Introduction to Integrability

Lecture Slides, Chapter 6

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The current version of this work can be found at: `http://people.phys.ethz.ch/~nbeisert/lectures/`.

6. Quantum Spin Chains

Class of QM models that display integrability

- form a large class of integrable models
- treated uniformly
- parameters to tune (preserve integrability)
- short chains will be genuine QM models
- long chains can approximate (1+1)-D QFT
- for large quantum numbers approach classical counterparts
- model magnetic materials

	classes	
nearby spins	ferromagnetic	anti-ferromagnetic
opp. alignment ↑↓	high energy	low energy
equal alig. ↑↑	low energy	high energy

two major models

Ising model : stat mech
lattice of ↑↓ : statistics
Heisenberg chain : QM model
Ham acts on neighbours
dynamics

6.1 Heisenberg Spin Chain

Setup : single spin $|\uparrow\rangle, |\downarrow\rangle \rightsquigarrow$ spin vector space $\mathbb{C}^2 = \mathbb{V}$

Spin chain of length L is L-fold tensor product of spins

Hilbert space $\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \dots \otimes \mathbb{V}_L$ $\quad$ dim $\mathbb{V}^{\otimes L} = 2^L$

Basis for $\mathbb{V}^{\otimes L}$ is given by pure states $|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow \dots \uparrow\rangle$

Hamiltonian op. $H : \mathbb{V}^{\otimes L} \to \mathbb{V}^{\otimes L}$ is homogeneous

$H = \sum_j \mathcal{H}_j \qquad \mathcal{H}_j : \mathbb{V}_j \otimes \mathbb{V}_{j+1} \to \mathbb{V}_j \otimes \mathbb{V}_{j+1}$ nearest neighbour

Pairwise Ham $\mathcal{H} = \lambda_0 \, 1 \otimes 1 + \lambda_x \, \sigma^x \otimes \sigma^x + \lambda_y \, \sigma^y \otimes \sigma^y + \lambda_z \, \sigma^z \otimes \sigma^z$

integrable for all values $\lambda_0, \lambda_{x,y,z}$. Three classes:

- most general one $\lambda_x < \lambda_y < \lambda_z$: called "XYZ" difficult!
- simplifications for $\lambda_x = \lambda_y \neq \lambda_z$ called "XXZ"
- enhanced symmetry for $\lambda_x = \lambda_y = \lambda_z$ called "XXX"

mainly consider XXX $\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z = \frac{1}{2}\lambda$

$$\mathcal{H}_j = \frac{1}{2}\left(1 \otimes 1 - \vec{\sigma} \otimes \vec{\sigma}\right) = \lambda\left(id_{j,j+1} - ex_{j,j+1}\right)$$, assume $\lambda > 0$ ferromagnetic reg.

$\lambda = 1 \quad \mathcal{H} = id - ex$

Boundary Conditions

various choices as for classical

→ • finite chains with closed boundaries, periodicity $\mathbb{V}_{L+j} \equiv \mathbb{V}_j$

• finite chain with open boundaries

↳ discrete, finite spectrum

• infinite chains with asymptotic boundaries → continuous spectrum

Symmetries

XXX has $su(2) \simeq so(3)$ symmetry: spin-$1/2$ rep of $su(2)$

$\vec{S}_j^2 = \frac{3}{4}\hbar^2 \, id$, $\vec{S}_j = \frac{1}{2}\hbar\vec{\sigma}_j$ Lie alg $[S^a_j, S^b_k] = i\hbar\,\delta_{j,k}\,\varepsilon^{abc} S^c_j$

total ang. mom. vect $\vec{J} = \sum_{j=1}^{L} \vec{S}_j = \sum_{j=1}^{L} \frac{1}{2}\hbar\,\vec{\sigma}_j$

$[\vec{J}, H] \overset{\text{symmetry!}}{=} 0 \rightsquigarrow$ spectrum arranges into spin-j rep of $su(2)$

spectrum of spin reps is predetermined

$L=1$ $(1/2)$

$L=2$ $(1)+(0)$

$L=3$ $(3/2)+2(1/2)$ ...

Higher Spin and Classical Limit

In order to obtain a classical limit per site, need to make a link between very discrete spins $\uparrow, \downarrow$ and continuous spins $\nearrow$

Generalise integrable spin chain to higher spin representations: spins at each site

$$s = 1/2 \quad \mathbb{V} = \mathbb{C}^2 \quad \longrightarrow \quad s \in \tfrac{1}{2}\mathbb{Z}^+ \quad \mathbb{V} = \mathbb{C}^{2s+1}$$

Spin operators $\vec{S}_j$ obeying $[S^a_j, S^b_k] = i\hbar \delta_{jk} \varepsilon^{abc} S^c_j \quad [so(3)]$

$\vec{S}_j^{\,2} = \hbar^2 s(s+1)$ (total spin op)

choose basis aligned with z-axis: $\vec{e}_z \cdot \vec{S}_j$ has ev. $-\hbar s \ldots +\hbar s$ in steps of $\hbar$

* nearest neighbour
* SO(3)

$$\left.\right\}\ \text{only use } f\left(\left(\vec{S}_j \cdot \vec{S}_{j+1}\right)\right) \sim \mathcal{H}_j$$

first introduce total spin (not squared)

$$J_{j,k} := \overbrace{\sqrt{\left(\vec{S}_j + \vec{S}_k\right)^2 + \frac{1}{4}\hbar^2}} - \frac{1}{2}\hbar$$

spectrum to range between 0 and $2s\hbar$

Integrable Ham turns out to be

$$\mathcal{H}_j = 2\psi(2s+1) - 2\psi\left(\hbar J_{j,j+1} + 1\right))$$

ψ is digamma function $\Psi(z) := d \log \Gamma(z) / dz$

harmonic series

$$\psi(z+1) = \psi(z) + \frac{1}{z} \qquad \psi(n+1) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k}$$

SO(3), spins s
XXX_s model

Two limiting cases for s: $s = 1/2$, $s \to \infty$

$s = 1/2$ spec J is $\{0, \hbar\}$ $\qquad\qquad$ $\Psi(2) = \psi(1) + 1$

$$J_{j,k} = \frac{3}{4}\hbar \; id_{j,k} + \frac{1}{4}\hbar\, \vec{\sigma}_j \cdot \vec{\sigma}_k = \frac{1}{2}\hbar \; id_{j,k} + \frac{1}{2}\hbar \; ex_{j,k}$$

$$\mathcal{H}_j = 2 - \frac{2}{\hbar} J_{j,j+1} = id_{j,j+1} - ex_{j,j+1}$$

$s \to \infty$: classical limit

$$\hbar = \frac{1}{s} \qquad \vec{S}_j^{\,2} \to 1 \qquad \vec{e}_z \cdot \vec{S}_j \text{ is cont between } -1 \text{ and } +1$$

Use large-z behaviour of $\psi(z) = \log z + O(1/z)$

$$\mathcal{H}_j^{qu} \to 2 \log \frac{J_{j,j+1}}{2} = -\log \frac{\vec{J}^{\,2}_{j,j+1}}{4} = -\log \frac{1 + \vec{S}_j \cdot \vec{S}_{j+1}}{2} = \mathcal{H}_j^{cl}$$

6.2 Spectrum of the Closed Chain

Conventional Strategy

2^L states in total

- Enumerate a basis of Hilbert space $\mathbb{V}^{\otimes L}$ $|\downarrow\ldots\downarrow\rangle, |\uparrow\downarrow\ldots\rangle, |\downarrow\uparrow\downarrow\ldots\downarrow\rangle\ldots$
- Evaluate H in this basis forming a $2^L \times 2^L$ matrix (integer coefficients, sparse)
- Solve eigenvalue problem...

can do in practice for $L<10$, $L<20,30$ by computer

~ will find su(2) multiplets of some spins

eg. find one state at $L=6$, $M=3$ up spins of $E=5+\sqrt{13}$

Bethe Equations consider a set of M alg. eq. (Bethe eq)
for M undetermined variables $u_k \in \mathbb{C}$ (Bethe roots):

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = \prod_{\substack{l=1\\l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad \text{for } k=1\dots M$$

Claim: for each eigenstate multiplet with ang mom $J=L/2-M$ of H there is a sol of above B. eq. with $M \leq L/2$ distinct Bethe roots u_k

Energy $$E = \sum_{k=1}^{M}\left(\frac{i}{u_k+i/2}-\frac{i}{u_k-i/2}\right).$$

eg. $L=6$, $M=3$ (su(2) singlet) $u_{1,2}=\pm\sqrt{-\frac{5}{12}+\frac{\sqrt{13}}{6}}$ $u_3=0$ $E=5+\sqrt{13}$

6.3 Coordinate Bethe Ansatz

Start with an infinite chain and investigate ferromag. vacuum + excitations
impose periodicity on wave function later to obtain spectrum of closed chain.
number M of up-spins is preserved by H

Vacuum State

ferromagnetic vacuum $|0\rangle := |\downarrow\downarrow \dots \downarrow\rangle$

state has energy zero $E=0$ by construction

$$\mathcal{H}_j |0\rangle = \mathrm{id}_{j,j+1} |0\rangle - ex_{j,j+1} |0\rangle = |0\rangle - |0\rangle = 0.$$

Solves spectral problem for $M=0$ (even at finite L)

Magnon States Flip one spin at pos j

$$|j\rangle := |\downarrow\dots\downarrow\overset{j}{\uparrow}\downarrow\dots\downarrow\rangle \qquad j\in\mathbb{Z}$$

Hamiltonian closes on this sector due to conserved $M=1$

Note: Ham is homogeneous along chain, commutes with elementary shift generated by $\exp(iP)$

Eigenstates of $\exp(iP)$ are plane waves

$$|p\rangle := \sum_j e^{ipj}|j\rangle \qquad (\text{Fourier transform of basis } |j\rangle)$$

magnon state with momentum p, note $p \equiv p+2\pi\mathbb{Z}$

act with H on $|p\rangle$

$$H|p\rangle = \sum_j e^{ipj} \left(\mathcal{H}_{j-1} |j\rangle + \mathcal{H}_j |j\rangle \right)$$

$$= \sum_j e^{ipj} \left(|j\rangle - |j-1\rangle + |j\rangle - |j+1\rangle \right)$$

$$= \sum_j e^{ipj} \left(1 - e^{ip} + 1 - e^{-ip} \right) |j\rangle$$

$$= e(p) \quad |p\rangle$$

magnon dispersion relation

$$e(p) = 2\left(1 - \cos p\right) = 4 \sin^2 \left(p/2\right)$$

Periodic chain: p is quantised $p = \frac{2\pi n}{L}$ $n = 0, \dots, L-1$ due to $e^{ipL} = 1$

problem solved for $M = 1$

Scattering Factor

$M=2$ two spin flips at j, k assume $j<k$

$$|j<k\rangle := |\downarrow\ldots\downarrow\overset{j}{\uparrow}\downarrow\ldots\downarrow\overset{k}{\uparrow}\downarrow\ldots\downarrow\rangle$$

Eigenstates of H within $M=2$ sector.

ansatz:
$$|p<q\rangle := \sum_{j<k=-\infty}^{+\infty} e^{ipj+iqk}\,|j<k\rangle$$

Partial eigenstate with total momentum $P=p+q$ (but not precise indiv. momenta)

should have $\overset{H}{\text{eigenvalue}}$ $E = e(p) + e(q)$

now act with $H-e(p)-e(q)$ on $|p<q\rangle$

$$\Rightarrow \quad (e^{ip+iq}-2e^{iq}+1)\sum_{j=-\infty}^{+\infty} \overbrace{|j<j+1\rangle}\, e^{i\overbrace{(p+q)}^{P}j}$$

want to compose exact eigenstate from partial eigenstates $|p<q)$

note: $(H-e(p)-e(q))$ acting on $|p<q)$ is symmetric in p,q up to factor

yes: $|q<p)$ yields $(e^{ip+iq}-2e^{ip}+1)\sum_{j=-\infty}^{+\infty} e^{i(p-q)j}\,|j<j+1)$

combine $|p<q)$, $|q<p)$ s.t. $H-e(p)-e(q)$ annihilates it

exact eigenstate $|p,q\rangle := |p<q) + S(p,q)\,|q<p)$

scattering factor S

$$S(p,q) := -\frac{e^{ip+iq}-2e^{iq}+1}{e^{ip+iq}-2e^{ip}+1}.$$

up to periodicity

M=2 is solved!

Factorised Scattering

$M=3$ has three magnons and $3!=6$ orderings

(asymptotic regions) ordered partial eigenstates $|p_1<p_2<p_3\rangle$, $|p_2<p_3<p_1\rangle$ ...

Exact eigenstate ansatz:

$$\begin{aligned}|p_1,p_2,p_3\rangle = {} & |p_1<p_2<p_3\rangle + S_{12}S_{13}S_{23}|p_3<p_2<p_1\rangle \\ & + S_{12}|p_2<p_1<p_3\rangle + S_{12}S_{23}|p_3<p_1<p_2\rangle \\ & + S_{23}|p_1<p_3<p_2\rangle + S_{12}S_{13}|p_2<p_3<p_1\rangle\end{aligned}$$

Conventionally expect:

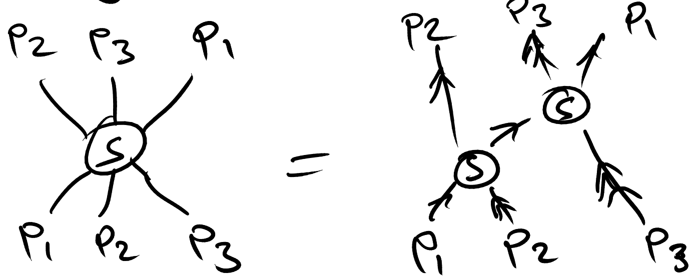
$$(H-E)|p_1p_2p_3\rangle \sim \sum_j e^{ip_j}|j<j+1<j+2\rangle$$ triple contact term.

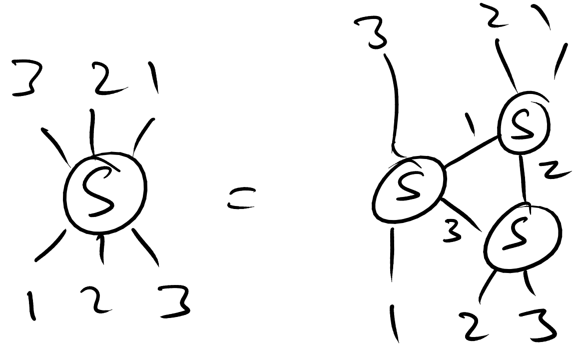
in integrable system find that coefficient = 0.

~ integrability, factorised scattering, no elem. 3 body interactions.

holds not only for $M=3$ but for all $M>3$ as well

Scattering factorises as follows





$S_\pi \qquad \pi \in S_M$

compose scattering factor between any two partial eigenstates

⇝ exact magnon eigenstates $(p_1 \dots p_M)$

Solution of the Infinite Chain

construction of generic magnon states

$|0\rangle = |\downarrow \ldots \downarrow\rangle$ $\qquad E = 0$

$|p\rangle = \sum_j e^{ipj} |\ldots \overset{j}{\uparrow} \ldots\rangle$ $\qquad E = e(p)$

$|p,q\rangle = |p<q\rangle + S(p,q)|q<p\rangle$ $\qquad E = e(p) + e(q)$

$|\{p_k\}\rangle = \sum_{\pi \in S_M} S_M |p_{\pi(1)} < \ldots < p_{\pi(M)}\rangle$ $\qquad E = \sum_j e(p_j)$

p_j are def. mod 2π

Some issues: $*$ $S(p,p) = -1$ $\sim$ Fermi statistics

$|\ldots, p, \ldots, p, \ldots\rangle = 0$ Pauli principle, momenta must be distinct!

* $S(p,0) = 1 = S(0,p)$ $\qquad e(0) = 0$

magnon corresponds to $su(2)$ ladder operator; note: bosonic more or less

Bound States ~ normalisable states on infinite chain

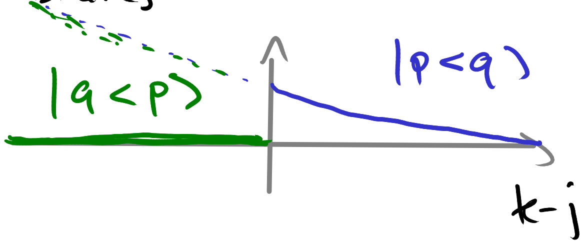
we may consider states with complex momenta

for $\operatorname{Im} p \neq 0$ $\quad e^{ipj}$ will exponentially decay at either $j \to \infty$, $j \to -\infty$

alone e^{ipj} will never be suitable for normalised states

but in scattering combination may work out:

if $S(p,q) = 0, \infty$



Bound states are compounds of two or more magnons where relative wave function decay exponentially with distance

6.4 Bethe Equations

6/2:17:05 – 6/2:39:43 (0:22:38)

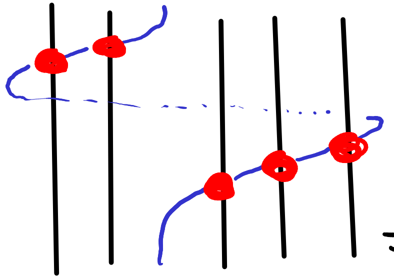
Closed Chains

Can use Coordinate Bethe Ansatz states for closed chains by imposing per. bdy.

" $\langle j_1, \ldots |\psi\rangle = \langle j_1+L, \ldots |\psi\rangle$ "

- * choose first magnon with momentum p_k
- * pick up factor e^{ip_kL}
- * pick up factor $S(p_k, p_\ell)$ for all $\ell \neq k$

$$\langle j_1, j_2, \ldots, j_M |\psi\rangle = \langle j_2, \ldots, j_M, j_1+L |\psi\rangle$$



Scat Fact. $\overset{!}{=} 1$

Bethe Equations $$e^{ip_kL}\prod_{\substack{\ell=1\\ \ell\neq k}}^{M} S(p_k, p_\ell) \overset{!}{=} 1 \qquad \text{for all } k=1\ldots M$$

One eq. for each d.o.f. p_k ~> quantises p_k

$$E=\sum_{k=1}^{M} e(p_k) \qquad P=\sum_{k=1}^{M} p_k \qquad e^{iPL}=1 \Rightarrow P\in\frac{2\pi}{L}\mathbb{Z}.$$

Rapidities

change variables p_k to u_k

$$p_k = 2\,\mathrm{arccot}(2u_k) \qquad u_k = \frac{1}{2}\cot(p_k/2) \qquad e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}$$

$$\rightsquigarrow \quad S(u,v) = \frac{u-v-i}{u-v+i} \qquad e(u) = \frac{i}{u+i/2} - \frac{i}{u-i/2}$$

Bethe Equations in rational form:

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2}$$

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1 \\ l\neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \quad \text{for } k = 1, \ldots, M \qquad E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right)$$

* u_k are real or complex conj. pairs

* u_k distinct except for $u_k = \infty$

* $u_k = \infty \Leftrightarrow p_k = 0 \Leftrightarrow$ su(2) ladder operator

Highest weight states have no $u_k = \infty$

* special values $u_k = \pm i/2$. careful!

6/2:39:43 – 6/3:07:45 (0:28:02)

6.5 Generalisations

Open Chains

$$H=\sum_{j=1}^{L-1}\mathcal{H}_j$$

need to generalise coord. B.A. to semi-infinite chains with reflection at bdy ferrom.

* vacuum is the same as before $\qquad |+p\rangle=\sum_k e^{ipk}|k\rangle$

* one magnon needs to reflect at bdy at first site 1

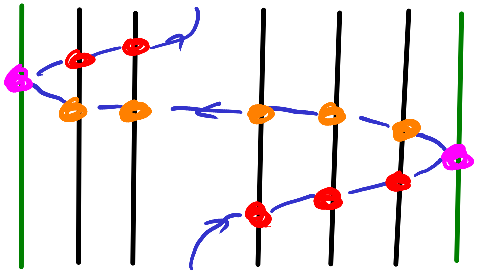
$$(H-e(p))\,|+p\rangle = (1-e^{+ip})\,|1\rangle \;\leftarrow \text{need to compensate}$$

consider magnon state $|-p\rangle$ for which $e(-p)=e(+p)$

$$(H-e(p))\,|-p\rangle = (1-e^{-ip})\,|1\rangle$$

proper eigenstate $\big|\,|p|\,\big\rangle_L = e^{-ip}|+p\rangle + e^{+ip}\,K_L(+p)\,|-p\rangle$

$$K_L(+p) = -e^{-2ip}\frac{1-e^{+ip}}{1-e^{-ip}} = e^{-ip}$$ boundary scattering factor



$$\frac{e^{i(L-1)(+p_k)}}{e^{i(L-1)(-p_k)}}\,\frac{K_R(+p_k)}{K_L(+p_k)}\prod_{\substack{l=1\\ l\neq k}}^{M}\frac{S(+p_k,p_l)}{S(-p_k,p_l)}=1$$

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^{2L}=\prod_{\substack{l=1\\ l\neq k}}^{M}\frac{u_k-u_l+i}{u_k-u_l-i}\,\frac{u_k+u_l+i}{u_k+u_l-i}$$

Higher Spins

$XXX_{1/2} \to XXX_s$ consider $s=1$ $\overset{\downarrow\downarrow}{|0\rangle}, \overset{\uparrow\downarrow}{|1\rangle}, \overset{\uparrow\uparrow}{|2\rangle}$

adjust coord. Bethe Ansatz

* ferromagn. vac $|0\rangle = |0 \dots 0\rangle$

* one magnon: $|p\rangle = \sum_j e^{ipj} |\dots \overset{j}{1} \dots\rangle$

* two magnons:

$$\left.\begin{aligned} |p<q\rangle &= \sum_{j<k} e^{ipj+iqk} |\dots \overset{j}{1} \dots \overset{k}{1} \dots\rangle \\ |p;2\rangle &= \sum_j e^{ipj} |\dots \overset{j}{2} \dots\rangle \end{aligned}\right\}$$ same sector expect mixing!

$$(H-E)|p<q\rangle = \sum_j e^{i(p+q)j} \left(* |\dots \overset{j}{1}\overset{j+1}{1} \dots\rangle + * |\dots \overset{j}{2} \dots\rangle \right)$$

$$(H-E)|p;2\rangle = \sum_j e^{ipj} \left(* |\dots 11 \dots\rangle + * |\dots 2 \dots\rangle \right)$$

true eigenstate:

$$|p,q\rangle = |p<q\rangle + \overset{\substack{\text{scattering factor (IR)}\\ \downarrow}}{S}\,|q<p\rangle + \overset{\substack{\text{contact term (UV)}\\ \downarrow}}{C}\,|p+q;2\rangle$$

$$\left(\frac{u_k+i}{u_k-i}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip} = \frac{u+i}{u-i} \qquad e(u) = -p'(u)$$

XXX_s

$$\left(\frac{u_k+is}{u_k-is}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip} = \frac{u+is}{u-is} \qquad e(u) = -p'(u)$$