Introduction to Integrability

Lecture Slides, Chapter 2

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The current version of this work can be found at:
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2 Algebraic Integrability

2.1 Spin Models

Rigid body in 3D $\xrightarrow{\text{reduce}}$ Spin d.o.f. is S^2

Spinning Top

Rigid body fixed at centre (origin of 3D)
d.o.f. are 3 angles of orientation of body in 3D space
coord. sys x, y, z fixed to principal axes of body, mom of inertia $\Omega_x, \Omega_y, \Omega_z$

Euler angles ϑ, φ, ψ to describe orientation in 3D space,
$\vec{S}$ angular momentum in body coord. system

$$S_x = -\Omega_x \left(\dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi \right)$$
$$S_y = -\Omega_y \left(\dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi \right)$$
$$S_z = -\Omega_z \left(\dot{\varphi} \cos\vartheta + \dot{\psi} \right)$$
$$L = \frac{S_x^2}{2\Omega_x} + \frac{S_y^2}{2\Omega_y} + \frac{S_z^2}{2\Omega_z}$$
$$\frac{d}{dt} S_x = \left(1/\Omega_y - 1/\Omega_z \right) \cdot S_y S_z$$
$$\frac{d}{dt} S_y = \left(1/\Omega_z - 1/\Omega_x \right) \cdot S_z S_x$$
$$\frac{d}{dt} S_z = \left(1/\Omega_x - 1/\Omega_y \right) \cdot S_x S_y$$

model is integrable: 6D phase sp.

4 cons. q.: H, $\dot{J}_x, J_y, J_z$

(in inertial frame)

Spin Model Reduce spinning top $\vartheta, \varphi, \psi, \dot\vartheta, \dot\varphi, \dot\psi$ to $(S_x, S_y, S_z) = \vec{S}$

However $\vec{S}$ has 3 d.o.f. : odd! no phase space!

one conserved qty is $\vec{S}^2$ can be fixed reduces $\mathbb{R}^3$ to S^2

Poisson brackets $\{S_j, S_k\} = \varepsilon_{jkl} S_l$ $\quad \{\vec{S}^2, \dots\} = 0$

$$H = \frac{1}{2} \vec{S}^T \Omega^{-1} \vec{S} \qquad \Omega = \operatorname{diag}(\Omega_x, \Omega_y, \Omega_z) \qquad \|\vec{S}\| = J = \text{const.}$$

$$\frac{d}{dt} \vec{S} = - \{H, \vec{S}\} = (\Omega^{-1} \vec{S}) \times \vec{S} \qquad (\text{later on } J = 1)$$

Poisson brackets: $\{F_1, F_2\} = \varepsilon_{jkl} S_l \dfrac{\partial F_1}{\partial S_j} \dfrac{\partial F_2}{\partial S_k}$.

Spin Parametrisations

* $\vec{S}$ with $\|\vec{S}\| =$ const or $\vec{S}^2 =$ const $\quad \{S_j, S_k\} = \varepsilon_{jkl} S_l$

* spherical coord. $\vec{S} = J \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$

  Poisson br. $\{\vartheta, \varphi\} = \dfrac{1}{J \sin\vartheta}$.

* stereographic proj : map S^2 to $\bar{\mathbb{C}} \ni \zeta$

$$\vec{S} = \frac{J}{1+|\zeta|^2} \begin{pmatrix} 2\,\mathrm{Re}\,\zeta \\ 2\,\mathrm{Im}\,\zeta \\ 1-|\zeta|^2 \end{pmatrix} \qquad \zeta = \tan(\vartheta/2)\exp(i\varphi) = \frac{S_x + i S_y}{J + S_z}$$

$\zeta = 0$ is N, $+\vec{e}_z$ $\zeta = \infty$ is S, $-\vec{e}_z$, $|\zeta| = 1$ is equator $S_z = 0$

$$\{\zeta, \zeta^*\} = \frac{i}{2J}\left(1+|\zeta|^2\right)^2.$$

* spinor parametrisation $\sim SO(3) \sim SU(2)$

translate to 2×2 complex matrices using Pauli matrices $\vec{\sigma}$

$\vec{S}\cdot\vec{\sigma}$ hermitian, eigenvalues $\pm J$

eigenvectors s, εs^* is called associated spinor $\quad \varepsilon = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$

$$(\vec{S}\cdot\vec{\sigma})s = +Js \qquad (\vec{S}\cdot\vec{\sigma})\varepsilon s^* = -J\varepsilon s^* \qquad \vec{S} = J\,\frac{s^\dagger\vec{\sigma}s}{s^\dagger s}$$

s is defined up to scale $\quad s \equiv \lambda s \leftarrow \mathbb{CP}^1$

N: $s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; S $= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \infty \end{pmatrix}$ $\quad$ related to stereogr. proj. $\quad s \equiv \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$.

Poisson bracket $F_{1,2}(\vec{s})$ $\quad F(s) = F(\lambda s) = F(s_2/s_1) = F(\zeta)$

$$\{F_1, F_2\} = -\frac{i}{2J} s^+ s \sum_{j=1,2} \left(\frac{\partial F_1}{\partial s_j} \cdot \frac{\partial F_2}{\partial s_j^*} - \frac{\partial F_1}{\partial s_j^*} \cdot \frac{\partial F_2}{\partial s_j} \right).$$

Altogether:

$$\mathbb{R}^3 \supset S^2 = \bar{\mathbb{C}} = \mathbb{CP}^1$$

$\vec{S}$ spherical coord.   stereo. proj   spinor

Classes of Solutions of spin model.

several distinct cases for dynamics:

- asymmetrical case $\Omega_x \neq \Omega_y \neq \Omega_z \neq \Omega_x$

solution in terms of elliptic fn

$$S_x = c_x \, \mathrm{cn}(\lambda t + \varphi, m)$$
$$S_y = c_y \, \mathrm{sn}(\lambda t + \varphi, m)$$
$$S_z = c_z \, \mathrm{dn}(\lambda t + \varphi, m)$$

λ, m are constants (in terms $E, J, \Omega_{x,y,z}$)

φ describes initial cond.

- symmetric top at $m=0$ $\quad \Omega_x = \Omega_y \neq \Omega_z$

$$S_x = c \cos(\lambda t + \varphi)$$
$$S_y = c \sin(\lambda t + \varphi)$$
$$S_z = \text{const.}$$

SO(2) symmetry

- spherical top $\Omega_x = \Omega_y = \Omega_z$ $SO(3)$ symmetry

$$H = \frac{1}{2}\Omega^{-1} J^2 = \text{const} \Rightarrow \text{no dynamics!}$$

mech sys. version of $\vec{S} = \text{const}$ $\|\vec{S}\| = J$

Relation to classes of integrable systems

type	rational	trigonometric	elliptic
symbol	XXX	XXZ	XYZ
top	spherical	symmetric	asymmetric
$\Omega_x \Omega_y \Omega_z$	$\Omega_x \Omega_x \Omega_x$	$\Omega_x \Omega_x \Omega_z$	$\Omega_x \Omega_y \Omega_z$
symmetry	$SO(3)$ n.a. Lie alg.	$SO(2)$ Cartan sub.	— $(\mathbb{Z}_2)$

2.2 Lax Pair

Algebraic Eq. of Motion for Spin Model

SO(3) / SU(2)

$\vec{S}$, $\{S_j, S_k\} = \varepsilon_{jkl} S_l$ ⇝ cast variables/equations into 2x2 matrices, 2vector

Pauli matrices σ_a $\qquad \vec{S}\cdot\vec{\sigma} = \begin{pmatrix} +S_z & S_x - iS_y \\ S_x + iS_y & -S_z \end{pmatrix}$ traceless hermitian

recall Pauli matrix relations: $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$, $[\vec{v}\cdot\vec{\sigma}, \vec{w}\cdot\vec{\sigma}] = 2i(\vec{v}\times\vec{w})\cdot\vec{\sigma}$.

$$\ldots \Rightarrow \frac{d}{dt}\vec{S}\cdot\vec{\sigma} = ((\Omega^{-1}\vec{S})\times\vec{S})\cdot\vec{\sigma} = -\tfrac{i}{2}[(\Omega^{-1}\vec{S})\cdot\vec{\sigma}, \vec{S}\cdot\vec{\sigma}]$$

define $T := \vec{S}\cdot\vec{\sigma}$, $M := -\tfrac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$ $\Rightarrow \dfrac{d}{dt}T = [M, T]$

Lax Pair

above: $\mathbb{C}^2$

Lax Pair (T, M) is pair of sqr. matrices $T, M \in \mathrm{End}(V)$
T: Lax matrix, M: (Lax) evolution matrix
if it satisfies the <u>lax eq.</u>

$$\frac{d}{dt} T := -\{H, T\} = [M, T]$$

- spectrum of T is (time) conserved; $\det(\gamma\,\mathrm{id} - T)$ as poly in γ is conserved

$$T(t) = g(t)\, T(t_0)\, g(t)^{-1}$$: time ev. of T by conjugation with $g(t)$ only.

T generates conserved charges as

$$F_k := \frac{1}{k} \mathrm{tr}(T^k) \qquad \frac{d}{dt} F_k = \mathrm{tr}\left([M, T]\, T^{k-1}\right) = 0$$

Complete Lax Pairs

Note: a generic lax pair for a given system:

- is not unique
- not necessarily useful
- there is universal recipe to construct
- There need not be a relation of lax pair to features of system, eg. dim

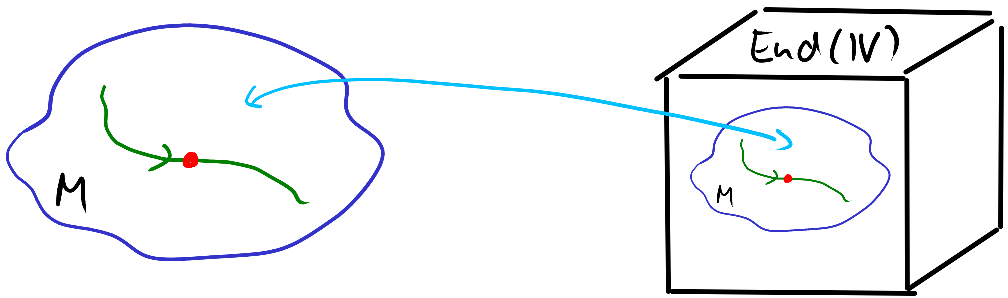
Generalise lax Pair for spin model?

- similarity transformation of T, M by a constant matrix
- generalise spin ½ rep of T, M to spin s dim: $(2s+1)$

add a constant trace $T = \vec{S} \cdot \vec{\sigma} + \underset{\downarrow}{\overset{\mathbb{C}}{}}\!\!v\, H \, id$ $\quad M = -\frac{i}{2} (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}$

same lax eq, spectrum is now: $\{ vH \pm J \}$

desirable property: Lax Matrix T should fully describe state of the system.
→ know state → know equations of motion.



Notion of complete Lax pairs: Lax Pair (T, M):

i) the pair obeys Lax equation $\frac{d}{dA} T = [M, T]$
ii) the Lax matrix T encodes all $2n$ phase space vars uniquely
iii) it is diagonalisable for almost all points in M
iv) its spectrum encodes n indep. variables
v) these variables are in involution.

} model is Liouville integrable!

2.3 Lax-Poisson Structure

Want to establish that $\{F_k, F_\ell\} = 0$

Lax-Poisson Equation

Want to specify Poisson brackets of all pairs of T_{jk} with $T_{\ell m}$. Lax-Poisson eq.

$$\{T_{jk}, T_{\ell m}\} = \sum_n R_{(j\ell)(nm)} T_{nk} - \sum_n T_{jn} R_{(n\ell)(km)}$$
$$- \sum_n R_{(\ell j)(nk)} T_{nm} + \sum_n T_{\ell n} R_{(nj)(mk)}$$

$R_{(j\ell)(km)}$ is a tensor operator of Rank 2 (tensor rank 4)
← Lax-Poisson matrix

- R expresses Poisson/symplectic structure on $\mathrm{End}(V)$
- this form implies that $\{F_k, F_\ell\} = 0$ for $F_k := \frac{1}{k} \mathrm{tr}(T^k)$.

Tensor Notation

Express a square matrix A (eg. T) through its components using a basis E_{jk}

$$A = \sum_{j,k} A_{j,k} E_{jk} \qquad (E_{jk})_{\ell m} = \delta_{j\ell}\,\delta_{km}$$

Poisson bracket of two Operators A, B whose components are phase space f^{n}:

$$\{A \overset{\otimes}{,} B\} := \sum_{jk\ell m} \{A_{jk}, B_{\ell m}\}\; E_{jk} \otimes E_{\ell m}$$

$$R := \sum_{jk\ell m} R_{(j\ell)(km)}\; E_{jk} \otimes E_{\ell m}$$

permutes sites

$$P(R) := \sum_{jk\ell m} R_{(j\ell)(km)}\; E_{\ell m} \otimes E_{jk} = \sum R_{(\ell j)(mk)}\, E_{jk} \otimes E_{\ell m}$$

Lax-Poisson eq: $\{T \overset{\otimes}{,} T\} = [R, T \otimes id] - [P(R), id \otimes T]$

Short-hand notation for tensors using site indices:

$$T_1 := T \otimes \mathrm{id} \qquad\qquad T_2 := \mathrm{id} \otimes T$$

$$R_{12} := R \qquad\qquad R_{21} := P(R)$$

$$\{T_1, T_2\} := \{T \overset{\otimes}{,} T\}$$

$$\Rightarrow \quad \{T_1, T_2\} = [R_{12}, T_1] - [R_{21}, T_2] .$$

Properties and Applications

Show that L.P. eq yields $\{F_k, F_\ell\} = 0$

$$\{F_j, F_k\} = \frac{1}{jk}\{\mathrm{tr}\,T^j, \mathrm{tr}\,T^k\} = \frac{1}{jk}\mathrm{tr}_{1,2}\{T_1^j, T_2^k\}$$

$$= \frac{1}{jk}\sum_{\ell=1}^{j}\sum_{m=1}^{k}\mathrm{tr}_{1,2}\left(T_1^{\ell-1}T_2^{m-1}\{T_1, T_2\}T_1^{j-\ell}T_2^{k-m}\right)$$

$$= \mathrm{tr}_{1,2}\left(T_1^{j-1}T_2^{k-1}\{T_1, T_2\}\right)$$

$$= \mathrm{tr}_{1,2}\left(\underbrace{T_1^{j-1}T_2^{k-1}[R_{12}, T_1]}_{=0} - \underbrace{T_1^{j-1}T_2^{k-1}[R_{21}, T_2]}_{=0}\right)$$

$$= 0 \, ,$$

Poisson brackets satify Jacobi, so R satisfies a Lax-Jacobi id.

$$0 = [T_1, [\![R,R]\!]_{123} + \{T_2, R_{13}\} + \{T_3, R_{12}\}] + 2 \text{ cyclic}$$

$[\![X,Y]\!]$ is a quadratic combination (non bilinear) of X, Y

$$[\![X,Y]\!]_{123} := [Y_{12}, Y_{13}] + [Y_{12}, X_{23}] + [X_{32}, Y_{13}]$$

special case: $[\![R,R]\!] = 0$ and R indep of M. $\Rightarrow$ L-J. holds.

Example: $T = \vec{S}\cdot\vec{\sigma} + u H \, id$

$$\{T_1, T_2\} = (\vec{\sigma}_1 \times \vec{\sigma}_2)\cdot\vec{S} + u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_1 - u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_2$$

$$R_{12} = -\frac{i}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2 - \frac{i}{2} u (\Omega^{-1}\vec{S})\cdot\vec{\sigma}_1$$

complete Lax-Poisson structure: (T, M, R)

i) pair obeys Lax eq $\frac{dT}{dt} = [M, T]$

ii) Lax matrix T encodes all $2n$ phase space vars.

iii) T is diagonalisable almost everywhere

iv) spectrum encodes n indep vars.

v) Lax-Poisson matrix R obeys Lax-Poisson eq.

Evolution from Lax-Poisson structure

We know that T encodes phase space, F_k encode conserved charges

$\Rightarrow$ we can write $\quad H = h(F_k)$

Lax Eq follows: $\frac{d}{dt} T = [M, T]$ with $M_1 = \sum_k \frac{\partial h}{\partial F_k} \mathrm{tr}_2 \left(T_2^{k-1} R_{12} \right)$

therefore (T, R) extends to (T, M, R) by means of $h(F)$

replace i) the Hamiltonian is given as a function of Lax traces F_k

ex: $T = \vec{S} \cdot \vec{\sigma} + u H \, \mathrm{id} \qquad F_1 = 2uH \qquad R_{12} = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u \left(\Omega^{-1} \vec{S} \right) \cdot \vec{\sigma}_1$

$H = \frac{\mathrm{tr}\, T}{2u} = \frac{F_1}{2u} \qquad M_1 = \frac{1}{2u} \mathrm{tr}_2 \, R_{12} = -\frac{i}{2} \left(\Omega^{-1} \vec{S} \right) \cdot \vec{\sigma}_1$

Parametric Lax Pairs

introduce parameter $u \in \mathbb{C}$ to allow for many more indep qty in T or in spect.

many equations are unchanged

$$T(u) = \vec{S}\cdot\vec{\sigma} + uH\,\mathrm{id} \qquad M(u) = -\frac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$$

$$\frac{d}{dt}T(u) = [M(u), T(u)]$$

$$F_1(u) = 2uH \qquad F_2(u) = J^2 + u^2H^2 \qquad F_3(u) = 2uH\left(J^2 + \frac{1}{3}u^2H^2\right)$$

However how about Poisson brackets? $\{T(u), T(v)\}$ need to extend to

$$\{T_1(u_1), T_2(u_2)\} = [R_{12}(u_1,u_2), T_1(u_1)] - [R_{21}(u_2,u_1), T_2(u_2)]$$

$$\Rightarrow \{F(u_1), F(u_2)\} = 0 \quad F_k(u) = \sum_j F_{kj}u^j \quad \Rightarrow \{F_{j,k}, F_{l,m}\} = 0$$

ex: $$R_{12}(u_1,u_2) = \frac{i}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2 - \frac{i}{2}u_2(\Omega^{-1}\vec{S})\cdot\vec{\sigma}_1$$

Classical r-matrix

r is a tensor operator as R but relation is ($\overset{cl.}{R}TT$ relation)

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T] \qquad \{T_1, T_2\} = [r_{12}, T_1 T_2]$$

$$r_{12} = -r_{21} \qquad \Rightarrow \qquad \{F_j, F_k\} = 0$$

Jacobi-id

$$0 = [[[r, r]], T_1 T_2 T_3] + ([\{r_{12}, T_3\}, T_1 T_2] + 2 \text{ cyclic})$$

but if r is indep of M: usually cl. Yang-Baxter-eq.

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$$

Ex:

$$r_{12}(u_1, u_2) = \frac{i}{2H} \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{u_1 - u_2} - \frac{i}{2} v_2 (\Omega^{-1} \vec{S}) \vec{\sigma}_1 \, T_2(u_2)^{-1} + \frac{i}{2} v_1 T_1(u_1)^{-1} (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_2$$