Introduction to Integrability

Lecture Slides, Chapter 8

ETH Zurich, 2024 HS

PROF. N. BEISERT

The current version of this work can be found at: http://people.phys.ethz.ch/~nbeisert/lectures/.

8. Quantum Integrability

8.1 R-Matrix Formalism

SU(N) spin chain

$$S^{cd}_{ab}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{(u-v)-i}$$

for flavours $a,b,c,d = 2,\dots,N$

satisfies Yang-Baxter Eq / Factorised scattering

turn into SU(N) fund. R-Matrix

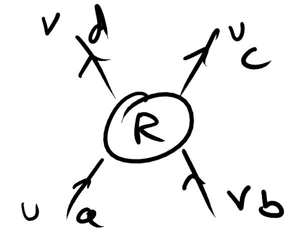
$$R^{cd}_{ab}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u-v+i}$$

$a,b,c,d = 1,\dots,N$ for SU(N) fund. rep

Rank-2 tensor operator $R: \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}$, $\mathbb{V} = \mathbb{C}^N$, $u,v \in \mathbb{C}$

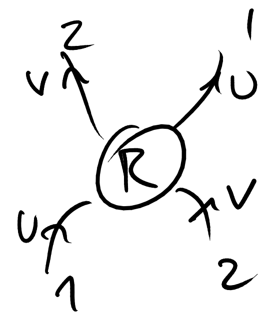
$$R(u,v) = \frac{(u-v)\,\mathrm{id} + i\,\mathrm{ex}}{u-v+i}$$

spectral parameters $= (E^a \otimes E^b)\, R^{cd}_{ab}\, (E_c \otimes E_d)$

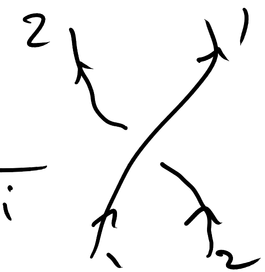


$R_{12} = R_{12}(u_1, v_2)$

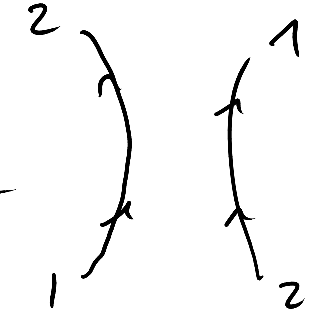
associated to spaces $\mathbb{V}_1, \mathbb{V}_2$



$$= \frac{u-v}{u-v+i}$$

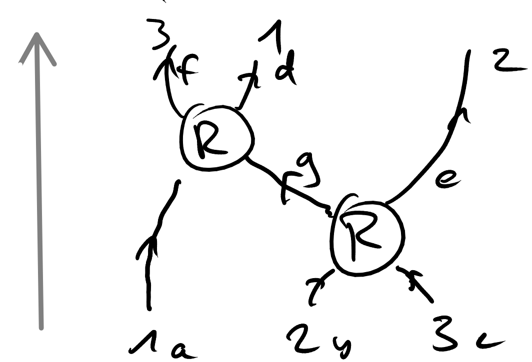


$$+ \frac{i}{u-v+i}$$



write composite objects (operator products) in components

$$R_{13} R_{23} =$$

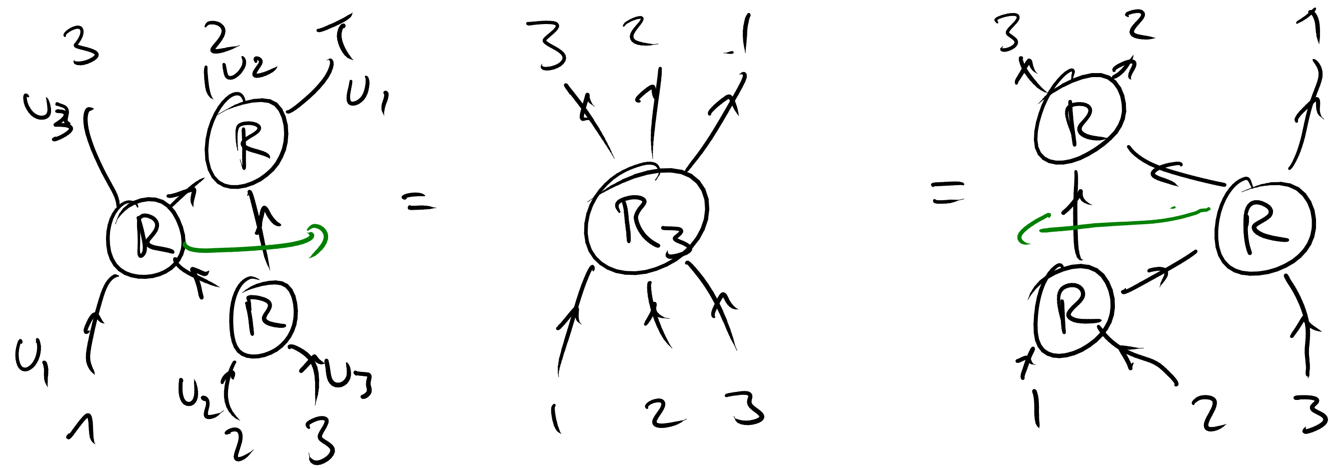


$$= R^{df}_{ag}(u_1, u_3)\, R^{eg}_{bc}(u_2, u_3)$$

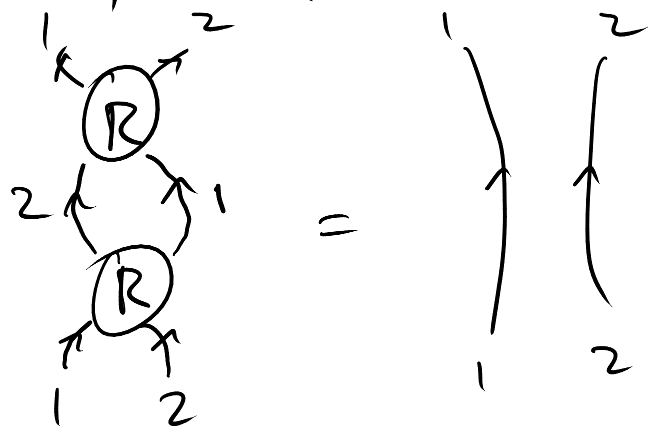
Properties of R-Matrices   Yang-Baxter Equation

$$R_{12}(u_1,u_2)\,R_{13}(u_1,u_3)\,R_{23}(u_2,u_3)$$
$$= R_{23}(u_2,u_3)\,R_{13}(u_1,u_3)\,R_{12}(u_1,u_2)$$

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12}$$



inverse property $\quad R_{21} R_{12} = \mathrm{id}_{12}$



here: exact

elsewhere: up to a factor

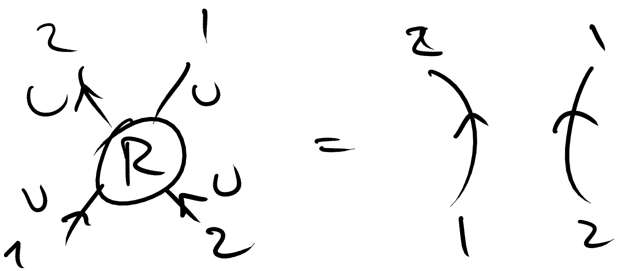
$$R_{12} = \frac{(u-v)\,\mathrm{id} + i\,\mathrm{ex}}{u-v+i}$$

$$\begin{aligned} R_{21} &:= R_{21}(v,u) \\ &= \mathrm{ex}_{12}\, R_{12}(v,u)\, \mathrm{ex}_{12} \\ &= \frac{(u-v)\,\mathrm{id} - i\,\mathrm{ex}}{u-v-i} \end{aligned}$$

two properties $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, $R_{21}R_{12} = \mathrm{id}$ realise perm. group S_*

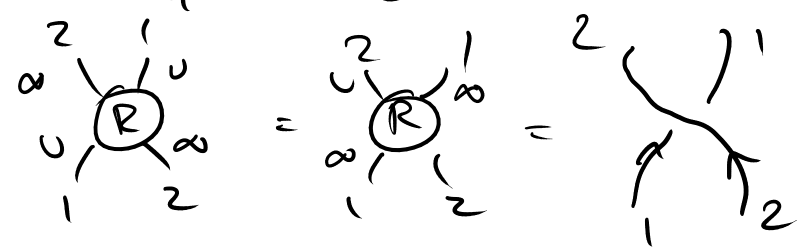
further properties for specific R

$R(u,u) = ex$



$R(u,\infty) = R(\infty,u) = id$

$\sim SU(N)$ symmetry



8.2 Charges

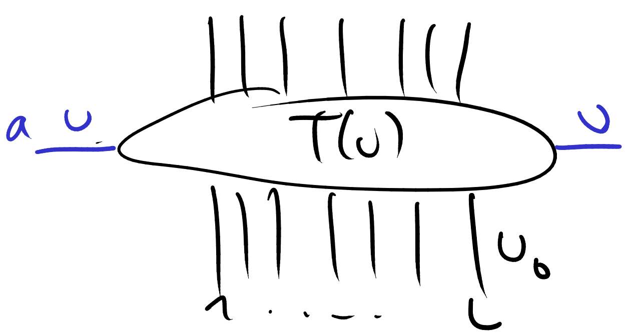
Monodromy and Traces

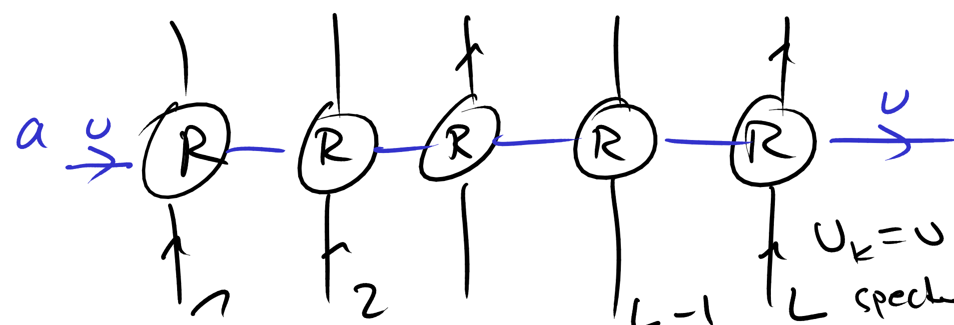
Lax transport → quantum operator $\quad \mathcal{L}(u) \to R_a(u-u_0)$

Monodromy $\quad T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdot \ldots \cdot R_{a,2} \cdot R_{a,1}$

$\mathbb{V}_a$ a, u is auxiliary space

$\mathbb{V}_k$ $u_k = u_0$ is quantum spin space at site k $\quad \mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_L$ is Hilb. spc.



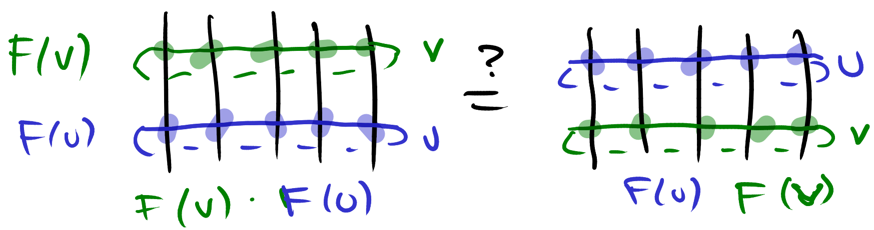


$u_k = u_0$ is spectral par in a hom. spin chain.

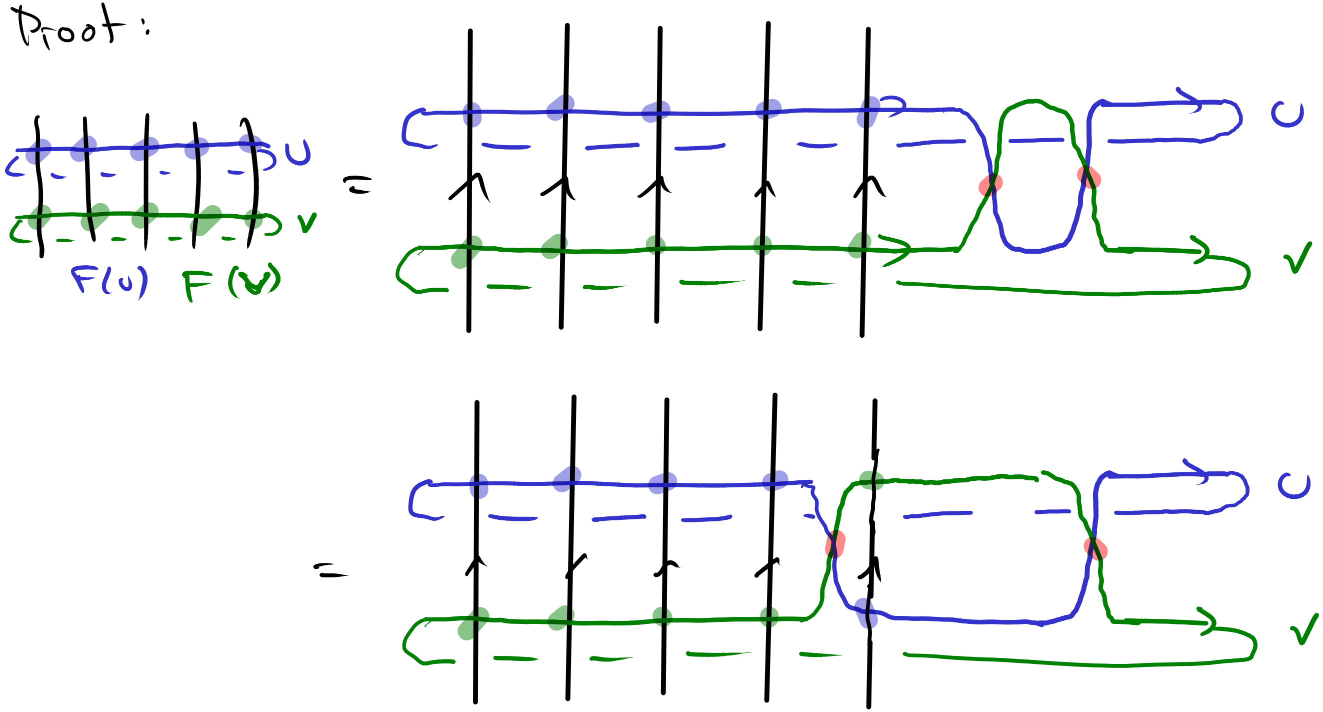
$u_0 = 0$

monodromy trace $\quad F(u) = \mathrm{tr}_a\, T_a(u) \quad$ for closed boundaries

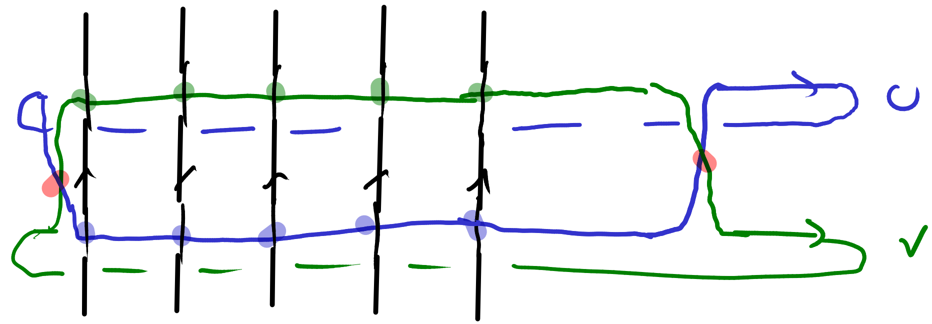
commutation relation $\quad [F(u), F(v)] = 0 \quad$ at all $u, v \in \mathbb{C}$



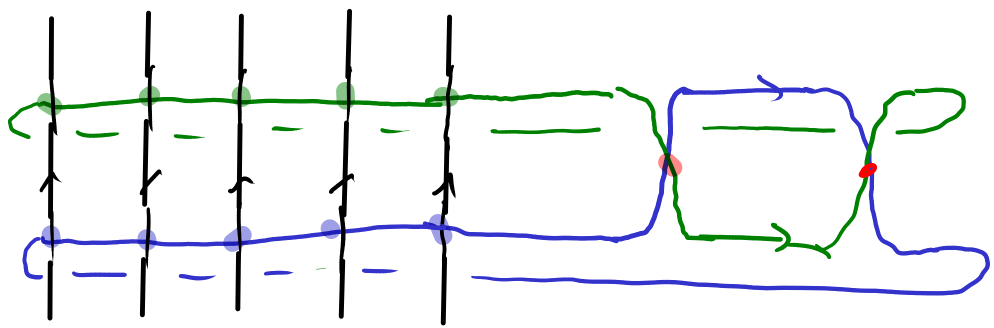
Proof:



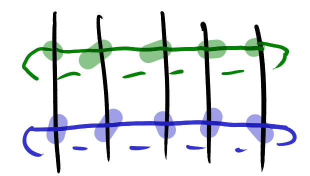
=



=



=



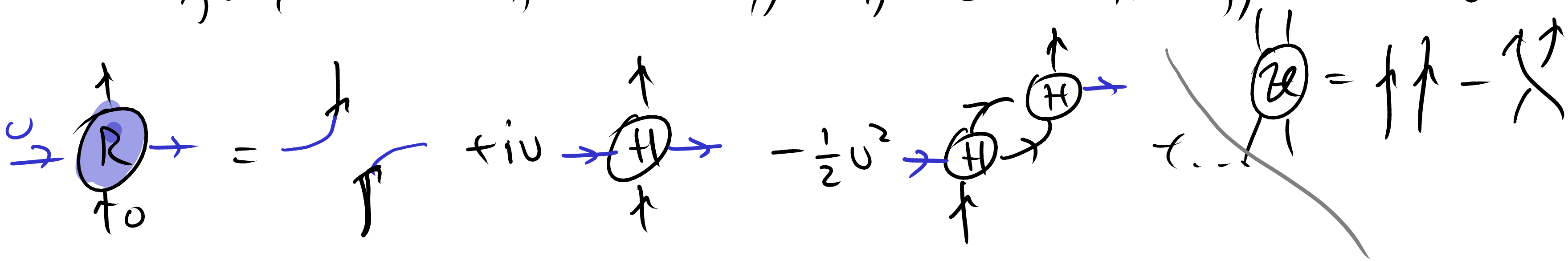
$F(v) \cdot F(u)$

Local Charges

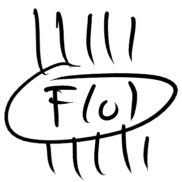
local charges are associated to point $u=0$

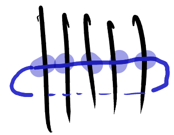
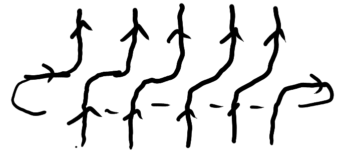
local Ham. density $\mathcal{H}_{k,l} = \mathrm{id}_{k,l} - \mathrm{ex}_{k,l}$

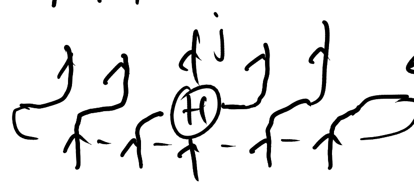
$$R_{a,j}(u,0) = \mathrm{ex}_{a,j} + iu\, \mathrm{ex}_{a,j} H_{a,j} - \frac{1}{2} u^2 \mathrm{ex}_{a,j} H_{a,j}^2 + \dots$$



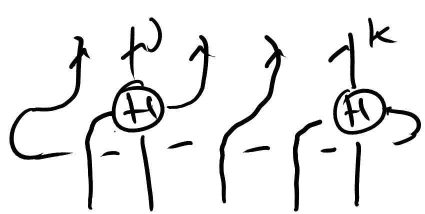
expand monodromy trace $F(u)$ at $u=0$

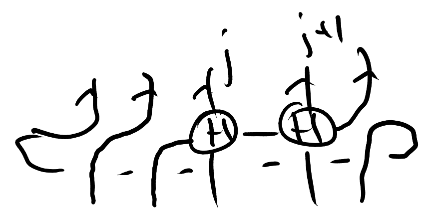


$=$

$=$

$\leftarrow \exp(iP)$

$+ \, iu \sum_{j=1}^{L}$


$\leftarrow iu \exp(iP) H$

$- u^2 \sum_{\substack{j<k=1 \\ |j-k|>1}}^{L}$


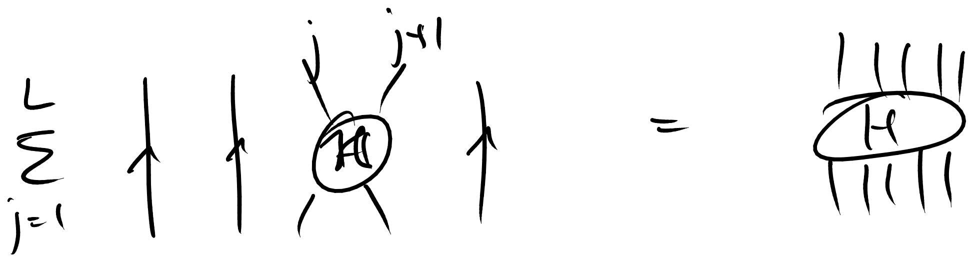
$- u^2 \sum_{j=1}^{L}$


$- \frac{1}{2} \sum_{j=1}^{L}$

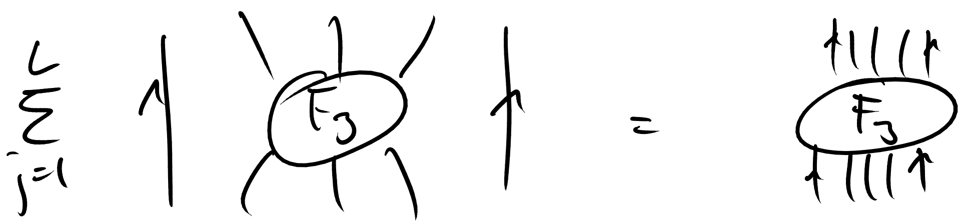

$\uparrow$ almost $\frac{1}{2} \exp(iP) H^2$

$- \frac{1}{2} u^2 \exp(iP) H^2 + iu^2 \exp(iP) F_3$

cyclic shift operator



local hom. N.N Hamiltonian



$$\mathcal{F}_{3,j} = \frac{i}{2}\left[\mathcal{H}_{j+1}, \mathcal{H}_j\right]$$

altogether expansion as: $F(u) = \exp(iP)\exp(iuH + iu^2 F_3 + \ldots)$

$$= \exp(iP + iuH + iu^2 F_3 + \ldots)$$

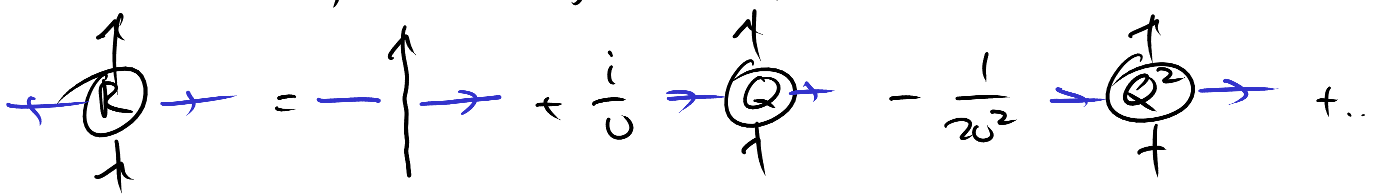
local operators

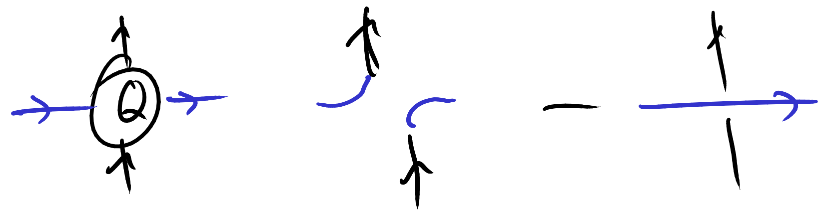
found/constructed local conserved commuting charges F_s $[F_s, F_r] = 0$

Multi-local Charges expansion at $u=\infty$

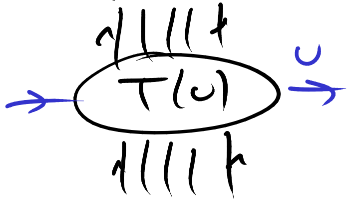
$$R_{a,j}(u,0) = \mathrm{id}_{a,j} + \frac{i}{u} Q_{a,j} - \frac{1}{2u^2} Q_{a,j}^2 + \ldots$$

$$Q_{a,j} := \mathrm{ex}_{a,j} - \mathrm{id}_{a,j}$$

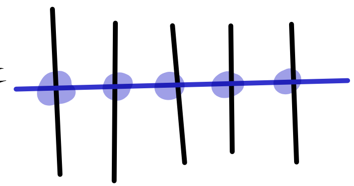




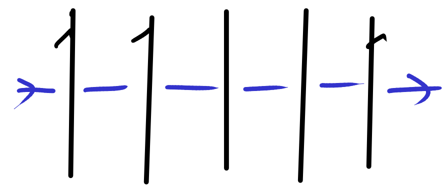
expand monodromy matrix $T_a(u)$ at $u=\infty$



$=$

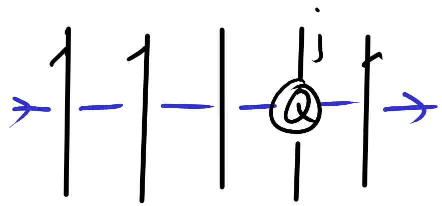


$=$



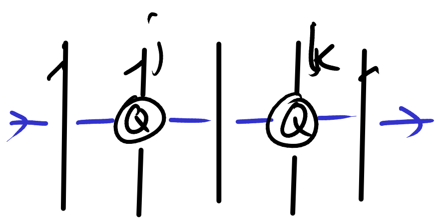
$\leftarrow$ id

$+\frac{i}{u}\sum_{j=1}^{L}$

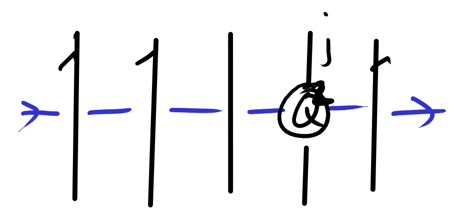


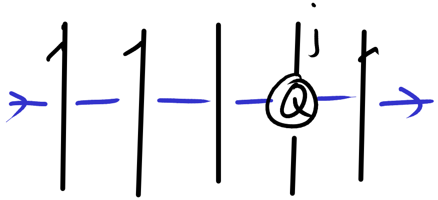
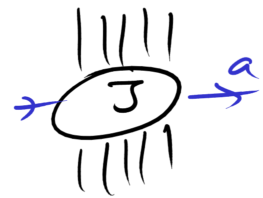
$\leftarrow \frac{i}{u} J$

$-\frac{1}{u^2}\sum_{j<k=1}^{L}$



$-\frac{1}{2u^2}\sum_{j=1}^{L}$



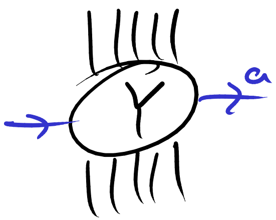
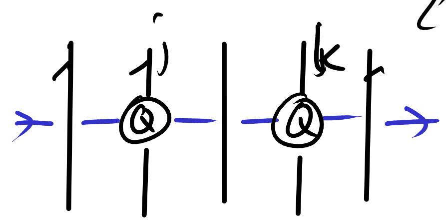
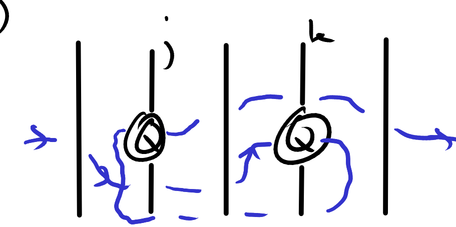
$\sum_{j=1}^{L}$  $=$ 

SU(N) symmetry generator

remaining term at $O(1/u^2)$ is almost $\left(\frac{i}{u}\right)^2 J^2$

difference is described by

bilocal

 $= \frac{i}{2} \sum_{j<k=1}^{L}$  $- \frac{i}{2} \sum_{j<k=1}^{L}$ 

$$T_a(u) = \exp\left(\frac{i}{u} J_a + \frac{i}{u^2} Y_a + \dots\right)$$

local — bilocal

extends SU(N) symmetry to ∞/L "copies"

⇝ Yangian algebra Y(SU(N))