Introduction to Integrability

Lecture Slides, Chapter 4

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4 Spectral Curves

expansions in spectral par $u \in \mathbb{C}$ $\rightsquigarrow$ complex analysis in u

obtain complex geometry: spectral curve ~ represents all conserved data.

Approach: given some state (solution of e.o.m. in terms of $\vec{S}$

transform this to monodromy $T(u)$

analyse thoroughly $T(u)$ as a function of u. (eigensystem!)

4.1 Spectral Curve

have S_i, $S_i(t)$ fixed $\rightsquigarrow$ monodromy $T(u)$

conserved quantities in $F(u)$

Eigenvalues

Lax eq. tells that time ev. of $T(u)$ is iso-spectral

eigenvalues $T_a(u)$ $a=1,2$ are conserved.

$\tau_a(u)$ are given by $F(u)$ as follows: $T(u)$ is 2×2 matrix

$$\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} \Rightarrow \det T(u) = \left(1 + \frac{1}{u^2}\right)^L = \tau_1(u)\cdot\tau_2(u)$$

$$\operatorname{tr} T(u) = F(u) = \tau_1(u) + \tau_2(u)$$

$$\Rightarrow \tau_{1,2}(u) = \frac{1}{2}F(u) \pm \sqrt{\frac{1}{4}F(u)^2 - \left(1+\frac{1}{u^2}\right)^L}$$

polynomial of degree L in $1/u$

Singularities

point $\tilde{u}=0$ is singular $T(u)$ is a polynomial of deg L in $1/u$

L-fold pole of $T(u)$ also $\tau_a(u)$ have L-fold poles at $u=\tilde{u}=0$

$T(u)$ is analytic for $u \neq \tilde{u} = 0$

$\Rightarrow$ eigenvalues $\tau_a(u)$ are analytic almost everywhere but not everywhere.

Potential square root singularities in $\tau_a(u)$ at radicand $= 0$ points $\hat{u}_j$

$$\frac{1}{4} F(\hat{u}_j)^2 = \left(1 + \frac{1}{\hat{u}_j^2}\right)^L \qquad \text{branch points}$$

$F(u)$ is pol. deg L in $1/u$: alg eq. of deg $2L$ in $1/u$ $\Rightarrow$ $2L$ solutions in $\mathbb{C}$

these are where $\tau_1(\hat{u}_j) = \tau_2(\hat{u}_j)$ eigenvalues degenerate

two $\hat{u}_j$ are fixed to $\hat{u} = \infty$ due to rel.

$$F(u) = 2 + \frac{0}{u} - \frac{1}{u^2}(J^2 - L) + \dots \quad \sim \text{related to } SO(3) \text{ sym.}$$

Simple States $L=2$, $L=3$ symmetric state

$L=2$: $$S_{1,2}(t) = \begin{pmatrix} 1 \\ \pm\tan(\vartheta/2)\cdot e^{-i\omega t}\end{pmatrix} \qquad \omega = \frac{2}{\cos\vartheta}$$

$$T(u) = \mathrm{id} + \frac{2i}{u}\cos\vartheta\,\sigma^z - \frac{1}{u^2}\begin{pmatrix}\cos(2\vartheta) & e^{+i\omega t}\sin(2\vartheta) \\ -e^{-i\omega t}\sin(2\vartheta) & \cos(2\vartheta)\end{pmatrix}$$

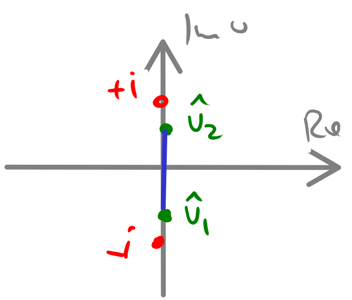
trace $$F(u) = \mathrm{tr}\, T(u) = 2 - \frac{2}{u^2}\cos(2\vartheta)$$

expansion around $u=\infty$ agrees with $so(3)$ symmetry charges $\vec{J} = 2\cos\vartheta\,\vec{e}_z$

$$H = -\log\frac{F(+i)\,F(-i)}{16} = -4\log|\cos\vartheta|.$$

Eigenvalues $$\tau_{1,2}(u) = 1 - \frac{\cos(2\vartheta)}{u^2} \pm \frac{2i\cos(\vartheta)}{u}\sqrt{1 + \frac{\sin^2\vartheta}{u^2}}$$

$$\hat{u}_{1,2} = \mp i\sin\vartheta$$



$L=3$; $F(u) = 2 + \frac{3-J^2}{u^2}$ $0 \leq J \leq 3$ $J=1$ singular; focus on $1<J<3$

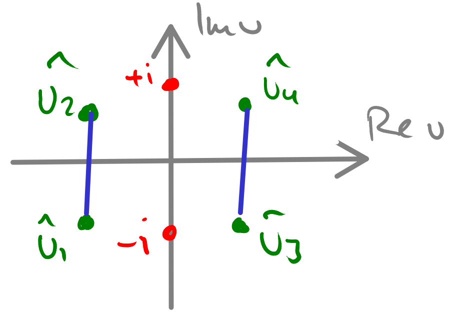
parametrisation for J in terms of (abstract) $\mu \in]0;\pi[$

$$J^2 = 5 - 4\cos\mu$$

branch pt of eigenvalues $\tau_{1,2}(u)$

$$\hat{U}_{1,4} = \pm \frac{e^{-i\mu}}{\sqrt{1-2e^{-i\mu}}}$$

$$\hat{U}_{2,3} = \pm \frac{e^{i\mu}}{\sqrt{1-2e^{i\mu}}} = \hat{U}^*_{1,4}$$



Spectral Curve square root sing $\hat{u}_k$ and neighbourhood in $\mathbb{C}$

Expand $\tau(u)$ around $u = \hat{u}$

$$\tau_{1,2}(u) = \frac{1}{2} F(u) \pm \hat{k}\sqrt{u-\hat{u}} + O(u-\hat{u})$$

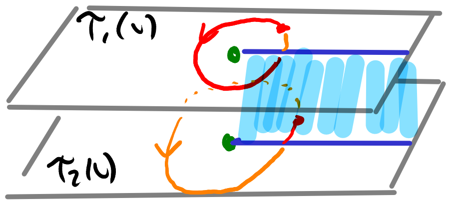
$$\hat{k} = \sqrt{\frac{1}{4} F(\hat{u}) F'(\hat{u}) + \frac{2L}{\hat{u}^3}\left(1+\frac{1}{\hat{u}^2}\right)^{L-1}}$$

Follow function analytically around $u = \hat{u}$ $\quad u(\sigma) = \hat{u} + \epsilon e^{i\sigma}$

$$\tau_1(u(\sigma)) = \frac{1}{2} F(\hat{u}) + \hat{k}\sqrt{\epsilon}\, e^{i\sigma/2} + O(\epsilon)$$

for $\Delta\sigma = 4\pi$ $\quad \tau_1(u(\sigma+4\pi)) = \tau_1(u(\sigma))$ $\quad$ 4π-periodic

for $\Delta\sigma = 2\pi$ $\quad \tau_1(u(\sigma+2\pi)) = \tau_2(u(\sigma))$ $\quad$ 2π rotation permutes eigenvalues!



a 2π rotation about a branch point $\hat{u}$ corresponds to exchange of eigenvalues, but overall spectrum $\{\tau_{1,2}(u)\}$ of $T(u)$ remains.

Introduce concept of spectral curve, Riemann sheet of function τ.

Function $f_a(u)$ with sheets $a = 1, 2, \ldots$ (eg $\tau_a(u)$ with $a = 1, 2$) can be viewed as a function $f(z)$ on a covering space (spectral curve) Γ with Riemann sheets labelled by $a = 1, 2, \ldots$

$$f(z) = f_{a(z)}(u(z))$$

$a(z) = 1, 2, \ldots$ is the sheet a pos z.
$u(z) \in \mathbb{C}$ is projection of u onto sheet.

$\Rightarrow$ Eigenvalue function is a analytic function on Riemann surface w/o singularities at $u = \hat{u}$

Riemann surface for eigenvalue problem is a so-called spectral curve

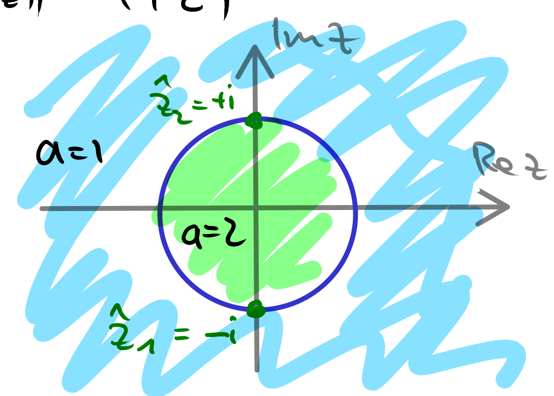
Curve: Embedding of $\mathbb{C}$ into $\bar{\mathbb{C}}^2 \ni (u,\tau)$:

$$\Gamma = \{ (u,\tau) \in \bar{\mathbb{C}}^2 ;\ \det (T(u) - \tau) = 0 \}$$

2×2 monodromy: introduce a permutation map $z \to z^\times$

$$u(z^\times) = u(z) \qquad \tau(z^\times) = \frac{\det T(u(z))}{\tau(z)} = F(u(z)) - \tau(z)$$

note: Branch pt $z = \hat{z}_n$ are fixed pts of $z \to z^\times$



Example $L=2$ $\quad u(z) = \frac{1}{2} \sin \vartheta \,(z - 1/z)$

$z^\times = -1/z$ $\qquad \tau(z) = \left(\frac{z + 1/z - 2i \cot \vartheta}{z - 1/z} \right)^2$

fixed pt $\hat{z}_{1,2} = \pm i$

General Picture arbitrary length L

discussion of $\tau(u)$ leads to $2L-2$ branch points $\hat{u}_n \rightsquigarrow L-1$ branch cuts

compact Riemann surface with 2 sheets and $L-1$ branch cuts has genus $g = L-2$

$L=2 \sim g=0$ simple, rational functions

$L=3 \sim g=1$ not simple, elliptic functions

$L>3 \sim g>1$ very non trivial, hyperelliptic functions

4.2 Ground State and Excitations

Explore spectral curve for small excitations of the ferromagnetic ground state

Ground State $\vec{S}_k = \vec{e}_z$ Lax transport $\mathcal{L}_k = \mathcal{L}(u) = \mathrm{id} + \frac{i}{u}\sigma^z$

two EV. $(u \pm i)/u$ therefore EV of $T(u) = \mathcal{L}(u)^L$

$$\tau_{1,2}(u) = \frac{(u \pm i)^L}{u^L}$$ has no branch points $\hat{u}$

generically expect curve of genus $g = L-2$. consider degeneracy

$$F(u) = \tau_1(u) + \tau_2(u) = \frac{(u+i)^L + (u-i)^L}{u^L}$$

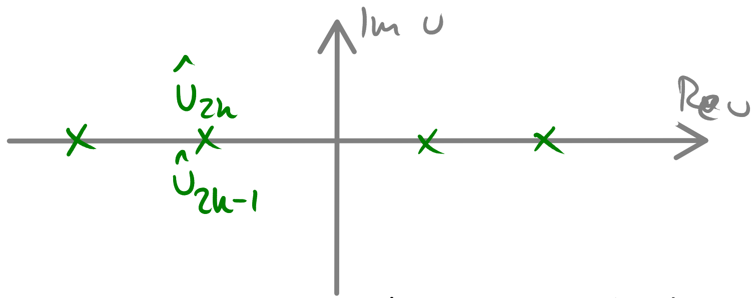
$2L$ potential branch points as zeros of radicand

$$\tau_{1,2}(u) = \frac{1}{2}F(u) \pm \sqrt{\frac{1}{4}F(u)^2 - \frac{(u^2+1)^L}{u^{2L}}} \overset{!}{=} 0$$

There are $2L-2$ double poles

at $\hat{u}_{2k-1} = \hat{u}_{2k} = -\cot\frac{\pi k}{L}$

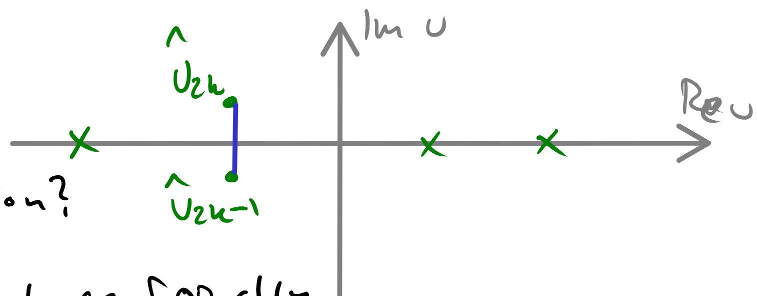
$k = 1, \dots, L-1$



Riemann surface of $g = L-2$ degenerates to two disc. sheets.

Single Excitation

pull two branch points apart by a bit



which variation of $F(u)$ induces this deformation?

$$\delta F(u) = -i\epsilon^2 \frac{(u+i)^L - (u-i)^L}{u^L (u - \hat{u}_{2k})}$$

analyse formally

$$F(\hat{u})^2 + 2F(\hat{u})\,\delta F(\hat{u}) + \dots = \frac{4(\hat{u}^2+1)^L}{\hat{u}^{2L}}$$

on double root splits up $\delta \hat{u}_{2n-1,2n} = \mp \dfrac{i\epsilon\sqrt{2/L}}{|\sin(\pi n/L)|}$

Relate expansion of (variation of) F at $u=\infty$ to (variation of) angular momentum

$$\delta F(u) = \frac{2L\epsilon^2}{u^2} + O(1/u^3) \sim \delta J^2 = -2L\epsilon^2 \sim \delta\vec{J} = -\epsilon^2 \vec{e}_z$$

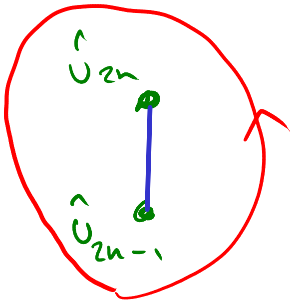
$$\vec{J} = J e_z$$

$$\delta H = -\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = \frac{2\epsilon^2}{\hat{u}_{2n}^2+1} = 2\epsilon^2 \sin^2\frac{\pi n}{L} = -2\,\delta J \sin^2\frac{\pi n}{L}$$

$$\delta P = +i\,\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = -\frac{2\epsilon^2 \hat{u}_{2n}}{\hat{u}_{2n}^2+1} = \epsilon^2 \sin\frac{2\pi n}{L}$$

ϵ^2 is related to action variable $I_n = \epsilon^2 + \dots$

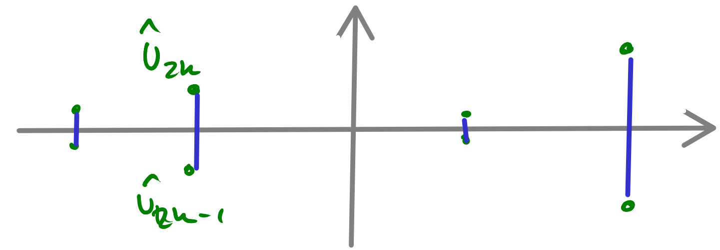
can be obtained from spectral curve



$$I_n = \frac{1}{2\pi} \oint_{\hat{u}_{2n}} dw \log \tau(u) = -\frac{1}{2\pi} \oint_{\hat{u}_{2n}} \frac{d\tau}{\tau} u = \epsilon^2 + \dots$$

$$\delta H = 2 \delta I_n \sin^2 \frac{\pi n}{L} + \dots \qquad \delta \vec{J} = -\delta I_n \vec{e}_z + \dots \qquad \frac{\partial H}{\partial I_n} = \omega_n \qquad \omega_n = 2 \sin^2 \frac{\pi n}{L}$$

Multiple Excitations

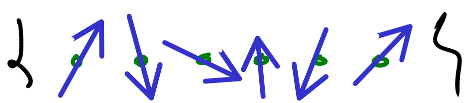


For small excitations of GS can apply perturbation series.

First order is linear! excitations superimpose, in part for δF

L-1 independent modes to split up L-1 double points ~ L-1 excitations of GS.

L-1 d.o.f. to change F 1:1 to conserved qty of an integrable system.
(plus to d.o.f. for $\vec{J}/J$ due to SO(3))



(L-1 conserved ch.)

$\longleftrightarrow$

(L-1) moduli $F(u)$

4.3 Dynamical Divisor

Singularities

monodromy eigenvector equation

$$\underbrace{T(u)}_{\text{monod.}}\, \psi_a(u) = \underbrace{\tau_a(u)}_{\text{e.val}}\, \underbrace{\psi_a(u)}_{\text{eigenvector}} \qquad a=1,2 \qquad 2\times 2 \text{ matrix}$$

$\psi_a(u)$ is mostly analytic because $T(u)$ is, $\tau_a(u)$ is.

three types of singularities

- singularities in $T(u) \leadsto \tau(u)$ eg. $u=0$
  $\leadsto$ no singularities in $\psi_a(u)$ (rescale singularity away in ev. eq.)
- square root branch points in $\tau(u)$ but not in $T(u)$
  eigenvect. eq implies a square root sing in $\psi_a(u)$
- normalisation of $\psi_a(u)$ can introduce / remove singularities in $\psi_a(u)$

Branch Points

Two eigenvectors $\psi_1(u), \psi_2(u)$ degenerate (collinear) at a branch point $u=\hat{u}$.
Same as for eigenvalues $\tau_1(\hat{u}) = \tau_2(\hat{u})$
due to non-diagonalisability of matrix $T(u)$ at $u=\hat{u}$.
Discuss a matrix $T(u)$ that becomes non-diag. at a certain $u=\hat{u}\in\mathbb{C}$

$$T(u) := \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$ A, B, C, D are analytic at $u=\hat{u}$

eigenvalues $$\tau_{1,2}(u) = \tfrac{1}{2}\left(A(u)+D(u)\right) \pm \sqrt{\tfrac{1}{4}\left(A(u)-D(u)\right)^2 + B(u)C(u)}$$

$$\tfrac{1}{4}(\hat{A}-\hat{D})^2 + \hat{B}\hat{C} = 0$$

assumption $\tau_1(\hat{u}) = \tau_2(\hat{u})$ need that radicand $=0$ at $u=\hat{u}$.

expand $\tau_{1,2}(u)$ around $u=\hat{u}$ $\quad \tau_{1,2}(u) = \tau(\hat{u}) \pm \hat{k}\sqrt{u-\hat{u}} + \dots$

$$\hat{k} = \sqrt{\tfrac{1}{2}(\hat{A}-\hat{D})(\hat{A}'-\hat{D}') + \hat{B}\hat{C}' + \hat{C}\hat{B}'} \neq 0$$ for a nontriv. square root sing.

assume further that $T(u)$ is diagonalisable at $u=\hat{u}$: $\tau_1 = \tau_2 = \tau$

$$T(\hat{u}) = U(\tau\, id)\, U^{-1} = \tau\, id \Rightarrow \hat{A} = \hat{D}, \quad \hat{B} = \hat{C} = 0 \quad \text{at } u=\hat{u}$$

implies $\hat{\kappa} = 0$ is contradiction $\Rightarrow$ $T(u)$ is non-diagonalisable.

Consider eigenvector function $\psi_a(u) \equiv \begin{pmatrix} -B(u) \\ A(u) - \tau_a(u) \end{pmatrix}$ both ψ_a coincide at $u=\hat{u}$ b/c τ_a coincide there.

$$\psi_1(\hat{u}) \equiv \psi_2(\hat{u})$$

Example $L=2$

$$\psi(z) = \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$$

Essential for establishing $\psi(z)$ as a function on Riemann surface Γ

$$\psi(z) \equiv \psi_{a(z)}(u(z))$$

$\tau(z)$, $\psi(z)$ are single-valued on Γ.

eigenvect. eq. on Γ: $T(u(z))\, \psi(z) = \tau(z)\, \psi(z)$

Dynamical Divisor

other type of pole in $\psi(z)$. $\psi(z)$ is meromorphic on Γ

$\psi(z)$ might be analytic at any point z through rescaling by scalar fn.

$$\psi(z) \equiv \lambda(z)\,\psi(z)$$

rescale globally only by meromorphic $\lambda(z)$. $\rightsquigarrow$ set of poles not universally defined.

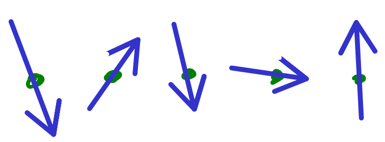
fix this by demanding further normalisation of $\psi(z)$

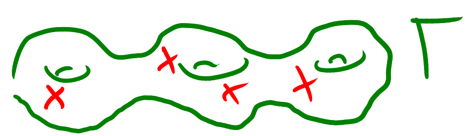
$$v_r \cdot \psi(z) \overset{!}{=} 1 \qquad \text{for some const. vector } v_r$$

this fixes scaling d.o.f. completely. thus fixes set of poles locations $\{\check{z}_k\}$

eg. for $v_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \psi(z) = \begin{pmatrix} 1 \\ \xi(z) \end{pmatrix}$ $\quad$ $\xi(z)$ is stereographic proj. variable for ψ

location of poles $\{\check{z}_k\}$ of $\xi(z)$ is dynamical divisor for state.





Argue that dyn. div. consists of precisely g+1 points on Γ g: genus of Γ

Consider $f(u) := \left(\underset{\text{normalised}}{\psi_1(u)^T \, \varepsilon \, \psi_2(u)}\right)^2 = \left(\xi_1(u) - \xi_2(u)\right)^2$

- $f(u)$ is a meromorphic function of $u \in \bar{\mathbb{C}}$.

  $\psi_{1,2}(u)$ are meromorphic almost everywhere

  around branch points ψ_1 and ψ_2 get interchanged but $f(u)$ returns to old value

  $\Rightarrow$ $f(u)$ has no branch points.

- branch points are zeros of $f(u)$. $f(u) = 0$ if $\psi_1(u)$ is coll to $\psi_2(u)$

  $f(u) \neq 0$ elsewhere because $\psi_1 \, \psi_2$ span $\mathbb{C}^2$

  square root behaviour implies that zeros of $f(u)$ are single.

  for Γ of genus g there are $2g+2$ branch points. $2g+2$ zeros in $f(u)$.

- $f(u)$ is meromorphic on $\bar{\mathbb{C}}$ has same no of poles as zeros. $2g+2$ poles

  all poles are double $\Rightarrow$ $f(u)$ has g+1 double poles $\Rightarrow$ ξ_1 and ξ_2 together have g+1 poles

  $\Rightarrow$ $\xi(z)$ has g+1 poles

locations of poles $\check{z}_k$ are dynamical $\check{z}_k(t)$ governed by diff. eq.

$$\frac{dT}{dt} = [M, T] \quad \rightsquigarrow \quad \frac{d\psi}{dt} = M\psi + \lambda\psi$$

λ governs scaling of ψ

through normalisation cond $v_r \cdot \psi = 1$ fix $\lambda = -v_r \cdot M\psi$

ex $L=2$: $\quad v_r = \begin{pmatrix} 1 \\ -1/\xi_r \end{pmatrix} \Rightarrow \psi(z) = \frac{1}{1 - i\xi_r^{-1} e^{-i\omega t} z} \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$

Divisor $\quad \check{z}(t) = -i\xi_r e^{i\omega t}$

other ref dir. $v_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \psi(z) = \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$

$\check{z}(t) = \infty$ static information is in coefficient of z at $z = \infty$

Symmetry impact of SO(3) symmetry on eigenvector function $\psi(z)$

superint system, $L-1$ dyn. d.o.f instead of L

lower generic genus from $g = L-1$ to $g = L-2$

always related to point $u = \infty$ double point

$$T(u) = \mathrm{id} + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \ldots \qquad T(\infty) = \mathrm{id}^{\checkmark} \to \text{eigenvector is not fixed}$$

at the two points $z = z_\infty$, $z = z_\infty^\times$ (both corr to $u = \infty$) the direction of $\psi(z)$ is fixed through analyticity

$$\tau(z) = 1 \pm \frac{iJ}{u(z)} + \ldots \qquad z = z_\infty, z_\infty^\times$$

$$(\vec{J} \cdot \vec{\sigma})\, \psi(z_\infty) = + J\, \psi(z_\infty)$$

$$(\vec{J} \cdot \vec{\sigma})\, \psi(z_\infty^\times) = - J\, \psi(z_\infty^\times) \qquad \text{recover } \frac{\vec{J}}{J} = \frac{\psi_\infty^{\times T} \varepsilon \vec{\sigma} \cdot \psi_\infty}{\psi_\infty^{\times T} \varepsilon \psi_\infty}$$

4.4 Construction of Solutions

we may now construct (spectral curve / divisor) / eigensystem functions from scratch.
eigensystem → reconstruct $T(u)$ → reconstruct state / solution.

Spectral Curve

construct $\tau(z)$ on a Riemann surface Γ

characteristic pol. eq. $\tau(z)^2 - F(u(z))\,\tau(z) + \det T(u(z)) = 0$

$F(u)$ is pol of deg L in $1/u$ with leading terms $F(u) = 2 + \frac{0}{u} + \dots$

$\det T(u) = (1 + 1/u^2)^L$ fixed. altogether $L-1$ d.o.f. in curve ~ moduli

char. eq. describes $2L-2$ branch points ⇒ genus $g = L-2$.

Dynamical Divisor

eigenvector function $\psi(z)$

normalised ev. function $\psi(z) = \begin{pmatrix} 1 \\ \xi(z) \end{pmatrix}$

$\xi(z)$ is a meromorphic (rational) function on T of degree $g+1$

$\xi(z)$ has $g+3$ moduli (deg. of freedom)

$\sim$ $\underbrace{g+1 \text{ locations of poles}}_{\text{dynamical divisor } \{\check{z}_a\}}$, $\underbrace{1 \text{ location of a zero},\ 1 \text{ overall scaling}}_{\text{direction of any mom } \vec{J}/J}$ $\rightsquigarrow$ always fix this to $+\vec{e}_z$

establishes integrability of Heisenberg chain:

- eigenvector function has no more than $g+3 = L+1$ d.o.f.
- $2L$ d.o.f in total
- at least $L-1$ d.o.f. encoded into spectral curve
- all eigensystem d.o.f. encode state fully.

eigenvector function takes the form:

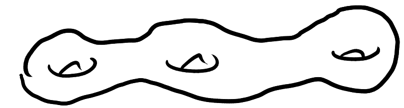
$$\xi(z) = \exp(i\Delta(z) + i\phi)\,\frac{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Delta} + \vec{\Phi})\,\Theta(\vec{\Phi} - \vec{\Delta})}{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Phi})\,\Theta(\vec{\Phi})}.$$

theta-fn Abel map

several functions θ, $\vec{\Omega}$ to be introduced, integral Δ + constants $\vec{\Delta}$

and get moduli $\vec{\Phi}$, ϕ ~ angle variables of solution (2π-periodic)

Complex Analysis on the spectral curve



Riemann surface of genus $g = L-2$ (typically) or $g < L-2$, def by alg. eq. $u \leftrightarrow \Gamma$

- Meromorphic (rational) functions on Γ: analytic except at isolated poles
- Meromorphic/abelian differentials $f(z)\,dz$ (one-form)
- Holomorphic fn/differentials: have no poles, analytic everywhere.

→ Hol. f^n are just constants.

→ Hol differentials ω_n exist, there are g for genus g

in our case $\quad \omega_n \in \operatorname{span}\{u(z)^{-3}, \dots, u(z)^{-L}\} \quad \dfrac{du(z)}{\tau(z) - \frac{1}{2}F(z)}$

→ mer differentials can be constructed for all desired locations of poles with specified sing. structure $\frac{*}{(z-z_0)^3} + \frac{*}{(z-z_0)^2} + \frac{*}{(z-z_0)} + \text{finite}$
(restriction: overall residue $\overset{!}{=} 0$)

→ mer functions exist for balanced set of zeros and poles.

$$\text{mer. funct} \underset{\text{integration}}{\overset{\text{diff}}{\rightleftarrows}} \text{mer. diff}$$

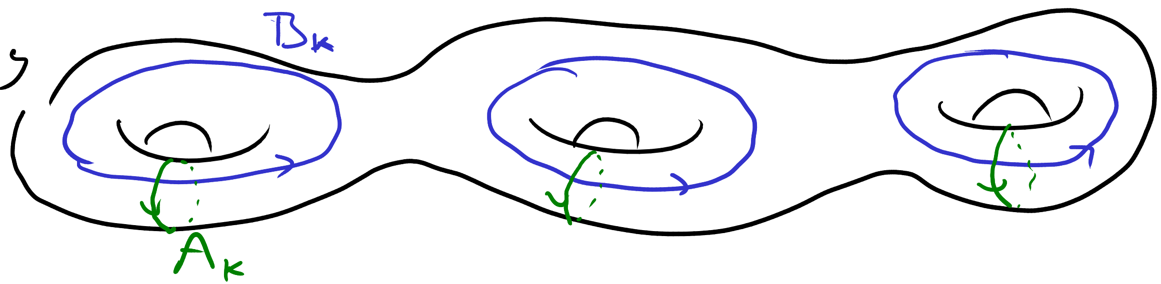
two obstacles for integration towards a mer. function:

- single pole in diff leads to a log-sing in function → $2\pi i \cdot$ res monodromy
  → not a pole
- non-trivial period $\oint_C \phi \neq 0$ → monodromy of function around C.

generically mer. diff integrate to abelian integrals.
some abelian int are in fact mer. functions

Hol. differentials ω_k have no poles $\Rightarrow$ $\int \omega_k$ can have non-trivial monodromy around C

surface genus g has $2g$ non-triv. cycles



A_k intersects B_k once (in same direction)

ω_k have non-triv periods $\oint_C \omega_k$ for all A_j, B_j

normalise ω_k s.t. $\quad \oint_{A_j} \omega_k \stackrel{!}{=} 2\pi \delta_{j,k}$

then eval B-periods $\quad \oint_{B_j} \omega_k =: 2\pi T_{j,k}$

period matrix
- symmetric
- $\mathrm{Im}\, T$ is pos.-def.

hol diff define Abel map

$$\Omega_k(z) := \int^z \omega_k \quad + \text{fixed constants}$$

$\vec{\Omega}$ not a function on Γ but rather

$$\Omega : \tilde{\Gamma} \to \mathbb{C}^g$$

A, B periods define a lattice

$$\Lambda : 2\pi \mathbb{Z}^g + 2\pi \tau \mathbb{Z}^g$$

can view $\vec{\Omega}$ mod Λ as a function on Γ Jacobian of Γ

$$\Omega : \Gamma \to T^g \qquad T^g := \mathbb{C}^g / \Lambda$$

complex g-Torus

real slice serves as part of Liouville torus $(\vec{\phi})$

finally abelian integrals are normalised s.t. A-periods are zero $\oint_A \phi = 0$ by subtracting lin. comb. of ω_k

Eigenvector Direction

$\xi(z)$ has $g+1$ poles/zeros

but choice $\vec{J} \sim \vec{e}_z$ fixes one zero/pole to $z_\infty, z_\infty^\times$ at $u=\infty$

g dynamical poles/zeros, treat them separately

Introduce theta-function: $\Theta : \mathbb{C}^g \to \mathbb{C}$ defined period matrix

$$\Theta(\vec{x}) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left(i\pi \vec{n}^T T \vec{n} + i \vec{x} \cdot \vec{n} \right)$$

three properties:

- $\operatorname{Im} T$ is pos.-def makes sum converge fast $\Rightarrow \Theta(\vec{x})$ is entire
- 2π periodicity in all g directions $\Theta(\vec{x} + 2\pi\vec{n}) = \Theta(\vec{x})$
- quasi periodicity in directions defined by T

$$\Theta(\vec{x} + 2\pi T\vec{n}) = \exp\left(-i\pi \vec{n}^T T \vec{n} - i \vec{x}\cdot\vec{n}\right) \Theta(\vec{x}) \qquad \vec{n} \in \mathbb{Z}^g.$$

θ likes to receive output of Abel map $\vec{\Omega}(z)$

$$f(z; \vec{\phi}) = \theta(\vec{\Omega}(z) + \vec{\phi})$$

entire function on $\tilde{\Gamma}$, • trivial monodromy under A-cycles
• multiplicative mon. under B-cycles.

- $f(z)$ has (typically) g zeros in each cell / on Γ
- location of zero is one-to-one with $\vec{\Phi} \in T^g$

Now consider:

$$\xi(z) = \exp(i\Delta(z) + i\phi)\,\frac{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Delta} + \vec{\Phi})\,\Theta(\vec{\Phi} - \vec{\Delta})}{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Phi})\,\Theta(\vec{\Phi})}.$$

$\Delta(z)$ is abelian integral of $d\Delta$ which is defined to have pole at $z = z_\infty$ res $-i$ zero at $z = z_\infty^\times$ res $+i$

$$d\Delta(z) \sim \frac{J\,v(z)^{-2}\,dv}{\tau(z) - \frac{1}{2}F(v(z))} + \text{hol. diff } \omega_u$$

$\vec{\Gamma} = \pi\vec{m} + \pi T\vec{m}'' \in \frac{1}{2}\Lambda$

$\vec{m}\cdot\vec{m}''$ is odd!

$\downarrow$ odd characteristic

log sing disappear in $\exp(i\Delta(z)) = \dfrac{\Xi(z, z_\infty)}{\Xi(z, z_\infty^\times)}$

normalise $\oint_{\vec{A}} d\Delta = 0$; $\oint_{\vec{B}} d\Delta =: \vec{\Delta}$

$\Xi(z, z') := \Theta(\vec{\Omega}(z) - \Omega(z') + \vec{\Gamma})$

1 zero at $z = z'$, $g-1$ zeros at pos z indep of z'

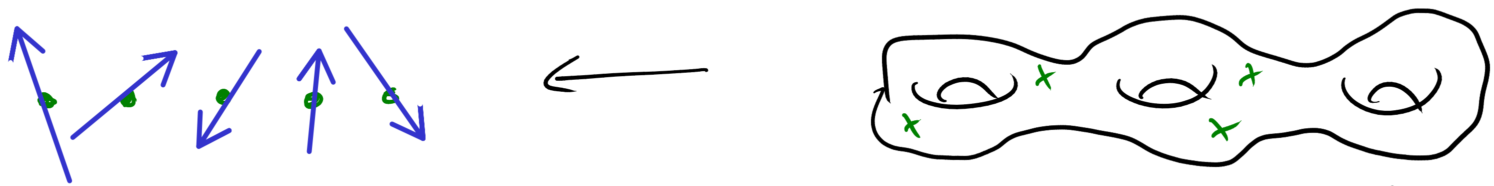
$\xi(z)$ is meromorphic!!

Reconstruction

have constructed T ← eigenvalue function $\tau(z)$ through an alg eq. $u \sim \tau$, specified by $F(u)$

constructed eigenvector $\xi(z)$ on T specified through $\vec{\Phi}$, ϵ (dir of $\vec{J}$)

- reconstruct Lax monodromy $T(u(z))$ through eigensystem.
- know how to construct J_j or $\vec{S}_j$ by recursive procedure



d.o.f $\quad 2L$ $\qquad\qquad$ $g \leq L-2$

$L-1$ dof for $\tau(z)$

$g+1 = L-1$ dof for $\xi(z)$

2 dof for $\vec{J}/J$

- reasonable to work with complexified vars.

Auxiliary Linear Problem

spectral curve

given a solution $\vec{S}_j(t)$; introduce auxiliary vector ($\mathbb{C}^2$) array $\psi_j(z;t)$.

ALP:
$$\psi_{j+1}(z;t) = \mathcal{L}_{j+1}(u(z);t)\,\psi_j(z,t)$$
$$\frac{d}{dt}\psi_j(z;t) = M_j(u(z);t)\,\psi_j(z;t)$$

compatibility between both by means of Lax transport eq.

- linear in ψ; 2-dim space of solutions eg specify $\psi_0(z;0)$

Furthermore impose eigenvalue property

$$\psi_{j+L}(z;t) = T_j(u(z);t)\,\psi_j(z;t) \overset{!}{=} \tau(z)\,\psi_j(z;t)$$

Propose solution:

$$\psi_{j,1}(z;t) = \exp\Big(ij\big(\Pi(z)-\Pi(z_\infty)\big) + it\big(\Sigma(z)-\Sigma(z_\infty)\big)\Big) \cdot \frac{\Theta\big(\vec{\Omega}(z)-\vec{\Omega}(z_\infty)+j\vec{\Pi}+t\vec{\Sigma}+\vec{\phi}_0\big)\,\Theta(\vec{\phi}_0)}{\Theta\big(j\vec{\Pi}+t\vec{\Sigma}+\vec{\phi}_0\big)\,\Theta\big(\vec{\Omega}(z)-\vec{\Omega}(z_\infty)+\vec{\phi}_0\big)}$$

$$\psi_{j,2}(z;t) = \exp\big(i\Delta(z)+i\phi_0\big)\,\frac{\Theta\big(\vec{\Omega}(z_\infty^{\times})-\vec{\Omega}(z_\infty)+\vec{\phi}_0\big)}{\Theta\big(\vec{\Omega}(z)-\vec{\Omega}(z_\infty)+\vec{\phi}_0\big)} \cdot \frac{\Theta\big(\vec{\Omega}(z)-\vec{\Omega}(z_\infty^{\times})+j\vec{\Pi}+t\vec{\Sigma}+\vec{\phi}_0\big)}{\Theta\big(j\vec{\Pi}+t\vec{\Sigma}+\vec{\phi}_0\big)} \cdot \exp\Big(ij\big(\Pi(z)-\Pi(z_\infty^{\times})\big)+it\big(\Sigma(z)-\Sigma(z_\infty^{\times})\big)\Big)$$

$\Pi(z)$, $\Sigma(z)$ are abelian integrals (like $\Delta(z)$)

$\uparrow_j$ $\uparrow t$ evolution

$d\Pi, d\Sigma$ has sing at $z = z_\pm, z_\pm^\times$ $(u = \pm i)$ and $z = z_0, z_0^\times$ $(u=0)$

reflect zeros/poles of $\mathcal{L}(u)$ $\mathcal{M}(u)$

$d\Pi$ has simple poles at $z = z_+^\times, z_-$ with res $-i$; $z = z_0, z_0^\times$ with res $+i$

$\hookrightarrow$ zeros $\hookrightarrow$ poles of $h_j(u)$

$$\exp(i\Pi(z)) = \frac{\Xi(z,z_-)\,\Xi(z,z_+^\times)}{\Xi(z,z_0)\,\Xi(z,z_0^\times)}$$

$\psi_j(z_+) = s_j$ $\psi_j(z_-) = s_{j+1}$

$\psi_j(z_-^\times) = \varepsilon s_j^*$ $\psi_j(z_+^\times) = \varepsilon s_{j+1}^*$

$d\Sigma$ has double poles at $z_\pm, z_\pm^\times$ without residues

integral $\Sigma(z)$ has single poles, no logs

$$\Sigma(z) = -\tfrac{1}{2}\psi(z,z_+) + \tfrac{1}{2}\psi(z,z_-) + \tfrac{1}{2}\psi(z,z_+^\times) - \tfrac{1}{2}\psi(z,z_-^\times) \qquad \psi(z,z') := \frac{i}{\Xi(z,z')}\frac{\partial\,\Xi(z,z')}{\partial u(z)}$$

A, B periods

A: $\oint_{\vec{A}} d\Pi = \oint_{\vec{A}} d\Sigma = 0$

B: $\oint_{\vec{B}} d\Pi =: \vec{\Pi} \qquad \oint_{\vec{B}} d\Sigma =: \vec{\Sigma}$

original spin d.o.f. (stereographic eig) $\zeta_j(t) = \dfrac{\varphi_{j,2}(z_+;t)}{\varphi_{j,1}(z_+;t)}$

evaluating for different values of j, t is computationally easy.

Properties of the Solution

- Monodromies of $\psi_j(z;t)$ around A, B cycles
- point $u=\infty$ (total ang momentum) $\psi_j(z_\infty, t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\psi_j(z;t) \sim u(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{at} \quad z = z_\infty^x.$$

- singularities at $u = \pm i$, $u = 0$ from $\Omega(u)$, $M(u)$
- periodicity along chain $\psi_{j+L}(z;t) = \tau(z)\, \psi_j(z;t)$

Θ argument must differ by lattice vector

$$\vec{\Pi} = \frac{2\pi \vec{n}}{L} + \frac{2\pi T \vec{n}'}{L} \qquad \tau(z) = e^{iq(z)} \qquad q(z) = L\left(\Pi(z) - \Pi(z_\infty)\right) - \left(\vec{\Omega}(z) - \vec{\Omega}(z_\infty)\right)\vec{n}'$$

$$\oint_{\vec{A}} dq = -i \oint_{\vec{A}} \frac{d\tau}{\tau} = \underset{\stackrel{!}{=}\, 0}{-2\pi \vec{n}'} \qquad \oint_{\vec{B}} dq = 2\pi \vec{n}$$

$\vec{n}$: mode numbers

additional integer $q(z_\infty^\times) = \int_{z_\infty}^{z_\infty^\times} dq = 2\pi n_0$

action vars $\vec{I} = \frac{1}{2\pi i} \oint_{\vec{A}} u\, dq$

• direction eigenvector $\xi_j(z;t) = \frac{\psi_{j,2}(z;t)}{\psi_{j,1}(z;t)} = \ldots$ form similar to $\xi(z)$

with $\phi_j(t) = \phi_0 + j\left(\pi(z_\infty) - \pi(z_\infty^\times)\right) + t\left(\Sigma(z_\infty) - \Sigma(z_\infty^\times)\right)$

$$- i \log \frac{\theta(\vec{\Phi}_j(t))\,\theta(\vec{\Phi}_0 - \vec{\Delta})}{\theta(\vec{\Phi}_0)\,\theta(\vec{\Phi}_0(t) - \vec{\Delta})}$$

$\vec{\Phi}_j(t) = \vec{\Phi}_0 + j\vec{\Pi} + t\vec{\Sigma}$ ← linear evolution on Liouville torus

Jacobian (curve)