Introduction to Integrability

Lecture Slides, Chapter 1

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

1. Integrable Mechanics

1.1 Hamiltonian Mechanics

System consists of a phase space M, dimension $2n$
Hamiltonian $H: M \to \mathbb{R}$ p.c. coordinates q_k, p_k $k=1,\dots,n$

Solutions, trajectories defined by curves $(q_k(t), p_k(t))$
satisfying the Hamiltonian eq. of motion

$$\dot{q}_k := \frac{dq_k}{dt} = +\frac{\partial H}{\partial p_k} \qquad \dot{p}_k := \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}$$

initial conditions at $t=t_0$: $(q_k(t_0), p_k(t_0))$ is some fixed values.

Poisson brackets: two phase space functions $F_{1,2}: M \to \mathbb{R}$

$$\{F_1, F_2\} := \sum_{k=1}^{n} \left(\frac{\partial F_1}{\partial q_k} \frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k} \frac{\partial F_2}{\partial q_k} \right)$$

is another function on phase space

- leibniz rule
- anti-symmetric
- bi-linear
- Jacobi-identity

any three $F_{1,2,3}$

$$\{\{F_1, F_2\}, F_3\} + 2 \text{ cyclic} = 0$$

Ham. eq. of motion $\dot{q}_k = -\{H, q_k\}$ $\dot{p}_k = -\{H, p_k\}$.

Poisson brackets express the time evolution of arbitrary qty in univ. way

Phase space function $F(q,p,t)$ evaluate on solution $(q_n(t), p_n(t))$

total time derivative $\dfrac{dF}{dt} = \dfrac{\partial F}{\partial t} - \{H, F\}$.

Mostly F has no explicit time dep. $F(q,p,t) = F(q,p)$

$$\frac{dF}{dt} = -\{H, F\}.$$

Poisson brackets equip p.s. with a symplectic structure

$$\hat{\omega} = \sum_k dq_k \wedge dp_k$$

closed 2-form on M

inverse of $\{\cdot,\cdot\}$

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ -1 & & & 1 \\ & \ddots & & \\ & & -1 & \end{pmatrix}$$

integral: canonical 1-form $\sum_k p_k dq_k$

Canonical transf. $(q,p) \to (\tilde{q},\tilde{p}) = (\tilde{q}(q,p), \tilde{p}(q,p))$

diffeomorphism preserving the Ham. eq. of motion, Poisson br. symplectic structure

want: $\{\tilde{q}_k, \tilde{p}_l\} = \delta_{k,l}$ $\quad \{\tilde{q}_k, \tilde{q}_l\} = 0 = \{\tilde{p}_k, \tilde{p}_l\}$

equivalently $\tilde{\hat{\omega}} = \sum_k d\tilde{q}_k \wedge d\tilde{p}_k \overset{!}{=} \sum_k dq_k \wedge dp_k = \hat{\omega}$

1.2 Integrals of Motion

For a time-independent Ham $H = H(q,p)$, $\partial H / \partial t = 0$

the value of H on a solution is constant

$$\frac{d}{dt} H = \frac{\partial H}{\partial t} - \{H, H\} = 0$$

Benefit: Solution remain on surfaces of constant $H(q,p) = E$.

Search for solution on hypersurface $M_E \subset M$,

E is given by initial value $E = H(q_0, p_0)$.

Further conserved phase space function $F_k(q,p)$ may exist s.t.

$$\frac{d}{dt} F_k = -\{H, F_k\} \overset{!}{=} 0$$

- integral of motion
- (conserved) charge, quantity.

Further cons. charges F_k constrain the submanifold for solutions to level set M_f

$$M_f := \{ (q,p) \in M ; F_k(q,p) = f_k \text{ for all } k \}$$

Moreover F_k can be used to generate further solutions from given ones. through Noether's Theorem.

$F(q,p)$ generate flow on phase space $-\{F, \cdot\}$
vector field on phase space

applied to solutions $(q_u(t), p_u(t))$ generates an infinitesimally close solution

$$(\hat{q}(t), \hat{p}(t)) = (q(t), p(t)) + \delta(q(t), p(t))$$

$$\delta q(t) = -\epsilon\{F, q(t)\}, \qquad \delta p(t) = -\epsilon\{F, p(t)\}$$

New solution $(\tilde{q}(t), \tilde{p}(t))$ with same energy E and same $F_k = f_k$
(but not necessarly for $F_\ell = f_\ell$ with $\ell \neq k$)

Additional simplifications arise from further demanding

$$\{F_k, F_\ell\} = 0 \quad \text{for all } k, \ell$$

- $\{F_k\}$ are in involution
- F_k (Poisson) commute

then H is among the conserved charges (functionally dependent)

- $H = F_1$ or $H = H(F_1, \ldots F_k)$.

To obtain (prove existance) of F_k is not trivial

- found by trial and error
- Noether's theorem ~ symmetries
- conserved charges $F_k \xrightarrow{\text{Noether}}$ symmetries (hidden)

2D Central Potential

Particle mass m in 2D central potential $V(r)$ with $r := \sqrt{x^2+y^2}$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - V\left(\sqrt{x^2+y^2}\right)$$

rotational symmetry $SO(2)$, switch to radial coordinates

$x = r\cos\varphi \quad y = r\sin\varphi \qquad SO(2)$ shifts φ

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - V(r)$$

symmetry: L does not depend on φ (only $\dot{\varphi}$)

Hamiltonian framework by Legendre transf. $p = m\dot{r} \quad \psi = m r^2 \dot{\varphi}$

$$H = \frac{p^2}{2m} + \frac{\psi^2}{2mr^2} + V(r)$$

φ is cyclic coordinate

Ham eq. of motion $\quad \dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial r} = \frac{\psi^2}{mr^3} - V'(r)$

$$\dot{\varphi} = \frac{\partial H}{\partial \psi} = \frac{\psi}{mr^2} \qquad \dot{\psi} = -\frac{\partial H}{\partial \varphi} = 0$$

H does not depend on φ (φ cyclic) $\Rightarrow$ $F = \psi =: J$ angular momentum

Solutions reside on level set determined by $H = E$, $\psi = J$.

use conservation to express p through E, J

$$P(r, E, J) = \sqrt{2m(E - V(r)) - J^2/r^2}$$

and also $\Psi(J) = J$

$$\frac{dr}{dt} = \frac{P}{m} \quad \text{sep. of var.} \quad \frac{m}{P} dr = dt \Rightarrow \int_{r_0}^{r} \frac{m\, dr'}{P(r', E, J)} = t - t_0$$

$t = t_0 + \int \ldots (r)$

Indirectly yields $r(t)$ depending on E, J and also r_0, t_0

Integrate the angular equation

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{J\, dt}{m\, r(t)^2} = \varphi_0 + \int_{r_0}^{r(t)} \frac{J\, dr'}{r'^2 P(r', E, J)}$$

Reduced finding solution of diff. Ham. eq. of motion

to solving relations and performing integrals.

Explicit solution for $V(r) \sim 1/r$ (Kepler) or $V(r) \sim r^2$ (Harm. osc).

1.3 Liouville Integrability

A classical mechanical system with $2n$-dimensional phase space M is called (Liouville) integrable if it has n phase space functions F_k, $k = 1 \dots n$

- all F_k are independent p.s.f^{n}s
- all F_k are everywhere differentiable
- F_k are conserved charges ("integrals of motion")
- F_k are in involution; Poisson commute $\{F_k, F_l\} = 0$ for all $k, l = 1 \dots n$

Such a system is called solvable by quadratures: any finite sequence of:

- resolving a set of coordinate relations on phase space (non-linear, but not differential or integral equation kind)
- calculating ordinary integrals

Phase Space Structure

In an integrable system phase space M splits into level sets M_f:

- M_f has codimension n => dimension n
- there are n independent commuting flows on the level set.

We can use flows to determine a complete coordinate system on M_f (locally)
Coordinates $G_k(q,p)$ are called flow functions (or time f^n).
Specified by diff. eq. $-\{F_k, G_\ell\} = \delta_{k,\ell}$; initial cond pick an origin for G_k on M_f
Further extend coordinate functions $G_k(q,p)$ across level sets by $\{G_k, G_\ell\} = 0$

Altogether: $\{G_k, F_\ell\} = \delta_{k\ell}$ $\quad \{F_k, F_\ell\} = 0 = \{G_k, G_\ell\}$

canonical transf. on M: $(q,p) \to (g,f) = (G(q,p), F(q,p))$.

Corollary: Dynamics in coordinates (g,f) is linear in time iff $H = H(f)$ is a function of f_k $k=1\dots n$ alone

Ham. eq of motion

$$\frac{d}{dt} F_k = -\{H, F_k\} = 0$$

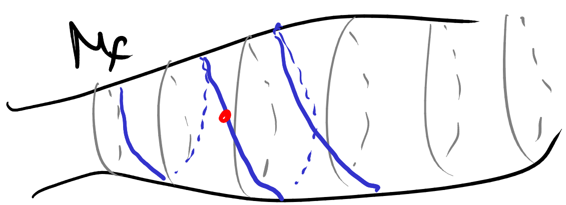
$$\frac{d}{dt} G_k = -\{H, G_k\} = -\sum_{k=1}^{n} \frac{\partial H}{\partial f_e}\{F_e, G_k\} = \frac{\partial H}{\partial f_k} = \text{const}$$

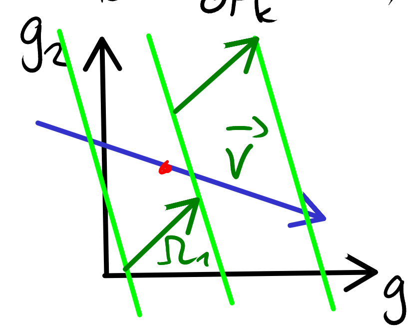
Solution

$$f_k(t) = f_{k,0} = \text{const}$$

$$g_k(t) = g_{k,0} + v_k(t - t_0)$$

$$v_k := \frac{\partial H}{\partial f_k}(f_0)$$





If level set has a non-triv. topology, i.e. non-trivial cycles $C_k(f)$

- ~~D~~iff eq define G_n up to a constant
- G_n can shift by a constant when moving around a cycle $C_\ell(f)$

G_n are multi-valued functions, shift of G_ℓ under cycle C_k is

$$\Omega_{k,\ell}(f) := \oint_{C_k(f)} dG_\ell \quad \text{is single-valued.}$$

Period matrix Ω

Charges

Given some integrable system M, H: have to establish conserved charges $F_k(q,p)$

Trade in momentum coordinates p_k for the cons. charge values f_k (q_k fixed)

$f_k = F_k(q,p)$ solve these for p_ℓ such that Funktion $P_\ell(q,f)$:

$$P_\ell(q, F(q,p)) = p_\ell \quad \text{or} \quad F_k(q, P(q,f)) = f_k$$

partial transformation $(q,p) \to (q,f)$

Generating Function

Establish flow functions Q_k by means of a generating function $S(q,f)$ for canonical transformations.

$$S(q,f) := \int_{\gamma(q,f)} \sum_k p_k \, dq_k' \qquad \gamma \text{ is a path on level set } M_f \text{ connecting a fixed point on } M_f \; q_0(f) \text{ to } q$$

Integral depends only on endpoints because integrand is closed 1-form.

$$S(q,f) = \int_{q_0(f)}^{q} \sum_k P_k(q',f)\, dq_k' \qquad \text{ordinary integral}$$

Show that integrand is closed when restricted to M_f

$$dP_k \wedge dq_k \overset{\sum_{k,e}}{=} \underbrace{df_e}_{=0 \text{ because on } M_f} \wedge dq_k \frac{\partial P_k}{\partial f_e} \overset{\sum_{k,e}}{+} \underbrace{dq_e \wedge dq_k}_{\text{anti-sym}} \frac{\partial P_k}{\partial q_e} = 0$$

$\frac{\partial P_k}{\partial q_e}$ ← matrix is symmetric consider $f_j = F_j(q,p)$ differentiate wrt. q_ℓ, multiply by $\partial F_m / \partial p_\ell$

$$0 = \frac{\partial F_j}{\partial q_\ell}\frac{\partial F_m}{\partial p_\ell} + \frac{\partial F_j}{\partial p_k}\frac{\partial p_k}{\partial q_\ell}\frac{\partial F_m}{\partial p_\ell}$$

subtract $m \leftrightarrow j$

$$0 = \frac{\partial F_m}{\partial p_\ell}\frac{\partial F_j}{\partial p_k}\left(\frac{\partial p_k}{\partial q_\ell} - \frac{\partial p_\ell}{\partial q_k}\right) + \{F_m, F_\ell\}$$

$= 0$ by assumption of integrability

F_k are indep $\Rightarrow$ matrices are invertible

$= 0$ if $\dfrac{\partial p_k}{\partial q_\ell}$ is symmetric

Flow Functions

Define flow functions through generating function technique

$$\frac{\partial S}{\partial q_k}(q,f) = p_k(q,f) \qquad G_k(q,p) = \frac{\partial S}{\partial f_k}(q, F(q,p))$$

canonical transf. $(q,p) \to (g,f)$

2D Central Potential

2D radial coordinates, radial potential $V(r)$

Hamiltonian $H = F_1 = E$, angular momentum $F_2 = \varphi = J$

$$P(r, \varphi, E, J) = \sqrt{2m(E - V(r)) - J^2/r^2} \qquad \Phi(r, \varphi, E, J) = J$$

Generating Function (r, φ)

$$S(r, \varphi, E, J) = \int_{(r_0, \varphi_0)}^{(r, \varphi)} \left(P(r', E, J)\, dr' + J\, d\varphi' \right)$$

$$= \int_{r_0}^{r} P(r', E, J)\, dr' + (\varphi - \varphi_0) J.$$

Flow coordinates

$$T := \frac{\partial S}{\partial E} = \int_{r_0}^{r} \frac{m\, dr'}{P(r', E, J)} = t - t_0$$

$$\Phi := \frac{\partial S}{\partial J} = \varphi - \varphi_0 - \int_{r_0}^{r} \frac{J\, dr'}{r'^2 P(r', E, J)} = \varphi(t_0) - \varphi_0$$

Compact Level Sets and Action-Angle Coordinates

If ~~level set~~ integrable system M_f is compact then it is diffeomorphic to n-dimensional torus $T^n = (S^1)^n$, the so-called Liouville torus

follows from existence of n commuting flows.

On a torus T^n we have n non-trivial cycles C_k.
We can define alternative, canonical integrals of motion

$$I_k(f) := \frac{1}{2\pi} \oint_{C_k(f)} \sum_j p_j \, dq_j$$ depend on level set ie on F_e alone

$I_k(f)$ $k=1\ldots n$ are generically equivalent to F_e
$\Rightarrow$ can use I_k as conserved charges to define integrability.

I_k are called action variables

θ_k dual to I_k are called angle variables, θ_k defined through gen. fund.

$$\theta_k = \frac{\partial S}{\partial I_k} \qquad S(q, I) = \int_{q_0(I)}^{q} \sum_j p_j(q' I)\, dq'_j$$

Gen function is multi-valued: extend path by cycle C_k

$$S \to S + \oint_{C_k(f)} \sum_j p_j\, dq_j = S + 2\pi I_k$$

also period matrix follows $\Omega_{k,l}(f) = \oint_{C_k(f)} d\theta_l = 2\pi \delta_{k,l}$.

can. transf. $\{\theta_j, I_k\} = \delta_{k,l} \quad \{I_i, I_n\} = 0 = \{\theta_j, \theta_n\}$

Solutions are linear trajectories

$I_k(t) = I_{0,k} = \text{const}$

ang. freq. $\omega_k(I_0) := \frac{\partial H}{\partial I_k}(I_0)$

$\theta_k(t) = \theta_{0,k} + (t - t_0)\, \omega_k(I_0)$

In each direction θ_k we have periodic motion with ang freq ω_k

Altogether: quasi-periodic solutions (unless super-integrability)

Towards quantisation: Action variables are quantised in units of $\hbar$

1.4 Variations of Integrability

Darboux Theorem

Mech sys $2n$ d.o.f., Ham H, symplectic form $\hat{\omega}$

Statement:

symplectic form $\hat{\omega}$ can be expressed in neighbourhood at any point as

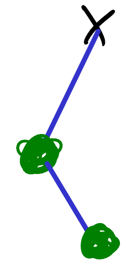
$$\hat{\omega} = \sum_{k=1}^{n} dG_k \wedge dF_k \qquad G_k, F_k \text{ are diff funct.}$$

In part: can always choose G_k, F_k s.t. $F_1 = H$. (conditions of Liouville integr., locally)

Insufficient Charges

For fewer than n charges in involution: system is not integrable!
it is rather chaotic.

Example: double pendulum

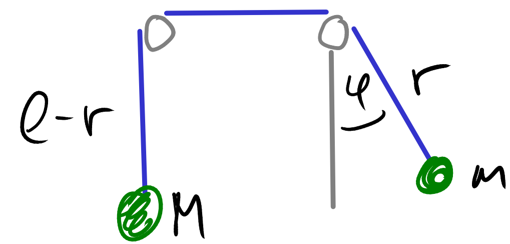


Broken Integrability

Kolmogorov-Arnold-Moser (KAM) thm.: non-integrable def of. integrable model

in short: quasi-per. orbits in integrable model deform almost classically to q.p. orbits of deformed model.

Example: swinging Atwood machine



$$L = \frac{1}{2}(M+m)\dot r^2 + \frac{1}{2} m r^2 \dot\varphi^2 - g r (M - m\cos\varphi)$$

$$H = \frac{p^2}{2(M+m)} + \frac{\psi^2}{2 m r^2} + g r (M - m \cos\varphi) \qquad p = (M+m)\dot r, \quad \psi = m r^2 \dot\varphi$$

dynamics qualitatively depends on ratio $\mu = M/m$.

System is integrable for $\mu = 3$

$$F = \frac{p\psi}{4m^2}\cos(\varphi/2) - \frac{\psi^2}{2m^2 r}\sin(\varphi/2) + g r^2 \sin(\varphi/2)\cos^2(\varphi/2)$$

$\{H, F\} = 0$ iff $\mu = M/m \overset{!}{=} 3$.

Super-Integrability

more than n conserved charges ⇒ only n are in involution. super-integrable

- implications: some of the periodicity freq. ω_k are rationally compatible

(maximal superintegrability if $2n-1$ cons. charges ⇒ truly periodic
(applies to all systems with 2D phase space $2n=2$).)

- action-angle coordinates are not uniquely defined.

Non-abelian Symmetries

Non-abelian sym are intrinsically linked to super-integrability

Ex: $SO(3)$ sym $\Rightarrow$ ang mom gen. J_x, J_y, J_z $\{H, J_k\} = 0$

3 cons. charges

but $\{J_k, J_l\} = \varepsilon_{klm} J_m$

so(3) Lie algebra J_k not all in involution

usual choice: $J_z, J^2 = J_x^2 + J_y^2 + J_z^2$: commute $\{J_z, J^2\} = 0$

but J_x or $\varphi = \arctan J_y/J_x$ $\{J_x, J_z\} \neq 0 \neq \{J_z, \varphi\}$