

Introduction to Integrability

Lecture Slides, Chapter 8

ETH Zurich, 2024 HS

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8. Quantum Integrability

8.1 R-Matrix Formalism

$$S_{ab}^{cd}(u, v) = \frac{(u-v)\delta_a^c \delta_b^d + i\delta_a^d \delta_b^c}{(u-v) - i}$$

for flavours $a, b, c, d = 2, \dots, N$ $SU(N)$ spin chain

satisfies Yang-Baxter Eq / Factorised scattering

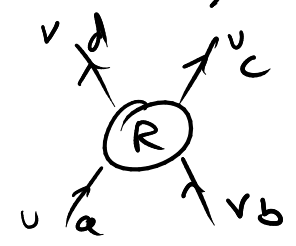
turn into $SU(N)$ fund. R-matrix

$$R_{ab}^{cd}(u, v) = \frac{(u-v)\delta_a^c \delta_b^d + i\delta_a^d \delta_b^c}{u-v+i}$$

$a, b, c, d = 1, \dots, N$ for $SU(N)$ fund. rep

Rank-2 tensor operator $R: V \otimes V \rightarrow V \otimes V$ $V = \mathbb{C}^N, u, v \in \mathbb{C}$

$$R(u, v) = \frac{(u-v)id + iex}{u-v+i}$$



$R_2 = R_2(u_1, u_2)$
 ↑
 associated to spaces V_1, V_2

spectral parameters = $(E^a \otimes E^b) R_{ab}^{cd} (E^c \otimes E^d)$

$$\begin{array}{c} \text{2} \\ \nearrow \\ \text{v} \\ \text{R} \\ \nwarrow \\ \text{u} \\ \text{1} \end{array} \begin{array}{c} \text{1} \\ \nearrow \\ \text{u} \\ \text{2} \\ \nwarrow \\ \text{v} \end{array} = \frac{u-v}{u-v+ti} \begin{array}{c} \text{2} \\ \nearrow \\ \text{1} \\ \nwarrow \\ \text{1} \end{array} \begin{array}{c} \text{1} \\ \nearrow \\ \text{2} \\ \nwarrow \\ \text{2} \end{array} + \frac{i}{u-v+ti} \begin{array}{c} \text{2} \\ \nearrow \\ \text{1} \\ \nwarrow \\ \text{1} \end{array} \begin{array}{c} \text{1} \\ \nearrow \\ \text{2} \\ \nwarrow \\ \text{2} \end{array}$$

write composite objects (operator products)

in components

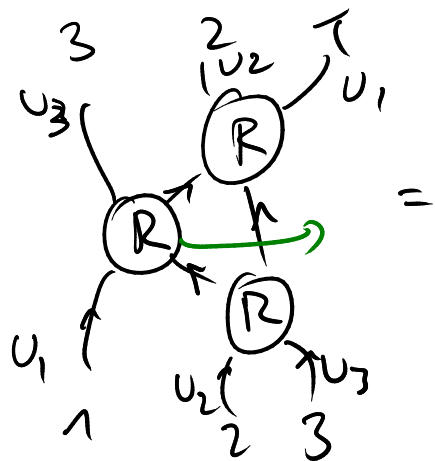
$$R_{13}R_{23} = \begin{array}{c} \text{3} \\ \nearrow \\ \text{1} \\ \text{R} \\ \nwarrow \\ \text{1a} \end{array} \begin{array}{c} \text{1} \\ \nearrow \\ \text{d} \\ \nwarrow \\ \text{2b} \end{array} \begin{array}{c} \text{2} \\ \nearrow \\ \text{e} \\ \nwarrow \\ \text{3c} \end{array} = R_{ag}^{df}(u_1, u_3) R_{bc}^{eg}(u_2, u_3)$$

Properties of R-Matrices

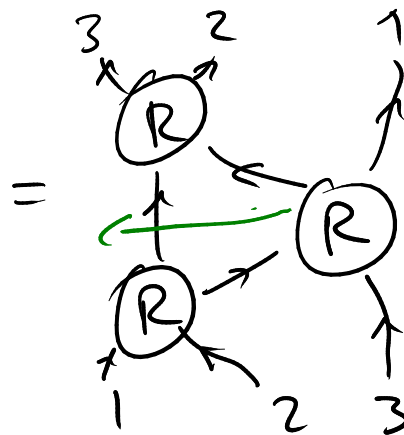
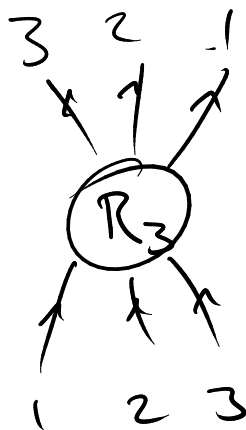
Yang-Baxter Equation

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2)$$

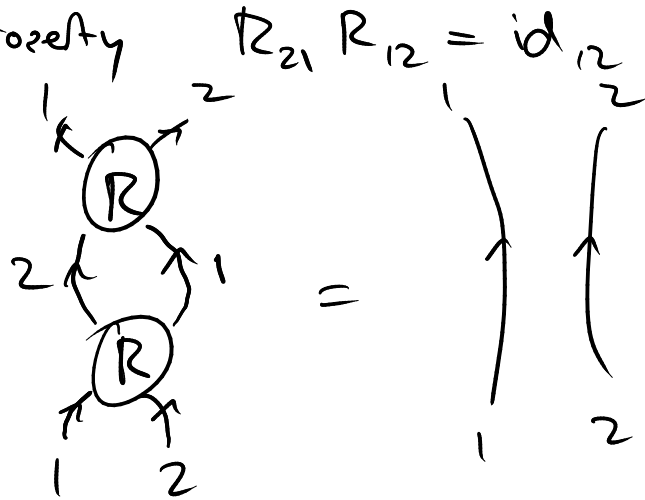
$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$



=



Inverse property



$$R_{12} = \frac{(u-v)id + iex}{u-v-i}$$

here: exact
 e(=)culture: up to a factor

$$\begin{aligned} R_{21} &:= R_{21}(v, u) \\ &= ex_{12} R_{12}(v, u) ex_{12} \\ &= \frac{(u-v)id - iex}{u-v-i} \end{aligned}$$

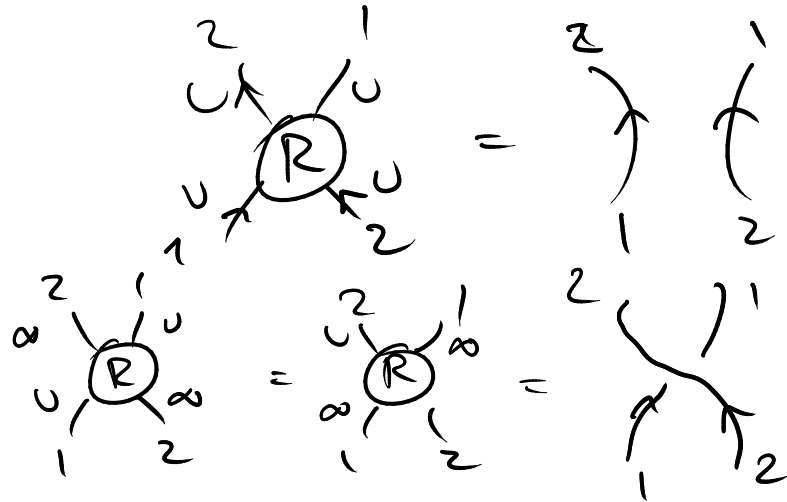
two properties $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, $R_{21}R_{12} = id$ realise perm. group S_*

further properties for specific R

$$R(u, u) = ex$$

$$R(u, \infty) = R(\infty, u) = id$$

$\sim SU(N)$ symmetry



8.2 Charges

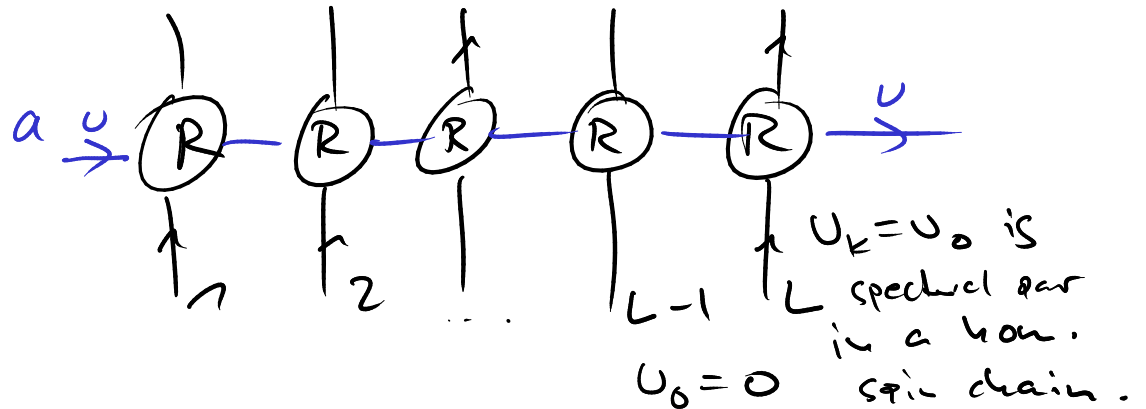
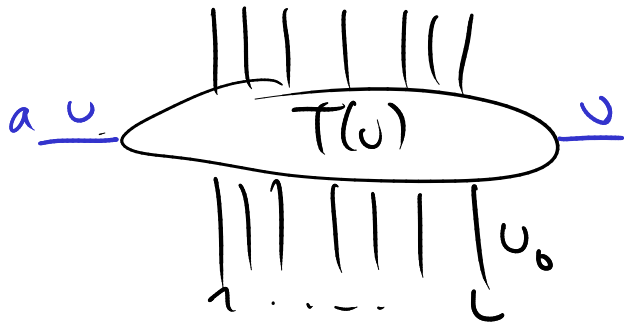
Monodromy and Traces

Lax transport \rightarrow quantum operators $\mathcal{L}(u) \rightarrow R_a(u-u_0)$

Monodromy $T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdots R_{a,2} \cdot R_{a,1}$

$\forall a, u$ is auxiliary space

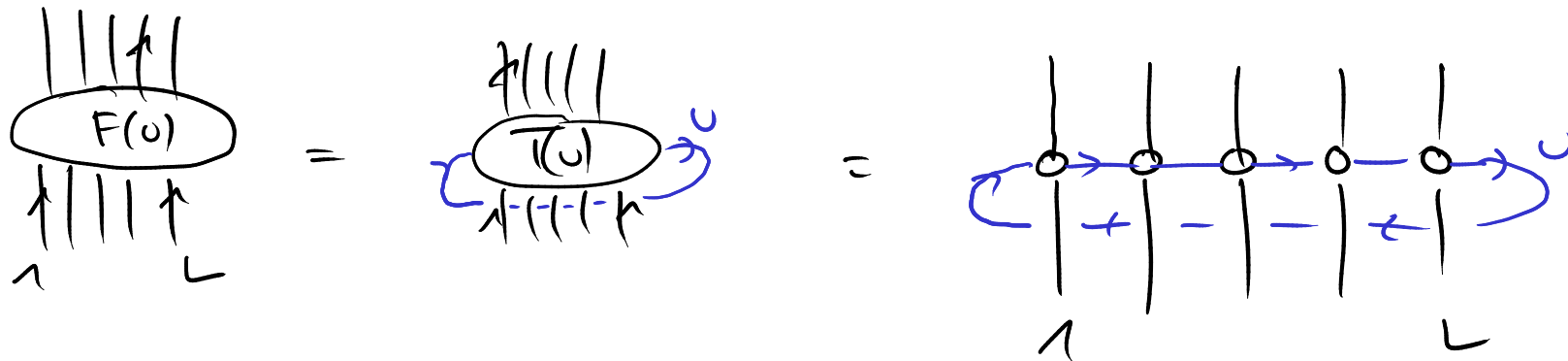
$\forall k, u_k = u_0$ is quantum spin space at site k $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L$ is fib. spc.



monodromy trace

$$F(u) = \text{tr}_a T_a(u)$$

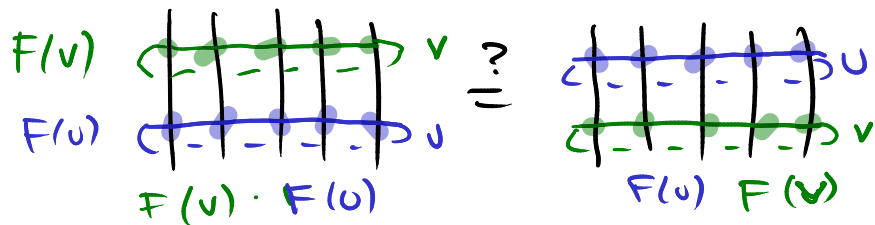
for closed boundaries



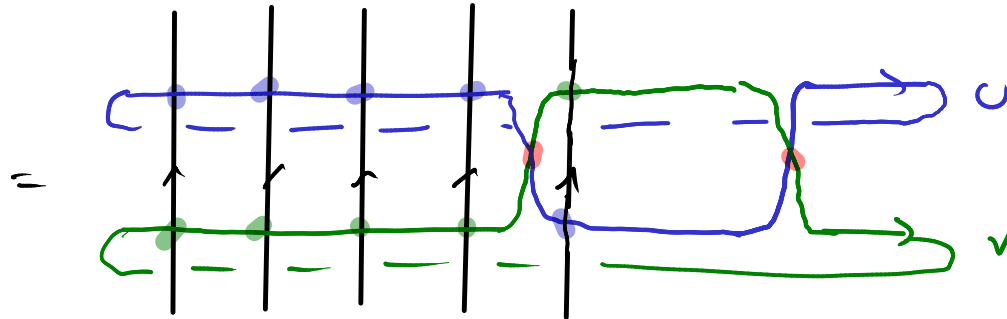
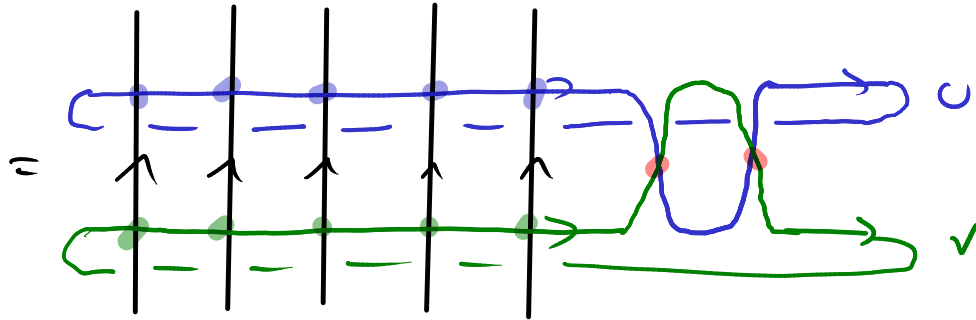
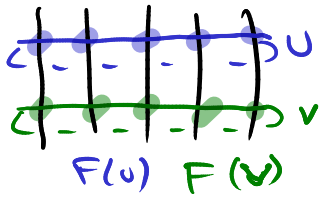
Commutation relation

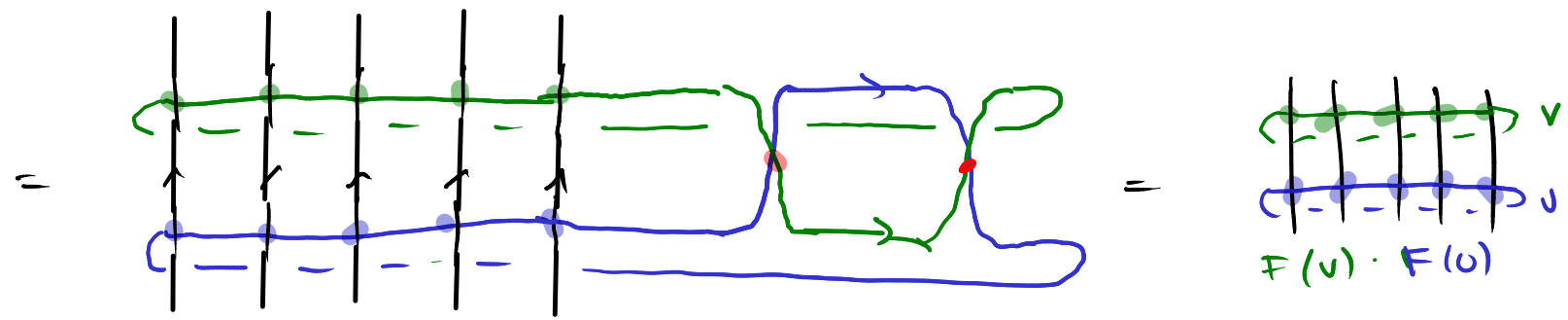
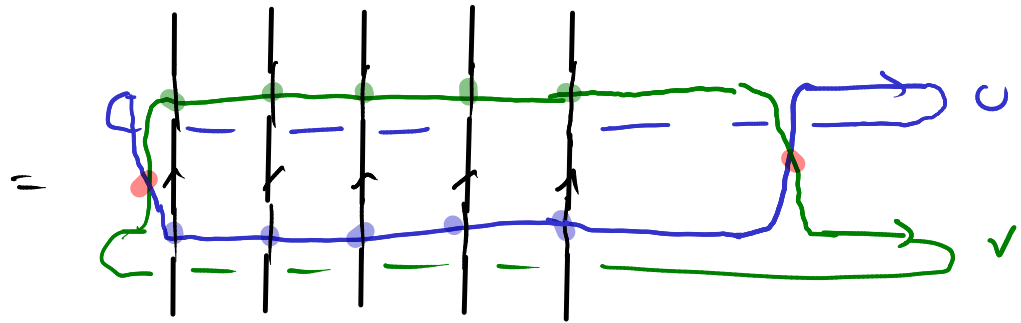
$$[F(u), F(v)] = 0$$

at all $u, v \in \bar{\mathbb{C}}$



Proof:

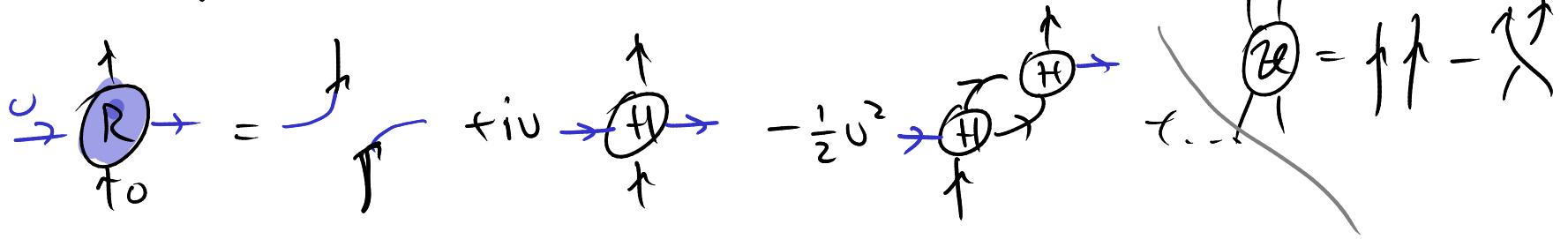




Local Charges

local charges are associated to point $u=0$
 ← local Ham. density $\mathcal{H}_{k,l} = \frac{1}{2} \dot{x}_{k,l}^2 - e x_{k,l}$

$$R_{a,j}(0,0) = e x_{a,j} + i v e x_{a,j} H_{a,j} - \frac{1}{2} v^2 e x_{a,j} H_{a,j}^2 \dots$$



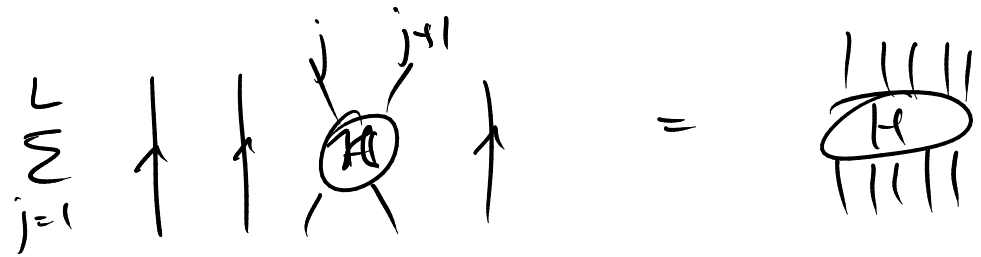
Expand monodromy trace $F(u)$ at $u=0$

$$\begin{aligned}
 \text{Tr} F(u) &= \text{Tr} \left(\text{Tr} \left(\prod_{j=1}^L T_j(u) \right) \right) \\
 &= \text{Tr} \left(\prod_{j=1}^L T_j(u) \right) \leftarrow \exp(iP) \\
 &+ iu \sum_{j=1}^L \text{Tr} \left(\text{Tr} \left(\prod_{k=1}^L T_k(u) \right) \right) \leftarrow iu \exp(iP) H
 \end{aligned}$$

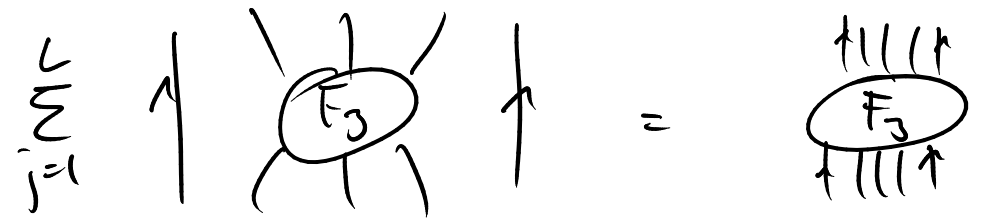
$$\begin{aligned}
 &- u^2 \sum_{\substack{j,k=1 \\ |j-k| \leq 1}}^L \text{Tr} \left(\text{Tr} \left(\prod_{l=1}^L T_l(u) \right) \right) \leftarrow \text{almost } \frac{1}{2} \exp(iP) H^2 \\
 &- u^2 \sum_{j=1}^L \text{Tr} \left(\text{Tr} \left(\prod_{k=1}^L T_k(u) \right) \right) \leftarrow -\frac{1}{2} u^2 \exp(iP) H^2 + iu^2 \exp(iP) \Gamma_3 \\
 &- \frac{1}{2} \sum_{j=1}^L \text{Tr} \left(\text{Tr} \left(\prod_{k=1}^L T_k(u) \right) \right) \leftarrow -\frac{1}{2} \sum_{j=1}^L \text{Tr} \left(\text{Tr} \left(\prod_{k=1}^L T_k(u) \right) \right)
 \end{aligned}$$



cyclic shift operator



local how. N.N Hamiltonian



$$F_3 = \frac{1}{2} (H \circ H - H \circ H)$$

$$F_{3,j} = \frac{1}{2} [H_{j+1}, H_j]$$

altogether expansion as:

$$F(u) = \exp(iP) \exp(iuH + iu^2 T_3 + \dots)$$

$$= \exp(iP + iuH + iu^2 T_3 + \dots)$$

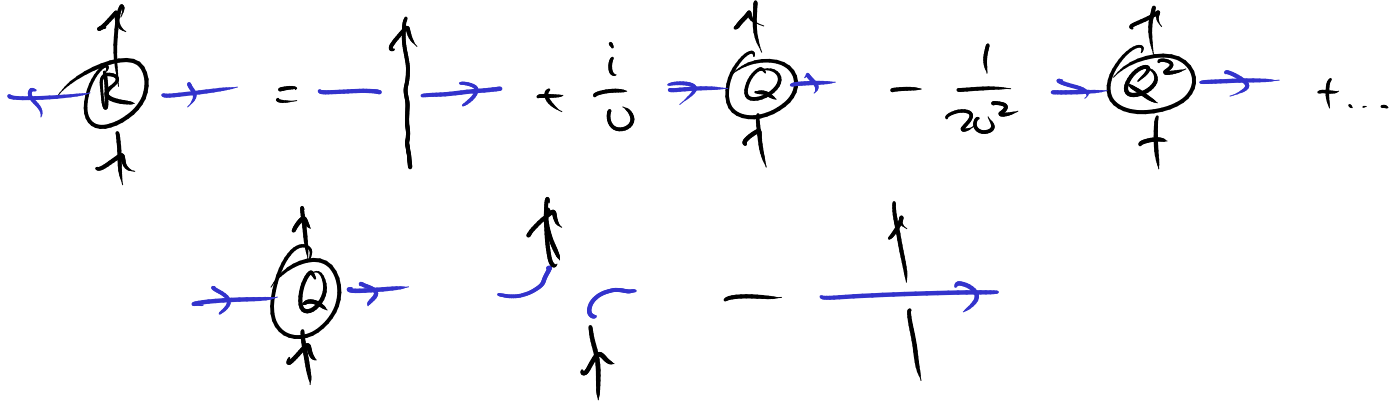
\uparrow \uparrow
 local operators

found / constructed local conserved counting charges F_s $[F_s, F_r] = 0$

Multi-local Charges expansion at $v = \infty$

$$R_{a,j}(v,0) = id_{a,j} + \frac{i}{v} Q_{a,j} - \frac{1}{2v^2} Q_{a,j}^2 + \dots$$

$$Q_{a,j} := ex_{a,j} - id_{a,j}$$

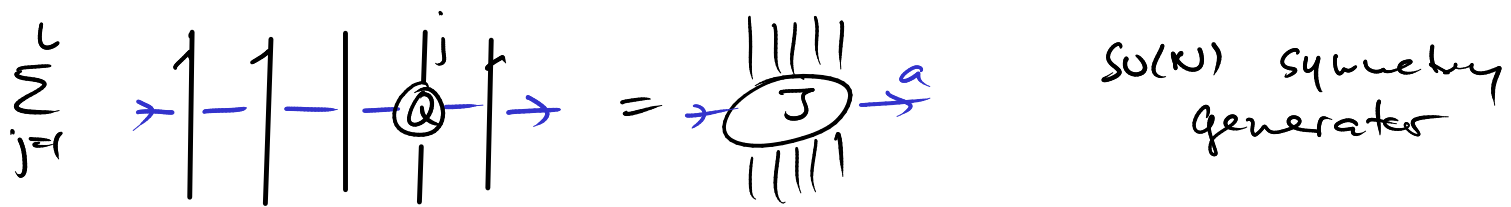


expand monodromy matrix $T_a(u)$ at $u = \infty$

$$\begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \\ \textcircled{T(u)} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \end{array} \xrightarrow{u} = \begin{array}{c} | \quad | \quad | \quad | \quad | \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \rightarrow 1 \quad 1 \quad 1 \quad 1 \quad 1 \rightarrow \end{array} \leftarrow \text{id}$$

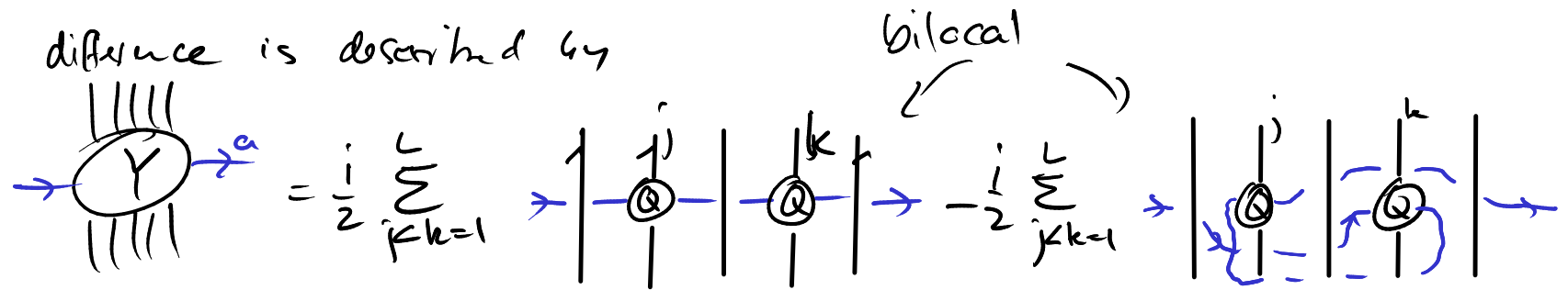
$$+ \frac{i}{u} \sum_{j=1}^L \begin{array}{c} | \quad | \quad | \quad | \quad | \\ \hline \rightarrow 1 \quad 1 \quad \textcircled{0} \quad 1 \quad 1 \rightarrow \end{array} \leftarrow \frac{i}{u} J$$

$$- \frac{1}{u^2} \sum_{j,k=1}^L \begin{array}{c} | \quad | \quad | \quad | \quad | \\ \hline \rightarrow 1 \quad \textcircled{0} \quad \textcircled{0} \quad 1 \quad 1 \rightarrow \end{array} \quad - \frac{1}{2u^2} \sum_{j=1}^L \begin{array}{c} | \quad | \quad | \quad | \quad | \\ \hline \rightarrow 1 \quad 1 \quad 1 \quad \textcircled{0} \quad 1 \rightarrow \end{array}$$



renaming term at $O(1/N^2)$ is almost $(\frac{i}{U})^2 J^2$

difference is described by



$$T_a(U) = \exp \left(\frac{i}{U} J_a + \frac{i}{U^2} \mathcal{Y}_a + \dots \right)$$

\uparrow local \uparrow bilocal

extends SU(N) symmetry to ∞/L^a copies
 \Rightarrow Yangian algebra $\Upsilon(SU(N))$