Introduction to Integrability

Lecture Slides, Chapter 3

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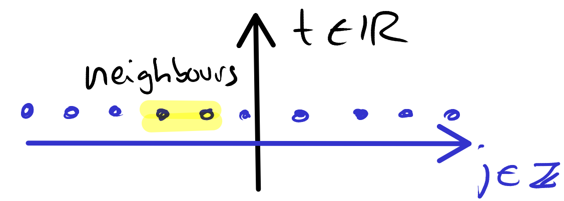
The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

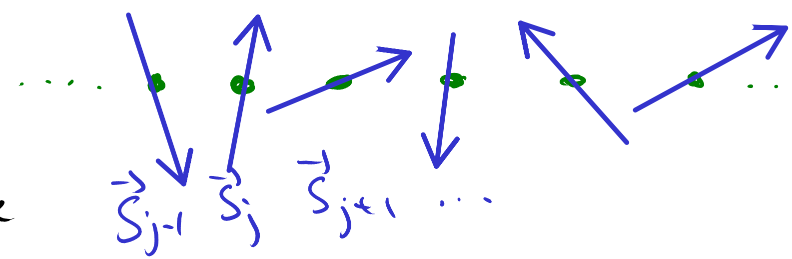
3. Classical Spin Chains



3.1 Heisenberg Spin Chain

Chain of elementary spin d.o.f. with interactions among nearest neighbours.

<u>Model</u> Hamiltonian mechanics



Phase space is copies of S^2

$\|\vec{S}_j\| = 1$ unit vectors; 2 dof per site

Poisson brackets $\qquad \{S_j^a, S_k^b\} = \delta_{jk}\, \varepsilon^{abc}\, S_k^c$

constraint $\vec{S}_k^2 = 1$ is compatible with Poisson structure $\qquad \{S_j, \vec{S}_k^2 - 1\} = 0$

Dynamics: based on sel alignment of neighbouring spins $\vec{S}_j$, $\vec{S}_{j+1}$

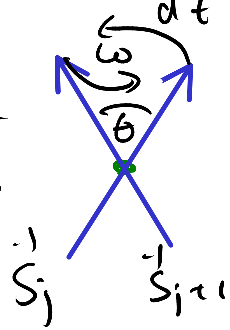
$$H = \sum_j \mathcal{H}_j \qquad \mathcal{H}_j = -\log \frac{1+\vec{S}_j\cdot\vec{S}_{j+1}}{2}.$$

homogeneous
symmetry SO(3) isotropic interaction depending on $\vec{S}_j\cdot\vec{S}_{j+1}$

Equations of motion

$$\frac{d\vec{S}_j}{dt} = -\{H, \vec{S}_j\} = -\frac{\vec{S}_{j-1}\times\vec{S}_j}{1+\vec{S}_{j-1}\cdot\vec{S}_j} + \frac{\vec{S}_j\times\vec{S}_{j+1}}{1+\vec{S}_j\cdot\vec{S}_{j+1}}$$

for nearest neighbours (only)



$$\omega = \frac{1}{\cos(\theta/2)}$$

Using stereographic projection var. $S^2 \ni \vec{S}_j \to \zeta_j \in \mathbb{C}$ or spinor var $s_j \in \mathbb{C}P^1$

$$\frac{1+\vec{S}_j\cdot\vec{S}_k}{2} = \frac{(1+\zeta_j\zeta_k^*)(1+\zeta_k\zeta_j^*)}{(1+|\zeta_j|^2)(1+|\zeta_k|^2)} = \frac{(s_j^\dagger s_k)(s_k^\dagger s_j)}{(s_j^\dagger s_j)(s_k^\dagger s_k)}$$

e.o.m.

$$\frac{d\zeta_j}{dt} = \frac{i}{2}\sum_{\pm}\frac{1+|\zeta_j|^2}{1+\zeta_{j\pm1}\zeta_j^*}\left(\zeta_{j\pm1}-\zeta_j\right)$$

governs the dynamics of the scale of s_j

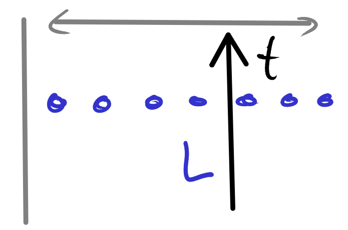
$$\frac{ds_j}{dt} = \frac{i}{2}\,\frac{s_j^\dagger s_j}{s_j^\dagger s_{j-1}}\,s_{j-1} + \frac{i}{2}\,\frac{s_j^\dagger s_j}{s_j^\dagger s_{j+1}}\,s_{j+1} + i\lambda_j s_j$$

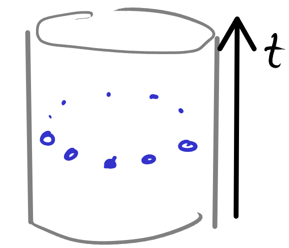
Boundary Conditions

there are different types of (integrable) boundary conditions
→ finite extent
→ infinite extent

closed boundary conditions

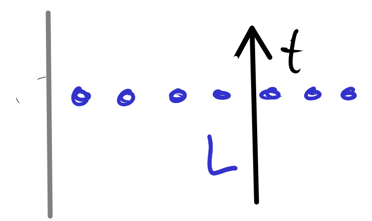
$$H=\sum_{j=1}^{L}\mathcal{H}_j \qquad \vec{S}_{j+L}=\vec{S}_j$$





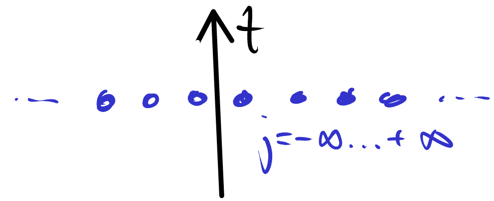
open boundary conditions

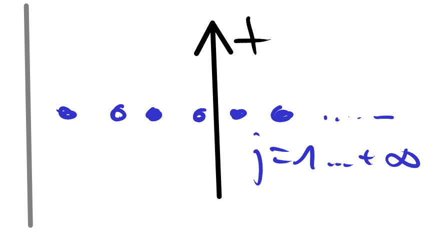
$$H=\sum_{j=1}^{L-1}\mathcal{H}_j$$



infinite boundary conditions

semi-infinite chains





asymptotic values for spin d.o.f.

$$H = \sum_{j=-\infty}^{\infty} \mathcal{H}_j \qquad \vec{S}_j \to \vec{S}_{L/R} \text{ for } j \to \mp\infty$$

deformations/twist of boundary conditions

boundary deg. of freedom.

Here: most undeformed closed chains.

Global Symmetries

has $SO(3)$ rotational symmetry for spin vectors $\vec{S}_j$

rotation for $\vec{S}_j$ generated by $\vec{S}_j$ — parametrises an inf. rot.

$$\delta \vec{S}_j = - \{ \delta \vec{X} \cdot \vec{S}_j , \vec{S}_j \}$$

global angular momentum charge

$$\vec{J} = \sum_j \vec{S}_j \qquad \delta \vec{X} \cdot \vec{J} \text{ for a global inf. rot.}$$

symmetry: $\{ H, \vec{J} \} = 0$

global cons. charge $\vec{J} = \sum_j \vec{Q}_j \qquad \vec{Q}_j = \vec{S}_j \qquad \vec{k}_j = \dfrac{\vec{S}_j \times \vec{S}_{j+1}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}}$

local cons. current Q_j, k_j disc. cons. eq. $\dfrac{d}{dt} \vec{Q}_j = - \{ H, \vec{Q}_j \} = \vec{k}_j - \vec{k}_{j-1}$ (e.o.m.)

SO(3) algebra: $\{J^a, J^b\} = \varepsilon^{abc} J^c$

Discrete symmetries:

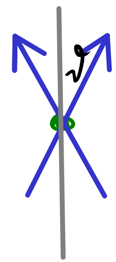
- $SO(3) \to O(3)$ by global refl. $\vec{S}_j \to -\vec{S}_j$
- chain symmetries translations $\vec{S}_j \to \vec{S}_{j\pm 1}$ } dihedral symmetry
  reflections $\vec{S}_j \to \vec{S}_{L+1-j}$ } D_{2L}

homogeneous, isotropic Hamiltonian (on chain)

Simple Solutions

- $L=1$ no dynamics $\vec{S}_1$ is constant

- $L=2$



two spins rotating around z-axis with $\omega = \dfrac{2}{\cos\vartheta}$

$$\vec{S}_{1,2}(t) = \begin{pmatrix} \pm\sin\vartheta \cos(-\omega t) \\ \pm\sin\vartheta \sin(-\omega t) \\ \cos\vartheta \end{pmatrix}$$

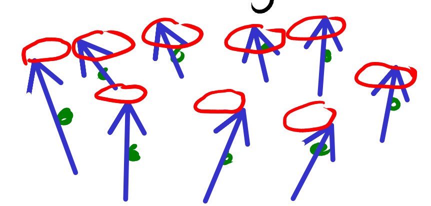
$$H = -4 \log|\cos\vartheta| \qquad \vec{J} = 2\cos\vartheta\, \vec{e}_z$$

generalizes to length L

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta \cos(2\pi n j/L - \omega t) \\ \sin\vartheta \sin(2\pi n j/L - \omega t) \\ \cos\vartheta \end{pmatrix}$$

$n \in \mathbb{Z}$ $0 \le n < L$

mode winding number



$$\omega = \frac{2\cos\vartheta \sin^2(\pi n/L)}{1-\sin^2\vartheta \sin^2(\pi n/L)}$$

$$H = -L \log\left(1-\sin^2\vartheta \sin^2(\pi n/L)\right)$$

$$\vec{J} = L\cos\vartheta\, \vec{e}_z$$

$L=3$ elliptic solutions in general

Special case, all spins are on a plane:

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta_j \cos(-\omega t) \\ \sin\vartheta_j \sin(-\omega t) \\ \cos\vartheta_j \end{pmatrix}$$

Plane rotates with ang. freq. ω

constraint

$\vec{J} = J\vec{e}_z$, $\sin\vartheta_1 + \sin\vartheta_2 + \sin\vartheta_3 = 0$

splits up phase space in $0<J<1$, $1<J<3$

$$H = -2\log\frac{|J^2-1|}{8} \qquad \omega = \frac{4J}{J^2-1} \qquad J^2 = 3 + 2\sum_j \cos(\vartheta_j - \vartheta_{j+1})$$

Excitations of the Ferromagnetic Ground State

Ground state $\vec{S}_j(t) = \vec{e}_z$ (all aligned) $\vec{J} = L\vec{e}_z$

discuss small excitations above ground state in pert. theory use stereographic var ζ_j

expand eom for ζ_j for small ζ_i (small dev from $\vec{S}_j = \vec{e}_z$)

$$\frac{d\zeta_j}{dt} = \frac{i}{2}\left(\zeta_{j-1} - 2\zeta_j + \zeta_{j+1}\right) + O(\zeta^3) \qquad \text{linear DE}$$

$$\Rightarrow \quad \zeta_j(t) = \epsilon\, a_n \exp\frac{2\pi i n j}{L}\, \exp(-i\omega_n t) + O(\epsilon^2) \qquad \omega_n = 2\sin^2\frac{\pi n}{L}$$

$$\vec{J} = \left(L - 2\epsilon^2 |a_n|^2 L\right)\vec{e}_z + O(\epsilon^3) \qquad H = 4\epsilon^2 |a_n|^2 L \sin^2\frac{\pi n}{L} + O(\epsilon^3)$$

use action angle variables ~ parametrise a_n through action var I_n

evaluate sympl. str. on solution

$$\hat{\omega} = \sum_j 2i\, dJ_j \wedge dJ_j^* = 2\epsilon |a_n|^2 L\, \omega_n\, dt \wedge d\epsilon + O(\epsilon)$$

← one period

$$\sim \quad dI_n = \frac{1}{2\pi} \oint \hat{\omega} = 4\, d\epsilon\, \epsilon\, |a_n|^2 L + O(\epsilon^3)$$

$$\Rightarrow \quad I_n = 2 |a_n|^2 \epsilon^2 L + O(\epsilon^3)$$

$$\vec{J} = (L - I_n)\vec{e}_z + O(I_n^2)$$

$$H = \omega_n I_n + O(I_m^2) \qquad \omega_n = \frac{\partial H}{\partial I_n}$$

3.2 Integrable Structure

want Lax pair T, M with $\frac{d}{dt}T = [M, T]$, T encodes phase space

Lax Transport

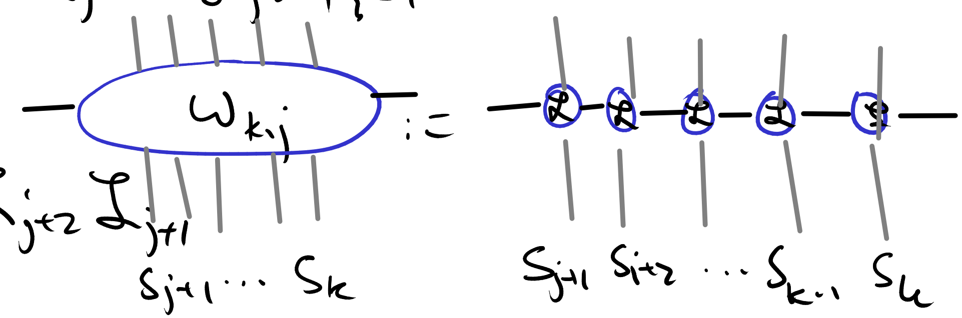
we introduce a local Lax transport $\mathcal{L}_j$, evolution matrix M_j

Lax transport eq.

$$\frac{d}{dt}\mathcal{L}_j = M_j \mathcal{L}_j - \mathcal{L}_j M_{j-1}$$

construct a composite Lax transport

$$W_{k,j} := \mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_{j+2} \mathcal{L}_{j+1}$$



evolution equation for $W_{k,j}$

$$\frac{d}{dt} W_{k,j} = \mathcal{M}_k W_{k,j} - W_{k,j} M_j$$

then introduce Lax Pair

monodromy $T = W_{L,0} = \mathcal{L}_L \mathcal{L}_{L-1} \cdots \mathcal{L}_2 \mathcal{L}_1 \qquad M = \mathcal{M}_L = \mathcal{M}_0$

$$\Rightarrow \quad \frac{d}{dt} T = [M, T]$$

For Heisenberg chain

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u} \vec{S}_j \vec{\sigma} \qquad M_j(u) = \frac{i}{u^2+1} \frac{\vec{S}_j + \vec{S}_{j+1} + u\, \vec{S}_j \times \vec{S}_{j+1}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}} \cdot \vec{\sigma}$$

confirm validity / eom.

$$(\vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma} = i\,(\vec{S}_{j+1} \cdot \vec{\sigma})(\vec{S}_j \cdot \vec{\sigma}) - i\,(\vec{S}_j \cdot \vec{S}_{j+1})\,\mathrm{id}$$

also (will show later) : $T(u)$ encodes all of phase space

lax pair

$$F_n(u) = \frac{1}{n}\,\mathrm{tr}\,T(u)^n$$ is conserved monodromy traces

we need only F_1 because $F_2 = \frac{1}{2}(F_1)^2 - \det T$ $\det T =$ fixed.

$$\det \mathcal{L}_j = 1 + \frac{1}{u^2} \Rightarrow \det T = \left(1 + \frac{1}{u^2}\right)^L$$

we need only $F(u) := F_1(u) = \mathrm{tr}\,T(u)$ contains all cons. charges.

Classical r-matrix

The elementary Lax transport admits a classical r-matrix (with parameter)
We want the classical RTT relation:

$$\{\mathcal{L}_j(u_1) \overset{\otimes}{,} \mathcal{L}_k(u_2)\} = \delta_{jk}\, r_j(u_1,u_2)\left(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2)\right) - \delta_{jk}\left(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2)\right) r_{j-1}(u_1,u_2).$$

Then it follows that monodromy $T = \mathcal{L}_L \cdots \mathcal{L}_1$ obeys classical RTT rel

$$\{T(u_1) \overset{\otimes}{,} T(u_2)\} = [r_L(u_1,u_2), T(u_1)\otimes T(u_2)]$$

$\Rightarrow$ all monodromy traces Poisson commute $\{F_m(u_1), F_n(u_2)\} = 0$
for all $u_1, u_2 \in \mathbb{C}$

Solution for r: $r_j(u_1,u_2) = r(u_1-u_2) = -\dfrac{\sigma^a\otimes\sigma^a}{2(u_1-u_2)}$. Satisfies CYBE $[[r,r]]=0$

3.3 Spectral Parameter

have introduced Lax Pair $T(u), M(u)$, $u \in \bar{\mathbb{C}}$

later: consider complex analytic structure of objects in terms of spectral par u.

Hamiltonian

All information on conserved qty should be contained in monod trace $F(u) = \operatorname{tr} T(u)$.

Complication: $\cdot F(u)$ is rather non-local wheras H is local $H = \sum \mathcal{H}_j$

we need a special point $u \in \mathbb{C}$ where $T(u)$ becomes more local. $\mathcal{L}_j$ must be special

$\mathcal{L}_j(u) = 1 + \frac{i}{u} \vec{\sigma} \cdot \vec{S}_j$ with three special points: $u = 0$ $\mathcal{L}_j$ is divergent.

$\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} = 0$ for $u = \pm i$ $\mathcal{L}_j$ is a singular matrix at $u = \pm i$

Final result for H $\quad H = -\log \frac{F(+i)\, F(-i)}{4^L}$.

properties of $\mathcal{L}_j$ at $u = \pm i$

$$\mathcal{L}_j(\pm i) = \mathrm{id} \pm \vec{S}_j \cdot \vec{\sigma}$$

is a matrix of lower rank, rank = 1, also have

$$\mathrm{tr}\, \mathcal{L}_j(\pm i) = 2 \qquad \mathcal{L}_j(\pm i)^\dagger = \mathcal{L}_j(\pm i)$$

thus we can write $\mathcal{L}_j$ in terms of spinor variables

$$\mathcal{L}_j(+i) = \frac{2}{s_j^\dagger s_j} s_j s_j^\dagger \qquad \mathcal{L}_j(-i) = \frac{2}{s_j^\dagger s_j} \varepsilon s_j^* s_j^T \varepsilon^{-1}.$$

Compose products for $T(\pm i) \to F(\pm i)$

$$F(+i) = 2^L \prod_{j=1}^{L} \frac{s_{j+1}^\dagger s_j}{s_j^\dagger s_j} \qquad F(-i) = 2^L \prod_{j=1}^{L} \frac{s_j^\dagger s_{j+1}}{s_j^\dagger s_j}$$

$$F(+i)\,F(-i) = 4^L \prod_{j=1}^{L} \frac{(s_{j+1}^+ s_j)(s_j^+ s_{j+1})}{(s_j^+ s_j)\,(s_{j+1}^+ s_{j+1})} = 2^L \prod_{j=1}^{L} \left(1 + \vec{S}_j \cdot \vec{S}_{j+1}\right)$$

$$\exp(-H) = \prod_{j=1}^{L} \frac{(s_{j+1}^+ s_j)(s_j^+ s_{j+1})}{(s_j^+ s_j)\,(s_{j+1}^+ s_{j+1})}.$$

$$\exp(iP) = \frac{F(-i)}{F(+i)} = \prod_{j=1}^{L} \frac{s_j^+ s_{j+1}}{s_{j+1}^+ s_j}$$ P: lattice momentum.

Note there are further local charges in expansion of $F(u)$ around $u = \pm i$

$$\frac{F(u)}{F(+i)} = 1 - \frac{i}{2} \sum_{j=1}^{L} \left(\frac{(s_{j+1}^+ s_{j-1})(s_j^+ s_j)}{(s_{j+1}^+ s_j)(s_j^+ s_{j+1})} - 2 \right) (u - i) + \dots$$

<u>Reconstruction</u> of phase space from $T(u)$

again: $T(u)$ is non-local but s_i, s_i^+ or $\vec{S}_j$ are local

need $u = \pm i$ again.

we can express $T(u)$ at $u = +i$

$$T(+i) = 2^L \frac{s_L s_1^+}{s_L^+ s_1} \prod_{j=1}^{L-1} \frac{s_{j+1} s_j^+}{s_j^+ s_j} = F(+i) \frac{s_L s_1^+}{s_1^+ s_L} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

two eigenvalues: 0, $F(+i)$ with eigenvectors εs_1^* and s_L

$$T(-i) = F(-i) \frac{\varepsilon s_L^* s_1^T \varepsilon^{-1}}{s_L^+ s_1} \qquad 0, F(-i) \text{ with } s_1 \text{ and } \varepsilon s_L^*$$

compose $\vec{S}_j = \dfrac{s_j^+ \vec{\sigma} s_j}{s_j^+ s_j}$ we know $\vec{S}_1$ (and $\vec{S}_L$)

To obtain other local variables $\vec{S}_j$ consider shifted monodromy

$$T_{j-1}(u) := \mathcal{L}_{j-1}(u) \dots \mathcal{L}_1(u)\, \mathcal{L}_L(u) \dots \mathcal{L}_j(u) \qquad T_L = T_0 = T$$

satisfy recursion relation $\quad T_j(u) = \mathcal{L}_j(u)\, T_{j-1}(u)\, \mathcal{L}_j(u)^{-1}$

if you know $T_0(u)$ you know $\vec{S}_1$, you know $\mathcal{L}_1(u) \Rightarrow$ know $T_1(u)$

$T_{j-1}(u)$ " $\vec{S}_j$ $\mathcal{L}_j(u) \Rightarrow$ " $T_j(u)$

recursion relation is singular at $u = \pm i$

$\rightsquigarrow$ need expansion of $T(u)$ at $u = \pm i$ at L orders

to recover all $\vec{S}_j$

Global Symmetry

SO(3) symmetry should be encoded by $T(u)$

turns out $\vec{J} = \sum_{j=1}^{L} \vec{S}_j$ is obtained at the point $u = \infty$

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u} \vec{S}_j \cdot \vec{\sigma} \left[+ \ldots \right]$$

Expand $T(u) = \prod \mathcal{L}_i$

$$T(u) = \mathrm{id} + \frac{1}{u} \sum_{j=1}^{L} \vec{S}_j \cdot \vec{\sigma} + \ldots = \mathrm{id} + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \ldots$$

also from monodromy trace

$$F(u) = 2 - \frac{1}{u^2} \left(J^2 - L \right) + \ldots$$

$\leftarrow$ total ang mom. magnitude

Multi-Local Generators

Expand $T(u)$ at $u=\infty$ to higher orders in $1/u$

$$T(u) = \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} - \frac{1}{u^2}\sum_{k=1}^{L}\sum_{j=1}^{k-1}\left(\vec{S}_k\cdot\vec{S}_j\,\mathrm{id} + i\,(\vec{S}_k\times\vec{S}_j)\cdot\vec{\sigma}\right) + \ldots$$

$$= \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} - \frac{J^2-L}{2u^2}\,\mathrm{id} + \frac{i}{u^2}\vec{Y}\cdot\vec{\sigma} + \ldots$$

$$\vec{Y} := \sum_{k=1}^{L}\sum_{j=1}^{k-1}\vec{S}_j\times\vec{S}_k$$ is a bi-local generator.

time Evolution $$\frac{d\vec{Y}}{dt} = \sum_{j=1}^{L}\sum_{k=j+1}^{L}\vec{S}_j\times(\vec{k}_k-\vec{k}_{k-1}) - \sum_{k=1}^{L}\sum_{j=1}^{k-1}\vec{S}_k\times(\vec{k}_j-\vec{k}_{j-1})$$

$$= \sum_{j=1}^{L}\vec{S}_j\times(\vec{k}_L-\vec{k}_j-\vec{k}_{j-1}+\vec{k}_0)$$

use identity $(\vec{S}_j + \vec{S}_{j+1}) \times \vec{k}_j = \ldots = \vec{S}_{j+1} - \vec{S}_j$

$$\frac{d}{dt}\vec{Y} = (\vec{J} - \vec{S}_L) \times \vec{k}_L - \vec{S}_L + (\vec{J} + \vec{S}_0) \times \vec{k}_0 + \vec{S}_0$$

on closed chain
$$= 2\vec{J} \times \vec{k}_L = \frac{2(\vec{S}_1 \times \vec{S}_L) \times \vec{J}}{1 + \vec{S}_1 \cdot \vec{S}_L}$$

later: build algebra on tower of $\vec{J}, \vec{Y}, \ldots$

also for $F(u) = 2 - \frac{1}{u^2}(J^2 - L) - \frac{2}{u^3}\vec{Y} \cdot \vec{J}$

$$\vec{Y} \cdot \vec{J} = \sum_{l>k>j=1}^{L} (\vec{S}_j \times \vec{S}_k) \cdot \vec{S}_l$$ tri-local, conserved

expansion of $T(u)$, $F(u)$ yield multi-local terms.