of the spectrum of groups from finite ones. In particular, they are of uncountable order, and for example it is impossible to define them in terms of generators and relations. As such they seem enormously complicated. However, because of the additional data of a manifold structure, it is nevertheless possible to study them quite easily in detail.

Lie groups represent a confluence of algebra and geometry, which accounts perhaps in part for their importance in modern mathematics; it also makes their analysis somewhat intimidating. Happily, because the algebra and the geometry of a Lie group are closely entwined, there is an object we can use to approach the study of Lie groups that extracts much of the structure of a Lie group (primarily its algebraic structure) while seemingly getting rid of the topological complexity. This is, of course, the **Lie algebra**. The Lie algebra is, at least according to its definition, a purely algebraic object, consisting simply of a vector space with a bilinear operation; and so it might appear that in associating to a Lie group its Lie algebra we are necessarily giving up a lot of information about the group. This is, in fact, not the case: as we shall see in many cases (and perhaps all of the most important ones), encoded in the algebraic structure of a Lie algebra is almost all the geometry of the group. In particular, there is a very close relationship between representations of the Lie group we start with, and representations of the Lie algebra we associate to it.

3.1 From the Lie group to the Lie algebra

Let G be a Lie group. We consider the set of curves $R(t)$, where t is a real parameter $t \in (-a, a)$ for some $a \in \mathbb{R}^+$, and each $R(t) \in G$. Let us assume that $R(0) = e$, the identity element of the group G. The tangent space of the identity contains then the derivatives

$$\Omega = \frac{d}{dt} R(t)\Big|_{t=0} . \tag{3.1.1}$$

[If G is a matrix group, then $R(t)$ is a matrix for each $t \in (-a, a)$; then the derivative is just the matrix consisting of the derivatives of each matrix element separately. In the general case of an abstract Lie group, the derivative can also be defined — this can be done for any map between differentiable manifolds — but for our purposes it will be sufficient to think of the derivative for the case of matrix groups.]

The set of all derivatives of the above class of curves defines a real vector space since

$$\alpha_1 \Omega_1 + \alpha_2 \Omega_2 = \frac{d}{dt} R_1(\alpha_1 t) R_2(\alpha_2 t)\Big|_{t=0} , \tag{3.1.2}$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$. This is the vector space underlying the Lie algebra, and we shall denote it by $\mathfrak{g}$. We shall use the convention that Lie groups will be denoted by capital letters, while the associated Lie algebras are denoted by the corresponding lower case letter (in $\mathfrak{mathfrak}$ font.)

The Lie algebra $\mathfrak{g}$ carries a representation of the Lie group G, since for any $R \in G$, we have the action

$$R\,\Omega_2 R^{-1} = \frac{d}{dt} R R_2(t) R^{-1}\Big|_{t=0} \;, \tag{3.1.3}$$

where $\Omega_2 = \frac{d}{dt}R_2(t)\big|_{t=0} \in \mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, it then follows that also

$$[\Omega_1, \Omega_2] \equiv \frac{d}{dt} R_1(t)\Omega_2 R_1(t)^{-1}\Big|_{t=0} \in \mathfrak{g} \;, \tag{3.1.4}$$

where $\Omega_1 = \frac{d}{dt}R_1(t)\big|_{t=0}$. Thus the Lie algebra $\mathfrak{g}$ is not just a vector space, but also possesses a bilinear product, the **Lie bracket** $[\cdot,\cdot]$. If we think of G as a matrix group, then we can use the product rule to rewrite the commutator as

$$[\Omega_1, \Omega_2] = \Omega_1\,\Omega_2 - \Omega_2\,\Omega_1 \;, \tag{3.1.5}$$

where we have used that $R_1(0) = e$, as well as

$$0 = \frac{d}{dt}\big(R_1(t)\,R_1(t)^{-1}\big)\Big|_{t=0} = \Omega_1 + \frac{d}{dt}\big(R_1(t)^{-1}\big)\Big|_{t=0} \quad\Longrightarrow\quad \frac{d}{dt}\big(R_1(t)^{-1}\big)\Big|_{t=0} = -\Omega_1 \;. \tag{3.1.6}$$

Thus the commutator of two Lie algebra generators is indeed the ‘commutator’; in particular it is anti-symmetric,

$$[\Omega_1, \Omega_2] = -[\Omega_2, \Omega_1] \;, \tag{3.1.7}$$

and it satisfies the Jacobi identity

$$\big[\Omega_1, [\Omega_2, \Omega_3]\big] + \big[\Omega_2, [\Omega_3, \Omega_1]\big] + \big[\Omega_3, [\Omega_1, \Omega_2]\big] = 0 \;, \tag{3.1.8}$$

as follows directly by plugging in (3.1.5).

As an example, let us consider the group $G \equiv \mathrm{SO}(3)$, i.e. the group of real orthogonal 3×3 matrices. Since each group element $R \in G$ is orthogonal, $R^T \cdot R = e$, it follows that

$$0 = \frac{d}{dt}\big(R(t)^T\,R(t)\big)\Big|_{t=0} = \Omega^T + \Omega \quad\Longrightarrow\quad \Omega^T = -\Omega \;, \tag{3.1.9}$$

i.e. the corresponding Lie algebra $\mathfrak{so}(3)$ consists of the real anti-symmetric 3×3 matrices. Every such matrix is of the form

$$\Omega(\vec{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \;, \tag{3.1.10}$$

and thus a basis for the corresponding vector space is given by $\Omega_i = \Omega(e_i)$, $i = 1, 2, 3$. One easily calculates that the Lie bracket has the form

$$[\Omega_1, \Omega_2] = \Omega_3 \qquad \text{(plus cyclic.)} \tag{3.1.11}$$

In particular, the commutator of two Lie algebra elements in $\mathfrak{so}(3)$ is again an element of the Lie algebra $\mathfrak{so}(3)$.

Up to now we have explained how to obtain the Lie algebra associated to a Lie group. However, as we have mentioned before, the Lie algebra encodes much of the structure of the Lie group. Indeed, we can associate to every element of the Lie algebra an element of the Lie group by the exponential map, that can (again for matrix groups) be explicitly defined by

$$\exp : \mathfrak{g} \to G \ , \qquad \Omega \mapsto e^{\Omega} = \mathbf{1} + \Omega + \frac{\Omega^2}{2!} + \frac{\Omega^3}{3!} + \cdots \ . \tag{3.1.12}$$

Here the products in the definition of the exponential map are just the usual matrix products, and it is clear that the infinite sum converges. The result is an invertible matrix since $e^{-\Omega}$ is the inverse matrix to e^{Ω}. Furthermore, Ω is the Lie algebra element associated to the curve $R(t) = e^{t\Omega}$. Finally, the product structure of the Lie group can be reconstructed from the Lie algebra since we have

$$\exp(\Omega_1) \cdot \exp(\Omega_2) = \exp(\Omega_1 \star \Omega_2) \ , \tag{3.1.13}$$

where $\Omega_1 \star \Omega_2$ is determined by the Baker-Campbell-Hausdorff formula (**Exercise**)

$$\Omega_1 \star \Omega_2 = \Omega_1 + \Omega_2 + \tfrac{1}{2}[\Omega_1, \Omega_2] + \tfrac{1}{12}[\Omega_1, [\Omega_1, \Omega_2]] + \tfrac{1}{12}[\Omega_2, [\Omega_2, \Omega_1]] + \cdots \ . \tag{3.1.14}$$

Because of the exponential map, much of the structure of the Lie group is captured by the Lie algebra, and instead of studying representations of the Lie group, we may analyse representations of the Lie algebra. This simplifies things enormously since the Lie algebra is a linear vector space (whereas the Lie group is in general a curved manifold).

3.1.1 Representations

We have already learned what a representation of a (finite) group is: it is a map ρ from the group G to the endomorphisms of some vector space V such that

$$\rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2) \ , \qquad \rho(e) = \mathbf{1} \ , \tag{3.1.15}$$

where $\circ$ denots the composition of endomorphisms, and $\mathbf{1}$ is the identity map on V. This definition applies equally well to Lie groups. Again, we shall only study finite-dimensional representations of the Lie group G, i.e. we shall take V to be a finite-dimensional vector space.

Given a representation of the Lie group, it induces a map from the Lie algebra $\mathfrak{g}$ to the endomorphisms of V as well. Indeed, we simply define

$$\rho(\Omega) = \frac{d}{dt}\rho(R(t))\bigg|_{t=0} \ , \tag{3.1.16}$$

where $\Omega = \frac{d}{dt}R(t)\big|_{t=0}$. It then follows from (3.1.4) that

$$\rho\left([\Omega_1, \Omega_2]\right) = \frac{d}{dt}\,\rho(R_1(t))\,\rho(\Omega_2)\,\rho(R_1(t)^{-1})\Big|_{t=0} = [\rho(\Omega_1), \rho(\Omega_2)]\ . \tag{3.1.17}$$

Thus we shall call a **representation of the Lie algebra** $\mathfrak{g}$ a map ρ from $\mathfrak{g}$ to the endomorphisms of V, $\rho : \mathfrak{g} \to \mathrm{End}(V)$, such that

$$\rho\left([\Omega_1, \Omega_2]\right) = [\rho(\Omega_1), \rho(\Omega_2)]\ . \tag{3.1.18}$$

With this definition any representation of a Lie group G gives rise to representation of the associated Lie algebra $\mathfrak{g}$.

Every Lie algebra $\mathfrak{g}$ possesses the adjoint representation, where the underlying vector space is $\mathfrak{g}$ itself, and the Lie algebra action ρ is defined via

$$\rho(t_1)\,(t_2) := [t_1, t_2]\ . \tag{3.1.19}$$

This satisfies the defining property of a Lie algebra representation because of the Jacobi identity,

$$\rho\left([t_1, t_2]\right)\,(t_3) = \big[[t_1, t_2], t_3\big] = \big[t_1, [t_2, t_3]\big] - \big[t_2, [t_1, t_3]\big] = \big[\rho(t_1), \rho(t_2)\big]\,(t_3)\ , \tag{3.1.20}$$

where we have used the Jacobi identity in the second step

$$0 = \big[[t_1, t_2], t_3\big] + \big[[t_2, t_3], t_1\big] + \big[[t_3, t_1], t_2\big] = \big[[t_1, t_2], t_3\big] - \big[t_1, [t_2, t_3]\big] + \big[t_2, [t_1, t_3]\big]\ . \tag{3.1.21}$$

Note that this is just the Lie algebra version of the group representation (3.1.3).

In the following we shall mainly concentrate on the Lie algebra $\mathfrak{g}$ and its representations; as we shall see later, we can effectively reconstruct the representations of G uniquely from those of $\mathfrak{g}$.

3.2 Lie algebras

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. We introduce a basis for the vector space underlying $\mathfrak{g}$ to consist of the vectors t_α, where $\alpha = 1, \ldots, \dim(\mathfrak{g})$. Then we can write the commutators as

$$[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma\, t_\gamma\ . \tag{3.2.1}$$

We call the numbers $f_{\alpha\beta}{}^\gamma$ the **structure constants** of the Lie algebra.

We call a Lie algebra **abelian** if all structure constants vanish, $f_{\alpha\beta}{}^\gamma = 0$; then all generators of the Lie algebra commute with one another. Furthermore, we say that $\mathfrak{h} \subset \mathfrak{g}$ is a **Lie subalgebra**, if $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, and if the commutators of any two elements in $\mathfrak{h}$ lie again in $\mathfrak{h}$, i.e. schematically if

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}\ . \tag{3.2.2}$$

Finally, we call $\mathfrak{h} \subset \mathfrak{g}$ an **invariant Lie subalgebra**, if $\mathfrak{h}$ is a Lie subalgebra and satisfies in addition that

$$[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h} \ . \tag{3.2.3}$$

i.e. any commutator involving an element from $\mathfrak{h}$ with any element from $\mathfrak{g}$ lies in $\mathfrak{h}$.

In this course we shall mainly concentrate on compact Lie algebras, i.e. Lie algebras that are associated to compact Lie groups. Compact Lie groups are the natural analogues of finite groups since for them all representations are completely reducible; in fact, the argument is essentially the same as the one given in section 1.3, the only difference being that the sum over the entire finite group in (1.3.1) is replaced by the integral over the compact group (which also converges). The situation for non-compact groups is much more complicated, and we shall not attempt to describe it here.

We shall furthermore exclude abelian factors; more precisely, we shall consider only **semi-simple** Lie algebras. We call a Lie algebra $\mathfrak{g}$ **semi-simple** if it contains no abelian invariant Lie subalgebra. Note that this definition does not exclude that $\mathfrak{g}$ contains a non-abelian invariant Lie subalgebra $\mathfrak{h}$. However, if $\mathfrak{g}$ is in addition compact (which in particular implies that all representations are completely reducible) then we can decompose the adjoint representation as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}' \ , \tag{3.2.4}$$

where

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h} \ , \qquad [\mathfrak{g}, \mathfrak{h}'] \subseteq \mathfrak{h}' \ . \tag{3.2.5}$$

Because of (3.2.4) it then follows that $[\mathfrak{h}, \mathfrak{h}'] \subseteq \mathfrak{h} \cap \mathfrak{h}' = 0$, and thus (3.2.4) decomposes $\mathfrak{g}$ into a direct sum of commuting Lie subalgebras. Proceeding recursively in this manner, we can therefore show that every compact semi-simple Lie algebra is a direct sum of pairwise commuting simple Lie algebras, where a **simple** Lie algebra is characterised by the property that it does not contain any invariant Lie subalgebra.

In the following we shall therefore restrict our attention to the analysis of compact simple Lie algebras; typical examples are the Lie algebras $\mathfrak{su}(N)$ associated to the groups $\mathrm{SU}(N)$, or the Lie algebras $\mathfrak{so}(N)$ associated to the groups $\mathrm{SO}(N)$. (**Exercise**: Find a basis for the Lie algebras $\mathfrak{so}(N)$ and determine their dimension.)

3.2.1 Killing Form

An important concept in the theory of Lie algebras is the Killing form. Suppose (V, ρ) is a representation of a Lie algebra $\mathfrak{g}$. We define the **Killing form** of $\mathfrak{g}$ via

$$\bar{g}_{\alpha\beta} \equiv \bar{g}(t_\alpha, t_\beta) = \mathrm{Tr}_V\big(\rho(t_\alpha)\,\rho(t_\beta)\big) \ . \tag{3.2.6}$$

For example, in the adjoint representation we have

$$g_{\alpha\beta} \equiv g(t_\alpha, t_\beta) = f_{\alpha\delta}{}^{\gamma}\, f_{\beta\gamma}{}^{\delta} \ , \tag{3.2.7}$$

where the sum over γ and δ is implicit.

Many of the structural properties of a Lie algebra can be read off from the Killing form. For example, a Lie algebra is *semi-simple* if the Killing form in the adjoint representation $g_{\alpha\beta}$, (3.2.7), is non-degenerate, i.e.

$$\det(g_{\alpha\beta}) \neq 0 \ . \tag{3.2.8}$$

Furthermore, a semi-simple Lie algebra is *compact* if the Killing form in the adjoint representation is negative definite.

It is sometimes awkward to work out the Killing form in the adjoint representation (since this may be quite large). However, for simple Lie algebras, the Killing form is the same in any representation, up to an overall proportionality factor.

The primary examples we shall consider in this course are the simple Lie algebras $\mathfrak{su}(N)$ associated to the compact group

$$\mathrm{SU}(N) = \left\{ M \in \mathrm{Mat}_N(\mathbb{C}) \ : \ M^\dagger = M^{-1} \text{ and } \det(M) = 1 \right\} \ . \tag{3.2.9}$$

(This group is indeed compact since all matrix entries must be in modulus less or equal than 1.) The condition that $\det(M) = 1$ implies that the corresponding Lie algebra element Ω is traceless, $\mathrm{Tr}(\Omega) = 0$, while $M^\dagger = M^{-1}$ implies that Ω is anti-hermitian, i.e. that $\Omega^\dagger = -\Omega$. Note that this last condition implies that the entries along the diagonal are purely imaginary, and the trace condition leads then to $N-1$ real parameters. The off-diagonal entries are complex, but $\Omega^\dagger = -\Omega$ determines the entries below the diagonal in terms of those above the diagonal; the total real dimension is therefore

$$\dim_{\mathbb{R}}(\mathfrak{su}(N)) = (N-1) + 2\frac{N(N-1)}{2} = (N-1)(N+1) = N^2 - 1 \ . \tag{3.2.10}$$

The simplest case arises for $\mathfrak{su}(2)$, in which case the Lie algebra is 3-dimensional. A basis may be taken to be given by the matrices

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ , \qquad M_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad \hat{M}_{12} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ . \tag{3.2.11}$$

The non-trivial commutation relations take the form

$$[D, M_{12}] = 2\hat{M}_{12} \ , \qquad [D, \hat{M}_{12}] = -2M_{12} \ , \qquad [M_{12}, \hat{M}_{12}] = 2D \ . \tag{3.2.12}$$

Note that this Lie algebra is precisely isomorphic to the Lie algebra of $\mathfrak{so}(3)$, defined above in (3.1.10) and (3.1.11), the identification being

$$\Omega_1 = \frac{1}{2}D \ , \qquad \Omega_2 = \frac{1}{2}M_{12} \ , \qquad \Omega_3 = \frac{1}{2}\hat{M}_{12} \ . \tag{3.2.13}$$

In the adjoint representation, the three generators are represented by the 3×3 matrices

$$D \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \ , \qquad M_{12} \cong \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \ , \qquad \hat{M}_{12} \cong \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{3.2.14}$$

where we take the basis to be given by $(D, M_{12}, \hat{M}_{12})$, in this order. Thus the Killing form, i.e. the trace of the products of these matrices, takes the form

$$g = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} . \tag{3.2.15}$$

The Killing form is indeed negative definite, thus reflecting that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ is a compact Lie algebra. Note that the definition of $\mathfrak{su}(2)$ in terms of 2×2 matrices in (3.2.11) implies that the Lie algebra possesses a 2-dimensional representation. We can therefore also evaluate the Killing form in this 2-dimensional representation, and we find that it equals

$$g_{2\mathrm{d}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{3.2.16}$$

Note that the two Killing forms are indeed proportional to one another, as must be the case for simple Lie algebras.

Another slightly more complicated example is provided by the Lie algebra $\mathfrak{su}(3)$. It is 8-dimensional, and a basis may be taken to be given by the matrices

$$D_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} , \tag{3.2.17}$$

as well as

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} , \tag{3.2.18}$$

and

$$\hat{M}_{12} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \hat{M}_{13} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \hat{M}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} . \tag{3.2.19}$$

One checks that, in the adjoint representation, the action of D_1 and D_2 is given by

$$D_1 : \; M_{12} \to 2\hat{M}_{12} , \quad \hat{M}_{12} \to -2M_{12} , \qquad D_2 : \; M_{12} \to -\hat{M}_{12} , \quad \hat{M}_{12} \to M_{12} , \tag{3.2.20}$$

$$D_1 : \; M_{13} \to \hat{M}_{13} , \quad \hat{M}_{13} \to -M_{13} , \qquad D_2 : \; M_{13} \to \hat{M}_{13} , \quad \hat{M}_{13} \to -M_{13} , \tag{3.2.21}$$

as well as

$$D_1 : \; M_{23} \to -\hat{M}_{23} , \quad \hat{M}_{23} \to M_{23} , \qquad D_2 : \; M_{23} \to 2\hat{M}_{23} , \quad \hat{M}_{23} \to -2M_{23} , \tag{3.2.22}$$

while D_1 and D_2 act trivially on D_i, $i = 1, 2$. Furthermore, $g(D_i, M_{jk}) = g(D_i, \hat{M}_{jk}) = 0$, and hence the Killing form, when restricted to D_1 and D_2, has the form

$$g(D_i, D_j) = \begin{pmatrix} -12 & 6 \\ 6 & -12 \end{pmatrix} \tag{3.2.23}$$

which is indeed negative definite.

Actually, writing the generators of $\mathfrak{su}(3)$ in terms of 3×3 matrices, automatically defines a 3-dimensional representation of $\mathfrak{su}(3)$. We can also evaluate the Killing form in that representation, and we find, again restricting attention to the generators D_1 and D_2,

$$\bar{g}(D_i, D_j) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} . \tag{3.2.24}$$

As before, the two Killing forms (at least when restricted to these generators) are proportional to one another, as must be the case for the simple Lie algebra $\mathfrak{su}(3)$.

3.2.2 Complexification

So far the Lie algebras we have discussed are real, i.e. the underlying vector space is a real vector space. (One should not get confused by the fact that some of the matrix entries are complex; the underlying vector space consists of real linear combinations of these generators, and the fact that the structure constants are real implies that the commutators lie again in the same real vector space.)

For much of the subsequent analysis of Lie algebras it is very convenient to consider the **complexification** of this Lie algebra, i.e. to consider the complex vector space with the same basis. Note that, upon complexification, different real Lie algebras can become the same complex Lie algebra. For example, consider the Lie algebra $\mathfrak{sl}(2)$ that is associated to the Lie group

$$\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathrm{Mat}_2(\mathbb{R}) \ : \ \det(A) = 1\} \ . \tag{3.2.25}$$

We note in passing that this Lie group is not compact since it contains the 'infinite' direction

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} , \qquad x \in \mathbb{R} \ . \tag{3.2.26}$$

The corresponding Lie algebra consists of the traceless real 2×2 matrices, for which we may take a basis to consist of

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{3.2.27}$$

The commutation relations are then

$$[H, E] = 2E \ , \qquad [H, F] = -2F \ , \qquad [E, F] = H \ . \tag{3.2.28}$$

If we complexify this Lie algebra, i.e. allow for complex linear combinations of the generators, then the Lie algebra is isomorphic to the complexification of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Indeed, we can write the generators of $\mathfrak{su}(2)$ as

$$D = iH \ , \qquad M_{12} = E - F \ , \qquad \hat{M}_{12} = i(E+F) \ . \tag{3.2.29}$$

However, the original real Lie algebras are not isomorphic, $\mathfrak{su}(2) \ncong \mathfrak{sl}(2)$. One way to see this is that in the adjoint representation of $\mathfrak{sl}(2)$ the generator H is diagonalisable with eigenvalues $0, \pm 2$, whereas in $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ the generators define rotations which are not diagonalisable over the reals.

Note that once we have complexified the Lie algebra, it does not make much sense any longer to talk about whether the Lie algebra is compact or not. (In particular, the condition whether the Killing form is negative definite is not invariant under multiplying the generators by complex numbers!) In fact, as we have just seen, a given complex Lie algebra can have (and actually will have) different 'real forms', i.e. different real Lie subalgebras, that may or may not be compact.

What is important for the following is that every complex semi-simple Lie algebra always possesses a real form that is compact. Thus, once we restrict our attention to semi-simple (and thus ultimately simple) complex Lie algebras, we may always think of them as being the complexification of a compact semi-simple (or simple) Lie algebra. In particular, complete decomposability and all the other nice properties will hold. Thus from now on we can just consider complex simple Lie algebras, and not worry any longer about their associated compact real form (although we know that it will exist, and we shall sometimes exhibit it).

4 Complex simple Lie algebras — the case of $\mathfrak{su}(2)$

In this chapter we want to begin and understand the structure of the complex simple Lie algebras. In particular, we want to introduce and motivate the so-called Cartan-Weyl basis, and explain the structure of the representation theory. Here we will be studying the familiar example of $\mathfrak{su}(2)$; the case of $\mathfrak{su}(3)$ will be covered in the following section, after which we turn to the generalisation to all simple Lie algebras.

4.1 The complexification of $\mathfrak{su}(2)$

Let us begin by studying the complexification of the simple Lie algebra $\mathfrak{su}(2)$. As we have seen above, this complexification is actually isomorphic to the complexification of the real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. In fact, it follows from the analysis of (3.2.27) that we can identify this complexification with the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ associated to the Lie group $\mathrm{SL}(2,\mathbb{C})$. So we should denote the complexification of $\mathfrak{su}(2)$ by $\mathfrak{sl}(2,\mathbb{C})$; we should warn the reader though that we shall sometimes follow physicist conventions in simply taking $\mathfrak{su}(2)$ also to mean its complexification.

As we have seen above, the complexified Lie algebra $\mathfrak{su}(2)$ (i.e. $\mathfrak{sl}(2,\mathbb{C})$) has a nice basis in which one generator, which we called H, is diagonal. Indeed, recall from (3.2.28) above that in the basis denoted by H, E, F the commutation relations are simply

$$[H,E] = 2E \ , \qquad [H,F] = -2F \ , \qquad [E,F] = H \ . \tag{4.1.1}$$

Thus we can think of E and F as raising and lowering operators that change the eigenvalue of H by ± 2.

This basis seems like a very natural basis to choose, but in fact the choice is dictated by more than aesthetics. There is, as we shall see, a nearly canonical way of choosing a basis of a simple Lie algebra (up to conjugation) which will yield this basis in the present circumstance and which will share many of the properties we describe below. This basis is called the Cartan-Weyl basis.

4.2 The representation theory

Part of the reason why this is a natural basis to consider is that it makes the analysis of the representation theory very simple. Suppose that V is an irreducible finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. One can show (although we shall not attempt to do this here) that the action of the generator H on V is diagonalisable. Thus we have a decomposition

$$V = \bigoplus_{\alpha} V_\alpha \ , \tag{4.2.1}$$

where α run over a collection of complex numbers, such that for any vector $v \in V_\alpha$ we have

$$H(v) = \alpha \cdot v \ . \tag{4.2.2}$$

Next we want to understand how E and F act on the various spaces V_α. We claim that E and F must each carry the subspaces V_α into other subspaces $V_{\alpha'}$. In order to see this, suppose that $v \in V_\alpha$. Then we find

$$\begin{aligned} HE(v) &= [H, E]v + EH(v) \\ &= 2E(v) + \alpha E(v) = (2+\alpha)E(v) \ , \end{aligned} \tag{4.2.3}$$

i.e. $E(v)$ is an eigenvector of H with eigenvalue $(2+\alpha)$; in other words,

$$E : V_\alpha \to V_{\alpha+2} \ . \tag{4.2.4}$$

Similarly, we can show that $F : V_\alpha \to V_{\alpha-2}$. Note that, as an immediate consequence of the irreducibility of V, the complex numbers α that appear in the decomposition (4.2.1) must be congruent to one another mod 2. Indeed, for any α_0 that actually occurs in (4.2.1), the subspace

$$\bigoplus_{n\in\mathbb{Z}} V_{\alpha_0+2n} \tag{4.2.5}$$

will be invariant under the action of $\mathfrak{sl}(2,\mathbb{C})$, and hence, because of irreducibility, must be all of V. Moreover, by the same token, the V_α that appear must form an unbroken string of numbers of the form $\beta, \beta+2, \beta+4, \ldots, \beta+2k$. We denote by n the last element in this sequence; at this point we just know that n is a complex number, but we will soon see that it must be an integer.

Now choose any $v \in V_n$. Since $V_{n+2} = \{0\}$, it follows that $E(v) = 0$. We ask now what happens when we apply F to the vector v, To begin with we claim

Claim: The vectors $\{v, F(v), F^2(v), \ldots\}$ span V.

From the irreducibility of V it is enough to show that the subspace $W \subset V$ spanned by these vectors is carried into itself under the action of $\mathfrak{sl}(2,\mathbb{C})$. Clearly F preserves W, and so does H (since it acts diagonally). To begin with we have $E(v) = 0$, so the first interesting cases are

$$\begin{aligned} EF(v) &= [E, F](v) + FE(v) \\ &= H(v) + 0 = nv \end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned} EF^2(v) &= [E, F]F(v) + FEF(v) \\ &= HF(v) + F(nv) \\ &= (n-2)F(v) + nF(v) = (2n-2)F(v) \ , \end{aligned} \tag{4.2.7}$$

where we have used the result of the first calculation in going to the second line. The pattern is now clear: E carries each vector in the sequence $v, F(v), F^2(v), \ldots$ into a multiple of the previous vector; explicitly we have

$$EF^m(v) = \Big((n-2m+2) + (n-2m+4) + \cdots (n-4) + (n-2) + n\Big)F^{m-1}(v) \tag{4.2.8}$$

or

$$EF^m(v) = m(n-m+1)\,F^{m-1}(v)\ , \tag{4.2.9}$$

as can readily be verified by induction noting that

$$(m+1)(n-(m+1)+1) = m(n-m)+(n-m) = m(n-m+1)+(n-2(m+1)+2)\ . \tag{4.2.10}$$

This then proves the claim.

Note that it follows from the above claim that all the eigenspaces V_α of H are one-dimensional. Furthermore, since we have in the course of the proof written down a basis for V and said exactly how each of H, E and F act on them, the representation V is completely determined by the one complex number n that we started with. To complete our analysis we have to use one more time the finite dimensionality of V. This tells us that there is a lower bound on the α for which $V_\alpha \neq 0$, so that $F^k v = 0$ for some sufficiently large k. Suppose then that m is the smallest power of F annihilating v, then from (4.2.9) it follows that

$$0 = EF^m(v) = m(n-m+1)\,F^{m-1}(v)\ . \tag{4.2.11}$$

Since m is the smallest power for which $F^m(v) = 0$, it follows that $F^{m-1}(v) \neq 0$, and hence $n-m+1 = 0$. In particular, we can therefore conclude that n is a non-negative integer. The picture is thus that the eigenvalues α of H on V form a string of integers differing by 2, and symmetric about the origin in $\mathbb{Z}$. For each integer n, there is a unique representation $V^{(n)}$ of dimension $n+1$, whose H-eigenvalues are $n, n-2, n-4, \ldots, -n+2, -n$.

Note that the existence part of this last statement may be deduced by checking that the actions of H, E and F as given above in terms of the basis $v, F(v), F^2(v), \ldots F^n(v)$ for V do indeed satisfy all the commutation relations of $\mathfrak{sl}(2,\mathbb{C})$. Alternatively, we will exhibit them shortly. Note that by the symmetry of the eigenvalues we may deduce a useful fact that any representation V of $\mathfrak{sl}(2,\mathbb{C})$ such that the eigenvalues of H all have the same parity and occur with multiplicity one is necessarily irreducible; more generally, the number of irreducible factors in an arbitrary representation V of $\mathfrak{sl}(2,\mathbb{C})$ is exactly the sum of the multiplicities of 0 and 1 as eigenvalues of H.

We can identify in these terms some of the standard representations of $\mathfrak{sl}(2,\mathbb{C})$. To begin with, the trivial one-dimensional representation $\mathbb{C}$ is clearly just $V^{(0)}$. As for the standard 2-dimensional representation in terms of 2×2 matrices, see (3.2.27) above, the eigenvalues of H are ± 1, and hence it can be identified with $V^{(1)}$. For the following it is convenient to denote the two basis vectors by x and y, with eigenvalues $H(x) = x$ and $H(y) = -y$.

4.3 The Clebsch-Gordan series

For the following it will be important to analyse tensor products of these representations. Recall that the action of the group G on a tensor product of two representations $V_1 \otimes V_2$ is defined by

$$\rho : G \to \mathrm{End}(V_1 \otimes V_2)\ , \quad g \mapsto \rho_1(g) \otimes \rho_2(g)\ . \tag{4.3.1}$$

We first want to understand what this implies for the action of the Lie algebra $\mathfrak{g}$. Recall from (3.1.16) that the action of the corresponding Lie algebra generator $\Omega = \frac{d}{dt}R(t)\big|_{t=0}$ is then defined by derivation; thus we conclude that the action of the Lie algebra on the tensor product is defined by

$$\rho : \mathfrak{g} \to \mathrm{End}(V_1 \otimes V_2) \ , \quad \Omega \mapsto \rho_1(\Omega) \otimes \mathbf{1} + \mathbf{1} \otimes \rho_2(\Omega) \ , \tag{4.3.2}$$

as follows directly from the product rule. (Here $\mathbf{1}$ is the identity map.)

With these preparations we can now consider tensor products of the representations $V^{(n)}$. Let us begin by studying the tensor product of $V \equiv V^{(1)}$ with itself. Using the notation x_i, y_i , $i = 1, 2$ for the basis vectors of the two vector spaces, the tensor product then consists of the vectors x_1x_2, x_1y_2, x_2y_1 and y_1y_2. Their H-eigenvalues are $+2$, 0, 0 and -2; thus it follows from the above considerations on the structure of the H-eigenvalues that

$$V \otimes V = V^{(2)} \oplus V^{(0)} \ . \tag{4.3.3}$$

Actually, $V^{(2)}$ can be identified with the symmetric square of this tensor product, i.e. $\mathrm{Sym}^2 V = V^{(2)}$.

Continuing in this manner, we can recursively study the tensor product of V with $V^{(n)}$; the H-eigenvalues of the tensor product are then

$$(n+1)\, ,\ 2\cdot(n-1)\, ,\ 2\cdot(n-3)\, , \ldots,\ 2\cdot(-n+3)\, ,\ 2\cdot(-n+1)\, ,\ -(n+1)\ . \tag{4.3.4}$$

Thus, by the above argument on the structure of the eigenvalues we conclude that

$$V \otimes V^{(n)} = V^{(n+1)} \oplus V^{(n-1)} \ . \tag{4.3.5}$$

In particular, this therefore proves by induction that all $V^{(n)}$ actually exist. It is not hard to deduce the general case in a similar manner (**Exercise**): we find

$$V^{(n)} \otimes V^{(m)} = \bigoplus_{l=|n-m|}^{n+m} V^{(l)} \ , \tag{4.3.6}$$

where the step size for l is two. Furthermore, one finds that the totally symmetric power of V, $\mathrm{Sym}^n V$, just consists of the top representation,

$$\mathrm{Sym}^n V = V^{(n)} \ . \tag{4.3.7}$$

Thus all irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ arise as a symmetric power of the standard $V \cong \mathbb{C}^2$ representation.

Much of the structure of this example, which is probably already familiar from the quantum mechanics course, can be generalised to the other Lie algebras as well. However, as an illustrative example, $\mathfrak{su}(2)$ is not really adequate as it does not exhibit many of the features that will be present in general. A somewhat more realistic example is the Lie algebra of $\mathfrak{su}(3)$ that we want to study next.

5 Complex simple Lie algebras — the case of $\mathfrak{su}(3)$

In this section we now want to generalise the results from the previous section to the case of $\mathfrak{su}(3)$.

5.1 The complexification of $\mathfrak{su}(3)$ and the Cartan-Weyl basis

The Lie algebra $\mathfrak{su}(3)$ was already introduced in section 3.2.1. Again, as before for the case of $\mathfrak{su}(2)$ we want to consider its complexification, i.e. the Lie algebra whose vector space consists of the complex linear combinations of the generators (3.2.17) – (3.2.19). This vector space can be identified with the vector space of traceless complex 3×3 matrices, and thus with the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$.

We will proceed by analogy with the analysis for $\mathfrak{sl}(2, \mathbb{C})$; in that case we started out with the special basis consisting of $\{H, E, F\}$, and then proceeded to decompose an arbitrary representation V of $\mathfrak{sl}(2, \mathbb{C})$ into a direct sum of eigenspaces for the action of H. What element of $\mathfrak{sl}(3, \mathbb{C})$ in particular will play the role of H? The answer – and this is the first and perhaps most wrenching change from the previous case — is that no one element really allows us to see what is going on. Instead, we have to replace the single element $H \in \mathfrak{sl}(2, \mathbb{C})$ with a **subspace** $\mathfrak{h} \subset \mathfrak{sl}(3, \mathbb{C})$, namely, the two-dimensional subspace of all diagonal matrices which is spanned by D_1 and D_2, see eq. (3.2.17). The idea is a basic one: it comes down to the observation that *commuting diagonalisable matrices are simultaneously diagonalisable*. This translates, in the present circumstance, to the statement that any finite-dimensional representation V of $\mathfrak{sl}(3, \mathbb{C})$ admits a decomposition $V = \oplus V_\alpha$, where every vector $v \in V_\alpha$ is an eigenvector for every element $H \in \mathfrak{h}$.

At this point some terminology is in order. To begin with, by an *eigenvector* for $\mathfrak{h}$ we will mean a vector $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. For such a vector v we can write

$$H(v) = \alpha(H) \cdot v \ , \tag{5.1.1}$$

where $\alpha(H)$ is a scalar depending linearly on H, i.e., α is an element of the dual space, $\alpha \in \mathfrak{h}^*$. This leads to our second notion: by an *eigenvalue* for the action of $\mathfrak{h}$ we will mean an element $\alpha \in \mathfrak{h}^*$ such that there exists a non-zero element $v \in V$ satisfying (5.1.1). Thus any finite-dimensional representation V of $\mathfrak{sl}(3, \mathbb{C})$ has a decomposition

$$V = \bigoplus_\alpha V_\alpha \ , \tag{5.1.2}$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$.

This is, in fact, a special case of a more general statement: for any semisimple Lie algebra $\mathfrak{g}$, we will be able to find an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the action of $\mathfrak{h}$ on any (finite-dimensional) $\mathfrak{g}$-module V will be diagonalisable, i.e., we will have a direct sum decomposition of V into eigenspaces V_α for $\mathfrak{h}$. The subalgebra $\mathfrak{h}$ will be called the **Cartan subalgebra** of $\mathfrak{g}$.

Having decided what the analogue for $\mathfrak{sl}(3,\mathbb{C})$ of $H \in \mathfrak{sl}(2,\mathbb{C})$ is, let us now consider what will play the role of E and F. The key here is to look at the commutation relations

$$[H,E] = 2E \ , \qquad [H,F] = -2F \ , \tag{5.1.3}$$

see (4.1.1). The correct way to interpret these is as saying that E and F are eigenvectors for the adjoint action of H on $\mathfrak{sl}(2,\mathbb{C})$. Thus, for the case of $\mathfrak{sl}(3,\mathbb{C})$, we should now look for eigenvectors (in the above sense) for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}(3,\mathbb{C})$. Put differently, we want to apply (5.1.2) to the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$ to obtain a decomposition

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{h} \oplus \Big(\bigoplus_\alpha \mathfrak{g}_\alpha \Big) \ , \tag{5.1.4}$$

where α ranges over a finite subset of $\mathfrak{h}^*$, and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multiplication, i.e., for any $H \in \mathfrak{h}$ and $E \in \mathfrak{g}_\alpha$

$$[H,E] = \mathrm{ad}(H)(E) = \alpha(H) \cdot E \ . \tag{5.1.5}$$

This is probably easier to carry out in practice than it is to say; we are being longwinded here because once this process is understood it will be straightforward to apply it to the other Lie algebras. In any case, to do it for $\mathfrak{sl}(3,\mathbb{C})$ we note that $\mathfrak{h}$ consists of the matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \ : \ a_1 + a_2 + a_3 = 0 \right\} \ . \tag{5.1.6}$$

The adjoint action of $\mathfrak{h}$ on the other generators of $\mathfrak{sl}(3,\mathbb{C})$ then takes the form

$$\begin{aligned} &\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 0 & e_{12} & e_{13} \\ e_{21} & 0 & e_{23} \\ e_{31} & e_{32} & 0 \end{pmatrix} - \begin{pmatrix} 0 & e_{12} & e_{13} \\ e_{21} & 0 & e_{23} \\ e_{31} & e_{32} & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (a_1-a_2)e_{12} & (a_1-a_3)e_{13} \\ (a_2-a_1)e_{21} & 0 & (a_2-a_3)e_{23} \\ (a_3-a_1)e_{31} & (a_3-a_2)e_{32} & 0 \end{pmatrix} \ . \end{aligned} \tag{5.1.7}$$

Thus the eigenvectors are simply the matrices E_{ij}, $i \neq j$, whose only non-zero matrix entry is a 1 in position (ij), and the corresponding eigenvalue is $(a_i - a_j)$. More formally, let us define the element L_i of $\mathfrak{h}^*$ by

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i \ , \qquad i = 1,2,3 \ . \tag{5.1.8}$$

Then the dual space $\mathfrak{h}^*$ is spanned by the L_i, subject to the relation $L_1 + L_2 + L_3 = 0$ (since this relation is true on any element of $\mathfrak{h}$). The linear functionals appearing in the decomposition (5.1.4) are then the six functionals $L_i - L_j$, where $i \neq j$, and the space

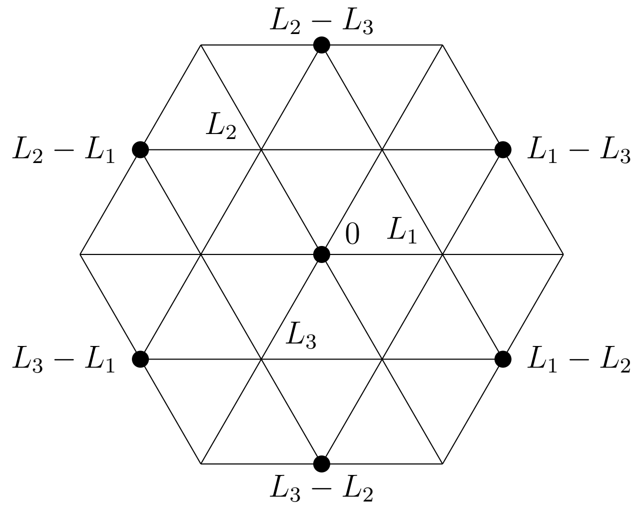


Figure 1: The roots of $\mathfrak{sl}(3,\mathbb{C})$ — usually the trivial eigenspace is not called a root.

$\mathfrak{g}_{L_i-L_j}$ is generated by the element E_{ij}. The structure of the eigenvalues is sketched in Figure 1.

The virtue of this decomposition is that we can read off from it pretty much the entire structure of the Lie algebra. By construction, the action of $\mathfrak{h}$ on $\mathfrak{g}$ is clear from the picture: $\mathfrak{h}$ carries each of the subspaces $\mathfrak{g}_\alpha$ into itself, acting on each $\mathfrak{g}_\alpha$ by scalar multiplication by the linear functional represented by α. In order to understand the structure of the other commutators, suppose that $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_\beta$. Then we find that their commutator satisfies for any $H \in \mathfrak{h}$

$$\begin{aligned} [H,[E,F]] &= [E,[H,F]] + [[H,E],F] \\ &= [E,\beta(H)\cdot F] + [\alpha(H)\cdot E,F] = \Big(\alpha(H)+\beta(H)\Big)\cdot[E,F]\ , \end{aligned} \tag{5.1.9}$$

where we have used the Jacobi identity in the first step. Thus we conclude that the commutator $[E,F]$ must lie in the eigenspace with eigenvalue $\alpha+\beta$,

$$[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}\ . \tag{5.1.10}$$

While this does not fix everything, e.g., we do not know the numerical factors that appear for these various commutators, etc., it does determine the structure of the Lie algebra to a large extent. As we have seen (and this will continue to hold in general) each of the eigenspaces $\mathfrak{g}_\alpha$ is actually 1-dimensional; the basis that consists of the generators of $\mathfrak{g}_\alpha$, as well as two arbitrary generators of $\mathfrak{h}$, is called the **Cartan-Weyl basis** of $\mathfrak{sl}(3,\mathbb{C})$.

Pretty much the same picture also applies to any other representation V of $\mathfrak{sl}(3,\mathbb{C})$. Again we start from the eigenspace decomposition $V=\bigoplus_\alpha V_\alpha$ for the action of $\mathfrak{h}$. Then the commutation relations for $\mathfrak{sl}(3,\mathbb{C})$ tell us exactly how the remaining summands of the decomposition of $\mathfrak{sl}(3,\mathbb{C})$, see (5.1.4), act on the space V, and again we will see that each of the spaces $\mathfrak{g}_\alpha$ acts by carrying an eigenspace V_β into another. Indeed, suppose that $E \in \mathfrak{g}_\alpha$ and $v \in V_\beta$. Then by an analogous calculation to (4.2.3) — in fact, also (5.1.9) is the same calculation in disguise, just applied to the adjoint representation — we find for any $H \in \mathfrak{h}$

$$\begin{aligned} H\,E(v) &= E\,H(v) + [H,E](v) \\ &= E\,(\beta(H)\cdot v) + (\alpha(H)\cdot E)(v) \\ &= \Big(\alpha(H)+\beta(H)\Big)\cdot E(v)\ . \end{aligned} \tag{5.1.11}$$

Thus $E(v)$ is again an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $\alpha+\beta$; in other words $\mathfrak{g}_\alpha$ carries V_β to $V_{\alpha+\beta}$,

$$\mathfrak{g}_\alpha : V_\beta \to V_{\alpha+\beta}\ . \tag{5.1.12}$$

Note that this implies, in particular, that the eigenvalues α that occur in an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ differ from one another by integral linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$.

At this point we should introduce a little bit of terminology. The vectors $L_i - L_j \in \mathfrak{h}^*$ generate a lattice in $\mathfrak{h}^*$ that, by definition, consists of all the integer linear combinations of $L_i - L_j \in \mathfrak{h}^*$; this lattice will be called the **root lattice** and denoted by Λ_R.

For a representation V, let $\alpha \in \mathfrak{h}^*$ be one of the eigenvalues that appears in the decomposition $V = \bigoplus_\alpha V_\alpha$. Then we call α a **weight** of the representation V. The corresponding eigenvectors in V_α are called **weight vectors**, and the spaces V_α themselves **weight spaces**.

The weights that appear in the adjoint representation are special; they are the **roots** of the Lie algebra, and the corresponding subspaces $\mathfrak{g}_\alpha \in \mathfrak{g}$ are called **root spaces**. By convention, zero is not a root. Then the root lattice is just the lattice generated by the roots of the Lie algebra.

5.2 The general representation theory

Next we want to understand general aspects of the representation theory of $\mathfrak{sl}(3,\mathbb{C})$ in some detail. In order to do so, let us go back to what we did for $\mathfrak{sl}(2,\mathbb{C})$. There, we started by considering an 'extremal' eigenspace, namely the one corresponding to the largest eigenvalue of H. The corresponding eigenvector then had the property that it was annihilated by the 'raising' operator E. Starting from this 'highest weight' state we then constructed the full representation by acting on it with the 'lowering' operators F.

What would be the appropriately analogous setup in the case of $\mathfrak{sl}(3,\mathbb{C})$? To start at the beginning, there is the question of what we mean by 'extremal': in the case of $\mathfrak{sl}(2,\mathbb{C})$ we knew that all the eigenvalues were scalars differing by integral multiple of 2, so there was not much of an ambiguity in what we meant by this. In the present case, however, this does involve a priori some choice (although, as we shall see, this choice does not affect the outcome). We have to choose a direction, and look for the largest α in that direction appearing in the decomposition of $\mathfrak{sl}(3,\mathbb{C})$ in (5.1.4). This is to say, we should choose a linear functional

$$l : \Lambda_R \to \mathbb{R}\ , \tag{5.2.1}$$

extend it by linearity to a linear functional $l : \mathfrak{h}^* \to \mathbb{C}$, and then for any representation V we should go to the eigenspace V_α for which the real part of $l(\alpha)$ is maximal. In order to avoid some ambiguity we should choose l to be irrational with respect to the lattice Λ_R, that is, to have no kernel in Λ_R.

What is the point of this? The answer is that, just as in the case of $\mathfrak{sl}(2,\mathbb{C})$, we will in this manner find a vector v that is an eigenvector for $\mathfrak{h}$, and at the same time in the kernel of the action of $\mathfrak{g}_\beta$ for every β for which $l(\beta) > 0$; in particular, it will therefore be killed by half the root spaces $\mathfrak{g}_\beta$. (These roots then play the role of E before.) Then we can generate the full representation by acting on this 'highest weight state' with the other roots, i.e., the root spaces $\mathfrak{g}_\beta$ with $l(\beta) < 0$.

Again, this is easier to understand if we carry it out explicitly. We make the ansatz for the functional l to take the form

$$l\big(c_1L_1 + c_2L_2 + c_3L_3\big) = r_1c_1 + r_2c_2 + r_3c_3\ , \tag{5.2.2}$$

for some real numbers r_1, r_2 and r_3. Since the root lattice is characterised by the condition $c_1 + c_2 + c_3 = 0$ we choose $r_1 + r_2 + r_3 = 0$. Furthermore, for concreteness we consider the case $r_1 > r_2 > r_3$, so that the spaces $\mathfrak{g}_\alpha \in \mathfrak{g}$ for which $l(\alpha) > 0$ are precisely $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$ and $\mathfrak{g}_{L_2-L_3}$, see Figure 2. Thus the generators E_{ij} with $i < j$ generate the positive root spaces, while the generators E_{ij} with $i > j$ generate the negative root spaces. We also define

$$H_{ij} = [E_{ij}, E_{ji}] = E_{ii} - E_{jj} \in \mathfrak{h}\ . \tag{5.2.3}$$

[In terms of our earlier notation, we therefore have $H_{12} = -iD_1$ and $H_{23} = -iD_2$, see eq. (3.2.17).]

Now let V be any irreducible finite-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$. Then consider a vector $v \in V$ that is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some α, and that has the property that v is annihilated by E_{12}, E_{13} and E_{23},

$$E_{12}(v) = E_{13}(v) = E_{23}(v) = 0\ . \tag{5.2.4}$$

For any representation V of $\mathfrak{sl}(3,\mathbb{C})$, a vector $v \in V$ with these properties is called a **highest weight vector**. It exists because V is finite-dimensional, and hence there is a weight α such that $l(\alpha)$ is maximal — v is then an element of that weight space. (In fact, as we shall see momentarily, since V is irreducible, the corresponding weight space must be one-dimensional.)

As for the case of $\mathfrak{sl}(2,\mathbb{C})$ we should now expect that we can generate the full representation by acting with the negative root spaces on v. In fact, since $[E_{32}, E_{21}] = E_{31}$, it is enough to use as 'creation' operators just E_{21} and E_{32}.

To see that this is indeed the case, we define (as before in the argument for $\mathfrak{sl}(2,\mathbb{C})$) the subspace $W \subseteq V$ that is spanned by the images of v under the subalgebra of $\mathfrak{sl}(3,\mathbb{C})$

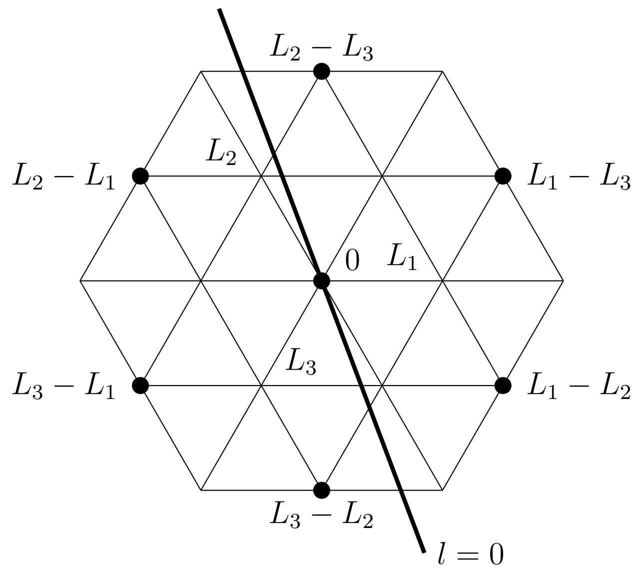


Figure 2: A choice of positive roots for $\mathfrak{sl}(3, \mathbb{C})$.

generated by the lower-triangular matrices E_{21}, E_{31} and E_{32}. Then we have to show that W is actually preserved under the action of the full Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, and hence must agree with V (since V is irreducible). So all we have to do is check that E_{12}, E_{23} and E_{13} carry W into itself — actually, it is sufficient to check this for the first two, since $E_{13} = [E_{12}, E_{23}]$.

To begin with let us check that $E_{21}(v)$ is kept in W; thus we have to calculate

$$\begin{aligned} E_{12}\big(E_{21}(v)\big) &= E_{21}\big(E_{12}(v)\big) + [E_{12}, E_{21}](v) \\ &= \alpha([E_{12}, E_{21}]) \cdot v \ , \end{aligned} \tag{5.2.5}$$

where we have used that $E_{12}(v) = 0$ and $[E_{12}, E_{21}] = H_{12} \in \mathfrak{h}$. Similarly, we calculate

$$\begin{aligned} E_{23}\big(E_{21}(v)\big) &= E_{21}\big(E_{23}(v)\big) + [E_{23}, E_{21}](v) \\ &= 0 \ , \end{aligned} \tag{5.2.6}$$

since $E_{23}(v) = 0$ and $[E_{23}, E_{21}] = 0$. A similar computation shows that $E_{32}(v)$ is also carried into W by the action of E_{12} and E_{23}.

We can now argue more generally by some sort of induction. Let w_n denote any word of length n or less in the letters E_{21} and E_{32}, and take W_n to be the vector space spanned by the vectors $w_n(v)$ for all such words. (Then W will be the union of all the spaces W_n.) We now claim that E_{12} and E_{23} carry W_n into W_{n-1}. To see this, we can write $w_n \in W_n$ as $w_n = E_{21}(w_{n-1})$ or as $w_n = E_{32}(w_{n-1})$, where in either case w_{n-1} is an eigenvector for

$\mathfrak{h}$ with some eigenvalue β. In the former case we then have

$$\begin{aligned} E_{12}w_n &= E_{12}\big(E_{21}(w_{n-1})\big) \\ &= E_{21}\big(E_{12}(w_{n-1})\big) + [E_{12}, E_{21}](w_{n-1}) \\ &= E_{21}(w_{n-2}) + \beta([E_{12}, E_{21}]) \cdot w_{n-1} \ , \end{aligned} \tag{5.2.7}$$

where $w_{n-2} \equiv E_{12}(w_{n-1})$ is by the induction hypothesis an element in W_{n-2}, and we have used that $[E_{12}, E_{21}] = H_{12} \in \mathfrak{h}$; and

$$\begin{aligned} E_{23}w_n &= E_{23}\big(E_{21}(w_{n-1})\big) \\ &= E_{21}\big(E_{23}(w_{n-1})\big) + [E_{23}, E_{21}](w_{n-1}) \\ &= E_{21}(\hat{w}_{n-2}) \ , \end{aligned} \tag{5.2.8}$$

where $\hat{w}_{n-2} \equiv E_{23}(w_{n-1})$ is again, by the induction hypothesis, an element in W_{n-2}. Essentially the same calculation covers the other case when $w_n = E_{32}(w_{n-1})$. In each case the right-hand side is then contained in W_{n-1}, thus proving the statement in question.

Actually, this argument shows more generally that any irreducible finite-dimensional representation of $\mathfrak{sl}(3, \mathbb{C})$ has a unique highest weight vector, up to scalars. Indeed, on every highest weight state we can build a subrepresentation W by applying successively the operators E_{21} and E_{23}, and if the representation V is irreducible, this must mean that $V = W$, i.e., that the space of highest weight states is 1-dimensional.

In order to understand the actual structure of the representation V that is generated from a highest weight state, it is very instructive to look at the strings of states $E_{21}^k(v)$. They generate vectors in the eigenspaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha+L_2-L_1}$, $\mathfrak{g}_{\alpha+2(L_2-L_1)}$, etc., that correspond to points on the boundary of the space of possible eigenvalues of V. We also know that they span an uninterrupted string of non-zero eigenspaces $\mathfrak{g}_{\alpha+k(L_2-L_1)} \cong \mathbb{C}$, $k = 0, 1, \ldots$, until we get to the first m such that $(E_{21})^m(v) = 0$; after that we have $\mathfrak{g}_{\alpha+k(L_2-L_1)} = \{0\}$ for all $k \geq m$.

The obvious question now is how long the string of non-zero eigenspaces is. One way to answer this would be to make a calculation analogous to the one we did for the case of $\mathfrak{sl}(2, \mathbb{C})$: use the above computation to determine explicitly the proportionality constant in $E_{12}(E_{21})^k(v) \cong (E_{21})^{k-1}(v)$; in particular for $k = m$ this constant must be zero since m is the first value for which $(E_{21})^m(v) = 0$. However it will be simpler — and indeed more instructive — if instead we use what we have already learned about representations of $\mathfrak{sl}(2, \mathbb{C})$. In fact, the elements

$$E = E_{12} \ , \qquad F = E_{21} \ , \qquad H = H_{12} = [E_{12}, E_{21}] \tag{5.2.9}$$

form a subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ that is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; we will denote this subalgebra by $\mathfrak{s}_{L_1-L_2}$. By the description we have already given of the action of $\mathfrak{sl}(3, \mathbb{C})$ on the representation V in terms of the decomposition $V = \bigoplus V_\alpha$, we see that the subalgebra $\mathfrak{s}_{L_1-L_2}$ will shift eigenspaces V_α only in the direction of $L_2 - L_1$; in particular, the direct sum of the eigenspaces in question, namely the subspace

$$W = \bigoplus_k V_{\alpha+k(L_2-L_1)} \tag{5.2.10}$$

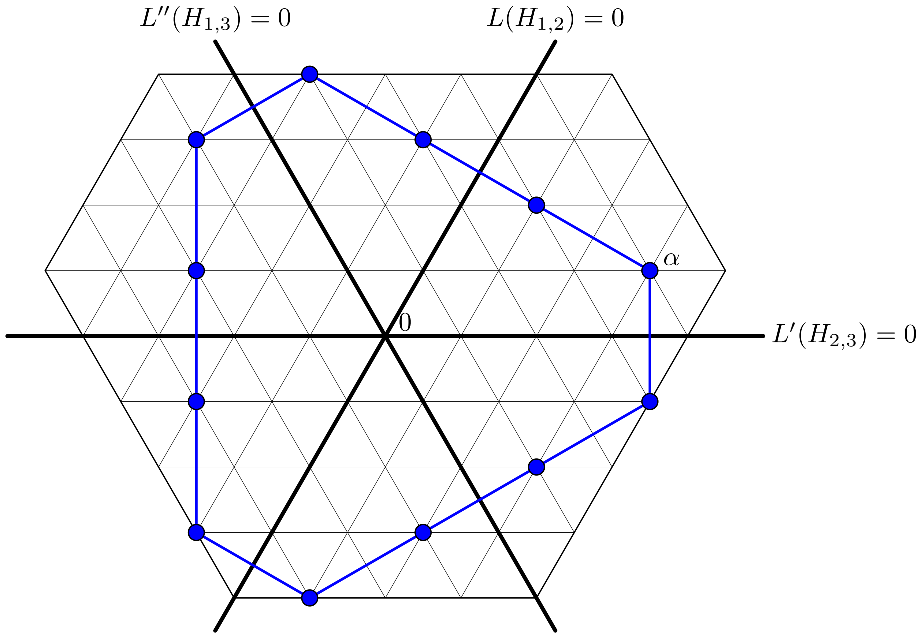


Figure 3: The weights of an $\mathfrak{sl}(3, \mathbb{C})$ representation.

of V will be preserved by the action of $\mathfrak{s}_{L_1-L_2}$. Thus the subspace W is a representation of $\mathfrak{s}_{L_1-L_2} \cong \mathfrak{sl}(2, \mathbb{C})$, and we may deduce from this that the eigenvalues of H_{12} on W are integral and symmetric with respect to zero. Leaving aside the integrality for the moment, this says that the string of eigenvalues must be symmetric with respect to the line L that is characterised by the condition $L(H_{12}) = 0$ in the plane $\mathfrak{h}^*$. Happily, (although by no means coincidentally), this line is perpendicular to the line spanned by $L_1 - L_2$. Thus we can conclude that the string of eigenvalues that appear in the weight space of V is symmetrical under the reflection in this line L.

Actually, this construction is not just possible for E_{12} and E_{21}: for any $i < j$, the elements

$$E = E_{ij} \ , \qquad F = E_{ji} \ , \qquad H = H_{ij} = [E_{ij}, E_{ji}] \tag{5.2.11}$$

form a subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ that is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, and that we shall denote by $\mathfrak{s}_{L_i-L_j}$. In particular, by analysing the subalgebra $\mathfrak{s}_{L_2-L_3}$ we can likewise show that the string of eigenvalues corresponding to the eigenspaces $V_{\alpha+k(L_3-L_2)}$ is preserved under reflection in the line $L^{'}$ in $\mathfrak{h}^*$ that contains the elements of $\mathfrak{h}^*$ with $L^{'}(H_{23}) = 0$.

Finally, let us consider the last vector in the first string of vectors, i.e., the vector $v' \equiv (E_{21})^{m-1}(v)$ with weight $\beta = \alpha + (m-1)(L_2 - L_1)$. (Remember that m is defined to be the first value for which $(E_{21})^m(v) = 0$.) By construction, v' is annihilated by E_{21}.

It is furthermore annihilated by E_{23} since

$$\begin{aligned} E_{23}v' &= E_{23}\,(E_{21})^{m-1}(v) = \sum_{l=0}^{m-2}(E_{21})^l\,[E_{23},E_{21}]\,(E_{21})^{m-2-l}(v) + (E_{21})^{m-1}\,E_{23}(v) \\ &= 0\ , \end{aligned} \tag{5.2.12}$$

since $E_{23}(v)=0$ and since the commutator $[E_{23},E_{21}]=0$. Furthermore, the same is true for

$$\begin{aligned} E_{13}v' &= E_{13}\,(E_{21})^{m-1}(v) = \sum_{l=0}^{m-2}(E_{21})^l\,[E_{13},E_{21}]\,(E_{21})^{m-2-l}(v) + (E_{21})^{m-1}\,E_{13}(v) \\ &= -\sum_{l=0}^{m-2}(E_{21})^l\,E_{23}\,(E_{21})^{m-2-l}(v) = 0\ , \end{aligned} \tag{5.2.13}$$

since $E_{13}(v)=0$ and since $[E_{13},E_{21}]=-E_{23}$. In the last step we have used the result from (5.2.12), which also holds if we replace m by any smaller non-negative integer.

Thus we conclude that v' is a highest weight state, but now with respect to the linear functional l with $r_2>r_1>r_3$ so that E_{21}, E_{13} and E_{23} are the positive roots. Indeed, if we had carried out the above analysis with respect to that choice of linear functional l, then v' would have played the role of v, and we would have considered the string of eigenvalues of V associated to the corresponding 'lowering' operators E_{12}, E_{31} and E_{32}. Note that $E_{32}=[E_{31},E_{12}]$, and thus it is sufficient to consider the generators E_{31} and E_{12}; thus we can conclude that the string of eigenvalues along L_3-L_1 and L_1-L_2 is symmetrical with respect to reflection along the lines $L''(H_{13})=0$ and $L(H_{12})=0$, respectively.

Needless to say, we can now continue the same game for the end-point of the string along the L_3-L_1 direction starting from v'; in this manner we will end up with a vector v'' that is annihilated by E_{31}, E_{21}, and E_{23}, and which is therefore highest weight with respect to the linear functional (5.2.2) with $r_2>r_3>r_1$. Continuing in this manner we will eventually end up with a hexagon of lines that bound the possible eigenvalues of the representation V; this hexagon is symmetric with respect to the reflection in each of the lines $L(H_{12})=0$, $L'(H_{23})=0$ and $L''(H_{13})=0$, see Figure 3. It is not hard to show (see the end of this section) that the actual set of eigenvalues includes all the points inside the hexagon that are congruent to α modulo the root lattice Λ_R. Furthermore, each eigenvalue along the boundary occurs with multiplicity one.

The use of the $\mathfrak{sl}(2,\mathbb{C})$ subalgebras $\mathfrak{s}_{L_i-L_j}$ does not stop here. In particular, we also know that the eigenvalues of the elements H_{ij} must be integers; if we write the weights in terms of the basis vectors L_i, i.e.,

$$\alpha = w_1L_1+w_2L_2+w_3L_3 \tag{5.2.14}$$

then this condition implies that $w_i-w_j\in\mathbb{Z}$. Since we may shift them by any multiple of $L_1+L_2+L_3\cong 0$, we may therefore take the w_i to be individually integers. Thus the L_i generate the **weight lattice**, i.e., the lattice Λ_W generated by the possible weights of any representation. Furthermore, within each irreducible representation, the weights must differ by elements in the root lattice Λ_R.

This is exactly analogous to the situation for $\mathfrak{sl}(2,\mathbb{C})$: there we saw that the eigenvalues of H in any irreducible finite-dimensional representation lie in the weight lattice $\Lambda_W \cong \mathbb{Z}$ of linear forms that are integral on H, and within an irreducible representation were congruent to one another modulo the sublattice $\Lambda_R = 2\cdot\mathbb{Z}$ generated by the eigenvalues of H under the adjoint representation. Note that in the case of $\mathfrak{sl}(2,\mathbb{C})$ we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}_2$, while in the present case we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}_3$ (since the quotient space is for example generated by L_1 and $2L_1$, whereas $3L_1 = (L_1 + L_2 + L_3) + (L_1 - L_2) + (L_1 - L_3) \in \Lambda_R$); we will see later how this reflects a general pattern.

To continue we can go into the interior of the eigenvalue diagram by observing that for any weight $\beta \in \mathfrak{h}^*$ appearing in the decomposition of V, the direct sum

$$W = \bigoplus_k V_{\beta+k(L_i-L_j)} \tag{5.2.15}$$

is a subrepresentation under the action of $\mathfrak{s}_{L_i-L_j}$ (although this representation needn't be irreducible). In particular, it follows that the values of k for which $V_{\beta+k(L_i-L_j)} \neq \{0\}$ form an unbroken string of integers. Thus we can conclude that all eigenvalues (not just those on the outer boundary) must by symmetrical with respect to the reflections along the lines $L(H_{12}) = 0$, $L'(H_{23}) = 0$ and $L''(H_{13}) = 0$.

5.3 Explicit examples of $\mathfrak{sl}(3,\mathbb{C})$ representations

In the previous subsection we have seen that the weights of any $\mathfrak{sl}(3,\mathbb{C})$ representation lie in the weight lattice that is generated by integer linear combinations of L_1, L_2 and L_3. Furthermore, for our choice of positive roots, see Figure 2, and Figure 3, the highest weight state lies in the first sextant of the plane that is bounded by the lines L and L'. Thus every highest weight can be written as

$$a\,L_1 - b\,L_3\ , \qquad a, b \in \mathbb{N}_0\ . \tag{5.3.1}$$

In fact, for each such weight there exists a (unique) irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$, and each irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ has a highest weight state of this form.

In order to get a feeling for this, let us begin by identifying some simple $\mathfrak{sl}(3,\mathbb{C})$ representations in this language. To start with we consider the standard (or so-called **fundamental**) representation of $\mathfrak{sl}(3,\mathbb{C})$ on $V \cong \mathbb{C}^3$. Of course, the eigenvectors for the action of $\mathfrak{h}$ are just the standard basis vectors e_1, e_2 and e_3. They have eigenvalues L_1, L_2 and L_3, respectively; the weight diagram for V is depicted in Figure 4. The highest weight vector therefore corresponds to L_1; in terms of the above parametrisation, see (5.3.1), this representation therefore correponds to the pair $[a, b] = [1, 0]$.

Next we consider the dual representation V^*. The eigenvalues of the dual of a representation of a Lie algebra are just the negatives of the eigenvalues of the original, so

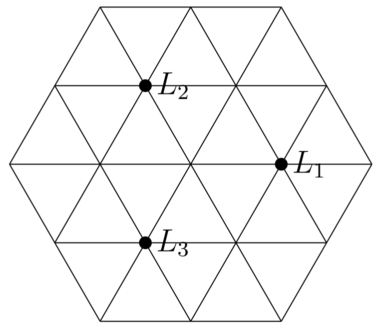


Figure 4: The weights of the standard representation of $\mathfrak{sl}(3,\mathbb{C})$.

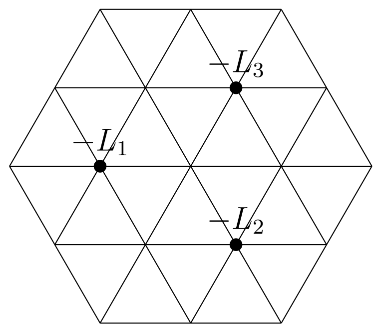


Figure 5: The weights of the dual of the standard representation of $\mathfrak{sl}(3,\mathbb{C})$.

the diagram for V^* has the form depicted in Figure 5. The highest weight vector now corresponds to $-L_3$, and hence V^* correponds to $[a,b]=[0,1]$.

Note that while in the case of $\mathfrak{sl}(2,\mathbb{C})$ the weights of any representation were symmetric about the origin, and correspondingly each representation was isomorphic to its dual, the same is not true here, in particular $V \not\cong V^*$.

Next we consider the tensor product of V with itself. The 9-dimensional vector space $V\otimes V$ splits up into

$$V\otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V \ , \tag{5.3.2}$$

where $\mathrm{Sym}^2 V$ consists of those vectors in $V\otimes V$ that are invariant under the exchange of the two vector spaces, while $\Lambda^2 V$ consists of those vectors that are odd under the exchange of the two vector spaces. (Thus $\mathrm{Sym}^2 V$ has dimension 6, while $\Lambda^2 V$ has dimension 3.) Note that since the action of the Lie algebra on the tensor product is invariant under the exchange of the two factors, both $\mathrm{Sym}^2 V$ and $\Lambda^2 V$ must be separately representations of $\mathfrak{sl}(3,\mathbb{C})$. In fact, both of them are irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$.

To begin with let us study $\Lambda^2 V$; its weights are the pairwise sums of the distinct weights of V, i.e. they are $L_1+L_2=-L_3$, $L_1+L_3=-L_2$ and $L_2+L_3=-L_1$. Thus we conclude that

$$\Lambda^2 V \cong V^* \ . \tag{5.3.3}$$

On the other hand, the weights of $\mathrm{Sym}^2 V$ are depicted in Figure 6. Note that it is clear from this picture that $\mathrm{Sym}^2 V$ must be irreducible since its collection of weights cannot be written as a union of two collections arising from $\mathfrak{sl}(3,\mathbb{C})$ representations. The highest weight state of this representation has weight $2L_1$, and hence the representation corresponds to $[a,b]=[2,0]$.

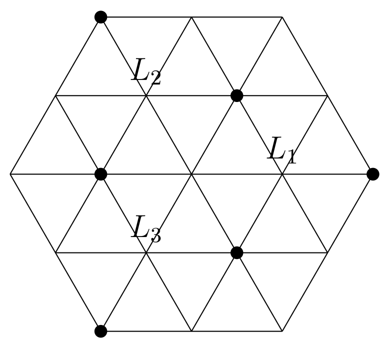


Figure 6: The weights of the 6-dimensional representation $\mathrm{Sym}^2 V$ of $\mathfrak{sl}(3,\mathbb{C})$.

The dual representation to $\mathrm{Sym}^2 V$ is obviously $\mathrm{Sym}^2 V^*$, whose weights are just the negatives of the weights of $\mathrm{Sym}^2 V$; its highest weight is $-2L_3$, and hence it is described by $[a,b]=[0,2]$.

Next we consider the tensor product $V \otimes V^*$, whose weights are just the sums of the weights $\{L_i\}$ of V with those $\{-L_i\}$ of V^*, that is, the linear functionals $L_i - L_j$, each occuring once, and 0 occuring with multiplicity three; the corresponding weight diagram is depicted in Figure 7, where the triple circle is intended to convey the fact that the weight space V_0 is 3-dimensional.

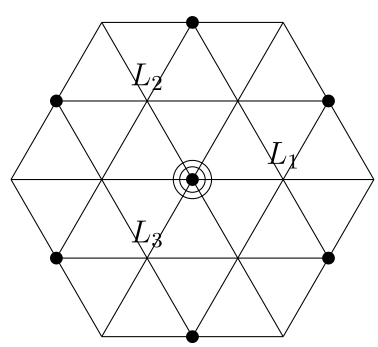


Figure 7: The weights of the adjoint and the trivial representation of $\mathfrak{sl}(3,\mathbb{C})$.

By contrast to the previous examples, this representation is not irreducible. In fact, there is a linear map

$$V \otimes V^* \to \mathbb{C} \ , \qquad v \otimes u^* \mapsto u^*(v) \ , \tag{5.3.4}$$

that is a map of $\mathfrak{sl}(3,\mathbb{C})$-modules, with $\mathbb{C}$ being the trivial representation of $\mathfrak{sl}(3,\mathbb{C})$. (In terms of the identification $V \otimes V^* \cong \mathrm{Hom}(V,V)$, this map is simply the **trace**.) The kernel of this map is then the subspace of $V \otimes V^*$ of traceless matrices, which is just the adjoint representation of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ and is irreducible. (Indeed, the unique highest weight state of the adjoint representation has weight $L_1 - L_3$, and hence the adjoint representation is just $[a,b] = [1,1]$.) Thus we have the decomposition

$$V \otimes V^* = \mathbf{1} \oplus \mathrm{adj} \ . \tag{5.3.5}$$

We should mention in passing that this adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$ is sometimes referred to as the 'eightfold way' describing the lightest mesons. (The representations V and V^* correspond to the quarks and anti-quarks, respectively; thus $V \otimes V^*$ describes the quark anti-quark pairs, i.e., the mesons.)

More generally, suppose that v and w are the highest weight states with weights α and β of two irreducible representations V and W, respectively, then $v \otimes w$ is a highest weight state for $V \otimes W$ with weight $\alpha + \beta$. In particular, by taking V to be the fundamental 3-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$, and W the representation corresponding to the pair $[a,b] = [n,0]$ — for $n = 1$, $W = V$; for $n = 2$, $W = \mathrm{Sym}^2(V)$; and the general case is obtained recursively — we deduce that

$$V \otimes [n,0] \supset [n+1,0] \ . \tag{5.3.6}$$

Thus we conclude that all representations associated to $[n,0]$ exist; in fact, it is not hard to see that the representation $[n,0]$ just corresponds to

$$[n,0] \cong \mathrm{Sym}^n(V) \ . \tag{5.3.7}$$

By the same token, we can also conclude that

$$[0,n] \cong \mathrm{Sym}^n(V^*) \ , \tag{5.3.8}$$

and thus, by the above argument, we also know that the representation

$$[n,m] \subset \mathrm{Sym}^n(V) \otimes \mathrm{Sym}^m(V^*) \tag{5.3.9}$$

exists. On the other hand, as we have seen before, starting from a highest weight, there is an unambiguous way of constructing the corresponding highest weight representation; it is therefore also clear that these representations are unique. Thus we conclude that the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ are precisely labelled by the pairs $[a,b]$ with $a,b \in \mathbb{N}_0$.

The above construction also implies that all of these representations are contained in suitable tensor products of the 3-dimensional fundamental representation V of $\mathfrak{sl}(3,\mathbb{C})$. Indeed, it is immediate that they appear in multiple tensor products of V and V^*, but since

$$V^* = \Lambda^2 V \tag{5.3.10}$$

we do not need to use V^* explicitly, but rather can just consider tensor products of V only. [Obviously, we may equivalently just consider tensor products of V^*.] This is actually a very useful and convenient point of view, as we shall now explain.

5.4 $\mathfrak{sl}(3,\mathbb{C})$ representations in terms of Young diagrams

Suppose we consider the N-fold tensor product of V. On this N-fold tensor product, the generators of the Lie algebra act by symmetrical sums, i.e., $\Omega \in \mathfrak{sl}(3,\mathbb{C})$ acts as

$$\rho(\Omega) = \sum_{i=1}^{N} \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\,\text{copies}} \otimes \rho_V(\Omega) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \; . \tag{5.4.11}$$

In particular, the action of $\mathfrak{sl}(3,\mathbb{C})$ on the tensor product therefore **commutes** with the permutation action of S_N on $V^{\otimes N}$. Thus we can simultaneously decompose $V^{\otimes N}$ into irreducible representations of S_N and $\mathfrak{sl}(3,\mathbb{C})$, i.e., we can write

$$V^{\otimes N} = \bigoplus_{(\Gamma,R)} \Gamma \otimes R \; , \tag{5.4.12}$$

where Γ is an irreducible representation of S_N, while R is an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$. As it turns out, S_N and $\mathfrak{sl}(3,\mathbb{C})$ are a 'dual pair', i.e., S_N is the maximal commutant for the action of $\mathfrak{sl}(3,\mathbb{C})$ on $V^{\otimes N}$ and vice versa.[3] As a consequence, each representation Γ and R appears at most once in this decomposition — if, say, a given Γ appeared twice, i.e., if the decomposition contained

$$V^{\otimes N} \supset (\Gamma \otimes R_1) \oplus (\Gamma \otimes R_2) \; , \tag{5.4.13}$$

then we can construct a linear operator on $V^{\otimes N}$ that acts as the identity on Γ, and maps a non-trivial vector in R_1 to a non-trivial vector in R_2. (We may choose some basis for R_1 and R_2, and simply map the first basis vector to the first basis vector, and map all other basis vectors to zero. We may furthermore continue this map to be trivial on the other summands in the decomposition.) This map then commutes by construction with the action of S_N, but it cannot be described by the action of (the universal enveloping algebra of) $\mathfrak{sl}(3,\mathbb{C})$ (or $\mathfrak{gl}(3,\mathbb{C})$) since the latter necessarily maps R_1 to itself. Thus, it follows that the commutant of S_N on $V^{\otimes N}$ is not just $\mathfrak{sl}(3,\mathbb{C})$, in contradistinction with the above assumption. Thus we conclude that each irreducible representation appears only once, i.e., each irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ (that is contained in this tensor product) appears together with a specific representation of S_N, and hence we may equivalently label the representations of $\mathfrak{sl}(3,\mathbb{C})$ in terms of the associated 'partner' representation of S_N.

We may construct this decomposition (and thereby establish the correspondence) quite explicitly, using the Young symmetrisers. Recall from section 2 that for each standard Young tableau, i.e., for each Young diagram Γ with a standard filling, we can apply the operator c_Γ to $V^{\otimes N}$, and the resulting image transforms in the irreducible representation of S_N that is associated to Γ. (Note that the image may also be trivial — in fact, this will be the case if Γ has a column whose length exceeds the dimension of V, i.e., $\dim(V) = 3$

[3]Strictly speaking, the maximal commutant for the action of S_N is the universal enveloping algebra of $\mathfrak{gl}(3,\mathbb{C})$, but this subtlety is not very important.

in our case.) By construction, each such subspace is invariant under the action of $\mathfrak{sl}(3,\mathbb{C})$, since the Young symmetriser $c_\Gamma \in \mathbb{C}(S_N)$, and the action of S_N and $\mathfrak{sl}(3,\mathbb{C})$ commute. As was also explained in section 2, in particular section 2.3, there are $\dim(\Gamma)$ standard fillings of Γ, and hence the given representation of $\mathfrak{sl}(3,\mathbb{C})$ appears with multiplicity $\dim(\Gamma)$, in perfect agreement with (5.4.12).

Let us exhibit this construction for the first few cases. To start with, i.e., for $N=1$, we just have the identification

$$\begin{array}{|c|}\hline \ \\ \hline\end{array} \longleftrightarrow [a,b]=[1,0]\ . \tag{5.4.14}$$

For $N=2$, we consider $V^{\otimes 2}$, which can be decomposed into the symmetric and anti-symmetric part, both of which are separately representations of $\mathfrak{sl}(3,\mathbb{C})$. Even more explicitly, if we choose a basis for V to consist of the vectors $\mathbf{e}_i$, $i=1,2,3$, then the symmetric part is spanned by the six vectors

$$\mathrm{Sym}^2 V: \quad (\mathbf{e}_i\otimes\mathbf{e}_j)+(\mathbf{e}_j\otimes\mathbf{e}_i)\ , \tag{5.4.15}$$

where $1\le i\le j\le 3$. On the other hand, the antisymmetric part is spanned by the three vectors

$$\Lambda^2 V: \quad (\mathbf{e}_i\otimes\mathbf{e}_j)-(\mathbf{e}_j\otimes\mathbf{e}_i)\ , \tag{5.4.16}$$

where $1\le i<j\le 3$. It is clear that both subspaces are separately representations of $\mathfrak{sl}(3,\mathbb{C})$ since $\Omega\in\mathfrak{sl}(3,\mathbb{C})$ acts by $(\Omega\otimes\mathbf{1})+(\mathbf{1}\otimes\Omega)$. In fact, the two subspaces define each an irreducible representation, as was already mentioned following (5.3.2): for the symmetric product the relevant representation is $[2,0]$, while for the antisymmetric product it is $[0,1]$. Now in terms of Young symmetrisers, the symmetric product corresponds to the Young diagram $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$, while the anti-symmetric product is associated to $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$. Thus we have the identification

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \longleftrightarrow [a,b]=[2,0]\ , \qquad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \longleftrightarrow [a,b]=[0,1]\ . \tag{5.4.17}$$

At the next step, i.e., at $N=3$, we find the decomposition

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array} \longleftrightarrow [a,b]=[3,0]\ , \qquad \begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longleftrightarrow [a,b]=[1,1]\ , \qquad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \longleftrightarrow [a,b]=[0,0]\ . \tag{5.4.18}$$

Note that there are two different fillings of $\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, and hence the corresponding representation of $\mathfrak{sl}(3,\mathbb{C})$ appears twice in $V^{\otimes 3}$, i.e., we have the decomposition

$$V^{\otimes 3} = [3,0]\oplus 2\cdot[1,1]\oplus[0,0]\ . \tag{5.4.19}$$

Again, if we write this out explicitly, then the states associated to $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$ are spanned by the 10 vectors of the form

$$\mathrm{Sym}^3 V: \quad (\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k)+\text{perm.}\ , \tag{5.4.20}$$

where $1 \leq i \leq i \leq k \leq 3$. For the states associated to $\yng(2,1)$ there are two Young symmetrisers we may use; if we consider the symmetriser (2.1.21), then the resulting space is spanned by the 8 vectors of the form

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \; : \qquad \begin{aligned} &(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) + (\mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_k) - (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) - (\mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j) \ , \\ &(\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) + (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) - (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) - (\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j) \ , \end{aligned}$$

while for the symmetriser (2.1.28) the resulting space is spanned by the 8 vectors of the form

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \; : \qquad \begin{aligned} &(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) + (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) - (\mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_k) - (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) \\ &(\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j) + (\mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i) - (\mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j) - (\mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_i) \ . \end{aligned}$$

Each such 8-dimensional space is separately invariant under the action of $\mathfrak{sl}(3,\mathbb{C})$, and defines the irreducible representation associated to $[1,1]$. Finally, the subspace associated to $\yng(1,1,1)$ is just 1-dimensional, namely the vector

$$\Lambda^3 V : \quad (\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3) \pm \text{perm.} \ , \tag{5.4.21}$$

Note that the corresponding representation of $\mathfrak{sl}(3,\mathbb{C})$ must therefore be the trivial, one-dimensional representation of $\mathfrak{sl}(3,\mathbb{C})$. Thus, for $\mathfrak{sl}(3,\mathbb{C})$, $\Lambda^3 V$ does not define a 'new' representation, and the same will be the case whenever the Young diagram contains 3 rows. (If the Young diagram contains more than 3 rows, then the corresponding representation will even vanish since $\Lambda^n V \cong 0$ for $n > 3$, see the comment above.) Thus we conclude that the Young diagrams that will label the irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$ are precisely those that have at most 2 rows. In fact, the correspondence between these Young diagrams and the $\mathfrak{sl}(3,\mathbb{C})$ representations is simply

$$\text{Young diagram with row lengths } (r_1, r_2) \ \longleftrightarrow \ [a,b] = [r_1 - r_2, r_2] \ . \tag{5.4.22}$$

Incidentally, an analogous description is also available for $\mathfrak{sl}(2,\mathbb{C})$. In this case, we only consider Young diagrams with at most 1 row, and then the correspondence is simply that the Young diagram with n horizontal boxes corresponds to the representation $V^{(n)}$. As we shall see soon, this description in fact generalises to all $\mathfrak{sl}(N,\mathbb{C})$ algebras.

It is also not very surprising that we can now describe various properties of the $\mathfrak{sl}(3,\mathbb{C})$ representations in terms of combinatorial data of the corresponding Young diagrams. For example, the dimension of the $\mathfrak{sl}(3,\mathbb{C})$ representation associated to the Young diagram Y equals

$$\dim_{\mathfrak{sl}(3,\mathbb{C})}(Y) = \text{number of Young tableau fillings of } Y \text{ with integers } \{1,2,3\}. \tag{5.4.23}$$

So for example, we then have

$$\dim_{\mathfrak{sl}(3,\mathbb{C})}(\square) = 3 \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array}\,, \tag{5.4.24}$$

and

$$\dim_{\mathfrak{sl}(3,\mathbb{C})}\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\right) = 3 \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\,, \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}\,, \tag{5.4.25}$$

as well as

$$\dim_{\mathfrak{sl}(3,\mathbb{C})}\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}\right) = 6 \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array}\,, \tag{5.4.26}$$

etc. More generally (see **exercise**), using the identification (5.4.22), we find that

$$\dim([a,b]) = \frac{1}{2}(a+1)(b+1)(a+b+2)\ . \tag{5.4.27}$$

Again, the corresponding statement then also holds for the case of $\mathfrak{sl}(2,\mathbb{C})$; there we have instead

$$\dim_{\mathfrak{sl}(2,\mathbb{C})}(Y) = \text{number of Young tableau fillings of } Y \text{ with integers } \{1,2\}. \tag{5.4.28}$$

Since we can only put 1s and 2s into these boxes, and since once we have put the first 2, all the boxes to the right of it have to be filled with 2s, it follows that there are $n+1$ such fillings if the Young diagram has n boxes, i.e.,

$$\dim_{\mathfrak{sl}(2,\mathbb{C})}(V^{(n)}) = n+1\ , \tag{5.4.29}$$

in agreement with what we had before.

The other very nice property one can read off from this Young diagram description, concerns the structure of the Clebsch-Gordan series, i.e., the behaviour of the representations under tensor products. (Let us first explain this for $\mathfrak{sl}(3,\mathbb{C})$, and then describe how the (obvious) generalisation for $\mathfrak{sl}(2,\mathbb{C})$ reproduces what we have found before.) Suppose we want to consider the tensor product of the representation W associated to the Young diagram Y with the 3-dimensional fundamental representation V that corresponds to a the Young diagram consisting of a single box; the general case (where V is not the fundamental representation but rather labelled by a more complicated Young diagram), can be described similarly, but is more complicated. Then the tensor product is the direct sum of representations that are associated to those Young diagrams one can obtain from Y upon attaching a single box — the box corresponding to V — to the given Young diagram in a way that is allowed by the usual Young diagram rules. In addition, any complete column of 3 boxes is simply removed. So the tensor products are then for example

$$\square \otimes \square = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\ , \tag{5.4.30}$$

thereby reproducing (5.3.2). Similarly, we have

$$\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1} \ , \tag{5.4.31}$$

and

$$\ydiagram{1,1} \otimes \ydiagram{1} \ = \ \ydiagram{2,1} \oplus \ydiagram{1,1,1} \ \cong \ \ydiagram{2,1} \oplus \mathbf{1} \ , \tag{5.4.32}$$

where we have used, in the last step, the fact that the a column of 3 boxes is simply removed. Note that these tensor product rules at least respect the dimension formula, since we have

$$\dim_{\mathfrak{sl}(3,\mathbb{C})}(\ydiagram{2}) = 6 \ , \quad \dim_{\mathfrak{sl}(3,\mathbb{C})}(\ydiagram{1,1}) = 3 \ , \quad \dim_{\mathfrak{sl}(3,\mathbb{C})}(\ydiagram{2,1}) = 8 \ , \quad \dim_{\mathfrak{sl}(3,\mathbb{C})}(\ydiagram{3}) = 10 \ . \tag{5.4.33}$$

Thus (5.4.31) corresponds, on the level of dimensions, to the identity

$$6 \times 3 = 10 + 8 \ , \tag{5.4.34}$$

while for the case of (5.4.31) we have

$$3 \times 3 = 8 + 1 \ . \tag{5.4.35}$$

We should also mention in passing that (5.4.19) reflects the dimension identity

$$3^3 = 27 = 10 + 2 \cdot 8 + 1 \ . \tag{5.4.36}$$

For the case of $\mathfrak{sl}(2,\mathbb{C})$, the corresponding rule is then simply that the tensor product with the 2-dimensional fundamental representation corresponds to adding the box to the n horizontal boxes of $V^{(n)}$ in one of the two possible ways, i.e., as

$$\ydiagram{6} \otimes \ydiagram{1} \ = \ \ydiagram{7} \oplus \ydiagram{6,1} \ \cong \ \ydiagram{7} \oplus \ydiagram{5} \ , \tag{5.4.37}$$

where in the second step we have now removed the full column of 2 boxes; translated into the previous conventions, this is then just the tensor product rule

$$V^{(n)} \otimes V^{(1)} = V^{(n+1)} \oplus V^{(n-1)} \ , \tag{5.4.38}$$

see eq. (4.3.5).

6 General simple Lie algebras

As we said before once the analysis of the representation theory of $\mathfrak{sl}(3,\mathbb{C})$ is understood, the analysis of the representations of any simple Lie algebra $\mathfrak{g}$ will be clear, at least in outline. In the following we would like to explain how this analysis works. We shall then specify more specifically how the resulting structure can be described for the case of $\mathfrak{sl}(N,\mathbb{C})$, and what the main features of the other simple Lie algebras are.

6.1 The general analysis

Given any simple Lie algebra $\mathfrak{g}$, the first step consists of finding an abelian subalgebra $\mathfrak{h}\subset\mathfrak{g}$ that acts diagonally on any (finite-dimensional) representation of $\mathfrak{g}$. For the case of $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{h}$ is simply the subalgebra generated by H, while for the case of $\mathfrak{sl}(3,\mathbb{C})$ it is the subalgebra generated by H_{12} and H_{23}. For the case of matrix algebras, we can always think of $\mathfrak{h}$ as being spanned by the diagonal matrices (that clearly commute with one another). As before for the case of $\mathfrak{sl}(3,\mathbb{C})$, we want to organise the whole Lie algebra in terms of eigenvectors of $\mathfrak{h}$. Thus, in order for this to be as useful as possible, we should take $\mathfrak{h}$ to be a **maximal abelian subalgebra** of $\mathfrak{g}$; such a subalgebra is then called a **Cartan subalgebra**.

The choice of a Cartan subalgebra is obviously not unique, but different choices only differ in some inconsequential manner — they are related to one another by the action of the group (the adjoint representation) — and we therefore need not worry about this problem. We should mention that the dimension of $\mathfrak{h}$ is called the **rank** of $\mathfrak{g}$; so $\mathfrak{sl}(2,\mathbb{C})$ has rank 1, while $\mathfrak{sl}(3,\mathbb{C})$ has rank 2.

Once we have found a Cartan subalgebra, we can decompose the whole adjoint representation under the action of $\mathfrak{h}$. By assumption, the action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalisable, and thus we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} \;=\; \mathfrak{h} \;\oplus\; \bigoplus_{\alpha} \mathfrak{g}_\alpha \ , \tag{6.1.39}$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$, and acts on it by scalar multiplication by the linear functional $\alpha\in\mathfrak{h}^*$. This is to say, for any $H\in\mathfrak{h}$ and any $X\in\mathfrak{g}_\alpha$, we have

$$[H,X]=\alpha(H)\cdot X \ . \tag{6.1.40}$$

The second direct sum in (6.1.39) is over a finite set of eigenvalues $\alpha\in\mathfrak{h}^*$; these eigenvalues are called the **roots** of the Lie algebra $\mathfrak{g}$, and the corresponding eigenspaces are called **root spaces**. Of course, $\mathfrak{h}$ itself is just the eigenspace for the action of $\mathfrak{h}$ corresponding to the eigenvalue 0, so in some contexts it is convenient to adapt the notation $\mathfrak{g}_0=\mathfrak{h}$. However, usually we will not count $0\in\mathfrak{h}^*$ as a root. The set of all roots will be denoted by $\Delta\subset\mathfrak{h}^*$. We also recall that, by the fundamental calculation, the adjoint action of $\mathfrak{g}_\alpha$ carries $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha+\beta}$, i.e.

$$[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta} \ . \tag{6.1.41}$$

Since the whole Lie algebra $\mathfrak{g}$ is finite dimensional, each root space $\mathfrak{g}_\alpha$ is finite-dimensional; in fact, it turns out that every $\mathfrak{g}_\alpha$ is **one-dimensional**. We also note that since $\mathfrak{h}$ has dimension $r = \dim(\mathfrak{h})$, each root α is an r-dimensional vector. The root lattice $\Lambda_R \subset \mathfrak{h}^*$ that is generated by these roots is thus a sublattice of $\mathbb{R}^r$; in fact, it turns out to have maximal rank, i.e., it is a lattice of rank r. Furthermore, one finds that Δ is symmetric about the origin, i.e., if $\alpha \in \Delta$, then $-\alpha \in \Delta$. These statements are just the obvious generalisation of the corresponding observations for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$; they are true for any simple Lie algebra, but we shall not attempt to prove this in generality here.

6.1.1 Representation theory

Next we want to understand the general structure of the representation theory of $\mathfrak{g}$. Suppose then that V is a finite-dimensional irreducible representation of $\mathfrak{g}$. By assumption, the generators of the Cartan subalgebra $\mathfrak{h}$ act diagonalisably on V, so we can write

$$V = \bigoplus_\alpha V_\alpha \ , \tag{6.1.42}$$

where the direct sum runs over a finite set of $\alpha \in \mathfrak{h}^*$, and $\mathfrak{h}$ acts diagonally on each V_α by multiplication by the eigenvalue α, i.e., for any $H \in \mathfrak{h}$ and $v \in V_\alpha$, we have

$$H(v) = \alpha(H) \cdot v \ . \tag{6.1.43}$$

The eigenvalues $\alpha \in \mathfrak{h}^*$ that appear in this direct sum decomposition are called the **weights** of V; the V_α are called the **weight spaces**, and the dimension of a weight space V_α will be called the **multiplicity** of the weight α in V. Again, each weight is an r-dimensional vector, where $r = \mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$.

The action of the rest of the Lie algebra on V can again be described in these terms: for any root $\beta \in \Delta$ we have (again by the fundamental calculation)

$$\mathfrak{g}_\beta : V_\alpha \to V_{\alpha+\beta} \ . \tag{6.1.44}$$

Thus it follows that all the weights of an irreducible representation are congruent to one another modulo the root lattice Λ_R; otherwise, for any weight α of V, the subspace

$$V' = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta} \tag{6.1.45}$$

would be a proper subrepresentation of V.

To understand precisely which weights appear in a representation, it is again useful to find distinguished subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{g}$. Recall that each $\mathfrak{g}_\alpha$ is one-dimensional, and that for each $\alpha \in \Delta$, also $-\alpha \in \Delta$. Furthermore, by the fundamental calculation, we know that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0 \cong \mathfrak{h}$. Thus we have a three-dimensional subalgebra of $\mathfrak{g}$ spanned by

$$\mathfrak{s}_\alpha \ : \quad \mathfrak{g}_\alpha \ \oplus \ \mathfrak{g}_{-\alpha} \ \oplus \ [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \ , \tag{6.1.46}$$

and it turns out that $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$; we denote the corresponding generators by $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha = [E_\alpha, F_\alpha]$ with $[H_\alpha, E_\alpha] = 2E_\alpha$ as well as $[H_\alpha, F_\alpha] = -2F_\alpha$.

Then we can again use the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ to conclude that the eigenvalues of $H_\alpha \in \mathfrak{h}$ have to be integer. Thus every eigenvalue $\beta \in \mathfrak{h}^*$ that appears in *any* representation V must be such that $\beta(H_\alpha) \in \mathbb{Z}$. Let us denote the set of linear functionals $\beta \in \mathfrak{h}^*$ that are integer valued on all H_α by Λ_W; Λ_W is a lattice that is called the 'weight lattice' of $\mathfrak{g}$, and all weights that appear in any representation of $\mathfrak{g}$ must lie in Λ_W. Obviously, by construction (since the adjoint representation is also a representation), $\Lambda_R \subset \Lambda_W$. In fact, since both lattices have rank r, Λ_R is a finite index sublattice of Λ_W, i.e., the quotient space

$$\Lambda_W/\Lambda_R = \mathcal{Z} \tag{6.1.47}$$

is a finite group. This group can actually be identified with the center of the (simply-connected) Lie group G associated to $\mathfrak{g}$.

Next, we want to find a convenient description of the different irreducible representations of $\mathfrak{g}$. To this end, we need again to define a suitable ordering on the set of roots. We choose a real linear functional l on the root space that is irrational with respect to the root lattice Λ_R. Then we get a decomposition of the roots into

$$\Delta = \Delta_+ \cup \Delta_- \ , \tag{6.1.48}$$

where the **positive roots** in Δ_+ are those roots α for which $l(\alpha) > 0$, while the negative roots $\beta \in \Delta_-$ satisfy $l(\beta) < 0$. The point of choosing a direction — and thereby an ordering of the roots into positive and negative roots — is, of course, to mimic the notion of highest weight vector that was so crucial in the analysis for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$. (Obviously, as became clear for the case of $\mathfrak{sl}(3,\mathbb{C})$, there is some arbitrariness in making this decomposition, but this is again largely irrelevant — so from now on we shall assume, that one such choice has been made.)

Suppose now that V is an irreducible finite-dimensional representation of $\mathfrak{g}$. Then it contains a **highest weight vector** $v \in V$ that is an eigenvector of $\mathfrak{h}$ and annihilated by all positive roots, i.e., by all elements in $\mathfrak{g}_\alpha$ for $\alpha \in \Delta_+$. In fact, if V is irreducible, the corresponding weight space is one-dimensional, i.e., the full irreducible representation V is obtained by the action of the negative roots, starting from a single highest weight vector. The weight of the highest weight vector will be called the **highest weight** of the corresponding representation, and it characterises it uniquely.

The highest weight of any irreducible representation obviously lies in the weight lattice Λ_W of $\mathfrak{g}$, but even more is true. In order to explain the relevant constraint in detail, let us choose a set of so-called simple (positive) roots so that any positive root is a non-negative integer linear combination of the simple roots. (For example, for the case of $\mathfrak{sl}(3,\mathbb{C})$, for which we can take the positive roots to be E_{12}, E_{23} amd E_{13}, the simple roots are E_{12} and E_{23} since the corresponding root vectors $L_1 - L_2$ and $L_2 - L_3$ generate in particular also the root vector of E_{13}, $L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3)$.) In general there are always $r = \mathrm{rank}(\mathfrak{g})$ simple roots and they form a basis for the root lattice. We shall denote them by α_i, $i = 1, \ldots, r$.

Then we consider the $\mathfrak{sl}(2,\mathbb{C})$ algebras associated to the simple roots, i.e., the analogues of $\mathfrak{s}_{L_1-L_2}$ and $\mathfrak{s}_{L_2-L_3}$ above. Since the highest weight β is in particular a highest weight for all of these $\mathfrak{sl}(2,\mathbb{C})$ algebras, it follows that β has the property that

$$\beta(H_i) \in \mathbb{N}_0 \ , \tag{6.1.49}$$

where H_i is the (canonically normalised) Cartan generator of the $\mathfrak{sl}(2,\mathbb{C})$ associated to α_i and $-\alpha_i$. (So if we denote the corresponding Lie algebra generators by $E_i \in \mathfrak{g}_{\alpha_i}$ and $F_i \in \mathfrak{g}_{-\alpha_i}$ then $H_i = [E_i, F_i]$ with $[H_i, E_i] = 2E_i$ and $[H_i, F_i] = -2F_i$.)

The simplest representations are those for which all but one $\beta(H_i) = 0$, with the remaining one equal to $\beta(H_j) = 1$. They are called the **fundamental weights**, and we denote them by ω_j, $j = 1, \ldots, r$, with

$$\omega_j(H_i) = \delta_{ij} \ . \tag{6.1.50}$$

The most general highest weight can then be written as a non-negative integer linear combination of these fundamental weights,

$$\beta = \sum_{i=1}^{r} m_i\, \omega_i \ . \tag{6.1.51}$$

Thus the irreducible representations of $\mathfrak{g}$ are parametrised by r-tuples of non-negative integers $[m_1, \ldots, m_r]$; the integers m_i are then called the **Dynkin labels** of the highest weight representation with highest weight β.

6.2 The case of $\mathfrak{sl}(N,\mathbb{C})$

Let us flesh out these ideas for the case of $\mathfrak{sl}(N,\mathbb{C})$, i.e., the complexification of the Lie algebra $\mathfrak{su}(N)$. Given that $\mathfrak{sl}(N,\mathbb{C})$ is a matrix algebra, a natural choice for the Cartan subalgebra is

$$\mathfrak{h} = \left\{ \sum_{i=1}^{N} a_i H_i \mid \sum_i a_i = 0 \right\} \ , \tag{6.2.52}$$

where H_i is the diagonal matrix whose only non-zero entry is a 1 in the (ii) position. In particular, it is clear from this that $\dim(\mathfrak{h}) = N-1$, so $\mathfrak{sl}(N,\mathbb{C})$ has rank $r \equiv N-1$.

As for the case of $\mathfrak{sl}(3,\mathbb{C})$, let us define the element $L_i \in \mathfrak{h}^*$ by $L_i(H_j) = \delta_{ij}$. Then it is easy to see that the roots of $\mathfrak{sl}(N,\mathbb{C})$ are described by the differences $L_i - L_j$, where $i \neq j$. (Indeed, the matrix whose only non-zero entry is in the (ij) position has this weight under the adjoint action of $\mathfrak{h}$.) As a consequence, the root lattice can therefore be described as

$$\Lambda_R = \left\{ \sum_{i=1}^{N} a_i L_i \mid a_i \in \mathbb{Z} \ , \ \ \sum_i a_i = 0 \right\} \ . \tag{6.2.53}$$

Note that Λ_R is indeed a lattice of rank $r = N-1$.

Next, we want to describe the weight lattice of $\mathfrak{sl}(N,\mathbb{C})$. In order to understand the relevant integrality conditions, we observe that the algebra $\mathfrak{s}_\alpha$ with $\alpha = L_i - L_j$ is generated by

$$E_{ij}\ ,\quad E_{ji}\ ,\quad [E_{ij}, E_{ji}] = H_i - H_j\ . \tag{6.2.54}$$

Any weight that appears in a representation of $\mathfrak{sl}(N,\mathbb{C})$ must therefore have integer eigenvalue with respect to $H_i - H_j$, and this is the only condition; thus we conclude that the weight lattice of $\mathfrak{sl}(N,\mathbb{C})$ is

$$\Lambda_W = \left\{ \sum_{i=1}^{N} a_i L_i \mid a_i \in \mathbb{Z} \right\} / \left\langle \sum_i L_i \right\rangle\ , \tag{6.2.55}$$

where the quotient is a consequence of the fact that $\sum_i L_i = 0$ in $\mathfrak{h}^*$. Note that

$$\Lambda_W / \Lambda_R = \mathbb{Z}_N\ ; \tag{6.2.56}$$

modulo Λ_R, the weight lattice is generated by L_1, say (since any other $L_i = (L_i - L_1) + L_1$), and we have the relation that

$$N L_1 = \sum_{i=1}^{N} (L_1 - L_i) + \sum_{i=1}^{N} L_i \in \Lambda_R\ . \tag{6.2.57}$$

We can introduce an ordering on the roots by saying that the root $L_i - L_j$ is positive (negative) if $i < j$ ($i > j$). Then the simple roots are

$$\alpha_i = L_i - L_{i+1}\ ,\quad i = 1, \ldots, N-1\ , \tag{6.2.58}$$

and the fundamental weights turn out to be

$$\omega_i = L_1 + \cdots + L_i = \sum_{j=1}^{i} L_j\ , \tag{6.2.59}$$

where $i = 1, \ldots, N-1$.

The fundamental (defining) N-dimensional representation V of $\mathfrak{sl}(N,\mathbb{C})$ corresponds to the fundamental weight $\omega_1 = L_1$, i.e., the Dynkin label $[1, 0, \ldots, 0]$. The conjugate representation has weight $\alpha = -L_N = \omega_{N-1} - \sum_{j=1}^{N} L_j$; thus it corresponds to the Dynkin label $[0, 0, \ldots, 0, 1]$. As for the case of $\mathfrak{sl}(3,\mathbb{C})$, any irreducible representation of $\mathfrak{sl}(N,\mathbb{C})$ occurs in a suitable tensor product of the fundamental N-dimensional representation V. As before for the case of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$, S_M commutes with the action of $\mathfrak{sl}(N,\mathbb{C})$ on $V^{\otimes M}$, and forms indeed a dual pair. Thus we can label the representations of $\mathfrak{sl}(N,\mathbb{C})$ by Young diagrams. Furthermore, since

$$\Lambda^N V \cong \mathbb{C} \tag{6.2.60}$$

is the trivial representation of $\mathfrak{sl}(N,\mathbb{C})$, the Young diagrams that occur for $\mathfrak{sl}(N,\mathbb{C})$ are those that do not contain any columns of length N (or greater). In fact, we find that the correspondence between Young diagrams and Dynkin labels is simply

$$\Big(\text{Young diagram with row length } r_i,\ i=1,\ldots,N-1\Big) \ \leftrightarrow\ [r_1-r_2, r_2-r_3, \ldots, r_{N-1}]\ . \tag{6.2.61}$$

Thus the fundamental representation corresponds again to the Young diagram $\square$, with $r_1 = 1$, $r_j = 0$ for $j \geq 2$. On the other hand, the conjugate representation of the fundamental representation has $r_1 = r_2 = \cdots = r_{N-1} = 1$, and hence corresponds to the Young diagram with only one column of length $N-1$.

As for the case of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$ there are now simple dimension formulae for these representations; in fact, we have the natural generalisation of (5.4.23)

$$\dim_{\mathfrak{sl}(N,\mathbb{C})}(Y) = \text{number of Young tableau fillings of } Y \text{ with integers } \{1,2,\ldots,N\}. \tag{6.2.62}$$

So for example, we have

$$\dim_{\mathfrak{sl}(N,\mathbb{C})}(\square) = N\ , \qquad \dim_{\mathfrak{sl}(N,\mathbb{C})}(\ydiagram{2}) = \tfrac{N(N+1)}{2}\ , \qquad \dim_{\mathfrak{sl}(N,\mathbb{C})}(\ydiagram{1,1}) = \tfrac{N(N-1)}{2}\ , \tag{6.2.63}$$

as well as

$$\dim_{\mathfrak{sl}(N,\mathbb{C})}(\ydiagram{2,1}) = \sum_{j=1}^{N}(N+1-j)(N-j) = \frac{1}{3}(N^3-N)\ , \tag{6.2.64}$$

and

$$\dim_{\mathfrak{sl}(N,\mathbb{C})}(\ydiagram{1,1,1}) = \sum_{j=1}^{N}\sum_{i=j+1}^{N}(N-i) = \frac{1}{6}N(N-1)(N-2)\ , \tag{6.2.65}$$

etc. Furthermore, the tensor product decomposition rules can be read off from the corresponding Young diagrams; so for example, we have again

$$\square \otimes \square = \ydiagram{2} \oplus \ydiagram{1,1}\ , \tag{6.2.66}$$

which in terms of dimensions is just the formula

$$N^2 = \frac{N(N+1)}{2} + \frac{N(N-1)}{2}\ . \tag{6.2.67}$$

Similarly we have

$$\ydiagram{1,1} \otimes \square = \ydiagram{2,1} \oplus \ydiagram{1,1,1}\ , \tag{6.2.68}$$

which in terms of dimensions corresponds to

$$N\frac{N(N-1)}{2} = \frac{1}{3}(N^3-N) + \frac{1}{6}N(N-1)(N-2)\ , \tag{6.2.69}$$

etc. We should also mention that the adjoint representation appears in $V \otimes V^*$, and thus corresponds to the Young diagram with row labels $r_1 = 2$, $r_2 = \cdots = r_{N-1} = 1$, i.e., to the Dynkin label $[1, 0, \ldots, 0, 1]$. In order to calculate its dimension we note that the box in the top left corner can either be filled by 1 or 2 only; if it is filled with 2, then the remaining boxes along the first column are uniquely fixed to be $3, 4, \ldots, N$, while there are $N - 1$ choices for the second box in the first row. On the other hand, if the box in the top left corner is filled by a 1, there are N different choices for filling the second box in the first row, and $N - 1$ different choices for filling the boxes along the first column. Altogether we therefore conclude that

$$\dim_{\mathfrak{sl}(N,\mathbb{C})}(\text{adj}) = N(N-1) + (N-1) = N^2 - 1 \ , \tag{6.2.70}$$

in agreement with what we had before.

6.3 Other simple Lie algebras

The simple complex Lie algebras have been classified, and apart from the $\mathfrak{sl}(N, \mathbb{C})$ series we have studied above, there are two (or rather really three) more infinite series, as well as a number of isolated (or exceptional) Lie algebras. The other two infinite families correspond to the complexifications of the Lie algebras

$$\mathfrak{so}(N) \ , \qquad \text{and} \qquad \mathfrak{sp}(2N) \ , \tag{6.3.71}$$

while the exceptional Lie algebras are denoted by

$$\mathfrak{g}_2 \ , \qquad \mathfrak{f}_4 \ , \qquad \mathfrak{e}_6 \ , \quad \mathfrak{e}_7 \ , \quad \mathfrak{e}_8 \ . \tag{6.3.72}$$

(Here the index always denotes the rank of the relevant Lie algebra.) Actually, the structure of the $\mathfrak{so}(N)$ algebras depends fairly crucially on the cardinality of N, so they really form two separate families corresponding to $\mathfrak{so}(2N)$ and $\mathfrak{so}(2N-1)$. In the mathematics literature, the corresponding Lie algebras are denoted by

$$\mathfrak{a}_{N-1} \cong \mathfrak{sl}(N, \mathbb{C}) \ , \qquad \mathfrak{d}_N \cong \mathfrak{so}(2N)_{\mathbb{C}} \ , \tag{6.3.73}$$

as well as

$$\mathfrak{b}_N \cong \mathfrak{so}(2N+1)_{\mathbb{C}} \ , \qquad \mathfrak{c}_N \cong \mathfrak{sp}(2N)_{\mathbb{C}} \ . \tag{6.3.74}$$

There are some low-level identifications, i.e.,

$$\mathfrak{d}_2 \cong \mathfrak{a}_1 \oplus \mathfrak{a}_1 \ , \qquad \mathfrak{d}_3 \cong \mathfrak{a}_3 \ , \tag{6.3.75}$$

as well as

$$\mathfrak{b}_1 \cong \mathfrak{c}_1 \cong \mathfrak{a}_1 \ , \qquad \mathfrak{b}_2 \cong \mathfrak{c}_2 \ , \tag{6.3.76}$$

but for larger values of N, these algebras are all inequivalent. For all of these algebras, the roots, as well as the root and weight lattices are known, and hence the representation theory is very well understood — explicit descriptions are for example given in the book of [FH].

The algebras corresponding to the $\mathfrak{a}$, $\mathfrak{d}$ and $\mathfrak{e}$ cases are in some sense simpler since they are **simply-laced**. In order to explain what this means, recall that the Killing form induces a natural inner product on $\mathfrak{h}$, and therefore also on its dual space $\mathfrak{h}^*$; in particular, since all of these Lie algebras are simple, we may simply take the Killing form to be the (suitably normalised) trace in the fundamental representation. For example, for the case of the $\mathfrak{sl}(N,\mathbb{C})$ algebras, the inner product is just given by — remember that H_i is the matrix with a single 1 in (ii) position, and L_i is the natural dual vector

$$(L_i, L_j) = \delta_{ij} \ . \tag{6.3.77}$$

The roots of $\mathfrak{sl}(N,\mathbb{C})$ are of the form $L_i - L_j$ with $i \neq j$, and thus we note that the length squared of each root is the same, namely equal to 2. Algebras for which this is the case, i.e., for which the length squared of every root is the same — after a suitable rescaling this length squared is conventionally taken to be equal to 2 — are called **simply-laced**.

Actually, we can characterise the Lie algebra completely by its roots, together with this inner product. In fact, we don't even need to specify all roots, but it is enough to consider the **simple roots**. (Recall that the simple roots have the property that (i) they generate the whole root lattice; and (ii) the positive roots — we assume we have made some choice of splitting the roots into positive and negative roots — are precisely non-negative integer combinations of the simple roots.) For example, for the case of $\mathfrak{sl}(N,\mathbb{C})$, with the choice of positive roots above, the (positive) simple roots are, see (6.2.58)

$$\alpha_i = L_i - L_{i+1} \ , \qquad i = 1, \ldots, N-1 \ . \tag{6.3.78}$$

Note that there are precisely $r = \mathrm{rank}(\mathfrak{g})$ simple roots. Instead of the inner products it is convenient to determine the so-called **Cartan matrix** which is defined by

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \ . \tag{6.3.79}$$

Note that, in the simply laced case and with the standard normalisation of the inner product, the Cartan matrix is just equal to $C_{ij} = (\alpha_i, \alpha_j)$; for the case of $\mathfrak{sl}(N,\mathbb{C})$ (or more specifically $\mathfrak{sl}(7,\mathbb{C})$) it takes the form

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \ . \tag{6.3.80}$$

In terms of this notation, the correct integrality condition for the weights is then that they live in the lattice

$$\Lambda_W = \left\{ x \in \mathbb{R}^r \ : \ \frac{2\,(x, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z} \ , \ \forall i = 1, \ldots, r \right\} \ . \tag{6.3.81}$$

For the simply-laced case (where $(\alpha_i, \alpha_i) = 2$), the weight lattice is then simply the dual lattice $\Lambda_W = \Lambda_R^*$; it is generated by the fundamental weights ω_i that are characterised by

$$(\omega_i, \alpha_j) = \delta_{ij} \ . \tag{6.3.82}$$

For the case of $\mathfrak{sl}(N, \mathbb{C})$, the fundamental weights can be taken to be (compare with (6.2.59))

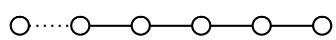
$$\omega_i = \sum_{j=1}^{i} L_i - \frac{i}{N} \sum_{j=1}^{N} L_j \ , \qquad i = 1, \ldots, N-1 \ , \tag{6.3.83}$$

where the last term is subtracted so as to ensure that also ω_i is orthogonal to $\sum_j L_j$. The inner product matrix of the fundamental weights is then the inverse of the Cartan matrix. The highest weight of a representation can be written in terms of a non-negative integer linear combination of the fundamental weights,

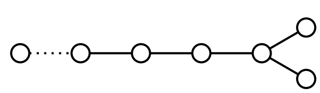
$$\lambda = \sum_{i=1}^{r} m_i \omega_i \ . \tag{6.3.84}$$

The $[m_1, \ldots, m_r]$ are then the Dynkin labels of the representation λ.

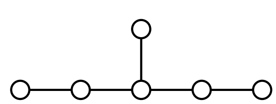
It turns out that the Cartan matrix determines the Lie algebra uniquely. In fact, since most of the entries of the Cartan matrix are actually zero, one can capture the whole information contained in the Cartan matrix more conveniently in terms of a so-called **Dynkin diagram**. To this end one draws a vertex for each simple root, and connects two vertices by $\max\{|C_{ij}|, |C_{ji}|\}$ lines. Furthermore — this only happens in the non simply-laced case — an arrow is added to the lines from the simple root i to the simple root j if $|C_{ij}| > |C_{ji}| > 0$. So, for the case of the $\mathfrak{sl}(N, \mathbb{C})$ algebras, the Dynkin diagram is simply described by

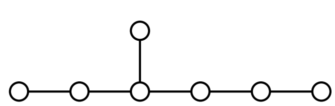
$\mathfrak{a}_n$: 

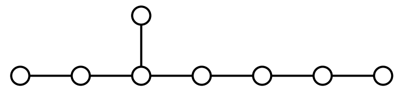
since there is just a single line between α_i and α_{i+1}. The Dynkin diagrams for the other simply-laced Lie algebras are

$\mathfrak{d}_n$: 

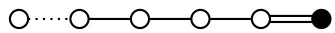
as well as

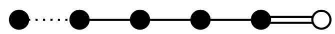
$\mathfrak{e}_6$: 

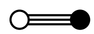
$\mathfrak{e}_7$: 

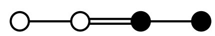
$\mathfrak{e}_8$: 

For completeness, we also give the Dynkin diagrams for the non simply-laced algebras (where instead of drawing an arrow the convention is that the arrow is from the unfilled vertex to the filled vertex); they are

$\mathfrak{b}_n$: 

$\mathfrak{c}_n$: 

$\mathfrak{g}_2$: 

$\mathfrak{f}_4$: 

For the $\mathfrak{d}_n$ case, the root lattice can be taken to be of the form

$$\Lambda_R = \left\{ \sum_{i=1}^{n} n_i \mathbf{e}_i \ : \ \sum_i n_i \in 2\mathbb{Z} \right\} , \tag{6.3.85}$$

where the $\mathbf{e}_i$ form an orthonormal basis with respect to the inner product (that is obtained from the Killing form). The simple roots may then be taken to be of the form

$$\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1} , \quad i = 1, \ldots, n-1 , \qquad \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n . \tag{6.3.86}$$

One easily checks that the corresponding Cartan matrix is then described by the above Dynkin diagram. The other roots are then simply of the form

$$\pm \mathbf{e}_i \pm \mathbf{e}_j , \qquad (i < j) , \tag{6.3.87}$$

and thus there are

$$\text{number of roots of } \mathfrak{d}_n = \mathfrak{so}(2n) \ = 4 \cdot \binom{n}{2} = 2n(n-1) . \tag{6.3.88}$$

Together with the n elements of the Cartan matrix we therefore obtain for the dimension of $\mathfrak{d}_n = \mathfrak{so}(2n)$

$$\dim(\mathfrak{so}(2n)) = n + 2n(n-1) = n(2n-1) = \frac{(2n)(2n-1)}{2} . \tag{6.3.89}$$

The corresponding fundamental weights are

$$\omega_i = \sum_{j=1}^{i} \mathbf{e}_j , \quad i = 1, \ldots, n-2 , \tag{6.3.90}$$

as well as

$$\omega_{n-1} = \frac{1}{2}\Big(\sum_{j=1}^{n-1} \mathbf{e}_j - \mathbf{e}_n\Big) \ , \qquad \omega_n = \frac{1}{2}\Big(\sum_{j=1}^{n-1} \mathbf{e}_j + \mathbf{e}_n\Big) \ . \tag{6.3.91}$$

Obviously, the root lattice is again a sublattice of the weight lattice, and the quotient group is

$$\Lambda_W/\Lambda_R \cong \begin{cases} \mathbb{Z}_4 & \text{if } n = 2m+1,\, m \in \mathbb{N} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n = 2m,\, m \in \mathbb{N}. \end{cases} \tag{6.3.92}$$

In particular, $2\omega_1 \in \Lambda_R$, and $2\omega_{n-1} \in \Lambda_R$ if $n = 2m$, while $2\omega_{n-1} \cong \omega_1$ modulo Λ_R if $n = 2m+1$.

The exceptional Lie algebra $\mathfrak{e}_6$ contains $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ as a subalgebra; in particular, the simple roots of $\mathfrak{e}_6$ may be taken to consist of the simple roots of $\mathfrak{so}(10)$,

$$\begin{aligned} \alpha_1 = (1,-1,0,0,0,0) \ , \quad \alpha_2 = (0,1,-1,0,0,0) \ , \quad \alpha_3 = (0,0,1,-1,0,0) \ , \\ \alpha_4 = (0,0,0,1,-1,0) \ , \quad \alpha_5 = (0,0,0,1,1,0) \ , \end{aligned} \tag{6.3.93}$$

together with

$$\alpha_6 = \Big(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\Big) \ . \tag{6.3.94}$$

Note that all of these roots have length squared two, and they give rise to the above Dynkin diagram for $\mathfrak{e}_6$. The Lie algebra of $\mathfrak{e}_6$ has dimension $\dim(\mathfrak{e}_6) = 78$; indeed, in addition to the 6-dimensional Cartan subalgebra, the Lie algebra has 72 roots — these are just the vectors of length squared two in the lattice generated by $\alpha_1, \ldots, \alpha_6$. In fact, $\mathfrak{e}_6$ is the extension of the Lie algebra $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ of dimension $\dim(\mathfrak{so}(10) \oplus \mathfrak{u}(1)) = 45 + 1 = 46$ by the two 16-dimensional spinor representations of $\mathfrak{so}(10)$; in the above language the corresponding roots are given by the 32 vectors of the form

$$\Big[\Big(\pm\frac{1}{2}\Big)^5, (-1)^{\delta+1}\frac{\sqrt{3}}{2}\Big] \ , \qquad \delta = \text{no. minus signs among the first five } \tfrac{1}{2}. \tag{6.3.95}$$

The center of the corresponding simply-connected Lie group is

$$\mathfrak{e}_6 : \qquad \Lambda_W/\Lambda_R \cong \mathbb{Z}_3 \ . \tag{6.3.96}$$

The exceptional Lie algebra $\mathfrak{e}_7$ contains $\mathfrak{so}(12) \oplus \mathfrak{u}(1)$ as a subalgebra; in particular, the simple roots of $\mathfrak{e}_7$ may be taken to consist of the simple roots of $\mathfrak{so}(12)$,

$$\begin{aligned} \alpha_1 = (1,-1,0,0,0,0,0) \ , \quad \alpha_2 = (0,1,-1,0,0,0,0) \ , \quad \alpha_3 = (0,0,1,-1,0,0,0) \ , \\ \alpha_4 = (0,0,0,1,-1,0,0) \ , \quad \alpha_5 = (0,0,0,0,1,-1,0) \ , \quad \alpha_6 = (0,0,0,0,1,1,0) \ , \end{aligned}$$

together with

$$\alpha_7 = \Big(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\Big) \ . \tag{6.3.97}$$

Again, all of these roots have length squared two, and they give rise to the above Dynkin diagram for $\mathfrak{e}_7$. The Lie algebra $\mathfrak{e}_7$ has dimension $\dim(\mathfrak{e}_7) = 133$, and the center of the corresponding simply-connected Lie group is

$$\mathfrak{e}_7 : \qquad \Lambda_W/\Lambda_R \cong \mathbb{Z}_2 \ . \tag{6.3.98}$$

Finally, the exceptional Lie algebra $\mathfrak{e}_8$ contains $\mathfrak{d}_8 = \mathfrak{so}(16)$; indeed the root lattice of $\mathfrak{e}_8$ is obtained from that of $\mathfrak{d}_8$ upon adjoining the 'spinor weight'

$$\omega_8 = (\underbrace{\tfrac{1}{2},\ldots,\tfrac{1}{2}}_{8 \text{ terms}}) \ . \tag{6.3.99}$$

Note that the length squared of this vector is also $8 \times \frac{1}{4} = 2$. The resulting root lattice is actually even (the length squared of any vector is even), and self-dual, i.e., the weight lattice (which is the dual of the root lattice) coincides with the root lattice; as a consequence the center of the associated simply-connected Lie group is trivial. The Lie algebra $\mathfrak{e}_8$ has dimension $\dim(\mathfrak{e}_8) = 248$.

We may mention in passing that the root lattice of $\mathfrak{e}_8$ is in fact the only even self-dual lattice in euclidean dimension 8. Even self-dual lattices in euclidean signature only exist in dimensions that are multiples of 8; in 16 dimensions there are 2 even self-dual lattices, the lattice $\Lambda_R(\mathfrak{e}_8) \oplus \Lambda_R(\mathfrak{e}_8)$ as well as the root lattice of $\Lambda_R(\mathfrak{d}_{16})$ extended by a spinor weight. In 24 dimensions, there are 24 even self-dual lattices, the 23 Niemeier lattices as well as the Leech lattice. The Leech lattice plays an important role, for example, in finite group theory, Monstrous Moonshine, etc.

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