Introduction to Integrability

Lecture Slides, Chapter 7

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The current version of this work can be found at:
`http://people.phys.ethz.ch/~nbeisert/lectures/`.

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7 Long Quantum Chains

$L \to \infty$

7.1 Magnon Spectrum

ferromag vac $|0\rangle$ has energy $E=0$ for all L as $L\to\infty$

consider finitely many magnons M (M fixed)

Mode Numbers take log of Bethe eq. log ambiguity, n_k is an integer number.

$$iL \log \frac{u_k + i/2}{u_k - i/2} - i\sum_{\substack{\ell=1\\ \ell\neq k}}^{M} \log \frac{u_k - v_\ell + i}{v_\ell - u_k - i} + 2\pi n_k = 0$$

assume Im log between $-\pi, +\pi$.

"mode numbers" n_k range between $-\frac{1}{2}L$ to $+\frac{1}{2}L$ (Fourier mode numbers)

Single Magnons $M=1$

mode #.

$$iL \log \frac{u+i/2}{u-i/2} + 2\pi n = 0 \Rightarrow u = \frac{1}{2}\cot\frac{\pi n}{L} \qquad p = \frac{2\pi n}{L} \qquad e = 4\sin^2\frac{\pi n}{L}$$

two cases $|n| \ll L$, $|n| \sim L$. assume n finite $\ll L$ b/c lower energies.

approximate $n \ll L \Rightarrow$ $u = \frac{L}{2\pi n}$ $\quad p = \frac{2\pi n}{L} \sim \frac{1}{L}$ $\quad e = \frac{4\pi^2 n^2}{L^2} \sim \frac{1}{L^2}$

Several Magnons

$M \ll L$ fixed $\quad n_k \ll L$ fixed $\quad$ distinguish $n_k \neq n_\ell$ $\quad n_k = n_\ell$

scattering phase $\quad$ first $n_k \neq n_\ell$

$$-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} \simeq -i \log \frac{L/2\pi n_k - L/2\pi n_\ell + i}{L/2\pi n_k - L/2\pi n_\ell - i} \simeq \frac{-2i}{L}\frac{1}{n_k - n_\ell} \quad \text{small!}$$

small scattering phase $\Rightarrow$ excitations interact weakly $\approx$ free magnons for equal mode numbers n_k analyze carefully

$$u_k = \frac{L}{2\pi n} + \delta u_k \qquad \text{for all Bethe roots with mode number } n.$$

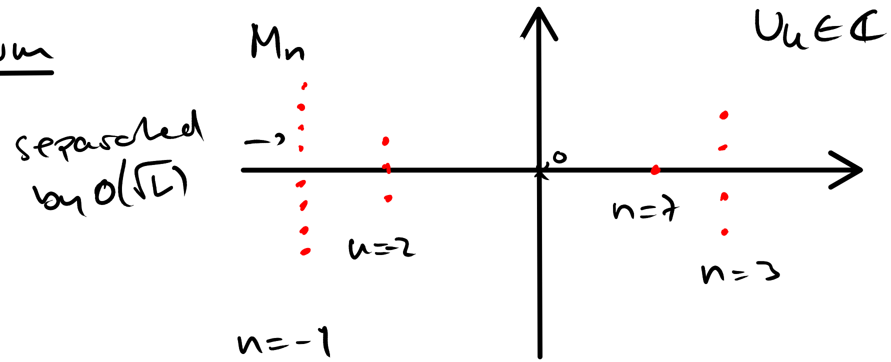
$$iL \log \frac{u_k + i/2}{u_k - i/2} = -2\pi n + \frac{4\pi^2 n^2}{L}\delta u_k + O(\delta u_k^2 / L^2)$$

$$-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} = \frac{2}{\delta u_k - \delta u_\ell} + O(1/\delta u_k^2)$$

together $$\frac{4\pi^2 n^2}{L}\delta u_k + \sum_{\substack{\ell=1\\ \ell\neq k}}^{M} \frac{2}{\delta u_k - \delta u_\ell} = 0$$

for proper solutions: needs $\delta u_k \sim \sqrt{\frac{L}{M}}\,\frac{1}{n}$

Magnon Spectrum



$$M = \sum_n M_n \qquad P = \sum_n M_n \frac{2\pi n}{L} \qquad E = \sum_n M_n \frac{4\pi^2 n^2}{L^2}$$

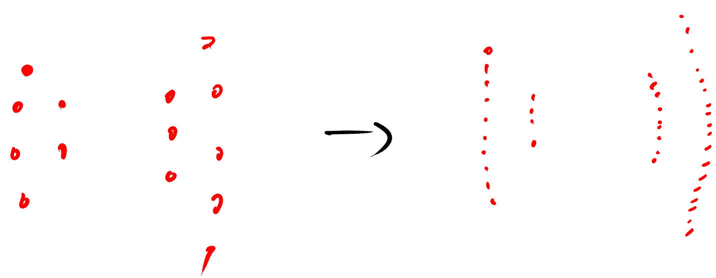
like free bosons on a circle

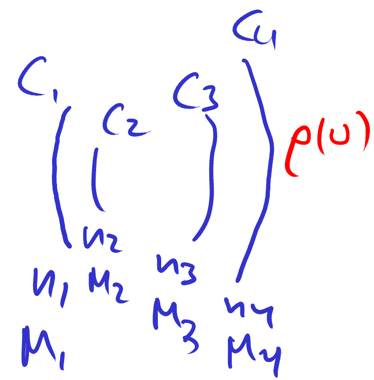
7.2 Ferromagnetic Continuum

what happens beyond $O(1/L^2)$

consider $E \sim \frac{Mn^2}{L^2}$ $\Rightarrow$ next higher states at $n \ll L$ finite, $M \sim L \to \infty$

coherent excitations of many magnons





density of Bethe roots

Rescaling of length $L^{org} \to \infty$. target length L

of spectral parameter (Bethe roots) $u^{org} = \frac{L^{org}}{2L} u$

Bethe root states represented by set $D = \cup_k D_k$ of contours D_k in $\mathbb{C}$

with a density function $\rho(u)$ defined on them

$$\sum_k \to \frac{L^{org}}{2L} \int du\, \rho(u)$$

apply to Bethe eq.

$$P \int_D \frac{2dv\, \rho(v)}{u-v} - \frac{2L}{u} + 2\pi n_k = 0 \qquad \text{for } u_k \in D_k$$

$$P = \int_D \frac{du\, \rho(u)}{u} \qquad E = \frac{2L}{L^{org}} \int_D \frac{du\, \rho(u)}{u^2} \qquad I_k = \int_{D_k} du\, \rho(u)$$

Spectral curve

Introduce quasi-momentum function $q(u) := \int_D \frac{dv\, \rho(v)}{u-v} + \frac{L}{u}$

$q(u)$ is analytic on $\mathbb{C}$ except for $u=0$ and $u \in D$

on D_k integral eq. $q(u+0)/q(u+i0)$

$$\lim_{\epsilon \to 0} \left(q(u+\epsilon) + q(u-\epsilon) \right) = 2\pi n_k \qquad \text{for } u \in D_k$$

$q(u)$ is one sheet of a two-sheeted fn. $-q(u)$ is other sheet

integral eq. determines continuation of $q(u)$ through a cut

$$q(u+\epsilon) = 2\pi n_k - q(u-\epsilon)$$

is continuous up to a shift by $2\pi n_k$ (compare to $e^{iq} = \tau$)

yields spectral curve for cont limit of Heisenberg spin chain.

<u>Hamiltonian Framework</u> how to obtain cont. limit of q. Heis. spin chain

you need coherent states: spin ½ state $|S\rangle \in \mathbb{C}^2$

relate $|S\rangle$ to a classical spin vector $\vec{S} \in S^2$ $\quad |S\rangle\langle S| = \frac{1}{2}(\mathrm{id} + \vec{S}\cdot\vec{\sigma})$

obtain $\vec{S}$ as exp. val. $\langle S|\vec{\sigma}|S\rangle = \vec{S}$

for arbit. observable X: $\langle X\rangle_S = \frac{1}{2}\operatorname{tr}\left((1+\vec{S}\cdot\vec{\sigma})X\right)$

for Ham. dens $\mathcal{H}_j$

$$\langle \mathcal{H}_j\rangle_S = \operatorname{tr}_{j,j+1}\left(\frac{1}{4}(1+\vec{S}_j\cdot\vec{\sigma}_j)(1+\vec{S}_{j+1}\cdot\vec{\sigma}_{j+1})(\mathrm{id}_{j,j+1} - \mathrm{ex}_{j,j+1})\right)$$

$$= \frac{1}{4}\operatorname{tr}(1+\vec{S}_j\cdot\vec{\sigma})\operatorname{tr}(1+\vec{S}_{j+1}\cdot\vec{\sigma}) - \frac{1}{4}\operatorname{tr}\left((1+\vec{S}_j\cdot\vec{\sigma})(1+\vec{S}_{j+1}\cdot\vec{\sigma})\right)$$

$$= 1 - \frac{1}{2} - \frac{1}{2}\vec{S}_j\cdot\vec{S}_{j+1} = \frac{1}{2}(1-\vec{S}_j\cdot\vec{S}_{j+1})$$

$$H = \frac{1}{2}\sum_j (1-\vec{S}_j\cdot\vec{S}_{j+1})$$

continuum limit of class. discrete model, site spacing $a = \frac{L}{L^{org}} \to 0$

smooth spin function $\vec{S}(x)$ from which $\vec{S}_j$ can be read off as

$$\vec{S}_j = \vec{S}(ja)$$

$$H = \frac{1}{a} \int dx \; \frac{1}{2}\left(1 - \vec{S} \cdot \left(\vec{S} + a\vec{S}' + \frac{1}{2} a^2 \vec{S}'' + \dots\right)\right)$$

$$= -\frac{1}{4} a \int dx \; \vec{S} \cdot \vec{S}'' + \dots = \frac{1}{4} a \int dx \, (\vec{S}')^2$$

Continuous Heisenberg model, integrable

7.3 Anti-Ferromagnetic Ground State

Entanglement

Ferromagnetic: all spins aligned $|\uparrow\uparrow\uparrow\uparrow$

anti-f. would like alternating spins $\uparrow\downarrow\uparrow\downarrow$

su(2) symmetry and for all pairs $|\uparrow\downarrow\rangle$ $|\downarrow\uparrow\rangle$

in QM can do linear comb. such as $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

look for energy eigenstates — spin-1 config $\sim |\uparrow\uparrow\rangle$, $E=0$ — spin-0 config., $E=2$

For more than 2 spins

can combine well spins into spin 1/2: ferrom. $\|$

cannot easily/globally make all pairs to be in spin 0 configuration $\Rightarrow$ real antiferrom. G.S.

$|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \ldots\rangle$ $|\downarrow\uparrow\downarrow\uparrow\ldots\rangle$ and all other states $|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\rangle$ $|\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\rangle$

to obtain precise linear comb. is huge comb. problem at $L\to\infty$ need Bethe Eq for progress.

Bethe Equations for anti-ferromag ground state

assume all Bethe roots u_k to be real for maximal energy

$$\log \frac{u+i}{u-i} = i\pi\, \mathrm{sgn}(u) - 2i \arctan u$$

with log branch at neg real axis such that Im log is between $-\pi, +\pi$

Bethe eq. $$2\arctan(2u_k) - \frac{2}{L}\sum_{l=1}^{M} \arctan(u_k - u_l) + \frac{2\pi \tilde{n}_k}{L} = 0$$

$$\tilde{n}_k = n_k + k - \tfrac{1}{2}M - \tfrac{1}{2} - \tfrac{1}{2}L\, \mathrm{sgn}(u_k)$$ are either integers or half integers.

n_k obey interesting statistics near a.f.g.s. $-\frac{L}{2} < n_k < +\frac{L}{2}$, $n_k = 0$ is $u_k = \infty$ su(2) descendant

* all other mode numbers occupied at most once
* neighbours $n_k \pm 1$ of occupied modes n_k <u>must not</u> be occupied.

Fill as many modes as possible: $M = L/2$ (assume L even)

$L = 14$ $-1 \quad -3 \quad -5 \quad \pm 7 \quad +5 \quad +3 \quad +1$

$M = 7$ ● ○ ● ○ ● ○ ● ○ ● ○ ● ○ ●

$$n_k = L\,\theta_{2k>M} - 2k + 1 \qquad \tilde{n}_k = \tfrac{1}{2}M - k + \tfrac{1}{2}$$

<u>Integral Equations</u> at $L \to \infty$

Bethe roots on real axis, introduce a density $\rho(u)$ (w/o rescaling u)

$$\rho(u) = \frac{1}{L}\frac{dk}{du} \qquad k(u) = L \int_{-\infty}^{u} dv\, \rho(v) \quad \text{counting function.}$$

Bethe eq at $L \to \infty$

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v) \arctan(u-v) - 2\pi \int_{-\infty}^{u} dv\, \rho(v) + \tfrac{1}{2}\pi$$

differentiate w.r.t. u $\quad \frac{4}{1+4u^2} - \int \frac{2dv\, \rho(v)}{1+(u-v)^2} - 2\pi\, \rho(u) = 0$

kernel $\frac{1}{1+(u-v)^2}$ has difference form $\Rightarrow$ Fourier transformation

$$\rho(u) = \int \frac{d\theta}{2\pi} e^{iu\theta} R(\theta), \qquad R(\theta) = \int du\, e^{-iu\theta} \rho(u)$$

use following integral $\quad \int \frac{du}{2\pi} \frac{2e^{-iu\theta}}{1+u^2} = e^{-|\theta|}$

$$e^{-|\theta|/2} - e^{-|\theta|} R(\theta) - R(\theta) = 0 \qquad R(\theta) = \frac{1}{2\cosh(\theta/2)}$$

$$\rho(u) = \frac{1}{2\cosh(\pi u)} \qquad k(u) = \frac{L}{4} + \frac{L}{\pi} \arctan \tanh\left(\tfrac{1}{2}\pi u\right)$$

Ground State Properties

$$E = L \int \frac{4\,du\,\rho(u)}{1+4u^2} = L \int d\theta\, e^{-|\theta|/2} R(\theta) = 2L \log 2 < 2L$$

$P = 0$ or $P = \pi$ established through parity argument

or mode numbers n_n contribute $p_n = 2\pi n_n / L$

$$P = \begin{cases} 0 & \text{for } M = L/2 \text{ even} \\ \pi & \text{for } M = L/2 \text{ odd} \end{cases}$$

Ang. mom $J = L/2 - M = 0$ spin-0 state

7.4 Spinons

Mode number occupation with one gap in perfect sequence. at $k \sim u_0$

consider differences to ground state in $\rho_0(u) + \delta\rho(u) = \rho(u)$

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\ \rho(v)\arctan(u-v) - 2\pi\int_{-\infty}^{u} du\ \rho(u) + \frac{\pi}{4} - \frac{\pi}{2L}\,\mathrm{sgn}(u-u_0)$$

differentiate wrt. u, consider $1/L$ terms only

$$-\int_{-\infty}^{+\infty} \frac{2dv\ \delta\rho(v)}{1+(u-v)^2} - 2\pi\,\delta\rho(u) - \frac{2\pi}{L}\,\delta(u-u_0) = 0$$

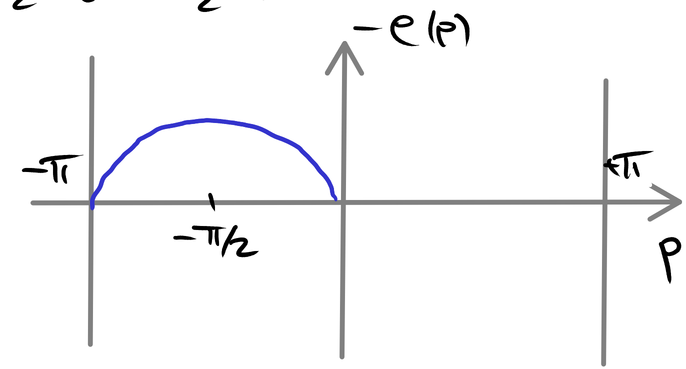
solved by $\delta R(\theta) = -\frac{1}{L}\ \frac{e^{|\theta|/2 - iu_0\theta}}{2\cosh(\theta/2)}$ depends on u_0 variable of gap.

Spinon Properties

energy shift $$e(u_0) = -\int \frac{d\theta\, e^{iu_0\theta}}{2\cosh(\theta/2)} = -\frac{\pi}{\cosh(\pi u_0)}$$

momentum shift $$p(u_0) = L\int du\, \delta\rho(u)\left(\pi - 2\arctan(2u)\right)$$
$$= 2\arctan \tanh\left(\frac{\pi}{2}u_0\right) - \frac{1}{2}\pi$$

dispersion relation of spinons $$e(p) = -\pi \sin(-p)$$



only covers $-\pi < p(u_0) < 0$

further curiosity $\delta J^z = L\left(\delta R(0) - \frac{1}{2}\right) = -\frac{1}{2}$ $\rightsquigarrow$ spinons have spin $1/2$

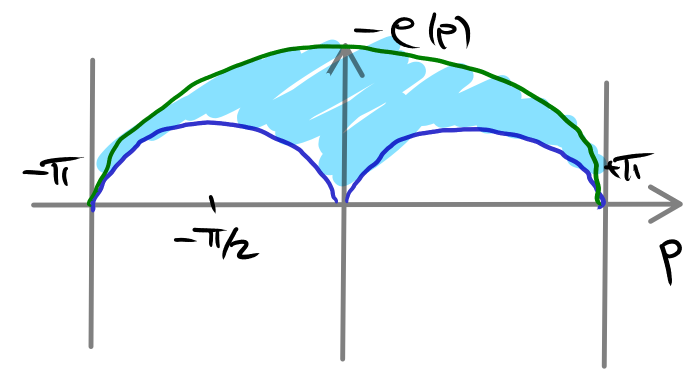
Physical Spinon States

Spinon is a collective elementary ex of g.s.

spinons can only come in pairs. spinon ~ 1/2 Bethe root

a pair of spinons has spin = 1

$$P = p_1 + p_2 \qquad E = e(p_1) + e(p_2)$$



consider occupation of mode numbers

$$\begin{array}{ccccccccccccc} -1 & & -3 & & & -6 & & +6 & & +4 & & & +1 \\ \bullet & \circ & \bullet & \circ & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \circ & \bullet \end{array}$$

spinon 1 spinon 2

one gap implies another gap

Odd Length

$L = 13$

$M = \frac{L-1}{2} = 6$

1 spinon at least!

odd length requires odd number of spinons

7.5 Spectrum overview

Distribution of energy eigenstates at large $L \to \infty$

ferromag. ground state $E = 0$ $P = 0$ $J = L/2$

finitely many magnons at finite mode numbers M_n are occupation numbers, bosonic

$$E = \sum_n M_n \frac{4\pi n^2}{L^2} \qquad P = \sum_n M_n \frac{2\pi n}{L} \qquad J^z = L/2 - \sum M_n$$

at infinitely many magnons $M \sim L$ at finite mode numbers

$$E \sim 1/L \qquad -\pi < P < +\pi \qquad J \sim L$$

$\vdots$ $\vdots$

spinon states: $E_0 - E = \sum_k \frac{2\pi^2 |u_k|}{L}$ $P = \pi \cancel{?} + \frac{2\pi n}{L}$ $J \leq \sum_k \frac{1}{2}$

antiferromagnetic ground state $E = E_0 = 2L \log 2$ $P \equiv \frac{1}{2}\pi L$ $J = 0$