Introduction to Integrability

Lecture Slides

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The current version of this work can be found at: `http://people.phys.ethz.ch/~nbeisert/lectures/`.

Contents

Chapter 1

Integrable Mechanics

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1. Integrable Mechanics

1.1 Hamiltonian Mechanics

System consists of a phase space M, dimension $2n$
Hamiltonian $H: M \to \mathbb{R}$ p.c. coordinates q_k, p_k $k=1,\dots,n$

Solutions, trajectories defined by curves $(q_k(t), p_k(t))$
satisfying the Hamiltonian eq. of motion

$$\dot{q}_k := \frac{dq_k}{dt} = +\frac{\partial H}{\partial p_k} \qquad \dot{p}_k := \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}$$

initial conditions at $t=t_0$: $(q_k(t_0), p_k(t_0))$ is some fixed values.

Poisson brackets: two phase space functions $F_{1,2} : M \to \mathbb{R}$

$$\{F_1, F_2\} := \sum_{k=1}^{n} \left(\frac{\partial F_1}{\partial q_k} \frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k} \frac{\partial F_2}{\partial q_k} \right)$$

is another function on phase space

→ leibniz rule

- anti-symmetric
- bi-linear
- Jacobi-identity

any three $F_{1,2,3}$

$$\{\{F_1, F_2\}, F_3\} + 2 \text{ cyclic} = 0$$

Ham. eq. of motion $\dot{q}_k = -\{H, q_k\}$ $\dot{p}_k = -\{H, p_k\}$.

Poisson brackets express the time evolution of arbitrary qty in univ. way

Phase space function $F(q,p,t)$ evaluate on solution $(q_n(t), p_n(t))$

total time derivative
$$\frac{dF}{dt} = \frac{\partial F}{\partial t} - \{H, F\}.$$

Mostly F has no explicit time dep. $F(q,p,t) = F(q,p)$

$$\frac{dF}{dt} = -\{H, F\}.$$

Poisson brackets equip p.s. with a symplectic structure

$$\hat{\omega} = \sum_k dq_k \wedge dp_k$$

closed 2-form on M
inverse of $\{\cdot,\cdot\}$

integral: canonical 1-form $\sum_k p_k dq_k$

$$\begin{pmatrix} & \begin{smallmatrix}1&&\\&\ddots&\\&&1\end{smallmatrix} \\ \begin{smallmatrix}-1&&\\&\ddots&\\&&-1\end{smallmatrix} & \end{pmatrix}$$

Canonical transf. $(q,p) \to (\tilde{q},\tilde{p}) = (\tilde{q}(q,p), \tilde{p}(q,p))$

diffeomorphism preserving the Ham. eq. of motion, Poisson br. symplectic structure

want: $\{\tilde{q}_k, \tilde{p}_\ell\} = \delta_{k,\ell}$ $\{\tilde{q}_k, \tilde{q}_\ell\} = 0 = \{\tilde{p}_k, \tilde{p}_\ell\}$

equivalently $\tilde{\hat{\omega}} = \sum_k d\tilde{q}_k \wedge d\tilde{p}_k \overset{!}{=} \sum_k dq_k \wedge dp_k = \hat{\omega}$

1.2 Integrals of Motion

For a time-independent Ham $H = H(q,p)$, $\partial H/\partial t = 0$

the value of H on a solution is constant

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} - \{H, H\} = 0$$

Benefit: solution remain on surfaces of constant $H(q,p) = E$.

search for solutions on hypersurface $M_E \subset M$,

E is given by initial value $E = H(q_0, p_0)$.

Further conserved phase space function $F_k(q,p)$ may exist s.t.

$$\frac{d}{dt}F_k = -\{H, F_k\} \overset{!}{=} 0$$

- integral of motion
- (conserved) charge, quantity.

Further cons. charges F_k constrain the submanifold for solutions to level set M_f

$$M_f := \{ (q,p) \in M ; F_k(q,p) = f_k \text{ for all } k \}$$

Moreover F_k can be used to generate further solutions from given ones through Noether's Theorem.

$F(q,p)$ generate flow on phase space $-\{F, \cdot\}$
vector field on phase space

applied to solutions $(q_u(t), p_u(t))$ generates an infinitesimally close solution

$$(\hat{q}(t), \hat{p}(t)) = (q(t), p(t)) + \delta(q(t), p(t))$$

$$\delta q(t) = -\epsilon \{F, q(t)\}, \qquad \delta p(t) = -\epsilon \{F, p(t)\}$$

New solution $(\tilde{q}(t), \tilde{p}(t))$ with same energy E and same $F_k = f_k$
(but not necessarily for $F_\ell = f_\ell$ with $\ell \neq k$)

Additional simplifications arise from further demanding

$$\{F_k, F_\ell\} = 0 \quad \text{for all } k, \ell$$

- $\{F_k\}$ are in involution
- F_k (Poisson) commute

then H is among the conserved charges (functionally dependent)

- $H = F_1$ or $H = H(F_1, \ldots F_k)$.

To obtain (prove existence) of F_k is not trivial

- found by trial and error
- Noether's theorem ~ symmetries
- conserved charges $F_k \xrightarrow{\text{Noether}}$ symmetries (hidden)

2D Central Potential

Particle mass m in 2D central potential $V(r)$ with $r := \sqrt{x^2+y^2}$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - V\left(\sqrt{x^2+y^2}\right)$$

rotational symmetry $SO(2)$, switch to radial coordinates

$x = r \cos\varphi \qquad y = r \sin\varphi \qquad SO(2)$ shifts φ

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - V(r)$$

symmetry: L does not depend on φ (only $\dot{\varphi}$)

Hamiltonian framework by Legendre transf. $p = m\dot{r} \quad \psi = m r^2 \dot{\varphi}$

$$H = \frac{p^2}{2m} + \frac{\psi^2}{2mr^2} + V(r)$$

φ is cyclic coordinate

Ham eq. of motion $\dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m}$ $\dot{p} = -\frac{\partial H}{\partial r} = \frac{\psi^2}{mr^3} - V'(r)$

$\dot{\varphi} = \frac{\partial H}{\partial \psi} = \frac{\psi}{mr^2}$ $\dot{\psi} = -\frac{\partial H}{\partial \varphi} = 0$

H does not depend on φ (φ cyclic) $\Rightarrow$ $F = \psi =: J$ angular momentum

Solutions reside on level set determined by $H = E$, $\psi = J$.

Use conservation to express p through E, J

$$P(r, E, J) = \sqrt{2m(E - V(r)) - J^2/r^2}$$

and also $\Psi(J) = J$

$$\frac{dr}{dt} = \frac{P}{m} \quad \text{sep. of var.} \quad \frac{m}{P} dr = dt \Rightarrow \int_{r_0}^{r} \frac{m\,dr'}{P(r', E, J)} = t - t_0 \qquad t = t_0 + \int \ldots (r)$$

Indirectly yields $r(t)$ depending on E, J and also r_0, t_0

Integrate the angular equation

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{J\,dt}{m\,r(t)^2} = \varphi_0 + \int_{r_0}^{r(t)} \frac{J\,dr'}{r'^2 P(r', E, J)}$$

Reduced finding solution of diff. Ham. eq. of motion to solving relations and performing integrals.

Explicit solution for $V(r) \sim 1/r$ (Kepler) or $V(r) \sim r^2$ (Harm. osc).

1.3 Liouville Integrability

A classical mechanical system with 2n-dimensional phase space M is called (Liouville) integrable if it has n phase space functions F_k, $k=1\dots n$

- all F_k are independent p.s.f^{n}s
- all F_k are everywhere differentiable
- F_k are conserved charges ("integrals of motion")
- F_k are in involution; Poisson commute $\{F_k, F_\ell\} = 0$ for all $k,\ell = 1\dots n$

Such a system is called solvable by quadratures: any finite sequence of:

- resolving a set of coordinate relations on phase space (non-linear, but not differential or integral equation kind)
- calculating ordinary integrals

Phase Space Structure

In an integrable system phase space M splits into level sets M_f:

- M_f has codimension n => dimension n
- there are n independent commuting flows on the level set.

We can use flows to determine a complete coordinate system on M_f (locally)

Coordinates $G_k(q,p)$ are called flow functions (or time fn).

Specified by diff. eq. $-\{F_k, G_l\} = \delta_{k,l}$; initial cond pick an origin for G_k on M_f

Further extend coordinate functions $G_k(q,p)$ across level sets by $\{G_k, G_l\} = 0$

Altogether: $\{G_k, F_l\} = \delta_{kl}$ $\quad \{F_k, F_l\} = 0 = \{G_k, G_l\}$

canonical transf. on M: $(q,p) \to (g,f) = \big(G(q,p), F(q,p)\big)$.

Corollary: Dynamics in coordinates (g, f) is linear in time

us $H = H(f)$ is a function of f_k $k = 1 \ldots n$ alone

Ham. eq of motion

$$\frac{d}{dt} F_k = -\{H, F_k\} = 0$$

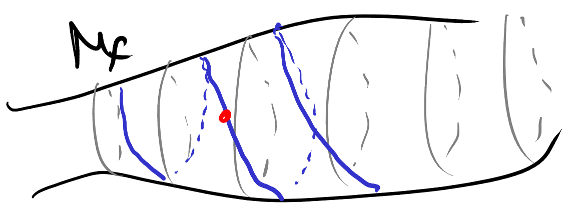
$$\frac{d}{dt} G_k = -\{H, G_k\} = -\sum_{l=1}^{n} \frac{\partial H}{\partial f_l} \{F_l, G_k\} = \frac{\partial H}{\partial f_k} = \text{const}$$

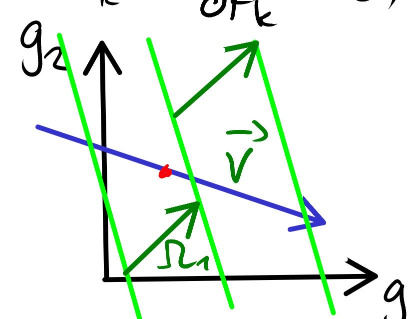
Solution

$$f_k(t) = f_{k,0} = \text{const}$$

$$g_k(t) = g_{k,0} + V_k (t - t_0)$$

$$V_k := \frac{\partial H}{\partial f_k}(f_0)$$





If level set has a non-triv. topology, ie. non-trivial cycles $C_k(f)$

- Diff eq defines G_n up to a constant
- G_n can shift by a constant when moving around a cycle $C_\ell(f)$

G_n are multi-valued functions, shift of G_ℓ under cycle C_k is

is single-valued.

$$\Omega_{k,\ell}(f) := \oint_{C_k(f)} dG_\ell$$

Period matrix Ω

Charges

Given some integrable system M, H: have to establish conserved charges $F_k(q,p)$

Trade in momentum coordinates p_k for the cons. charge values f_k (q_k fixed)

$f_k = F_k(q,p)$ solve these for p_e such that Funktion $P_e(q,f)$:

$$P_e(q, F(q,p)) = p_e \quad \text{or} \quad F_k(q, P(q,f)) = f_k$$

partial transformation $(q,p) \to (q,f)$

Generating Function

Establish flow functions G_k by means of a generating function $S(q,f)$ for canonical transformations.

$$S(q,f) := \int_{\gamma(q,f)} \sum_k p_k \, dq_k' \qquad \gamma \text{ is a path on level set } M_f \text{ connecting a fixed point on } M_f \text{ to } q$$

$q_0(f)$

Integral depends only on endpoints because integrand is closed 1-form.

$$S(q,f) = \int_{q_0(f)}^{q} \sum_k P_k(q',f)\, dq_k' \qquad \text{ordinary integral}$$

Show that integrand is closed when restricted to M_f

$$dP_k \wedge dq_k \overset{\sum_{k,\ell}}{=} \underset{\uparrow = 0 \text{ because on } M_f}{df_\ell} \wedge dq_k \frac{\partial P_k}{\partial f_\ell} \overset{\sum_{k,\ell}}{+} \underbrace{dq_\ell \wedge dq_k}_{\text{anti-sym}} \frac{\partial P_k}{\partial q_\ell} = 0$$

$\leftarrow$ matrix is symmetric

consider $f_j = F_j(q,p)$ differentiate wrt. q_ℓ, multiply by $\partial F_m / \partial p_\ell$

$$0 = \frac{\partial F_j}{\partial q_\ell}\frac{\partial F_m}{\partial p_\ell} + \frac{\partial F_j}{\partial p_k}\frac{\partial p_k}{\partial q_\ell}\frac{\partial F_m}{\partial p_\ell}$$

subtract $m \leftrightarrow j$

$$0 = \frac{\partial F_m}{\partial p_\ell}\frac{\partial F_j}{\partial p_k}\left(\frac{\partial p_k}{\partial q_\ell} - \frac{\partial p_\ell}{\partial q_k}\right) + \underbrace{\{F_m, F_\ell\}}_{=0}$$

by assumption of integrability

F_k are indep
$\Rightarrow$ matrices are invertible

$=0$ if $\dfrac{\partial p_k}{\partial q_\ell}$ is symmetric

Flow Functions

Define flow functions through generating function technique

$$\frac{\partial S}{\partial q_k}(q,f) = p_k(q,f) \qquad G_k(q,p) = \frac{\partial S}{\partial f_k}(q, F(q,p)) \qquad \underset{\text{canonical transf.}}{(q,p) \to (g,f)}$$

2D Central Potential

2D radial coordinates, radial potential $V(r)$

Hamiltonian $H = F_1 = E$, angular momentum $F_2 = \varphi = J$

$$P(r,\varphi,E,J) = \sqrt{2m(E-V(r)) - J^2/r^2} \qquad \Phi(r,\varphi,E,J) = J$$

Generating Function (r,φ)

$$S(r,\varphi,E,J) = \int_{(r_0,\varphi_0)}^{(r,\varphi)} \left(P(r',E,J)\,dr' + J\,d\varphi' \right)$$

$$= \int_{r_0}^{r} P(r',E,J)\,dr' + (\varphi-\varphi_0)J.$$

flow coordinates

$$T := \frac{\partial S}{\partial E} = \int_{r_0}^{r} \frac{m\,dr'}{P(r',E,J)} = t - t_0$$

$$\Phi := \frac{\partial S}{\partial J} = \varphi - \varphi_0 - \int_{r_0}^{r} \frac{J\,dr'}{r'^2 P(r',E,J)} = \varphi(t_0) - \varphi_0$$

Compact Level Sets and Action-Angle Coordinates

integrable system

If level set M_f is compact then it is diffeomorphic to n-dimensional torus $T^n = (S^1)^n$, the so-called Liouville torus

follows from existence of n commuting flows.

On a torus T^n we have n non-trivial cycles C_k.

We can define alternative, canonical integrals of motion

$$I_k(f) := \frac{1}{2\pi} \oint_{C_k(f)} \sum_j p_j \, dq_j$$

depend on level set
ie on F_ℓ alone

$I_k(f)$ $k=1\ldots n$ are generically equivalent to F_ℓ

$\Rightarrow$ can use I_k as conserved charges to define integrability.

I_k are called action variables

θ_k dual to I_k are called angle variables, θ_k defined through gen. fund.

$$\theta_k = \frac{\partial S}{\partial I_k} \qquad S(q,I) = \int_{q_0(I)}^{q} \sum_j p_j(q', I)\, dq'_j$$

Gen function is multi-valued: extend path by cycle C_k

$$S \to S + \oint_{C_k(f)} \sum_j p_j\, dq_j = S + 2\pi I_k$$

also period matrix follows $\Omega_{k,l}(f) = \oint_{C_k(f)} d\theta_l = 2\pi \delta_{k,l}$.

can. transf. $\{\theta_j, I_k\} = \delta_{k,l} \quad \{I_i, I_n\} = 0 = \{\theta_j, \theta_n\}$

Solutions are linear trajectories

$I_k(t) = I_{0,k} = \text{const}$

$\theta_k(t) = \theta_{0,k} + (t-t_0)\,\omega_k(I_0)$

ang. freq. $\omega_k(I_0) := \frac{\partial H}{\partial I_k}(I_0)$

In each direction θ_k we have periodic motion with ang freq ω_k

Altogether: quasi-periodic solutions (unless super-integrability)

Towards quantisation: Action variables are quantised in units of $\hbar$

1.4 Variations of Integrability

Darboux Theorem

Mech sys $2n$ d.o.f., Ham H, symplectic form $\hat{\omega}$

Statement:

symplectic form $\hat{\omega}$ can be expressed in neighbourhood at any point as

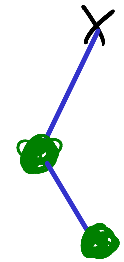
$$\hat{\omega} = \sum_{k=1}^{n} dG_k \wedge dF_k \qquad G_k, F_k \text{ are diff funct.}$$

In part: can always choose G_k, F_k s.t. $F_1 = H$. (conditions of Liouville integr., locally)

Insufficient Charges

For fewer than n charges in involution: system is not integrable!
it is rather chaotic.

Example: double pendulum

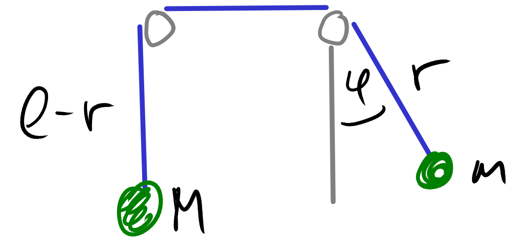


Broken Integrability

Kolmogorov-Arnold-Moser (KAM) thm.: non-integrable def. of integrable model

in short: quasi-per. orbits in integrable model deform almost classically to q.p. orbits of deformed model.

Example: swinging Atwood machine



$$L = \frac{1}{2}(M+m)\dot r^2 + \frac{1}{2} m r^2 \dot\varphi^2 - g r (M - m\cos\varphi)$$

$$H = \frac{p^2}{2(M+m)} + \frac{\psi^2}{2mr^2} + g r (M - m\cos\varphi) \qquad p = (M+m)\dot r\,, \quad \psi = m r^2 \dot\varphi$$

dynamics qualitatively depends on ratio $\mu = M/m$.

System is integrable for $\mu = 3$

$$F = \frac{p\psi}{4m^2}\cos(\varphi/2) - \frac{\psi^2}{2m^2 r}\sin(\varphi/2) + g r^2 \sin(\varphi/2)\cos^2(\varphi/2)$$

$\{H, F\} = 0$ iff $\mu = M/m \overset{!}{=} 3$.

Super-Integrability

more than n conserved charges $\Rightarrow$ only n are in involution. super-integrable

- implications: some of the periodicity freq. ω_k are rationally compatible

(maximal superintegrability if $2n-1$ cons. charges $\Rightarrow$ truly periodic
(applies to all systems with 2D phase space $2n=2$).)

- action-angle coordinates are not uniquely defined.

Non-abelian Symmetries

Non-abelian sym are intrinsically linked to super-integrability

Ex: $SO(3)$ sym $\Rightarrow$ ang mom gen. J_x, J_y, J_z $\{H, J_k\} = 0$

3 cons. charges

but $\{J_k, J_l\} = \varepsilon_{klm} J_m$ $so(3)$ Lie algebra J_k not all in involution

usual choice: J_z, $J^2 = J_x^2 + J_y^2 + J_z^2$: commute $\{J_z, J^2\} = 0$

but J_x or $\varphi = \arctan J_y/J_x$ $\{J_x, J_z\} \neq 0 \neq \{J_z, \varphi\}$

Chapter 2

Algebraic Integrability

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2 Algebraic Integrability

2.1 Spin Models

Rigid body in 3D $\xrightarrow{\text{reduce}}$ Spin d.o.f. is S^2

Spinning Top

Rigid body fixed at centre (origin of 3D)
d.o.f. are 3 angles of orientation of body in 3D space
coord. sys x, y, z fixed to pricipal axes of body, mom of inertia $\Omega_x, \Omega_y, \Omega_z$

Euler angles ϑ, φ, ψ to discribe orientation in 3D space,
$\vec{S}$ angular momentum in body coord. system

$$S_x = -\Omega_x \left(\dot{\varphi} \sin\vartheta \sin\psi + \dot{\vartheta} \cos\psi \right)$$
$$S_y = -\Omega_y \left(\dot{\varphi} \sin\vartheta \cos\psi - \dot{\vartheta} \sin\psi \right)$$
$$S_z = -\Omega_z \left(\dot{\varphi} \cos\vartheta + \dot{\psi} \right)$$

$$L = \frac{S_x^2}{2\Omega_x} + \frac{S_y^2}{2\Omega_y} + \frac{S_z^2}{2\Omega_z}$$

$$\frac{d}{dt} S_x = \left(1/\Omega_y - 1/\Omega_z \right) \cdot S_y S_z$$
$$\frac{d}{dt} S_y = \left(1/\Omega_z - 1/\Omega_x \right) \cdot S_z S_x$$
$$\frac{d}{dt} S_z = \left(1/\Omega_x - 1/\Omega_y \right) \cdot S_x S_y$$

model is integrable: 6D phase sp.

4 cons. qu.: H, J_x, J_y, J_z

(in inertial frame)

Spin Model Reduce spinning top $\vartheta, \varphi, \psi, \dot\vartheta, \dot\varphi, \dot\psi$ to $(S_x, S_y, S_z) = \vec{S}$

However $\vec{S}$ has 3 d.o.f. : odd! no phase space!

one conserved qty is $\vec{S}^2$ can be fixed reduces $\mathbb{R}^3$ to S^2

Poisson brackets $\{S_j, S_k\} = \varepsilon_{jkl} S_l \qquad \{\vec{S}^2, \dots\} = 0$

$$H = \frac{1}{2} \vec{S}^T \Omega^{-1} \vec{S} \qquad \Omega = \mathrm{diag}(\Omega_x, \Omega_y, \Omega_z) \qquad \|\vec{S}\| = J = \text{const.}$$

$$\frac{d}{dt} \vec{S} = -\{H, \vec{S}\} = (\Omega^{-1} \vec{S}) \times \vec{S} \qquad (\text{later on } J = 1)$$

Poisson brackets: $\{F_1, F_2\} = \varepsilon_{jkl} S_l \frac{\partial F_1}{\partial S_j} \frac{\partial F_2}{\partial S_k}$.

Spin Parametrisations

* $\vec{S}$ with $\|\vec{S}\| =$ const or $\vec{S}^2 =$ const $\qquad \{S_j, S_k\} = \varepsilon_{jkl} S_l$

* spherical coord. $\qquad \vec{S} = J \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$

  Poisson br. $\qquad \{\vartheta, \varphi\} = \dfrac{1}{J \sin\vartheta}$.

* stereographic proj : map S^2 to $\bar{\mathbb{C}} \ni \zeta$

$$\vec{S} = \frac{J}{1+|\zeta|^2} \begin{pmatrix} 2\,\mathrm{Re}\,\zeta \\ 2\,\mathrm{Im}\,\zeta \\ 1-|\zeta|^2 \end{pmatrix} \qquad \zeta = \tan(\vartheta/2)\exp(i\varphi) = \frac{S_x + iS_y}{J+S_z}$$

$\zeta = 0$ is $N, +\vec{e}_z$ $\zeta = \infty$ is $S, -\vec{e}_z$, $|\zeta| = 1$ is equator $S_z = 0$

$$\{\zeta, \zeta^*\} = \frac{i}{2J}\left(1+|\zeta|^2\right)^2.$$

* spinor parametrisation $\sim SO(3) \sim SU(2)$

translate to 2×2 complex matrices using Pauli matrices $\vec{\sigma}$

$\vec{S}\cdot\vec{\sigma}$ hermitian, eigenvalues $\pm J$

eigenvectors s, εs^* is called associated spinor $\qquad \varepsilon = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$

$$(\vec{S}\cdot\vec{\sigma})\, s = +J s \qquad (\vec{S}\cdot\vec{\sigma})\, \varepsilon s^* = -J \varepsilon s^* \qquad \vec{S} = J\,\frac{s^\dagger \vec{\sigma} s}{s^\dagger s}$$

s is defined up to scale $\quad s \equiv \lambda s \leftarrow \mathbb{C}P^1$

$N: s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \infty \end{pmatrix}$ $\qquad$ related to stereogr. proj. $s \equiv \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$.

Poisson bracket $F_{1,2}(\vec{s})$ $F(s) = F(\lambda s) = F(s_2/s_1) = F(\zeta)$

$$\{F_1, F_2\} = -\frac{i}{2J} s^+ \sum_{j=1,2} s \left(\frac{\partial F_1}{\partial s_j} \cdot \frac{\partial F_2}{\partial s_j^*} - \frac{\partial F_1}{\partial s_j^*} \cdot \frac{\partial F_2}{\partial s_j} \right).$$

Altogether:

$$\underset{\substack{\uparrow \\ \vec{c}^{\,-1}}}{\mathbb{R}^3} \supset \underset{\substack{\uparrow \\ \text{spherical} \\ \text{coord.}}}{S^2} = \underset{\substack{\uparrow \\ \text{stereo.} \\ \text{proj}}}{\bar{\mathbb{C}}} = \underset{\substack{\uparrow \\ \text{spinor}}}{\mathbb{C}P^1}$$

Classes of Solutions of spin model.

several distinct cases for dynamics:

- asymmetrical case $\Omega_x \neq \Omega_y \neq \Omega_z \neq \Omega_x$

  solution in terms of elliptic fn

  $$S_x = c_x \,\mathrm{cn}\,(\lambda t + \varphi, m)$$
  $$S_y = c_y \,\mathrm{sn}\,(\lambda t + \varphi, m)$$
  $$S_z = c_z \,\mathrm{dn}\,(\lambda t + \varphi, m)$$

  λ, m are constants (in terms $E, J, \Omega_{x,y,z}$)

  φ describes initial cond.

- symmetric top at $m=0$ $\quad \Omega_x = \Omega_y \neq \Omega_z$

  $$S_x = c \cos(\lambda t + \varphi)$$
  $$S_y = c \sin(\lambda t + \varphi)$$
  $$S_z = \text{const.}$$

  SO(2) symmetry

- spherical top $\Omega_x = \Omega_y = \Omega_z$ $\quad SO(3)$ symmetry

$$H = \frac{1}{2}\Omega^{-1} J^2 = \text{const} \quad \Rightarrow \text{no dynamics!}$$

mech sys. version of $\vec{S} = \text{const}$ $\quad \|\vec{S}\| = J$

Relation to classes of integrable systems

type	rational	trigonometric	elliptic
symbol	XXX	XXZ	XYZ
top	spherical	symmetric	asymmetric
$\Omega_x \Omega_y \Omega_z$	$\Omega_x \Omega_x \Omega_x$	$\Omega_x \Omega_x \Omega_z$	$\Omega_x \Omega_y \Omega_z$
symmetry	$SO(3)$ n.a. Lie alg.	$SO(2)$ Cartan sub.	— ($\mathbb{Z}_2$)

2.2 Lax Pair

Algebraic Eq. of Motion for Spin Model

$SO(3)\ /\ SU(2)$

$\vec{S}$, $\{S_j, S_k\} = \varepsilon_{jkl} S_l$ ⇝ cast variables/equations into 2x2 matrices, 2 vector

Pauli matrices σ_a

$$\vec{S}\cdot\vec{\sigma} = \begin{pmatrix} +S_z & S_x - iS_y \\ S_x + iS_y & -S_z \end{pmatrix}$$

traceless
hermitian

recall Pauli matrix relations: $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$, $[\vec{v}\cdot\vec{\sigma}, \vec{w}\cdot\vec{\sigma}] = 2i(\vec{v}\times\vec{w})\cdot\vec{\sigma}$.

$$\ldots \Rightarrow \frac{d}{dt}\vec{S}\cdot\vec{\sigma} = ((\Omega^{-1}\vec{S})\times\vec{S})\cdot\vec{\sigma} = -\tfrac{i}{2}[(\Omega^{-1}\vec{S})\cdot\vec{\sigma}, \vec{S}\cdot\vec{\sigma}]$$

define $T := \vec{S}\cdot\vec{\sigma}$, $M := -\frac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$ $\Rightarrow$ $\frac{d}{dt}T = [M, T]$

Lax Pair

above: $\mathbb{C}^2$

Lax Pair (T, M) is pair of sqr. matrices $T, M \in \mathrm{End}(\mathbb{V})$

T: Lax matrix, M: (Lax) evolution matrix

if it satisfies the Lax eq.

$$\frac{d}{dt} T := -\{H, T\} = [M, T]$$

· spectrum of T is (time) conserved; $\det(\gamma\,\mathrm{id} - T)$ as poly in T is conserved

$$T(t) = g(t)\, T(t_0)\, g(t)^{-1}$$: time ev. of T by conjugation with $g(t)$ only.

T generates conserved charges as

$$F_k := \frac{1}{k} \mathrm{tr}(T^k) \qquad \frac{d}{dt} F_k = \mathrm{tr}\left([M, T]\, T^{k-1}\right) = 0$$

Complete Lax Pairs

Note: a generic lax pair for a given system:

- is not unique
- not necessarily useful
- there is universal recipe to construct
- There need not be a relation of lax pair to features of system, eg. dim

Generalise lax Pair for spin model?

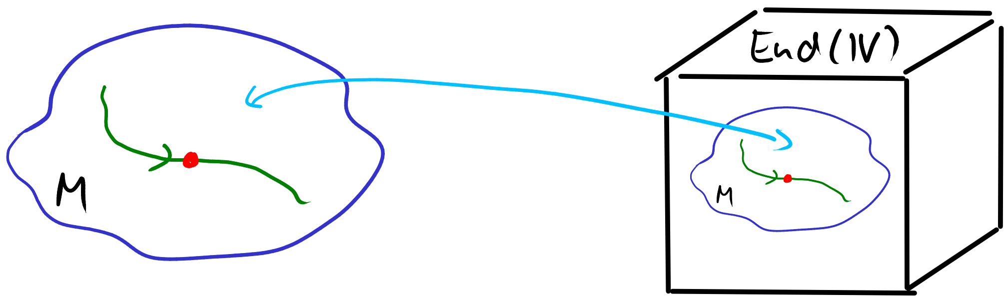
- similarity transformation of T, M by a constant matrix
- generalise spin ½ rep of T, M to spin s dim: $(2s+1)$

add a constant trace $T = \vec{S} \cdot \vec{\sigma} + \overset{\mathbb{C}}{\overset{\downarrow}{v}} H \, \mathrm{id}$ $\quad M = -\frac{i}{2} (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}$

same lax eq, spectrum is now: $\{ vH \pm J \}$

desirable property: Lax Matrix T should fully describe state of the system.

→ know state → know equations of motion.



Notion of complete Lax pairs: Lax Pair (T, M):

i) the pair obeys Lax equation $d/dA\, T = [M, T]$

ii) the Lax matrix T encodes all $2n$ phase space vars uniquely

iii) it is diagonalisable for almost all points in M

iv) its spectrum encodes n indep. variables

v) these variables are in involution.

} model is Liouville integrable!

2.3 Lax-Poisson Structure

Want to establish that $\{F_k, F_e\} = 0$

Lax-Poisson Equation

Want to specify Poisson brackets of all pairs of T_{jk} with T_{em}. Lax-Poisson eq.

$$\{T_{jk}, T_{em}\} = \sum_n R_{(je)(nm)} T_{nk} - \sum_n T_{jn} R_{(ne)(km)}$$
$$- \sum_n R_{(ej)(nk)} T_{nm} + \sum_n T_{en} R_{(nj)(mk)}$$

$R_{(je)(km)}$ is a tensor operator of rank 2 (tensor rank 4)
← Lax-Poisson matrix

- R expresses Poisson/symplectic structure on $\mathrm{End}(V)$
- this form implies that $\{F_k, F_e\} = 0$ for $F_k := \frac{1}{k} \operatorname{tr}(T^k)$.

Tensor Notation

Express a square matrix A (eg. T) through its components using a basis E_{jk}

$$A = \sum_{j,k} A_{j,k} E_{jk} \qquad (E_{jk})_{\ell m} = \delta_{j\ell}\,\delta_{km}$$

Poisson bracket of two operators A, B whose components are phase space f^{ns}:

$$\{A \overset{\otimes}{,} B\} := \sum_{jk\ell m} \{A_{jk}, B_{\ell m}\}\; E_{jk} \otimes E_{\ell m}$$

$$R := \sum_{jk\ell m} R_{(j\ell)(km)}\; E_{jk} \otimes E_{\ell m}$$

permutes sites

$$P(R) := \sum_{jk\ell m} R_{(j\ell)(km)}\; E_{\ell m} \otimes E_{jk} = \sum R_{(\ell j)(mk)}\, E_{jk} \otimes E_{\ell m}$$

Lax-Poisson eq: $\{T \overset{\otimes}{,} T\} = [R, T \otimes \mathrm{id}] - [P(R), \mathrm{id} \otimes T]$

Short-hand notation for tensors using site indices:

$$T_1 := T \otimes \mathrm{id} \qquad\qquad T_2 := \mathrm{id} \otimes T$$

$$R_{12} := R \qquad\qquad R_{21} := P(R)$$

$$\{T_1, T_2\} := \{T \overset{\otimes}{,} T\}$$

$$\Rightarrow \quad \{T_1, T_2\} = [R_{12}, T_1] - [R_{21}, T_2] .$$

Properties and Applications

Show that L.P. eq yields $\{F_k, F_\ell\} = 0$

$$\{F_j, F_k\} = \frac{1}{jk}\{\mathrm{tr}\,T^j, \mathrm{tr}\,T^k\} = \frac{1}{jk}\,\mathrm{tr}_{1,2}\{T_1^j, T_2^k\}$$

$$= \frac{1}{jk}\sum_{\ell=1}^{j}\sum_{m=1}^{k} \mathrm{tr}_{1,2}\left(T_1^{\ell-1} T_2^{m-1}\{T_1, T_2\} T_1^{j-\ell} T_2^{k-m}\right)$$

$$= \mathrm{tr}_{1,2}\left(T_1^{j-1} T_2^{k-1}\{T_1, T_2\}\right)$$

$$= \mathrm{tr}_{1,2}\left(\underbrace{T_1^{j-1} T_2^{k-1} [R_{12}, T_1]}_{=0} - \underbrace{T_1^{j-1} T_2^{k-1} [R_{21}, T_2]}_{=0}\right)$$

$$= 0 \, .$$

Poisson brackets satify Jacobi, so R satisfies a Lax-Jacobi id.

$$0 = [T_1, [\![R,R]\!]_{123} + \{T_2, R_{13}\} + \{T_3, R_{12}\}] + 2 \text{ cyclic}$$

$[\![X,Y]\!]$ is a quadratic combination (non bilinear) of X, Y

$$[\![X,Y]\!]_{123} := [Y_{12}, Y_{13}] + [Y_{12}, X_{23}] + [X_{32}, Y_{13}]$$

special case: $[\![R,R]\!] = 0$ and R indep of M. ⇒ L-J. holds.

Example: $T = \vec{S}\cdot\vec{\sigma} + u H \, id$

$$\{T_1, T_2\} = (\vec{\sigma}_1 \times \vec{\sigma}_2)\cdot\vec{S} + u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_1 - u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_2$$

$$R_{12} = -\frac{i}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2 - \frac{i}{2}u(\Omega^{-1}\vec{S})\cdot\vec{\sigma}_1$$

complete Lax-Poisson structure: (T, M, R)

i) pair obeys Lax eq $\frac{dT}{dt} = [M, T]$

ii) Lax matrix T encodes all 2n phase space vars.

iii) T is diagonalisable almost everywhere

iv) spectrum encodes n indep vars.

v) Lax-Poisson matrix R obeys Lax-Poisson eq.

Evolution from Lax-Poisson structure

We know that T encodes phase space, F_k encode conserved charges

$\Rightarrow$ we can write $\quad H = h(F_k)$

Lax Eq follows: $\frac{d}{dt} T = [M, T]$ with $\quad M_1 = \sum_k \frac{\partial h}{\partial F_k} \mathrm{tr}_2 \left(T_2^{k-1} R_{12} \right)$

therefore (T, R) extends to (T, M, R) by means of $h(F)$

replace i) the Hamiltonian is given as a function of Lax traces F_k

ex: $T = \vec{S} \cdot \vec{\sigma} + u H \, \mathrm{id} \qquad F_1 = 2uH \qquad R_{12} = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u \, (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_1$

$H = \frac{\mathrm{tr}\, T}{2u} = \frac{F_1}{2u} \qquad M_1 = \frac{1}{2u} \mathrm{tr}_2 \, R_{12} = -\frac{i}{2} \, (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_1$

Parametric Lax Pairs

introduce Parameter $u \in \mathbb{C}$ to allow for many more indep qty in T or in spect.

many equations are unchanged

$$T(u) = \vec{S} \cdot \vec{\sigma} + uH\, id \qquad M(u) = -\frac{i}{2} (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}$$

$$\frac{d}{dt} T(u) = [M(u), T(u)]$$

$$F_1(u) = 2uH \qquad F_2(u) = J^2 + u^2 H^2 \qquad F_3(u) = 2uH \left(J^2 + \frac{1}{3} u^2 H^2 \right)$$

However how about Poisson brackets? $\{T(u), T(v)\}$ need to extend to

$$\{T_1(u_1), T_2(u_2)\} = [R_{12}(u_1, u_2), T_1(u_1)] - [R_{21}(u_2, u_1), T_2(u_2)]$$

$$\Rightarrow \{F(u_1), F(u_2)\} = 0 \quad F_k(u) = \Sigma_j F_{kj} u^j \quad \Rightarrow \{F_{j,k}, F_{l,m}\} = 0$$

ex: $R_{12}(u_1, u_2) = \frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u_2 (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_1$

Classical r-matrix

r is a tensor operator as R but relation is (cl. RTT relation)

$$\{T \overset{\otimes}{,} T\} = [r, T\otimes T] \qquad \{T_1, T_2\} = [r_{12}, T_1 T_2]$$

$$r_{12} = -r_{21} \qquad \Rightarrow \qquad \{F_j, F_k\} = 0$$

Jacobi-id

$$0 = [\,[\![r,r]\!], T_1 T_2 T_3\,] + ([\{r_{12}, T_3\}, T_1 T_2] + 2\text{ cyclic})$$

but if r is indep of M: usually cl. Yang-Baxter-eq.

$$[\![r,r]\!] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$$

Ex:

$$r_{12}(u_1,u_2) = \frac{i}{2H}\frac{\vec\sigma_1\cdot\vec\sigma_2}{u_1-u_2} - \frac{i}{2}v_2(\Omega^{-1}\vec S)\vec\sigma_1\, T_2(u_2)^{-1} + \frac{i}{2}v_1 T_1(u_1)^{-1}(\Omega^{-1}\vec S)\cdot\vec\sigma_2$$

Chapter 3

Classical Spin Chains

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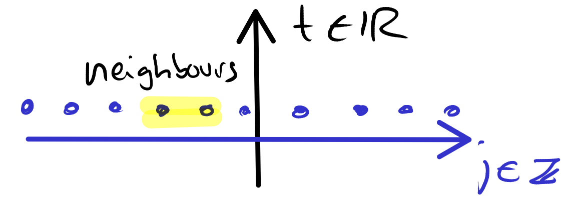
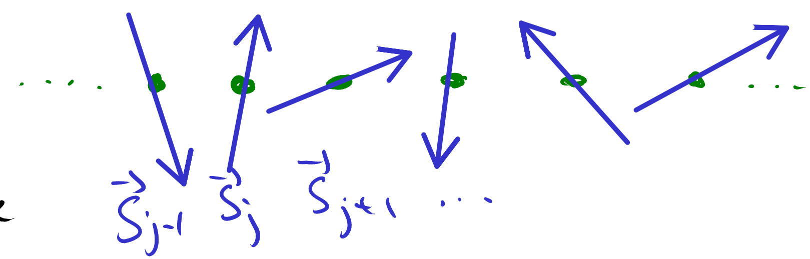
3. Classical Spin Chains



3.1 Heisenberg Spin Chain

Chain of elementary spin d.o.f. with interactions among nearest neighbours.



Model Hamiltonian mechanics

Phase space is copies of S^2

$\|\vec{S}_j\| = 1$ unit vectors; 2 dof per site

Poisson brackets $\{S^a_j, S^b_k\} = \delta_{jk}\varepsilon^{abc}\, S^c_k$

constraint $\vec{S}_k^2 = 1$ is compatible with Poisson structure $\{S_j, \vec{S}_k^2 - 1\} = 0$

Dynamics: based on sel alignment of neighbours spins $\vec{S}_j, \vec{S}_{j+1}$

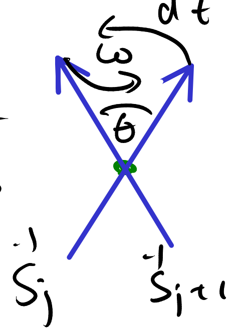
$$H = \sum_j \mathcal{H}_j \qquad \mathcal{H}_j = -\log \frac{1+\vec{S}_j\cdot\vec{S}_{j+1}}{2}.$$

homogeneous symmetry $SO(3)$ isotropic interaction depending on $\vec{S}_j\cdot\vec{S}_{j+1}$

Equations of motion

$$\frac{d\vec{S}_j}{dt} = -\{H, \vec{S}_j\} = -\frac{\vec{S}_{j-1}\times\vec{S}_j}{1+\vec{S}_{j-1}\cdot\vec{S}_j} + \frac{\vec{S}_j\times\vec{S}_{j+1}}{1+\vec{S}_j\cdot\vec{S}_{j+1}}$$

for nearest neighbours (only)



$$\omega = \frac{1}{\cos(\theta/2)}$$

Using stereographic projection $\vec{S} \in S^2 \Rightarrow \vec{S}_j \to \zeta_j \in \mathbb{C}$ or spinor var $s_j \in \mathbb{CP}^1$

$$\frac{1+\vec{S}_j\cdot\vec{S}_k}{2} = \frac{(1+\zeta_j\zeta_k^*)(1+\zeta_k\zeta_j^*)}{(1+|\zeta_j|^2)(1+|\zeta_k|^2)} = \frac{(s_j^+ s_k)(s_k^+ s_j)}{(s_j^+ s_j)(s_k^+ s_k)}$$

e.o.m.

$$\frac{d\zeta_j}{dt} = \frac{i}{2}\sum_{\pm}\frac{1+|\zeta_j|^2}{1+\zeta_{j\pm1}\zeta_j^*}(\zeta_{j\pm1}-\zeta_j)$$

governs the dynamics of the scale of s_j

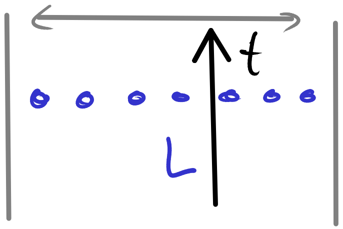
$$\frac{ds_j}{dt} = \frac{i}{2}\frac{s_j^+ s_j}{s_j^+ s_{j-1}}s_{j-1} + \frac{i}{2}\frac{s_j^+ s_j}{s_j^+ s_{j+1}}s_{j+1} + i\lambda_j s_j$$

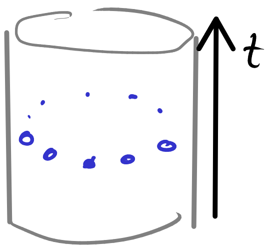
Boundary Conditions

there are different types of (integrable) boundary conditions

→ finite extent

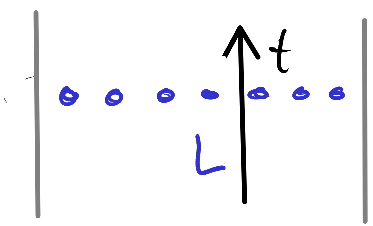
→ infinite extent





closed boundary conditions

$$H = \sum_{j=1}^{L} \mathcal{H}_j \qquad \vec{S}_{j+L} = \vec{S}_j$$

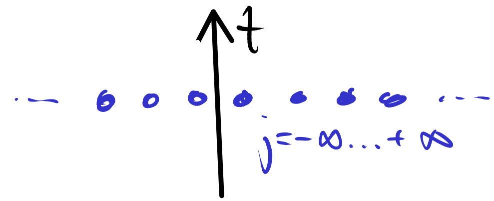


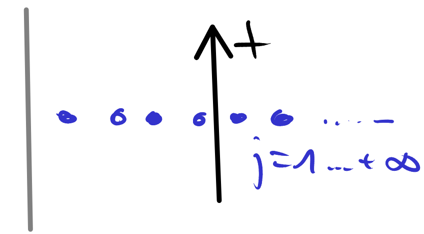
open boundary conditions

$$H = \sum_{j=1}^{L-1} \mathcal{H}_j$$

infinite boundary conditions

semi-infinite chains





asymptotic values for spin d.o.f.

$$H = \sum_{j=-\infty}^{\infty} \mathcal{H}_j \qquad \vec{S}_j \to \vec{S}_{L/R} \text{ for } j \to \mp\infty$$

deformations/twist of boundary conditions

boundary deg. of freedom.

Here: most undeformed closed chains.

Global Symmetries

has SO(3) rotational symmetry for spin vectors $\vec{S}_j$

rotation for $\vec{S}_j$ generated by $\vec{S}_j$ parametrises an inf. rot.

$$\delta \vec{S}_j = -\{\delta\vec{X}\cdot\vec{S}_j, \vec{S}_j\}$$

global angular momentum charge

$$\vec{J} = \sum_j \vec{S}_j \qquad \delta\vec{X}\cdot\vec{J} \text{ for a global inf. rot.}$$

symmetry: $\{H, \vec{J}\} = 0$

global cons. charge $\vec{J} = \sum_j \vec{Q}_j \qquad \vec{Q}_j = \vec{S}_j \qquad \vec{k}_j = \dfrac{\vec{S}_j \times \vec{S}_{j+1}}{1 + \vec{S}_j\cdot\vec{S}_{j+1}}$

local cons. current Q_j, k_j disc. cons. eq. $\dfrac{d}{dt}\vec{Q}_j = -\{H, \vec{Q}_j\} = \vec{k}_j - \vec{k}_{j-1}$ (e.o.m.)

SO(3) algebra: $\{J^a, J^b\} = \varepsilon^{abc} J^c$

Discrete symmetries:

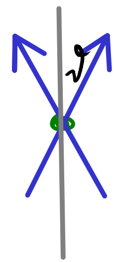
- $SO(3) \to O(3)$ by global refl. $\vec{S}_j \to -\vec{S}_j$
- chain symmetries translations $\vec{S}_j \to \vec{S}_{j\pm 1}$; reflections $\vec{S}_j \to \vec{S}_{L+1-j}$ $\}$ dihedral symmetry D_{2L}

homogeneous, isotropic Hamiltonian (on chain)

Simple Solutions

- $L=1$ no dynamics $\vec{S}_1$ is constant

- $L=2$



two spins rotating around z-axis with $\omega = \dfrac{2}{\cos\vartheta}$

$$\vec{S}_{1,2}(t) = \begin{pmatrix} \pm \sin\vartheta \cos(-\omega t) \\ \pm \sin\vartheta \sin(-\omega t) \\ \cos\vartheta \end{pmatrix}$$

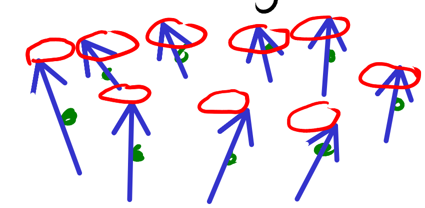
$$H = -4 \log|\cos\vartheta| \qquad \vec{J} = 2\cos\vartheta\, \vec{e}_z$$

generalises to length L

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta \cos(2\pi n j/L - \omega t) \\ \sin\vartheta \sin(2\pi n j/L - \omega t) \\ \cos\vartheta \end{pmatrix}$$

$n \in \mathbb{Z}$ $0 \le n < L$

mode winding number



$$\omega = \frac{2\cos\vartheta \sin^2(\pi n/L)}{1 - \sin^2\vartheta \sin^2(\pi n/L)}$$

$$H = -L \log\left(1 - \sin^2\vartheta \sin^2(\pi n/L)\right)$$

$$\vec{J} = L\cos\vartheta\, \vec{e}_z$$

$L=3$ elliptic solutions in general

Special case, all spins are on a plane,

Plane rotates with ang. freq. ω

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta_j \cos(-\omega t) \\ \sin\vartheta_j \sin(-\omega t) \\ \cos\vartheta_j \end{pmatrix}$$

constraint

$\vec{J} = J\vec{e}_z$, $\quad \sin\vartheta_1 + \sin\vartheta_2 + \sin\vartheta_3 = 0$

splits up phase space in $0<J<1$
$1<J<3$

$$H = -2\log\frac{|J^2-1|}{8} \qquad \omega = \frac{4J}{J^2-1} \qquad J^2 = 3+2\sum_j \cos(\vartheta_j - \vartheta_{j+1})$$

Excitations of the Ferromagnetic Ground State

Ground state $\vec{S}_j(t) = \vec{e}_z$ (all aligned) $\vec{J} = L\vec{e}_z$

discuss small excitations above ground state in pert. theory use stereographic var ζ_j

expand eom for ζ_j for small ζ_i (small dev from $\vec{S}_j = \vec{e}_z$)

$$\frac{d\zeta_j}{dt} = \frac{i}{2}\left(\zeta_{j-1} - 2\zeta_j + \zeta_{j+1}\right) + O(\zeta^3)$$ linear DE

$$\Rightarrow \quad \zeta_j(t) = \epsilon\, a_n \exp\frac{2\pi i n j}{L}\exp(-i\omega_n t) + O(\epsilon^2) \qquad \omega_n = 2\sin^2\frac{\pi n}{L}$$

$$\vec{J} = \left(L - 2\epsilon^2|a_n|^2 L\right)\vec{e}_z + O(\epsilon^3) \qquad H = 4\epsilon^2|a_n|^2 L\sin^2\frac{\pi n}{L} + O(\epsilon^3)$$

use action angle variables ~ parametrise a_n through action var I_n
evaluate sympl. str. on solution

$$\hat{\omega} = \sum_j 2i\, dJ_j \wedge dJ_j^* = 2\epsilon |a_n|^2 L\, \omega_n\, dt \wedge d\epsilon + O(\epsilon)$$

← one period

$$\sim\ dI_n = \frac{1}{2\pi} \oint \hat{\omega} = 4\, d\epsilon\ \epsilon\ |a_n|^2 L + O(\epsilon^3)$$

$$\Rightarrow\quad I_n = 2|a_n|^2 \epsilon^2 L + O(\epsilon^3)$$

$$\vec{J} = (L - I_n)\vec{e}_z + O(I_n^2)$$

$$H = \omega_n I_n + O(I_m^2) \qquad\qquad \omega_n = \frac{\partial H}{\partial I_n}$$

3.2 Integrable Structure

want Lax pair T, M with $\frac{d}{dt}T = [M,T]$, T encodes phase space

Lax Transport

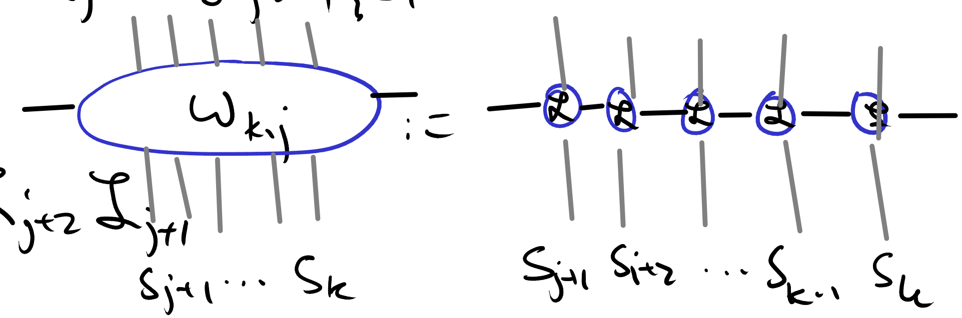
we introduce a local Lax transport $\mathcal{L}_j$, evolution matrix M_j

Lax transport eq.

$$\frac{d}{dt}\mathcal{L}_j = M_j \mathcal{L}_j - \mathcal{L}_j M_{j-1}$$

Construct a composite Lax transport

$$W_{k,j} := \mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_{j+2} \mathcal{L}_{j+1}$$



evolution equation for $W_{k,j}$

$$\frac{d}{dt} W_{k,j} = M_k W_{k,j} - W_{k,j} M_j$$

then introduce Lax Pair

monodromy $T = W_{L,0} = \mathcal{L}_L \mathcal{L}_{L-1} \cdots \mathcal{L}_2 \mathcal{L}_1 \qquad M = M_L = M_0$

$$\Rightarrow \quad \frac{d}{dt} T = [M, T]$$

For Heisenberg chain

$$\mathcal{L}_j(u) = id + \frac{i}{u} \vec{S}_j \vec{\sigma} \qquad M_j(u) = \frac{i}{u^2+1} \frac{\vec{S}_j + \vec{S}_{j+1} + u \vec{S}_j \times \vec{S}_{j+1}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}} \cdot \vec{\sigma}$$

confirm validity / eom.

$$(\vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma} = i\,(\vec{S}_{j+1} \cdot \vec{\sigma})(\vec{S}_j \cdot \vec{\sigma}) - i\,(\vec{S}_j \cdot \vec{S}_{j+1})\,\mathrm{id}$$

also (will show later): $T(u)$ encodes all of phase space

lax pair

$$F_n(u) = \frac{1}{n} \operatorname{tr} T(u)^n$$ is conserved monodromy traces

we need only F_1 because $F_2 = \frac{1}{2}(F_1)^2 - \det T$ $\quad \det T = $ fixed.

$$\det \mathcal{L}_j = 1 + \frac{1}{u^2} \Rightarrow \det T = \left(1 + \frac{1}{u^2}\right)^L$$

we need only $F(u) := F_1(u) = \operatorname{tr} T(u)$ contains all cons. charges.

Classical r-matrix

The elementary Lax transport admits a classical r-matrix (with parameter)
We want the classical RTT relation:

$$\{\mathcal{L}_j(u_1) \overset{\otimes}{,} \mathcal{L}_k(u_2)\} = \delta_{jk}\, r_j(u_1,u_2)\left(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2)\right) - \delta_{jk}\left(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2)\right) r_{j-1}(u_1,u_2).$$

Then it follows that monodromy $T = \mathcal{L}_L \cdots \mathcal{L}_1$ obeys classical RTT rel

$$\{T(u_1) \overset{\otimes}{,} T(u_2)\} = [r_L(u_1,u_2), T(u_1)\otimes T(u_2)]$$

$\Rightarrow$ all monodromy traces Poisson commute $\{F_m(u_1), F_n(u_2)\} = 0$ for all $u_1, u_2 \in \mathbb{C}$

Solution for r: $r_j(u_1,u_2) = r(u_1-u_2) = -\dfrac{\sigma^a\otimes\sigma^a}{2(u_1-u_2)}$. Satisfies CYBE $[[r,r]]=0$

3.3 Spectral Parameter

have introduced Lax Pair $T(u), M(u)$, $u \in \bar{\mathbb{C}}$

later: consider complex analytic structure of objects in terms of spectral par u.

Hamiltonian

All information on conserved qty should be contained in monod trace $F(u) = \operatorname{tr} T(u)$.

Complication: • $F(u)$ is rather non-local whereas H is local $H = \sum \mathcal{H}_j$

we need a special point $u \in \mathbb{C}$ where $T(u)$ becomes more local. $\mathcal{L}_j$ must be special

$\mathcal{L}_j(u) = 1 + \frac{i}{u} \vec{\sigma} \cdot \vec{S}_j$ with three special points: $u = 0$ $\mathcal{L}_j$ is divergent.

$\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} = 0$ for $u = \pm i$ $\mathcal{L}_j$ is a singular matrix at $u = \pm i$

Final result for H $\quad H = -\log \frac{F(+i)\, F(-i)}{4^L}$.

properties of $\mathcal{L}_j$ at $u = \pm i$

$$\mathcal{L}_j(\pm i) = id \pm \vec{S}_j \cdot \vec{\sigma}$$

is a matrix of lower rank, rank = 1, also have

$$\operatorname{tr} \mathcal{L}_j(\pm i) = 2 \qquad \mathcal{L}_j(\pm i)^+ = \mathcal{L}_j(\pm i)$$

thus we can write $\mathcal{L}_j$ in terms of spinor variables

$$\mathcal{L}_j(+i) = \frac{2}{s_j^+ s_j}\, s_j s_j^+ \qquad \mathcal{L}_j(-i) = \frac{2}{s_j^+ s_j}\, \varepsilon s_j^* s_j^T \varepsilon^{-1}.$$

Compose Products for $T(\pm i) \to F(\pm i)$

$$F(+i) = 2^L \prod_{j=1}^{L} \frac{s_{j+1}^+ s_j}{s_j^+ s_j} \qquad F(-i) = 2^L \prod_{j=1}^{L} \frac{s_j^+ s_{j+1}}{s_j^+ s_j}$$

$$F(+i)F(-i) = 4^L \prod_{j=1}^{L} \frac{(s_{j+1}^{+} s_j)(s_j^{+} s_{j+1})}{(s_j^{+} s_j)(s_{j+1}^{+} s_{j+1})} = 2^L \prod_{j=1}^{L} (1 + \vec{S}_j \cdot \vec{S}_{j+1})$$

$$\exp(-H) = \prod_{j=1}^{L} \frac{(s_{j+1}^{+} s_j)(s_j^{+} s_{j+1})}{(s_j^{+} s_j)(s_{j+1}^{+} s_{j+1})} .$$

$$\exp(iP) = \frac{F(-i)}{F(+i)} = \prod_{j=1}^{L} \frac{s_j^{+} s_{j+1}}{s_{j+1}^{+} s_j}$$

P: lattice momentum.

Note there are further local charges in expansion of $F(u)$ around $u = \pm i$

$$\frac{F(u)}{F(+i)} = 1 - \frac{i}{2} \sum_{j=1}^{L} \left(\frac{(s_{j+1}^{+} s_{j-1})(s_j^{+} s_j)}{(s_{j+1}^{+} s_j)(s_j^{+} s_{j+1})} - 2 \right) (u - i) + \ldots$$

<u>Reconstruction</u> of phase space from $T(u)$

again: $T(u)$ is non-local but s_i, s_i^+ or $\vec{S}_j$ are local

need $u = \pm i$ again.

we can express $T(u)$ at $u = +i$

$$T(+i) = 2^L \frac{s_L s_1^+}{s_L^+ s_1} \prod_{j=1}^{L-1} \frac{s_{j+1} s_j^+}{s_j^+ s_j} = F(+i) \frac{s_L s_1^+}{s_1^+ s_L} \qquad \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

two eigenvalues: $0, F(+i)$ with eigenvectors εs_1^* and s_L

$$T(-i) = F(-i) \frac{\varepsilon s_L^* s_1^T \varepsilon^{-1}}{s_L^+ s_1} \qquad 0, F(-i) \text{ with } s_1 \text{ and } \varepsilon s_L^*$$

compose $\vec{S}_j = \dfrac{s_j^+ \vec{\sigma} s_j}{s_j^+ s_j}$ we know $\vec{S}_1$ (and $\vec{S}_L$)

To obtain other local variables $\vec{S}_j$ consider shifted monodromy

$$T_{j-1}(u) := \mathcal{L}_{j-1}(u) \dots \mathcal{L}_1(u)\, \mathcal{L}_L(u) \dots \mathcal{L}_j(u) \qquad T_L = T_0 = T$$

satisfy recursion relation $\quad T_j(u) = \mathcal{L}_j(u)\, T_{j-1}(u)\, \mathcal{L}_j(u)^{-1}$

if you know $T_0(u)$ you know $\vec{S}_1$, you know $\mathcal{L}_1(u) \Rightarrow$ know $T_1(u)$

$T_{j-1}(u)$ " $\vec{S}_j$ $\mathcal{L}_j(u) \Rightarrow$ " $T_j(u)$

recursion relation is singular at $u = \pm i$

$\rightsquigarrow$ need expansion of $T(u)$ at $u = \pm i$ at L orders

to recover all $\vec{S}_i$

Global Symmetry

$SO(3)$ symmetry should be encoded by $T(u)$

turns out $\vec{J} = \sum_{j=1}^{L} \vec{S}_j$ is obtained at the point $u = \infty$

$$\mathcal{L}_j(u) = id + \frac{i}{u} \vec{S}_j \cdot \vec{\sigma} \left[+ \dots \right]$$

Expand $T(u) = \prod_i \mathcal{L}_i$

$$T(u) = id + \frac{1}{u} \sum_{j=1}^{L} \vec{S}_j \cdot \vec{\sigma} + \dots = id + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \dots$$

also from monodromy trace

$$F(u) = 2 - \frac{1}{u^2} \left(J^2 - L \right) + \dots$$

$\leftarrow$ total ang mom. magnitude

Multi-Local Generators

Expand $T(u)$ at $u=\infty$ to higher orders in $1/u$

$$T(u) = \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} - \frac{1}{u^2}\sum_{k=1}^{L}\sum_{j=1}^{k-1}\left(\vec{S}_k\cdot\vec{S}_j\,\mathrm{id} + i\,(\vec{S}_k\times\vec{S}_j)\cdot\vec{\sigma}\right)+\dots$$

$$= \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} - \frac{J^2-L}{2u^2}\,\mathrm{id} + \frac{i}{u^2}\vec{Y}\cdot\vec{\sigma} + \dots$$

$\vec{Y} := \sum_{k=1}^{L}\sum_{j=1}^{k-1} \vec{S}_j\times\vec{S}_k$ is a bi-local generator.

time Evolution $$\frac{d\vec{Y}}{dt} = \sum_{j=1}^{L}\sum_{k=j+1}^{L} \vec{S}_j\times(\vec{k}_k-\vec{k}_{k-1}) - \sum_{k=1}^{L}\sum_{j=1}^{k-1}\vec{S}_k\times(\vec{k}_j-\vec{k}_{j-1})$$

$$= \sum_{j=1}^{L}\vec{S}_j\times(\vec{k}_L-\vec{k}_j-\vec{k}_{j-1}+\vec{k}_0)$$

use identity $(\vec{S}_j + \vec{S}_{j+1}) \times \vec{k}_j = \dots = \vec{S}_{j+1} - \vec{S}_j$

$$\frac{d}{dt}\vec{Y} = (\vec{J} - \vec{S}_L) \times \vec{k}_L - \vec{S}_L + (\vec{J} + \vec{S}_0) \times \vec{k}_0 + \vec{S}_0$$

on closed chain

$$= 2\vec{J} \times \vec{k}_L = \frac{2(\vec{S}_1 \times \vec{S}_L) \times \vec{J}}{1 + \vec{S}_1 \cdot \vec{S}_L}$$

later: build algebra on tower of $\vec{J}, \vec{Y}, \dots$

also for $F(u) = 2 - \frac{1}{u^2}(J^2 - L) - \frac{2}{u^3}\vec{Y} \cdot \vec{J}$

$$\vec{Y} \cdot \vec{J} = \sum_{l>k>j=1}^{L} (\vec{S}_j \times \vec{S}_k) \cdot \vec{S}_l \quad \text{tri-local, conserved}$$

expansion of $T(u), F(u)$ yield multi-local terms.

Chapter 4

Spectral Curves

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4 Spectral Curves

expansions in spectral par $u \in \mathbb{C}$ $\rightsquigarrow$ complex analysis in u

obtain complex geometry: spectral curve ~ represents all conserved data.

Approach: given some state (solution of e.o.m. in terms of $\vec{S}$
transform this to monodromy $T(u)$
analyse thoroughly $T(u)$ as a function of u. (eigensystem!)

4.1 Spectral Curve

have S_i, $S_i(t)$ fixed $\rightsquigarrow$ monodromy $T(u)$
conserved quantities in $F(u)$

Eigenvalues

Lax eq. tells that time ev. of $T(u)$ is iso-spectral

eigenvalues $\tau_a(u)$ $a = 1,2$ are conserved.

$\tau_a(u)$ are given by $F(u)$ as follows: $T(u)$ is 2×2 matrix

$$\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} \Rightarrow \det T(u) = \left(1 + \frac{1}{u^2}\right)^L = \tau_1(u)\cdot\tau_2(u)$$

$$\operatorname{tr} T(u) = F(u) = \tau_1(u) + \tau_2(u)$$

$$\Rightarrow \tau_{1,2}(u) = \frac{1}{2}F(u) \pm \sqrt{\frac{1}{4}F(u)^2 - \left(1+\frac{1}{u^2}\right)^L}$$

polynomial of degree L in $1/u$

Singularities

point $\tilde{u}=0$ is singular $T(u)$ is a polynomial of deg L in $1/u$

L-fold pole of $T(u)$ also $\tau_a(u)$ have L-fold poles at $u=\tilde{u}=0$

$T(u)$ is analytic for $u\neq\tilde{u}=0$

$\Rightarrow$ eigenvalues $\tau_a(u)$ are analytic almost everywhere but not everywhere.

potential square root singularities in $\tau_a(u)$ at radicand $= 0$ points $\hat{u}_j$

$$\frac{1}{4} F(\hat{u}_j)^2 = \left(1 + \frac{1}{\hat{u}_j^2}\right)^L$$

branch points

$\downarrow$

$F(u)$ is pol. deg L in $1/u$: alg eq. of deg $2L$ in $1/u$ $\Rightarrow$ $2L$ solutions in $\mathbb{C}$

these are where $\tau_1(\hat{u}_j) = \tau_2(\hat{u}_j)$ eigenvalues degenerate

two $\hat{u}_j$ are fixed to $\hat{u} = \infty$ due to rel.

$$F(u) = 2 + \frac{0}{u} - \frac{1}{u^2}(J^2 - L) + \dots$$ $\sim$ related to $SO(3)$ sym.

Simple States $L=2$, $L=3$ symmetric state

$L=2$: $\quad s_{1,2}(t) = \left(\pm \tan(\vartheta/2) \cdot e^{-i\omega t} \right)$ $\qquad \omega = \dfrac{2}{\cos\vartheta}$

$$T(u) = \mathrm{id} + \frac{2i}{u}\cos\vartheta\, \sigma^z - \frac{1}{u^2}\begin{pmatrix} \cos(2\vartheta) & e^{+i\omega t}\sin(2\vartheta) \\ -e^{-i\omega t}\sin(2\vartheta) & \cos(2\vartheta) \end{pmatrix}$$

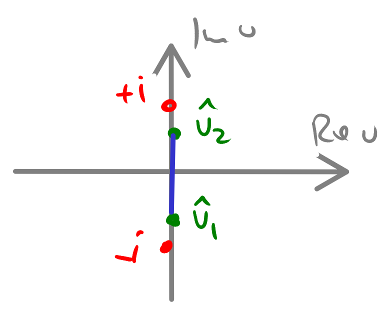
trace $\quad F(u) = \mathrm{tr}\, T(u) = 2 - \dfrac{2}{u^2}\cos(2\vartheta)$

expansion around $u=\infty$ agrees with $so(3)$ symmetry charges $\vec{J} = 2\cos\vartheta\, \vec{e}_z$

$$H = -\log \frac{F(+i)\,F(-i)}{16} = -4\log|\cos\vartheta|.$$

Eigenvalues $\quad \tau_{1,2}(u) = 1 - \dfrac{\cos(2\vartheta)}{u^2} \pm \dfrac{2i\cos(\vartheta)}{u}\sqrt{1 + \dfrac{\sin^2\vartheta}{u^2}}$

$\hat{u}_{1,2} = \mp i \sin\vartheta$



$L=3$; $F(u) = 2 + \frac{3-J^2}{u^2}$ $0 \leq J \leq 3$ $J=1$ singular; focus on $1<J<3$

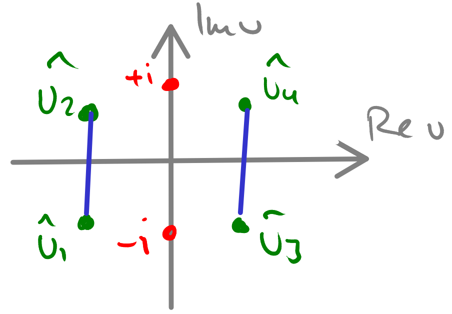
parametrisation for J in terms of (abstract) $\mu \in]0;\pi]$

$$J^2 = 5 - 4\cos\mu$$

branch pt of eigenvalues $\tau_{1,2}(u)$

$$\hat{u}_{1,4} = \pm \frac{e^{-i\mu}}{\sqrt{1-2e^{-i\mu}}}$$

$$\hat{u}_{2,3} = \pm \frac{e^{i\mu}}{\sqrt{1-2e^{i\mu}}} = \hat{u}^*_{1,4}$$



<u>Spectral Curve</u> square root sing $\hat{u}_k$ and neighbourhood in $\mathbb{C}$

Expand $\tau(u)$ around $u = \hat{u}$

$$\tau_{1,2}(u) = \frac{1}{2} F(u) \pm \hat{k} \sqrt{u - \hat{u}} + O(u - \hat{u})$$

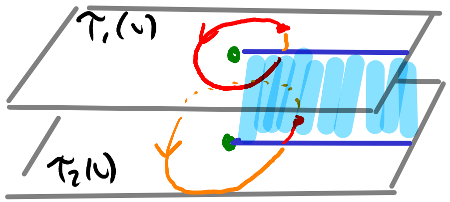
$$\hat{k} = \sqrt{\frac{1}{4} F(\hat{u}) F'(\hat{u}) + \frac{2L}{\hat{u}^3} \left(1 + \frac{1}{\hat{u}^2}\right)^{L-1}}$$

Follow function analytically around $u = \hat{u}$ $\quad u(\sigma) = \hat{u} + \epsilon e^{i\sigma}$

$$\tau_1(u(\sigma)) = \frac{1}{2} F(\hat{u}) + \hat{k} \sqrt{\epsilon}\, e^{i\sigma/2} + O(\epsilon)$$

for $\Delta\sigma = 4\pi$ $\quad \tau_1(u(\sigma + 4\pi)) = \tau_1(u(\sigma))$ $\quad$ 4π - periodic

for $\Delta\sigma = 2\pi$ $\quad \tau_1(u(\sigma + 2\pi)) = \tau_2(u(\sigma))$ $\quad$ 2π rotation permutes eigenvalues!



a 2π rotation about a branch point $\hat{u}$ corresponds to exchange of eigenvalues, but overall spectrum $\{\tau_{1,2}(u)\}$ of $T(u)$ remains.

Introduce concept of spectral curve, Riemann sheet of function τ.

Function $f_a(u)$ with sheets $a=1,2,\ldots$ (eg $\tau_a(u)$ with $a=1,2$) can be viewed as a function $f(z)$ on a covering space (spectral curve) Γ with Riemann sheets labelled by $a=1,2,\ldots$

$$f(z) = f_{a(z)}(u(z))$$

$a(z)=1,2,\ldots$ is the sheet a pos z.
$u(z)\in\mathbb{C}$ is projection of u onto sheet.

$\Rightarrow$ Eigenvalue function is a analytic function on Riemann surface w/o singularities at $u=\hat{u}$

Riemann surface for eigenvalue problem is a so-called spectral curve

Curve: Embedding of $\mathbb{C}$ into $\bar{\mathbb{C}}^2 \ni (u,\tau)$;

$$\Gamma = \{ (u,\tau) \in \bar{\mathbb{C}}^2 ; \ \det (T(u) - \tau) = 0 \}$$

2×2 monodromy: introduce a permutation map $z \to z^\times$

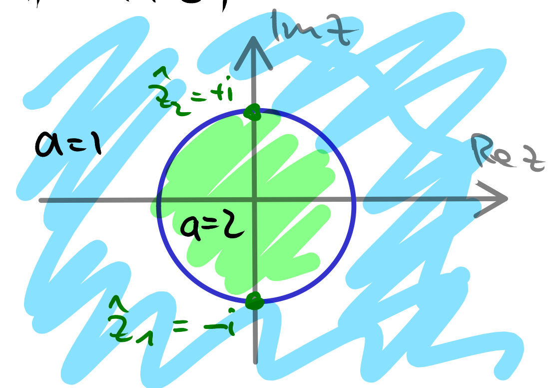
$$u(z^\times) = u(z) \qquad \tau(z^\times) = \frac{\det T(u(z))}{\tau(z)} = F(u(z)) - \tau(z)$$

note: Branch pt $z = \hat{z}_n$ are fixed pts of $z \to z^\times$

Example $L=2$ $\quad u(z) = \frac{1}{2} \sin\vartheta \,(z - 1/z)$

$z^\times = -1/z$ $\qquad \tau(z) = \left(\dfrac{z + 1/z - 2i\cot\vartheta}{z - 1/z} \right)^2$

fixed pt $\hat{z}_{1,2} = \pm i$



General Picture arbitrary length L

discussion of $\tau(u)$ leads to $2L-2$ branch points $\hat{u}_n \rightsquigarrow L-1$ branch cuts

compact Riemann surface with 2 sheets and $L-1$ branch cuts has genus $g = L-2$

$L=2 \sim g=0$ simple, rational functions

$L=3 \sim g=1$ not simple, elliptic functions

$L>3 \sim g>1$ very non trivial, hyperelliptic functions

4.2 Ground State and Excitations

Explore spectral curve for small excitations of the ferromagnetic ground state

Ground State $\quad \vec{S}_k = \vec{e}_z \quad$ Lax transport $\mathcal{L}_k = \mathcal{L}(u) = \mathrm{id} + \frac{i}{u}\sigma^z$

two EV. $(u \pm i)/u$ therefore EV of $T(u) = \mathcal{L}(u)^L$

$$\tau_{1,2}(u) = \frac{(u \pm i)^L}{u^L} \quad \text{has no branch points in } \hat{u}$$

generically expect curve of genus $g = L-2$. consider degeneracy

$$F(u) = \tau_1(u) + \tau_2(u) = \frac{(u+i)^L + (u-i)^L}{u^L}$$

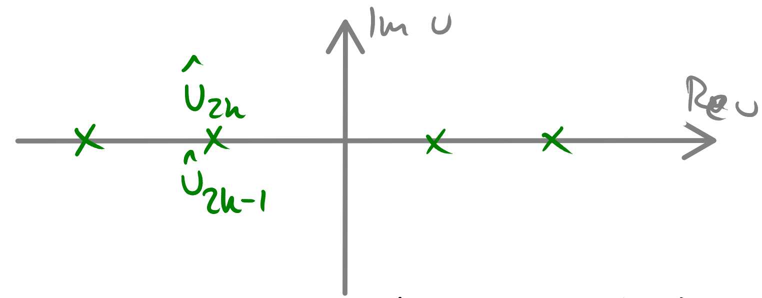
2L potential branch points as zeros of radicand

$$\tau_{1,2}(u) = \frac{1}{2}F(u) \pm \sqrt{\frac{1}{4}F(u)^2 - \frac{(u^2+1)^L}{u^{2L}}} \quad \overset{!}{=} 0$$

There are $2L-2$ double roots

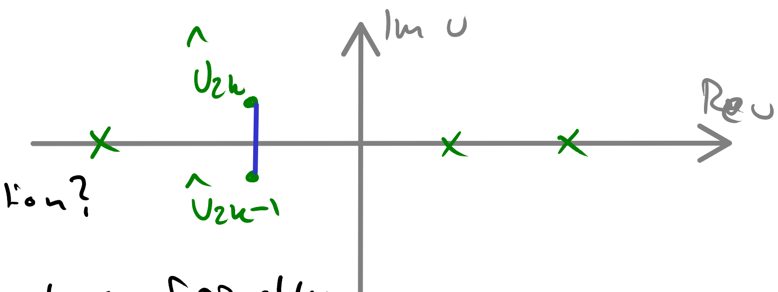
at $\hat{u}_{2k-1} = \hat{u}_{2k} = -\cot\frac{\pi k}{L}$

$k = 1, \ldots, L-1$



Riemann surface of $g = L-2$ degenerates to two disc. sheets.

Single Excitation



pull two branch points apart by a bit

which variation of $F(u)$ induces this deformation?

$$\delta F(u) = -i\epsilon^2 \frac{(u+i)^L - (u-i)^L}{u^L(u-\hat{u}_{2k})}$$

analyse formally

$$F(\hat{u})^2 + 2F(\hat{u})\,\delta F(\hat{u}) + \ldots = \frac{4(\hat{u}^2+1)^L}{\hat{u}^{2L}}$$

on double root splits up $\delta\hat{u}_{2n-1,2n} = \mp \frac{i\epsilon\sqrt{2/L}}{|\sin(\pi n/L)|}$

Relate expansion of (variation of) F at $u=\infty$ to (variation of) angular momentum

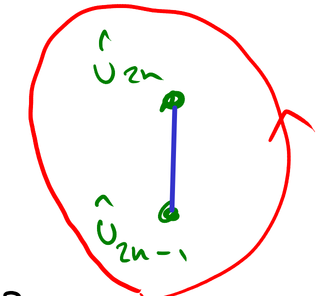
$$\delta F(u) = \frac{2L\epsilon^2}{u^2} + O(1/u^3) \sim \delta J^2 = -2L\epsilon^2 \sim \delta\vec{J} = -\epsilon^2\vec{e}_z$$

$$\vec{J} = J e_z$$

$$\delta H = -\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = \frac{2\epsilon^2}{\hat{u}_{2n}^2+1} = 2\epsilon^2 \sin^2\frac{\pi n}{L} = -2\,\delta J \sin^2\frac{\pi n}{L}$$

$$\delta P = +i\,\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = -\frac{2\epsilon^2\hat{u}_{2n}}{\hat{u}_{2n}^2+1} = \epsilon^2\sin\frac{2\pi n}{L}$$

ϵ^2 is related to action variable $I_n = \epsilon^2 + \dots$

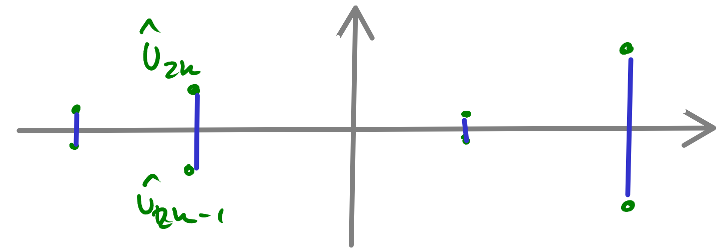


can be obtained from spectral curve

$$I_n = \frac{1}{2\pi} \oint_{\hat{u}_{2n}} d\omega \, \log \tau(\omega) = -\frac{1}{2\pi} \oint_{\hat{u}_{2n}} \frac{d\tau}{\tau} \, \omega = \epsilon^2 + \dots$$

$$\delta H = 2 \delta I_n \sin^2 \frac{\pi n}{L} + \dots \qquad \delta \vec{J} = -\delta I_n \vec{e}_z + \dots \qquad \frac{\partial H}{\partial I_n} = \omega_n \qquad \omega_n = 2 \sin^2 \frac{\pi n}{L}$$

Multiple Excitations

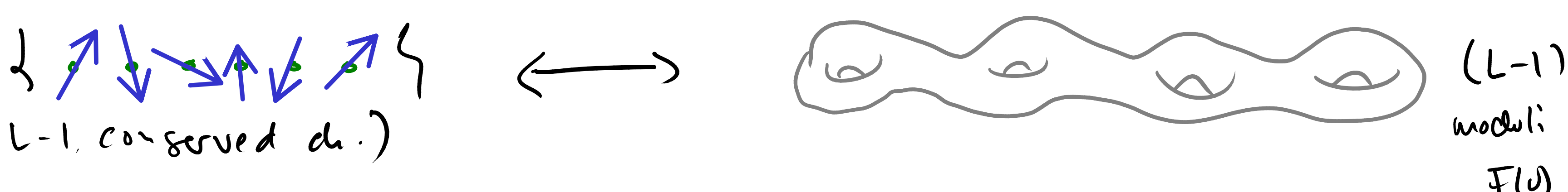


For small excitations of GS can apply perturbation series.

First order is linear! excitations superimpose, in part for δF

L-1 independent modes to split up L-1 double points ~ L-1 excitations of GS.

L-1 d.o.f. to change F 1:1 to conserved qty of an integrable system.
(plus to d.o.f for $\vec{J}/J$ due to SO(3))



4.3 Dynamical Divisor

Singularities

monodromy eigenvector equation

$$\underset{\text{monod.}}{T(u)}\,\psi_a(u) = \underset{\text{e.val}}{\tau_a(u)}\,\underset{\text{eigenvector}}{\psi_a(u)} \qquad a=1,2 \qquad 2\times 2 \text{ matrix}$$

$\psi_a(u)$ is mostly analytic because $T(u)$ is, $\tau_a(u)$ is.

three types of singularities

- singularities in $T(u)$ $\rightsquigarrow$ $\tau(u)$ eg. $u=0$
  $\rightsquigarrow$ no singularities in $\psi_a(u)$ (rescale singularity away in ev. eq.)
- square root branch points in $\tau(u)$ but not in $T(u)$
  eigenvect. eq implies a square root sing in $\psi_a(u)$
- normalisation of $\psi_a(u)$ can introduce/remove singularities in $\psi_a(u)$

Branch Points

Two eigenvectors $\psi_1(u), \psi_2(u)$ degenerate (collinear) at a branch point $u=\hat{u}$.
Same as for eigenvalues $\tau_1(\hat{u}) = \tau_2(\hat{u})$
due to non-diagonalisability of matrix $T(u)$ at $u=\hat{u}$.
Discuss a matrix $T(u)$ that becomes non-diag. at a certain $u=\hat{u}\in\mathbb{C}$

$$T(u) := \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$ A, B, C, D are analytic at $u=\hat{u}$

eigenvalues $$\tau_{1,2}(u) = \frac{1}{2}\left(A(u)+D(u)\right) \pm \sqrt{\frac{1}{4}\left(A(u)-D(u)\right)^2 + B(u)C(u)}$$

$$\frac{1}{4}(\hat{A}-\hat{D})^2 + \hat{B}\hat{C} = 0$$

assumption $\tau_1(\hat{u}) = \tau_2(\hat{u})$ need that radicand $=0$ at $u=\hat{u}$.

expand $\tau_{1,2}(u)$ around $u=\hat{u}$ $\quad \tau_{1,2}(u) = \tau(\hat{u}) \pm \hat{k}\sqrt{u-\hat{u}} + \ldots$

$$\hat{k} = \sqrt{\frac{1}{2}(\hat{A}-\hat{D})(\hat{A}'-\hat{D}') + \hat{B}\hat{C}' + \hat{C}\hat{B}'} \neq 0$$ for a nontriv. square root sing.

assume further that $T(u)$ is diagonalisable at $u = \hat{u}$: $\tau_1 = \tau_2 = \tau$

$$T(\hat{u}) = U(\tau \, id) U^{-1} = \tau \, id \Rightarrow \hat{A} = \hat{D}, \quad \hat{B} = \hat{C} = 0 \quad \text{at } u = \hat{u}$$

implies $\hat{\kappa} = 0$ is contradiction $\Rightarrow$ $T(u)$ is non-diagonalisable.

Consider eigenvector function $\psi_a(u) \equiv \begin{pmatrix} -B(u) \\ A(u) - \tau_a(u) \end{pmatrix}$ both ψ_a coincide at $u = \hat{u}$ b/c τ_a coincide there.

$$\psi_1(\hat{u}) \equiv \psi_2(\hat{u})$$

Example $L=2$

$$\psi(z) = \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$$

Essential for establishing $\psi(z)$ as a function on Riemann surface Γ

$$\psi(z) \equiv \psi_{a(z)}(u(z))$$

$\tau(z), \psi(z)$ are single-valued on Γ.

eigenvect. eq. on Γ: $T(u(z)) \psi(z) = \tau(z) \psi(z)$

Dynamical Divisor

other type of pole in $\psi(z)$. $\psi(z)$ is meromorphic on Γ

$\psi(z)$ might be analytic at any point z through rescaling by scalar fn.

$$\psi(z) \equiv \lambda(z)\, \psi(z)$$

rescale globally only by meromorphic $\lambda(z)$. $\rightsquigarrow$ set of poles not universally defined.

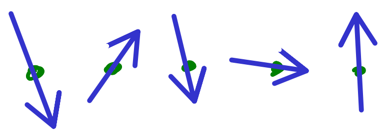
fix this by demanding further normalisation of $\psi(z)$

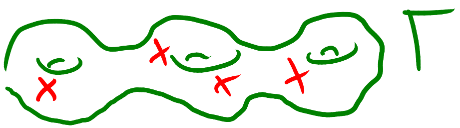
$$v_r \cdot \psi(z) \overset{!}{=} 1 \qquad \text{for some const. vector } v_r$$

this fixes scaling d.o.f. completely. thus fixes set of poles locations $\{\check{z}_k\}$

eg. for $v_r = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \psi(z) = \begin{pmatrix}1\\ \xi(z)\end{pmatrix}$ $\quad \xi(z)$ is stereographic proj. variable for ψ

location of poles of $\xi(z)$ is dynamical divisor for state. $\{\check{z}_k\}$





Argue that dyn. div. consists of precisely $g+1$ points on Γ g: genus of Γ

Consider $f(u) := \big(\underset{\text{normalised}}{\psi_1(u)^T \varepsilon \psi_2(u)} \big)^2 = \big(\xi_1(u) - \xi_2(u) \big)^2$

- $f(u)$ is a meromorphic function of $u \in \bar{\mathbb{C}}$.

  $\psi_{1,2}(u)$ are meromorphic almost everywhere

  around branch points ψ_1 and ψ_2 get interchanged but $f(u)$ returns to old value

  $\Rightarrow$ $f(u)$ has no branch points.

- branch points are zeros of $f(u)$. $f(u) = 0$ if $\psi_1(u)$ is coll to $\psi_2(u)$

  $f(u) \neq 0$ elsewhere because $\psi_1 \psi_2$ span $\mathbb{C}^2$

  square root behaviour implies that zeros of $f(u)$ are single.

  for Γ of genus g there are $2g+2$ branch points. $2g+2$ zeros in $f(u)$.

- $f(u)$ is meromorphic on $\bar{\mathbb{C}}$ has same no of poles as zeros. $2g+2$ poles

  all poles are double $\Rightarrow$ $f(u)$ has $g+1$ double poles $\Rightarrow$ ξ_1 and ξ_2 together have $g+1$ poles

  $\Rightarrow$ $\xi(z)$ has $g+1$ poles

locations of poles $\check{z}_k$ are dynamical $\check{z}_k(t)$ governed by diff. eq.

$$\frac{dT}{dt} = [M, T] \quad \rightsquigarrow \quad \frac{d\psi}{dt} = M\psi + \lambda\psi$$

governs scaling of ψ

through normalisation cond $v_r \cdot \psi = 1$ fix $\lambda = -v_r \cdot M\psi$

ex $L=2$: $\quad v_r = \begin{pmatrix} 1 \\ -1/\xi_r \end{pmatrix} \Rightarrow \psi(z) = \dfrac{1}{1 - i\xi_r^{-1} e^{-i\omega t} z} \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$

Divisor $\quad \check{z}(t) = -i\xi_r e^{i\omega t}$

other ref dir. $v_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \psi(z) = \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$

$\check{z}(t) = \infty \quad$ static $\quad$ information is in coefficient of z at $z = \infty$

<u>Symmetry</u> impact of $SO(3)$ symmetry on eigenvector function $\psi(z)$

superint system, $L-1$ dyn. d.o.f instead of L

lower generic genus from $g = L-1$ to $g = L-2$

always related to point $u = \infty$ double point

$$T(u) = \mathrm{id} + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \dots \qquad T(\infty) = \mathrm{id} \checkmark \to \text{eigenvector is not fixed}$$

at the two points $z = z_\infty$, $z = z_\infty^\times$ (both corr to $u = \infty$) The direction of $\psi(z)$ is fixed through analyticity

$$\tau(z) = 1 \pm \frac{iJ}{u(z)} + \dots \qquad z = z_\infty, z_\infty^\times$$

$$(\vec{J} \cdot \vec{\sigma}) \, \psi(z_\infty) = + J \, \psi(z_\infty)$$

$$(\vec{J} \cdot \vec{\sigma}) \, \psi(z_\infty^\times) = - J \, \psi(z_\infty^\times) \qquad \text{recover } \frac{\vec{J}}{J} = \frac{\psi_\infty^{\times T} \varepsilon \vec{\sigma} \cdot \psi_\infty}{\psi_\infty^{\times T} \varepsilon \psi_\infty}$$

4.4 Construction of Solutions

we may now construct (spectral curve / divisor) / eigensystem functions from scratch.
eigensystem → reconstruct $T(u)$ → reconstruct state / solution.

Spectral Curve

construct $\tau(z)$ on a Riemann surface Γ

characteristic pol. eq. $\tau(z)^2 - F(u(z))\,\tau(z) + \det T(u(z)) = 0$

$F(u)$ is pol of deg L in $1/u$ with leading terms $F(u) = 2 + \frac{0}{u} + \dots$

$\det T(u) = (1 + 1/u^2)^L$ fixed. altogether $L-1$ d.o.f. in curve ~ moduli

char. eq. describes $2L-2$ branch points ⇒ genus $g = L-2$.

Dynamical Divisor

eigenvector function $\psi(z)$

normalised ev. function $\psi(z) = \begin{pmatrix} 1 \\ \xi(z) \end{pmatrix}$

$\xi(z)$ is a meromorphic (rational) function on Γ of degree $g+1$

$\xi(z)$ has $g+3$ moduli (deg. of freedom)

$\sim$ $\underbrace{g+1 \text{ locations of poles}}_{\text{dynamical divisor } \{\check{z}_a\}}$, $\underbrace{1 \text{ location of a zero}, 1 \text{ overall scaling}}_{\text{direction of any mom } \vec{J}/J}$ $\rightarrow$ always fix this to $+\vec{e}_z$

establishes integrability of Heisenberg chain:

- eigenvector function has no more than $g+3 = L+1$ d.o.f.
- $2L$ d.o.f in total
- at least $L-1$ d.o.f. encoded into spectral curve
- all eigensystem d.o.f. encode state fully.

eigenvector function takes the form:

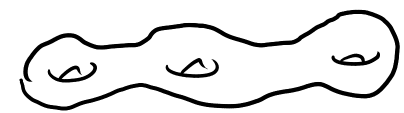
$$\xi(z) = \exp(i\Delta(z) + i\phi)\,\frac{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Delta} + \vec{\Phi})\,\Theta(\vec{\Phi} - \vec{\Delta})}{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Phi})\,\Theta(\vec{\Phi})}.$$

theta-fn Abel map

several functions θ, $\vec{\Omega}$ to be introduced, integral Δ + constants $\vec{\Delta}$

and get moduli $\vec{\Phi}$, ϕ ~ angle variables of solution (2π-periodic)

Complex Analysis on the spectral curve



Riemann surface of genus $g = L-2$ (typically) or $g < L-2$, def by alg. eq. $u \leftrightarrow \tau$

- Meromorphic (rational) functions on Γ: analytic except at isolated poles
- Meromorphic/abelian differentials $f(z)\,dz$ (one-form)
- Holomorphic fn/differentials: have no poles, analytic everywhere.

→ Hol. f^n are just constants.

→ Hol differentials ω_k exist, there are g for genus g

in our case $\omega_k \in \text{span}\{u(z)^{-3}, \dots, u(z)^{-L}\} \dfrac{du(z)}{\tau(z) - \frac{1}{2}F(z)}$

→ mer differentials can be constructed for all desired locations of poles with specified sing. structure $\frac{*}{(z-z_0)^3} + \frac{*}{(z-z_0)^2} + \frac{*}{(z-z_0)} + \text{finite}$

(restriction: overall residue $\stackrel{!}{=} 0$)

→ mer fuctions exist for balanced set of zeros and poles.

$$\text{mer. funct} \underset{\text{integration}}{\overset{\text{diff}}{\rightleftarrows}} \text{mer. diff}$$

two obstacles for integration towards a mer. function:

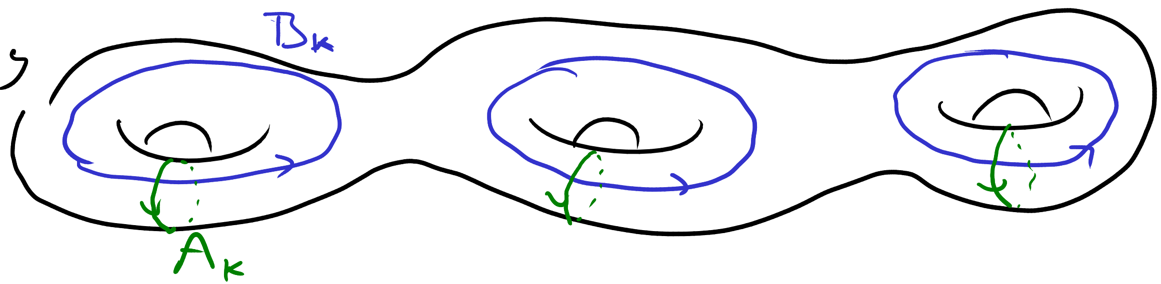
- single pole in diff leads to a log-sing in function → $2\pi i \cdot \text{res}$ monodromy (→ not a pole)
- non-trivial period $\oint_C \phi \neq 0$ → monodromy of function around C.

generically mer. diff integrate to abelian integrals.

some abelian int are in fact mer. functions

Hol. differentials ω_k have no poles $\Rightarrow$ $\int \omega_k$ can have non-trivial monodromy around C

surface genus g
has $2g$
non-triv. cycles



A_k intersects B_k once (in same direction)

ω_k have non-triv periods $\oint_C \omega_k$ for all A_j, B_j

normalise ω_k s.t. $$\oint_{A_j} \omega_k \overset{!}{=} 2\pi \delta_{j,k}$$

then eval B-periods $$\oint_{B_j} \omega_k =: 2\pi T_{j,k}$$

period matrix
- symmetric
- $\operatorname{Im} T$ is pos. def.

hol diff define Abel map

$$\Omega_k(z) := \int^z \omega_k \quad + \text{fixed constants}$$

$\vec{\Omega}$ not a function on Γ but rather

$$\Omega : \tilde{\Gamma} \to \mathbb{C}^g$$

A, B periods define a lattice

$$\Lambda : 2\pi \mathbb{Z}^g + 2\pi \tau \mathbb{Z}^g$$

can view $\vec{\Omega}$ mod Λ as a function on Γ Jacobian of Γ

$$\Omega : \Gamma \to T^g \qquad T^g := \mathbb{C}^g / \Lambda$$

complex g-Torus
real slice serves as part of Liouville torus ($\vec{\phi}$)

finally abelian integrals are normalised s.t. A-periods are zero $\oint_A \phi = 0$ by subtracting lin. comb. of ω_k

Eigenvector Direction

$\xi(z)$ has $g+1$ poles/zeros

but choice $\vec{J} \sim \vec{e}_z$ fixes one zero/pole to z_∞, z_∞^x at $u=\infty$

g dynamical poles/zeros, treat them separately

Introduce theta-function: $\Theta : \mathbb{C}^g \to \mathbb{C}$ defined period matrix

$$\Theta(\vec{x}) := \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left(i\pi \vec{n}^T T \vec{n} + i\vec{x}\cdot\vec{n} \right)$$

three properties:

- $\mathrm{Im}\, T$ is pos.-def makes sum converge fast $\Rightarrow$ $\Theta(\vec{x})$ is entire
- 2π periodicity in all g directions $\Theta(\vec{x}+2\pi\vec{n}) = \Theta(\vec{x})$
- quasi periodicity in directions defined by T

$$\Theta(\vec{x}+2\pi T\vec{n}) = \exp\left(-i\pi \vec{n}^T T\vec{n} - i\vec{x}\cdot\vec{n}\right) \Theta(\vec{x}) \qquad \vec{n}\in\mathbb{Z}^g.$$

θ likes to receive output of Abel map $\vec{\Omega}(z)$

$$f(z;\vec{\phi}) = \theta(\vec{\Omega}(z) + \vec{\phi})$$

entire function on $\tilde{\Gamma}$, • trivial monodromy under A-cycles
• multiplicative mon. under B-cycles.

- $f(z)$ has (typically) g zeros in each cell / on Γ
- location of zero is one-to-one with $\vec{\Phi} \in T^g$

Now consider:

$$\xi(z) = \exp(i\Delta(z) + i\phi)\,\frac{\Theta(\vec\Omega(z) - \vec\Omega(z_\infty) + \vec\Delta + \vec\Phi)\,\Theta(\vec\Phi - \vec\Delta)}{\Theta(\vec\Omega(z) - \vec\Omega(z_\infty) + \vec\Phi)\,\Theta(\vec\Phi)}.$$

$\Delta(z)$ is abelian integral of $d\Delta$ which is defined to have pole at $z = z_\infty$ res $-i$, zero at $z = z_\infty^\times$ res $+i$

$$d\Delta(z) \sim \frac{J\,v(z)^{-2}\,dv}{\tau(z) - \frac{1}{2}F(v(z))} + \text{hol. diff } \omega_u$$

$\vec\Gamma = \pi\vec m + \pi T\vec m'' \in \frac{1}{2}\Lambda$

$\vec m \cdot \vec m''$ is odd!

$\downarrow$ odd characteristic

log sing disappear in $\exp(i\Delta(z)) = \dfrac{\Xi(z, z_\infty)}{\Xi(z, z_\infty^\times)}$

normalise $\oint_{\vec A} d\Delta = 0$; $\oint_{\vec B} d\Delta =: \vec\Delta$

$\Xi(z, z') := \Theta(\vec\Omega(z) - \Omega(z') + \vec\Gamma)$

1 zero at $z = z'$, $g-1$ zeros at pos z indep of z'

$\xi(z)$ is meromorphic!!

Reconstruction

have constructed Γ ← eigenvalue function $\tau(z)$ through an alg eq. $u \sim \tau$, specified by $F(u)$

constructed eigenvector $\xi(z)$ on Γ specified through $\vec{\Phi}$, ϵ (dir of $\vec{J}$)

- reconstruct Lax monodromy $T(u(q))$ through eigensystem.
- know how to construct ζ_j or $\vec{S}_j$ by recursive procedure

d.o.f $\quad 2L$

$g \leq L-2$

$L-1$ dof for $\tau(z)$

$g+1 = L-1$ dof for $\xi(z)$

2 dof for $\vec{J}/J$

- reasonable to work with complexified vars.

Auxiliary Linear Problem

spectral curve

given a solution $\vec{S}_j(t)$; introduce auxiliary vector ($\mathbb{C}^2$) array $\psi_j(z;t)$.

ALP:
$$\psi_{j+1}(z;t) = \mathcal{L}_{j+1}(u(z);t)\,\psi_j(z;t)$$
$$\frac{d}{dt}\psi_j(z;t) = M_j(u(z);t)\,\psi_j(z;t)$$

compatibility between both by means of Lax transport eq.

- linear in ψ ; 2-dim space of solutions eg specify $\psi_0(z;0)$

Furthermore impose eigenvalue property

$$\psi_{j+L}(z;t) = T_j(u(z);t)\,\psi_j(z;t) \stackrel{!}{=} \tau(z)\,\psi_j(z;t)$$

Propose solution:

$$\psi_{j,1}(z;t) = \exp\Big(ij\,(\Pi(z) - \Pi(z_\infty)) + it\,(\Sigma(z) - \Sigma(z_\infty))\Big)$$

$$\cdot\ \frac{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + j\vec{\Pi} + t\vec{\Sigma} + \vec{\Phi}_0)\ \Theta(\vec{\Phi}_0)}{\Theta(j\vec{\Pi} + t\vec{\Sigma} + \vec{\Phi}_0)\ \Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Phi}_0)}$$

$$\psi_{j,2}(z;t) = \exp(i\Delta(z) + i\phi_0)\ \frac{\Theta(\vec{\Omega}(z_\infty^\times) - \vec{\Omega}(z_\infty) + \vec{\Phi}_0)}{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty) + \vec{\Phi}_0)}$$

$$\cdot\ \frac{\Theta(\vec{\Omega}(z) - \vec{\Omega}(z_\infty^\times) + j\vec{\Pi} + t\vec{\Sigma} + \vec{\Phi}_0)}{\Theta(j\vec{\Pi} + t\vec{\Sigma} + \vec{\Phi}_0)}$$

$$\cdot\ \exp\Big(ij\,(\Pi(z) - \Pi(z_\infty^\times)) + it\,(\Sigma(z) - \Sigma(z_\infty^\times))\Big)$$

$\Pi(z)$, $\Sigma(z)$ are abelian integrals (like $\Delta(z)$)
$\uparrow j$ $\uparrow t$ evolution

$d\Pi, d\Sigma$ has sing at $z = z_\pm, z_\pm^\times$ $(u = \pm i)$ and $z = z_0, z_0^\times$ $(u=0)$
reflect zeros/poles of $\mathcal{L}(u)$ $\mathcal{M}(u)$

$d\Pi$ has simple poles at $z = z_+^\times, z_-$ with res $-i$; $z = z_0, z_0^\times$ with res $+i$
$\hookrightarrow$ zeros $\hookrightarrow$ poles of $h_j(u)$

$$\exp(i\Pi(z)) = \frac{\Xi(z,z_-)\,\Xi(z,z_+^\times)}{\Xi(z,z_0)\,\Xi(z,z_0^\times)}$$

$\psi_j(z_+) = s_j$ $\psi_j(z_-) = s_{j+1}$
$\psi_j(z_-^\times) = \varepsilon s_j^*$ $\psi_j(z_+^\times) = \varepsilon s_{j+1}^*$

$d\Sigma$ has double poles at $z_\pm, z_\pm^\times$ without residues
integral $\Sigma(z)$ has single poles, no logs

$$\Sigma(z) = -\tfrac{1}{2}\psi(z,z_+) + \tfrac{1}{2}\psi(z,z_-) + \tfrac{1}{2}\psi(z,z_+^\times) - \tfrac{1}{2}\psi(z,z_-^\times) \qquad \psi(z,z') := \frac{i}{\Xi(z,z')}\frac{\partial\,\Xi(z,z')}{\partial u(z)}$$

A, B periods

A: $\oint_{\vec{A}} d\Pi = \oint_{\vec{A}} d\Sigma = 0$

B: $\oint_{\vec{B}} d\Pi =: \vec{\Pi}$ $\oint_{\vec{B}} d\Sigma =: \vec{\Sigma}$

original spin d.o.f. (stereographic eig) $\xi_j(t) = \dfrac{\varphi_{j,2}(z_+;t)}{\varphi_{j,1}(z_+;t)}$

evaluating for different values of j, t is computationally easy.

Properties of the Solution

- Monodromies of $\psi_j(z;t)$ around A, B cycles
- point $u=\infty$ (total ang momentum) $\psi_j(z_\infty, t) = \begin{pmatrix}1\\0\end{pmatrix}$

$$\psi_j(z;t) \sim u(z) \begin{pmatrix}0\\1\end{pmatrix} \quad \text{at} \quad z = z_\infty^{\times}.$$

- singularities at $u = \pm i$, $u=0$ from $f(u)$, $M(u)$
- periodicity along chain $\psi_{j+L}(z;t) = \tau(z)\, \psi_j(z;t)$

Θ argument must differ by lattice vector

$$\vec{\Pi} = \frac{2\pi \vec{n}}{L} + \frac{2\pi T \vec{n}'}{L} \qquad \tau(z) = e^{iq(z)} \qquad q(z) = L\left(\Pi(z) - \Pi(z_\infty)\right) - \left(\vec{\Omega}(z) - \vec{\Omega}(z_\infty)\right)\vec{n}'$$

$$\oint_{\vec{A}} dq = -i \oint_{\vec{A}} \frac{d\tau}{\tau} = \underset{\stackrel{!}{=}0}{-2\pi \vec{n}'} \qquad \oint_{\vec{B}} dq = 2\pi \vec{n}$$

mode numbers

additional integer $\quad q(z_\infty^\times) = \int_{z_\infty}^{z_\infty^\times} dq = 2\pi n_0$

action vars $\quad \vec{I} = \frac{1}{2\pi i} \oint_{\vec{A}} v\, dq$

, direction eigenvector $\quad \xi_j(z;t) = \dfrac{\psi_{j,2}(z;t)}{\psi_{j,1}(z;t)} = \ldots$ form similar to $\xi(z)$

with $\quad \phi_j(t) = \phi_0 + j\left(\pi(z_\infty) - \pi(z_\infty^\times)\right) + t\left(\Sigma(z_\infty) - \Sigma(z_\infty^\times)\right)$

$$- i \log \frac{\theta(\vec{\Phi}_j(t))\, \theta(\vec{\Phi}_0 - \vec{\Delta})}{\theta(\vec{\Phi}_0)\, \theta(\vec{\Phi}_0(t) - \vec{\Delta})}$$

$\vec{\Phi}_j(t) = \vec{\Phi}_0 + j\vec{\Pi} + t\vec{\Sigma} \quad \leftarrow$ linear evolution on Liouville torus

Jacobian (curve)

Chapter 6

Quantum Spin Chains

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6. Quantum Spin Chains

Class of QM models that display integrability

- form a large class of integrable models
- treated uniformly
- parameters to tune (preserve integrability)
- short chains will be genuine QM models
- long chains can approximate (1+1)-D QFT
- for large quantum numbers approach classical counterparts
- model magnetic materials

classes

nearby spins	ferromagnetic	anti-ferromagnetic
opp. alignment ↑↓	high energy	low energy
equal alig. ↑↑	low energy	high energy

two major models

Ising model : stat mech
lattice of ↑↓ : statistics
Heisenberg chain : QM model
Ham acts on neighbours
dynamics

6.1 Heisenberg Spin Chain

Setup : single spin $|\uparrow\rangle, |\downarrow\rangle \rightsquigarrow$ spin vector space $\mathbb{C}^2 = \mathbb{V}$

Spin chain of length L is L-fold tensor product of spins

Hilbert space $\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_L$ $\quad \dim \mathbb{V}^{\otimes L} = 2^L$

Basis for $\mathbb{V}^{\otimes L}$ is given by pure states $|\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow \ldots \uparrow\rangle$

Hamiltonian op. $H : \mathbb{V}^{\otimes L} \to \mathbb{V}^{\otimes L}$ is homogeneous

$H = \sum_j \mathcal{H}_j \qquad \mathcal{H}_j : \mathbb{V}_j \otimes \mathbb{V}_{j+1} \to \mathbb{V}_j \otimes \mathbb{V}_{j+1}$ nearest neighbour

Pairwise Ham $\mathcal{H} = \lambda_0 \, 1 \otimes 1 + \lambda_x \, \sigma^x \otimes \sigma^x + \lambda_y \, \sigma^y \otimes \sigma^y + \lambda_z \, \sigma^z \otimes \sigma^z$

integrable for all values $\lambda_0, \lambda_{x,y,z}$. Three classes:

- most general one $\lambda_x < \lambda_y < \lambda_z$: called "XYZ" difficult!
- simplifications for $\lambda_x = \lambda_y \neq \lambda_z$ called "XXZ"
- enhanced symmetry for $\lambda_x = \lambda_y = \lambda_z$ called "XXX"

mainly consider XXX $\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z = \frac{1}{2}\lambda$

$$\mathcal{H}_j = \frac{1}{2}\left(1 \otimes 1 - \vec{\sigma} \otimes \vec{\sigma}\right) = \lambda\left(id_{j,j+1} - ex_{j,j+1}\right), \text{ assume } \lambda > 0$$

ferromagnetic reg.

$\lambda = 1 \quad \mathcal{H} = id - ex$

Boundary Conditions

various choices as for classical

→ • finite chains with closed boundaries, periodicity $\mathbb{V}_{L+j} \equiv \mathbb{V}_j$

• finite chain with open boundaries → discrete, finite spectrum

• infinite chains with asymptotic boundaries → continuous spectrum

Symmetries

XXX has $su(2) \simeq so(3)$ symmetry : spin-1/2 rep of $su(2)$

$\vec{S}_j^2 = \frac{3}{4}\hbar^2 \, id_j$ $\quad \vec{S}_j = \frac{1}{2}\hbar \vec{\sigma}_j$ $\quad$ Lie alg $[S_j^a, S_k^b] = i\hbar \delta_{j,k} \varepsilon^{abc} S_j^c$

total ang. mom vect $\vec{J} = \sum_{j=1}^{L} \vec{S}_j = \sum_{j=1}^{L} \frac{1}{2}\hbar \vec{\sigma}_j$ $\quad [\vec{J}, H] \overset{\text{symmetry!}}{=} 0 \rightsquigarrow$ spectrum arranges into spin-j rep of $su(2)$

spectrum of spin reps is predetermined

$L=1 \quad (1/2)$

$L=2 \quad (1) + (0)$

$L=3 \quad (3/2) + 2(1/2) \quad \dots$

Higher Spin and Classical Limit

In order to obtain a classical limit per site, need to make a link between very discrete spins $\uparrow, \downarrow$ and craft spins $\nearrow$

Generalise integrable spin chain to higher spin representations: spins at each site

$$s = 1/2 \quad \mathbb{V} = \mathbb{C}^2 \quad \longrightarrow \quad s \in \tfrac{1}{2}\mathbb{Z}^+ \quad \mathbb{V} = \mathbb{C}^{2s+1}$$

Spin operators $\vec{S}_j$ obeying $[S_j^a, S_k^b] = i\hbar\, \delta_{jk} \varepsilon^{abc} S_j^c$ $\quad [so(3)]$

$$\vec{S}_j^{\,2} = \hbar^2 s(s+1) \quad (\text{total spin op})$$

choose basis aligned with z-axis: $\vec{e}_z \cdot \vec{S}_j$ has ev. $-\hbar s \ldots +\hbar s$ in steps of $\hbar$

* nearest neighbour
* SO(3)

$\Big\}$ only use $f\left(\vec{S}_j \cdot \vec{S}_{j+1}\right) \sim \mathcal{H}_j$

first introduce total spin (not squared)

$$J_{j,k} := \sqrt{(\vec{S}_j + \vec{S}_k)^2 + \tfrac{1}{4}\hbar^2} - \tfrac{1}{2}\hbar$$ spectrum to range between 0 and $2s\hbar$

Integrable Ham turns out to be

$$\mathcal{H}_j = 2\psi(2s+1) - 2\psi(\hbar J_{j,j+1} + 1)$$

ψ is digamma function $\Psi(z) := d \log\Gamma(z)/dz$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \qquad \psi(n+1) = \psi(1) + \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{\text{harmonic series}}$$

SO(3), spins
XXX_s model

Two limiting cases for s: $s = 1/2$, $s \to \infty$

$s = 1/2$ spec J is $\{0, \hbar\}$ $\Psi(2) = \psi(1) + 1$

$$J_{j,k} = \frac{3}{4}\hbar \; id_{j,k} + \frac{1}{4}\hbar \, \vec{\sigma}_j \cdot \vec{\sigma}_k = \frac{1}{2}\hbar \; id_{j,k} + \frac{1}{2}\hbar \; ex_{j,k}$$

$$\mathcal{H}_j = 2 - \frac{2}{\hbar} J_{j,j+1} = id_{j,j+1} - ex_{j,j+1}$$

$s \to \infty$: classical limit $\hbar = \frac{1}{s}$ $\vec{S}_j^{\,2} \to 1$ $\vec{e}_z \cdot \vec{S}_j$ is cont between -1 and $+1$

use large-z behaviour of $\psi(z) = \log z + O(1/z)$

$$\mathcal{H}_j^{qu} \to 2 \log \frac{J_{j,j+1}}{2} = -\log \frac{\vec{J}_{j,j+1}^{\,2}}{4} = -\log \frac{1 + \vec{S}_j \cdot \vec{S}_{j+1}}{2} = \mathcal{H}_j^{cl}$$

6.2 Spectrum of the Closed Chain

Conventional Strategy

2^L states in total

* Enumerate a basis of Hilbert space $\mathbb{V}^{\otimes L}$ $|\downarrow\ldots\downarrow\rangle, |\uparrow\downarrow\ldots\rangle, |\downarrow\uparrow\downarrow\ldots\downarrow\rangle\ldots$
* Evaluate H in this basis forming a $2^L \times 2^L$ matrix (integer coefficients, sparse)
* Solve eigenvalue problem...

can do in practice for $L<10$, $L<20,30$ by computer

~ will find su(2) multiplets of some spins

eg. find one state at $L=6$, $M=3$ up spins of $E=5+\sqrt{13}$

Bethe Equations consider a set of M alg. eq. (Bethe eq)
for M undetermined variables $u_k \in \mathbb{C}$ (Bethe roots):

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad \text{for } k = 1 \dots M$$

Claim: for each eigenstate multiplet with ang. mom. $J = L/2 - M$ of H there is a sol of above B. eq. with $M \leq L/2$ distinct Bethe roots u_k

Energy
$$E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2} \right).$$

eg. $L = 6$, $M = 3$ (su(2) singlet) $u_{1,2} = \pm\sqrt{-\frac{5}{12} + \frac{\sqrt{13}}{6}}$ $u_3 = 0$ $E = 5 + \sqrt{13}$

6.3 Coordinate Bethe Ansatz

Start with an infinite chain and investigate ferromag. vacuum + excitations
impose periodicity on wave function later to obtain spectrum of closed chain.
number M of up-spins is preserved by H

Vacuum State

ferromagnetic vacuum $\quad |0\rangle := |\downarrow\downarrow \dots \downarrow\rangle$

state has energy zero $E=0$ by construction

$$\mathcal{H}_j |0\rangle = \mathrm{id}_{j,j+1} |0\rangle - \mathrm{ex}_{j,j+1} |0\rangle = |0\rangle - |0\rangle = 0.$$

Solves spectral problem for $M=0$ (even at finite L)

Magnon States Flip one spin at pos j

$$|j\rangle := |\downarrow \ldots \downarrow \overset{j}{\uparrow} \downarrow \ldots \downarrow\rangle \qquad j \in \mathbb{Z}$$

Hamiltonian closes on this sector due to conserved $M=1$

Note: Ham is homogeneous along chain, commutes with elementary shift generated by $\exp(iP)$

Eigenstates of $\exp(iP)$ are plane waves

$$|p\rangle := \sum_j e^{ipj} |j\rangle \qquad \text{(Fourier transform of basis } |j\rangle)$$

magnon state with momentum p, note $p \equiv p + 2\pi\mathbb{Z}$

act with H on $|p\rangle$

$$H|p\rangle = \sum_j e^{ipj}\left(\mathcal{H}_{j-1}|j\rangle + \mathcal{H}_j|j\rangle\right)$$

$$= \sum_j e^{ipj}\left(|j\rangle - |j-1\rangle + |j\rangle - |j+1\rangle\right)$$

$$= \sum_j e^{ipj}\left(1 - e^{ip} + 1 - e^{-ip}\right)|j\rangle$$

$$= e(p)\quad |p\rangle$$

magnon dispersion relation

$$e(p) = 2\left(1 - \cos p\right) = 4\sin^2\left(p/2\right)$$

Periodic chain: p is quantised $p = \frac{2\pi n}{L}$ $n = 0, \dots, L-1$ due to $e^{ipL} = 1$

problem solved for $M=1$

Scattering Factor

$M=2$ two spin flips at j, k

$$|j<k\rangle := |\downarrow\ldots\downarrow\overset{j}{\uparrow}\downarrow\ldots\downarrow\overset{k}{\uparrow}\downarrow\ldots\downarrow\rangle \qquad \text{assume } j<k$$

Eigenstates of H within $M=2$ sector.

ansatz: $$|p<q\rangle := \sum_{j<k=-\infty}^{+\infty} e^{ipj+iqk}\,|j<k\rangle$$

Partial eigenstate with total momentum $P=p+q$ (but not precise indiv. momenta)

should have H eigenvalue $E = e(p) + e(q)$

now act with $H - e(p) - e(q)$ on $|p<q\rangle$

$$\Rightarrow \quad (e^{ip+iq} - 2e^{iq} + 1)\sum_{j=-\infty}^{+\infty} |j<j+1\rangle\, e^{i\overbrace{(p+q)}^{P}j}$$

want to compose exact eigenstate from partial eigenstates $|p<q)$

note: $(H-e(p)-e(q))$ acting on $|p<q)$ is symmetric in p,q up to factor

yes: $|q<p)$ yields $\left(e^{ip+iq}-2e^{ip}+1\right)\sum_{j=-\infty}^{+\infty} e^{i(p-q)j}\,|j<j+1)$

combine $|p<q)$, $|q<p)$ s.t. $H-e(p)-e(q)$ annihilates it

exact eigenstate $|p,q\rangle := |p<q\rangle + S(p,q)\,|q<p\rangle$

scattering factor S

$$S(p,q) := -\frac{e^{ip+iq}-2e^{iq}+1}{e^{ip+iq}-2e^{ip}+1}.$$

up to periodicity

$M=2$ is solved!

Factorised Scattering $M=3$ has three magnons and $3!=6$ orderings

(asymptotic regions) ordered partial eigenstates $|p_1<p_2<p_3\rangle$, $|p_2<p_3<p_1\rangle$ ...

Exact eigenstate ansatz:

$$\begin{aligned}|p_1,p_2,p_3\rangle = {} & |p_1<p_2<p_3\rangle + S_{12}S_{13}S_{23}\,|p_3<p_2<p_1\rangle \\ & + S_{12}\,|p_2<p_1<p_3\rangle + S_{12}S_{23}\,|p_3<p_1<p_2\rangle \\ & + S_{23}\,|p_1<p_3<p_2\rangle + S_{12}S_{13}\,|p_2<p_3<p_1\rangle\end{aligned}$$

Conventionally expect:

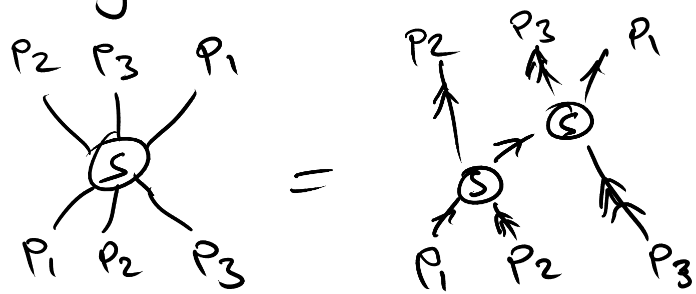
$$(H-E)|p_1 p_2 p_3\rangle \sim \sum_j e^{ip_j}\,|j<j+1<j+2\rangle \quad \text{triple contact term.}$$

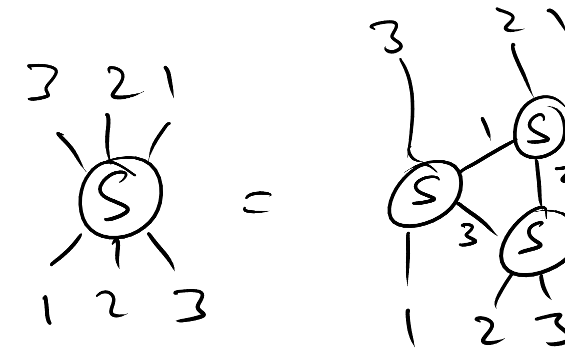
in integrable system find that coefficient = 0

$\sim$ integrability, factorised scattering, no elem. 3 body interactions.

holds not only for $M=3$ but for all $M>3$ as well

Scattering factorises as follows





$S_\pi \qquad \pi \in S_M$

compose scattering factor between any two partial eigenstates

$\rightsquigarrow$ exact magnon eigenstates $(p_1 \dots p_M)$

Solution of the Infinite Chain

construction of generic magnon states

$$|0\rangle = |\downarrow \ldots \downarrow\rangle \qquad E=0$$

$$|p\rangle = \sum_j e^{ipj} |\ldots \overset{j}{\uparrow} \ldots\rangle \qquad E = e(p)$$

$$|p,q\rangle = |p<q\rangle + S(p,q)|q<p\rangle \qquad E = e(p) + e(q)$$

$$|\{p_k\}\rangle = \sum_{\pi \in S_M} S_M |p_{\pi(1)} < \ldots < p_{\pi(M)}\rangle \qquad E = \sum_j e(p_j)$$

p_j are def. mod 2π

Some issues: * $S(p,p) = -1$ ~ Fermi statistics

$|\ldots, p, \ldots, p, \ldots\rangle = 0$ Pauli principle, momenta must be distinct!

* $S(p,0) = 1 = S(0,p)$ $\qquad e(0) = 0$

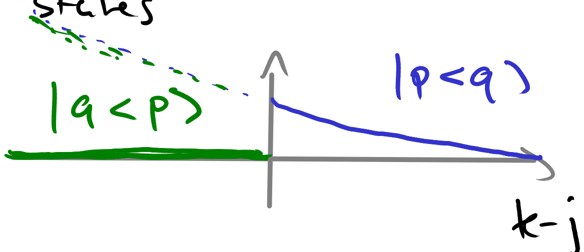
magnon corresponds to $su(2)$ ladder operator; note: bosonic more or less

Bound States ~ normalisable states on infinite chain

we may consider states with complex momenta

for $\operatorname{Im} p \neq 0$ $\; e^{ipj}$ will exponentially decay at either $j \to \infty$, $j \to -\infty$

alone e^{ipj} will never be suitable for normalised states



but in scattering combination may work out:

if $S(p,q) = 0, \infty$

Bound states are compounds of two or more magnons where relative wave function decay exponentially with distance

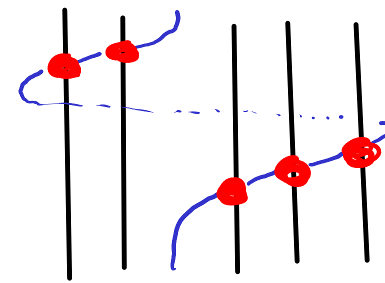
6.4 Bethe Equations

Closed Chains

Can use Coordinate Bethe Ansatz states for closed chains by imposing per. bdy.

" $\langle j, \ldots | \psi \rangle = \langle j+L, \ldots | \psi \rangle$ "

* choose first magnon with momentum p_k
* pick up factor $e^{ip_k L}$



* pick up factor $S(p_k, p_\ell)$ for all $\ell \neq k$

$$\langle j_1, j_2, \ldots, j_M | \psi \rangle = \langle j_2, \ldots, j_M, j_1 + L | \psi \rangle$$

Scat. Fact. $\stackrel{!}{=} 1$

Bethe Equations $\quad e^{ip_k L} \prod_{\substack{\ell=1 \\ \ell \neq k}}^{M} S(p_k, p_\ell) \stackrel{!}{=} 1 \quad$ for all $k = 1 \ldots M$

one eq. for each d.o.f. p_k $\rightsquigarrow$ quantises p_k

$$E = \sum_{k=1}^{M} e(p_k) \qquad P = \sum_{k=1}^{M} p_k \qquad e^{iPL} = 1 \Rightarrow P \in \frac{2\pi}{L} \mathbb{Z}.$$

Rapidities

change variables p_k to u_k

$$p_k = 2\,\mathrm{arccot}(2u_k) \qquad u_k = \tfrac{1}{2}\cot(p_k/2) \qquad e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}$$

$$\rightsquigarrow \quad S(u,v) = \frac{u-v-i}{u-v+i} \qquad e(u) = \frac{i}{u+i/2} - \frac{i}{u-i/2}$$

Bethe Equations in rational form:

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1 \\ l\neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \quad \text{for } k = 1, \dots, M$$

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} \qquad E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right)$$

* u_k are real or complex conj. pairs
* u_k distinct except for $u_k = \infty$
* $u_k = \infty \iff p_k = 0 \iff$ su(2) ladder operator
  Highest weight states have no $u_k = \infty$
* special values $u_k = \pm i/2$. careful!

6.5 Generalisations

Open Chains

$$\mathcal{H} = \sum_{j=1}^{L-1} \mathcal{H}_j$$

need to generalise coord. B.A. to semi-infinite chains with reflection at bdy
ferron.

* vacuum is the same as before

$$|+p\rangle = \sum_k e^{ipk} |k\rangle$$

* one magnon needs to reflect at bdy at first site 1

$$(H - e(p))\,|+p\rangle = (1 - e^{+ip})\,|1\rangle$$

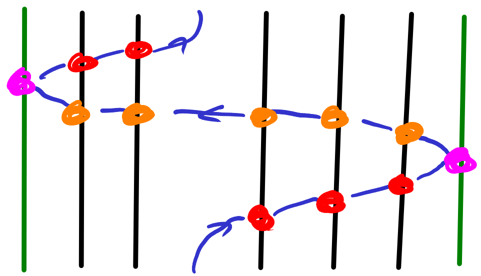
need to compensate

consider magnon state $|-p\rangle$ for which $e(-p) = e(+p)$

$$(H - e(p))\,|-p\rangle = (1 - e^{-ip})\,|1\rangle$$

proper eigenstate $|\,|p|\,\rangle_L = e^{-ip}|+p\rangle + e^{+ip} K_L(+p)|-p\rangle$

$$K_L(+p) = -e^{-2ip}\frac{1-e^{+ip}}{1-e^{-ip}} = e^{-ip}$$ boundary scattering factor



$$\frac{e^{i(L-1)(+p_k)}}{e^{i(L-1)(-p_k)}} \frac{K_R(+p_k)}{K_L(+p_k)} \prod_{\substack{l=1\\l\neq k}}^{M} \frac{S(+p_k,p_l)}{S(-p_k,p_l)} = 1$$

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^{2L} = \prod_{\substack{l=1\\l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i}\,\frac{u_k+u_l+i}{u_k+u_l-i}$$

Higher Spins

$XXX_{1/2} \to XXX_s$ consider $s=1$ $\quad \overset{\downarrow}{|0\rangle}, \overset{\uparrow\downarrow}{|1\rangle}, \overset{\uparrow}{|2\rangle}$

adjust coord. Bethe Ansatz

* ferromagn. vac $\quad |0\rangle = |0\dots0\rangle$
* one magnon: $\quad |p\rangle = \sum_j e^{ipj} |\dots \overset{j}{1} \dots\rangle$
* two magnons:

$$|p<q\rangle = \sum_{jk} e^{ipj+iqk} |\dots \overset{j}{1} \dots \overset{k}{1} \dots\rangle$$

$$|p;2\rangle = \sum_j e^{ipj} |\dots \overset{j}{2} \dots\rangle$$

same sector
expect mixing!

$$(H-E)|p<q\rangle = \sum_j e^{i(p+q)j} \left(* |\dots \overset{j\,j+1}{11} \dots\rangle + * |\dots \overset{j}{2} \dots\rangle \right)$$

$$(H-E)|p;2\rangle = \sum_j e^{ipj} \left(* |\dots 11 \dots\rangle + * |\dots 2 \dots\rangle \right)$$

true eigenstate:

scattering factor (IR) $\quad$ contact term (UV)

$$|p,q\rangle = |p<q\rangle + \underset{\text{scattering factor (IR)}}{S}\,|q<p\rangle + \underset{\text{contact term (UV)}}{C}\,|p=q;2\rangle$$

$$\left(\frac{u_k+i}{u_k-i}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip} = \frac{u+i}{u-i} \qquad e(u) = -p'(u)$$

XXX_s

$$\left(\frac{u_k+is}{u_k-is}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip} = \frac{u+is}{u-is} \qquad e(u) = -p'(u)$$

Chapter 7

Long Spin Chains

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7 Long Quantum Chains

$L \to \infty$

7.1 Magnon Spectrum

ferromag vac $|0\rangle$ has energy $E=0$ for all L as $L \to \infty$

consider finitely many magnons M (M fixed)

Mode Numbers take log of Bethe eq. log ambiguity, n_k is an integer number.

$$iL \log \frac{u_k + i/2}{u_k - i/2} - i \sum_{\substack{\ell=1 \\ \ell \neq k}}^{M} \log \frac{u_k - v_\ell + i}{v_\ell - u_k - i} + 2\pi n_k = 0$$

assume Im log between $-\pi, +\pi$.

"mode numbers" n_k range between $-\frac{1}{2}L$ to $+\frac{1}{2}L$ (Fourier mode numbers)

Single Magnons $M=1$

mode #.

$$iL \log \frac{u+i/2}{u-i/2} + 2\pi n = 0 \Rightarrow u = \frac{1}{2}\cot\frac{\pi n}{L} \qquad p = \frac{2\pi n}{L} \qquad e = 4\sin^2\frac{\pi n}{L}$$

two cases $|n| \ll L$, $|n| \sim L$. assume n finite $\ll L$ b/c lower energies.

approximate $n \ll L \Rightarrow$ $u = \frac{L}{2\pi n}$ $\quad p = \frac{2\pi n}{L} \sim \frac{1}{L}$ $\quad e = \frac{4\pi^2 n^2}{L^2} \sim \frac{1}{L^2}$

Several Magnons $\quad M \ll L$ fixed $\quad n_k \ll L$ fixed $\quad$ distinguish $n_k \neq n_l$ $\quad n_k = n_l$

scattering phase $\quad$ first $n_k \neq n_l$

$$-i \log \frac{u_k - u_l + i}{u_k - u_l - i} \simeq -i \log \frac{L/2\pi n_k - L/2\pi n_l + i}{L/2\pi n_k - L/2\pi n_l - i} \simeq \frac{-2i}{L}\,\frac{1}{n_k - n_l} \quad \text{small!}$$

small scattering phase $\Rightarrow$ excitations interact weakly $\approx$ free magnons for equal mode numbers n_k analyze carefully

$$u_k = \frac{L}{2\pi n} + \delta u_k \qquad \text{for all Bethe roots with mode number } n.$$

$$iL \log \frac{u_k + i/2}{u_k - i/2} = -2\pi n + \frac{4\pi^2 n^2}{L} \delta u_k + O(\delta u_k^2 / L^2)$$

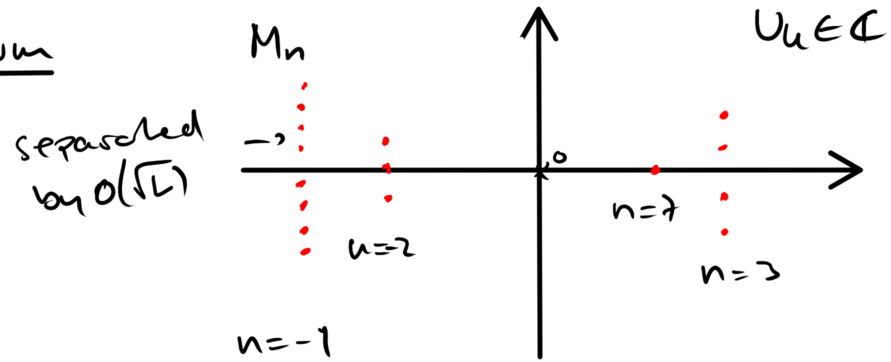
$$-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} = \frac{2}{\delta u_k - \delta u_\ell} + O(1/\delta u_k^2)$$

together $$\frac{4\pi^2 n^2}{L} \delta u_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{M} \frac{2}{\delta u_k - \delta u_\ell} = 0$$

for proper solutions:

needs $\delta u_k \sim \sqrt{\frac{L}{M}} \, \frac{1}{n}$

Magnon Spectrum



$$M = \sum_n M_n \qquad P = \sum_n M_n \frac{2\pi n}{L} \qquad E = \sum_n M_n \frac{4\pi^2 n^2}{L^2}$$

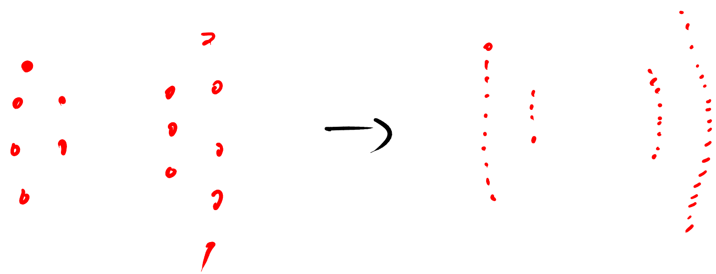
like free bosons on a circle

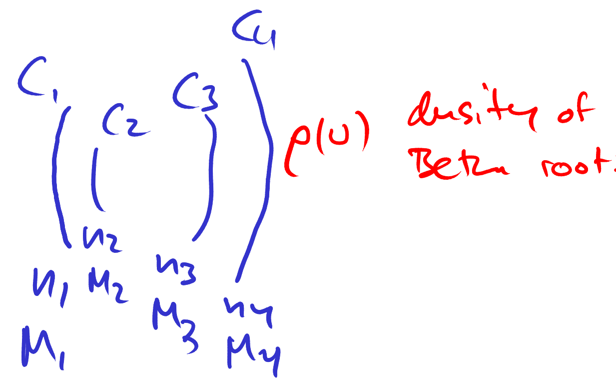
7.2 Ferromagnetic Continuum

what happens beyond $O(1/L^2)$

consider $E \sim \frac{M n^2}{L^2}$ $\Rightarrow$ next higher states at $n \ll L$ finite, $M \sim L \to \infty$

coherent excitations of many magnons





Rescaling of length $L^{org} \to \infty$. target length L

of spectral parameter (Bethe roots) $u^{org} = \frac{L^{org}}{2L} u$

Bethe root states represented by set $D = \bigcup_k D_k$ of contours D_k in $\mathbb{C}$

with a density function $\rho(u)$ defined on them

$$\sum_k \to \frac{L^{org}}{2L} \int du\, \rho(u)$$

apply to Bethe eq.

$$P \int_D \frac{2dv\, \rho(v)}{u - v} - \frac{2L}{u} + 2\pi n_k = 0 \qquad \text{for } u_k \in D_k$$

$$P = \int_D \frac{du\, \rho(u)}{u} \qquad E = \frac{2L}{L^{org}} \int_D \frac{du\, \rho(u)}{u^2} \qquad I_k = \int_{D_k} du\, \rho(u)$$

Spectral curve

Introduce quasi-momentum function $q(u) := \int_D \frac{dv\, \rho(v)}{u-v} + \frac{L}{u}$

$q(u)$ is analytic on $\mathbb{C}$ except for $u=0$ and $u \in D$

on D_k integral eq. $q(u+0)/q(u+i0)$

$$\lim_{\varepsilon \to 0} \left(q(u+\varepsilon) + q(u-\varepsilon) \right) = 2\pi n_k \qquad \text{for } u \in D_k$$

$q(u)$ is one sheet of a two-sheeted fn. $-q(u)$ is other sheet

integral eq. determines continuation of $q(u)$ through a cut

$$q(u+\varepsilon) = 2\pi n_k - q(u-\varepsilon)$$

is continuous up to a shift by $2\pi n_k$ (compare to $e^{iq} = \tau$)

yields spectral curve for cont limit of Heisenberg spin chain.

Hamiltonian Framework how to obtain cont. limit of q. Heis. spin chain

you need coherent states: spin 1/2 state $|S\rangle \in \mathbb{C}^2$

relate $|S\rangle$ to a classical spin vector $\vec{S} \in S^2$ $\quad |S\rangle\langle S| = \frac{1}{2}(\mathrm{id} + \vec{S}\cdot\vec{\sigma})$

obtain $\vec{S}$ as exp. val. $\langle S|\vec{\sigma}|S\rangle = \vec{S}$

for arbit. observable X: $\langle X\rangle_S = \frac{1}{2}\mathrm{tr}\left((1+\vec{S}\cdot\vec{\sigma})X\right)$

for Ham. dens $\mathcal{H}_j$

$$\langle \mathcal{H}_j\rangle_S = \mathrm{tr}_{j,j+1}\left(\frac{1}{4}(1+\vec{S}_j\cdot\vec{\sigma}_j)(1+\vec{S}_{j+1}\cdot\vec{\sigma}_{j+1})(\mathrm{id}_{j,j+1} - \mathrm{ex}_{j,j+1})\right)$$

$$= \frac{1}{4}\mathrm{tr}(1+\vec{S}_j\cdot\vec{\sigma})\,\mathrm{tr}(1+\vec{S}_{j+1}\cdot\vec{\sigma}) - \frac{1}{4}\mathrm{tr}\left((1+\vec{S}_j\cdot\vec{\sigma})(1+\vec{S}_{j+1}\cdot\vec{\sigma})\right)$$

$$= 1 - \frac{1}{2} - \frac{1}{2}\vec{S}_j\cdot\vec{S}_{j+1} = \frac{1}{2}(1-\vec{S}_j\cdot\vec{S}_{j+1})$$

$$H = \frac{1}{2}\sum_j (1-\vec{S}_j\cdot\vec{S}_{j+1})$$

continuum limit of class. discrete model, site spacing $a = \frac{L}{L^{org}} \to 0$
smooth spin function $\vec{S}(x)$ from which $\vec{S}_j$ can be read off as

$$\vec{S}_j = \vec{S}(ja)$$

$$H = \frac{1}{a} \int dx \, \frac{1}{2}\left(1 - \vec{S} \cdot \left(\vec{S} + a\vec{S}' + \frac{1}{2} a^2 \vec{S}'' + \dots\right)\right)$$

$$= -\frac{1}{4} a \int dx \, \vec{S} \cdot \vec{S}'' + \dots = \frac{1}{4} a \int dx \, (\vec{S}')^2$$

Continuous Heisenberg model, integrable

7.3 Anti-Ferromagnetic Ground State

Entanglement

Ferromagnetic: all spins aligned $|\uparrow\uparrow\uparrow\uparrow$

anti-f. would like alternating spins $\uparrow\downarrow\uparrow\downarrow$

su(2) symmetry and for all pairs $|\uparrow\downarrow\rangle$ $|\downarrow\uparrow\rangle$

in QM can do linear comb. such as $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

look for energy eigenstates

spin-1 config $\sim |\uparrow\uparrow\rangle$ || $E=0$

spin-0 config. $E=2$

For more than 2 spins

can combine well spins into spin ½: ferrom.

cannot easily/globally make all pairs to be in spin 0 configuration ⇒ real antiferrom. G.S.

$|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\ldots\rangle$ $|\downarrow\uparrow\downarrow\uparrow\ldots\rangle$ and all other states $|\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\rangle$ $|\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\rangle$

to obtain precise linear comb. is huge comb. problem at $L\to\infty$ need Bethe Eq for progress.

Bethe Equations for anti-ferromagnetic ground state

assume all Bethe roots u_k to be real for maximal energy

$$\log \frac{u+i}{u-i} = i\pi \,\mathrm{sgn}(u) - 2i \arctan u$$

with log branch at neg real axis such that Im log is between $-\pi, +\pi$

Bethe eq.
$$2\arctan(2u_k) - \frac{2}{L}\sum_{l=1}^{M} \arctan(u_k - u_l) + \frac{2\pi \tilde{n}_k}{L} = 0$$

$$\tilde{n}_k = n_k + k - \tfrac{1}{2}M - \tfrac{1}{2} - \tfrac{1}{2}L\,\mathrm{sgn}(u_k)$$ are either integers or half integers.

n_k obey interesting statistics near a.f.g.s. $-\frac{L}{2} < n_k < +\frac{L}{2}$, $n_k = 0$ is $u_k = \infty$ su(2) descendant

* all other mode numbers occupied at most once
* neighbours $n_k \pm 1$ of occupied modes n_k <u>must not</u> be occupied.

Fill as many modes as possible: $M = L/2$ (assume L even)

$L = 14$

$M = 7$

$$\begin{array}{ccccccccccccc} -1 & & -3 & & -5 & & \pm 7 & & +5 & & +3 & & +1 \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array}$$

$$n_k = L\,\theta_{2k>M} - 2k + 1 \qquad \tilde{n}_k = \tfrac{1}{2}M - k + \tfrac{1}{2}$$

Integral Equations at $L \to \infty$

Bethe roots on real axis, introduce a density $\rho(u)$ (w/o rescaling u)

$$\rho(u) = \frac{1}{L}\frac{dk}{du} \qquad k(u) = L\int_{-\infty}^{u} dv\, \rho(v) \quad \text{counting function.}$$

Bethe eq at $L \to \infty$

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v) \arctan(u-v) - 2\pi \int_{-\infty}^{u} dv\, \rho(v) + \tfrac{1}{2}\pi$$

differentiate w.r.t. u $\frac{4}{1+4u^2} - \int \frac{2dv\,\rho(v)}{1+(u-v)^2} - 2\pi\,\rho(u) = 0$

kernel $\frac{1}{1+(u-v)^2}$ has difference form $\Rightarrow$ Fourier transformation

$$\rho(u) = \int \frac{d\theta}{2\pi}\, e^{iu\theta}\, R(\theta)\,, \qquad R(\theta) = \int du\, e^{-iu\theta}\, \rho(u)$$

use following integral $\int \frac{du}{2\pi}\, \frac{2e^{-iu\theta}}{1+u^2} = e^{-|\theta|}$

$$e^{-|\theta|/2} - e^{-|\theta|}\, R(\theta) - R(\theta) = 0 \qquad R(\theta) = \frac{1}{2\cosh(\theta/2)}$$

$$\rho(u) = \frac{1}{2\cosh(\pi u)} \qquad k(u) = \frac{L}{4} + \frac{L}{\pi}\arctan\tanh\left(\tfrac{1}{2}\pi u\right)$$

Ground State Properties

$$E = L \int \frac{4\,du\,\rho(u)}{1+4u^2} = L \int d\theta\, e^{-|\theta|/2} R(\theta) = 2L \log 2 < 2L$$

$P = 0$ or $P = \pi$ established through parity argument

or mode numbers n_n contribute $p_n = 2\pi n_n / L$

$$P = \begin{cases} 0 & \text{for } M = L/2 \text{ even} \\ \pi & \text{for } M = L/2 \text{ odd} \end{cases}$$

Ang. mom $J = L/2 - M = 0$ spin-0 state

7.4 Spinons

Mode number occupation with one gap in perfect sequence. at $k \sim u_0$

consider differences to ground state in $\rho_0(u) + \delta\rho(u) = \rho(u)$

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v) \arctan(u-v) - 2\pi \int_{-\infty}^{u} du\, \rho(u) + \frac{\pi}{4} - \frac{\pi}{2L}\operatorname{sgn}(u-u_0)$$

differentiate wrt. u, consider $1/L$ terms only

$$-\int_{-\infty}^{+\infty} \frac{2dv\, \delta\rho(v)}{1+(u-v)^2} - 2\pi\, \delta\rho(u) - \frac{2\pi}{L}\delta(u-u_0) = 0$$

solved by $\delta R(\theta) = -\dfrac{1}{L}\dfrac{e^{|\theta|/2 - iu_0\theta}}{2\cosh(\theta/2)}$ depends on u_0 variable of gap.

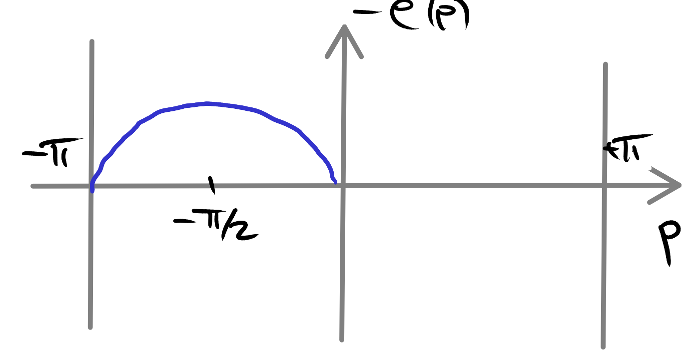
Spinon Properties

energy shift $\quad e(v_0) = -\int \frac{d\theta\, e^{i v_0 \theta}}{2\cosh(\theta/2)} = -\frac{\pi}{\cosh(\pi v_0)}$

momentum shift $\quad p(v_0) = L \int dv\, \delta\rho(v) \left(\pi - 2\arctan(2v)\right)$

$$= 2\arctan\tanh\left(\frac{\pi}{2} v_0\right) - \frac{1}{2}\pi$$

dispersion relation of spinons $\quad e(p) = -\pi \sin(-p)$



only covers $\quad -\pi < p(v_0) < 0$

further curiosity $\quad \delta J^z = L\left(\delta R(0) - \frac{1}{2}\right) = -\frac{1}{2} \leadsto$ spinons have spin $1/2$

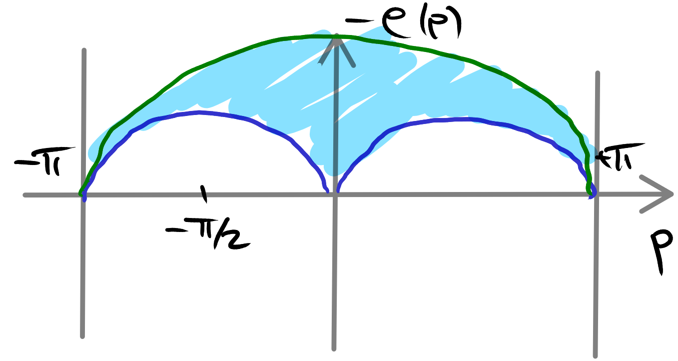
Physical Spinon States

spinon is a collective elementary ex of g.s.

spinons can only come in pairs. spinon ~ 1/2 Bethe root

a pair of spinons has spin = 1

$$P = p_1 + p_2 \qquad E = e(p_1) + e(p_2)$$



consider occupation of mode numbers

-1 -3 -6 $+6$ $+4$ $+1$

● ○ ● ○ ○ ● ○ ● ○ ● ○ ○ ●

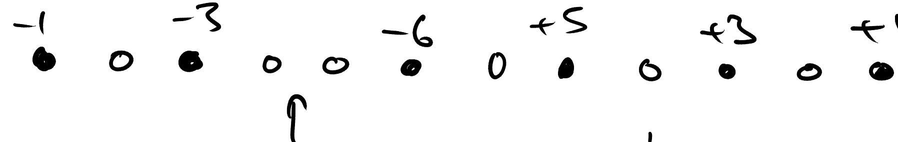
spinon 1   spinon 2

one gap implies another gap

Odd Length

$L = 13$

$M = \frac{L-1}{2} = 6$



1 spinon at least!

odd length requires odd number of spinons

7.5 Spectrum overview

Distribution of energy eigenstates at large $L \to \infty$

ferromag. ground state $E = 0$ $P = 0$ $J = L/2$

finitely many magnons at finite mode numbers

M_n are occupation numbers, bosonic

$$E = \sum_n M_n \frac{4\pi n^2}{L^2} \qquad P = \sum_n M_n \frac{2\pi n}{L} \qquad J^z = L/2 - \sum M_n$$

at infinitely many magnons $M \sim L$ at finite mode numbers

$E \sim 1/L$ $\quad -\pi < P < +\pi$ $\quad J \sim L$

$\vdots$ $\quad \vdots$

spinon states: $E_0 - E = \sum_k \frac{2\pi^2 |u_k|}{L}$ $\quad P = \pi \cancel{?} + \frac{2\pi n}{L}$ $\quad J \leq \sum_k \frac{1}{2}$

antiferromagnetic ground state $E = E_0 = 2L \log 2$ $\quad P \equiv \frac{1}{2}\pi L$ $\quad J = 0$

Chapter 8

Quantum Integrability

duration: 1:38:45

8. Quantum Integrability

8.1 R-Matrix Formalism

SU(N) spin chain

$$S_{ab}^{cd}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\,\delta_a^d\delta_b^c}{(u-v) - i}$$ for flavours $a,b,c,d = 2,\dots,N$

satisfies Yang-Baxter Eq / Factorised scattering

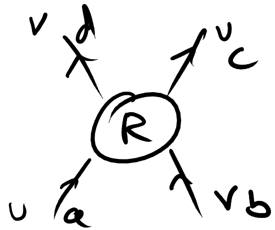
turn into SU(N) fund. R-Matrix

$$R_{ab}^{cd}(u,v) = \frac{(u-v)\,\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u-v+i}$$ $a,b,c,d = 1,\dots,N$ for SU(N) fund. rep

Rank-2 tensor operator $R : \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}$ $\quad \mathbb{V} = \mathbb{C}^N, \quad u,v\in\mathbb{C}$

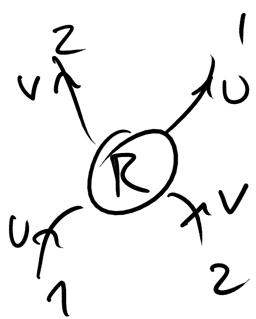
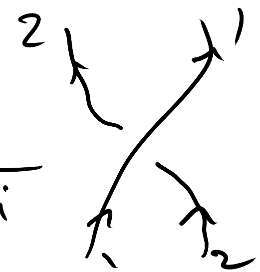
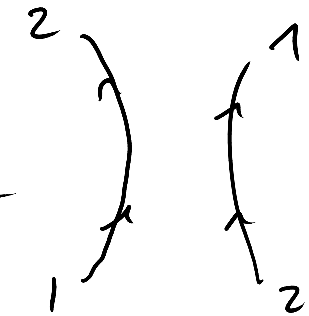
$$R(u,v) = \frac{(u-v)\,\mathrm{id} + i\,\mathrm{ex}}{u-v+i}$$

spectral parameters $= (E^a\otimes E^b)\,R_{ab}^{cd}\,(E_c\otimes E_d)$

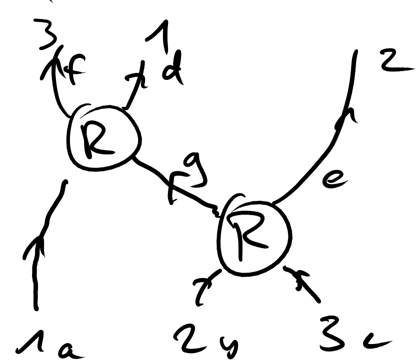


$R_{12} = R_{12}(u_1, u_2)$

associated to spaces $\mathbb{V}_1, \mathbb{V}_2$

 $= \dfrac{u-v}{u-v+i}$  $+ \dfrac{i}{u-v+i}$ 

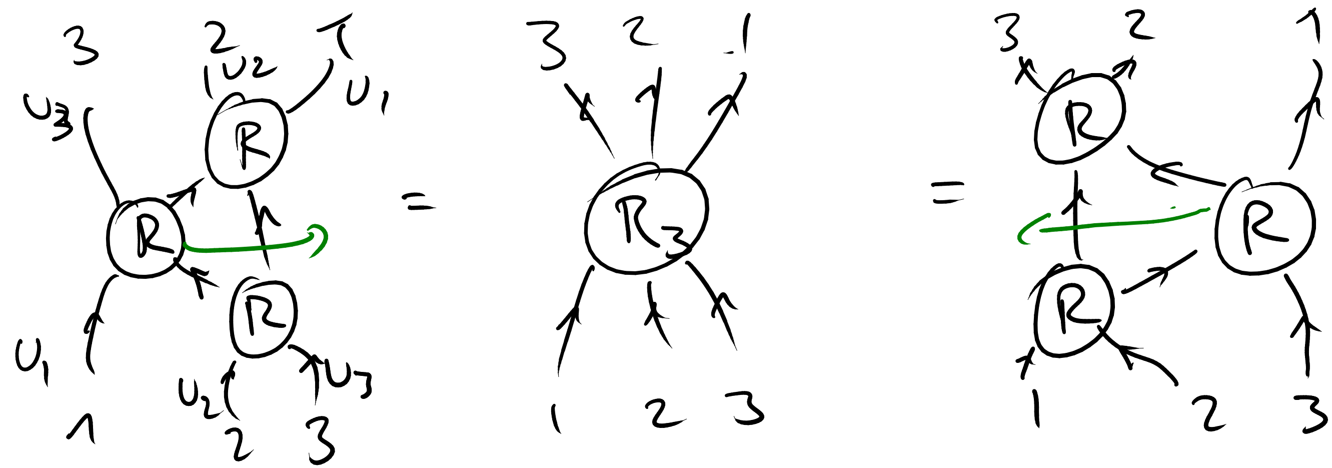
write composite objects (operator products) in components

$R_{13}R_{23} =$  $= R^{df}_{ag}(u_1,u_3)\, R^{eg}_{bc}(u_2,u_3)$

Properties of R-Matrices   Yang-Baxter Equation

$$R_{12}(u_1,u_2)\,R_{13}(u_1,u_3)\,R_{23}(u_2,u_3)$$
$$= R_{23}(u_2,u_3)\,R_{13}(u_1,u_3)\,R_{12}(u_1,u_2)$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



Inverse property $R_{21} R_{12} = id_{12}$

here: exact

elsewhere: up to a factor

$$R_{21} := R_{21}(v,u)$$
$$= ex_{12} \, R_{12}(v,u) \, ex_{12}$$
$$= \frac{(u-v)\, id - i\, ex}{u-v-i}$$

$$R_{12} = \frac{(u-v)\, id + i\, ex}{u-v+i}$$

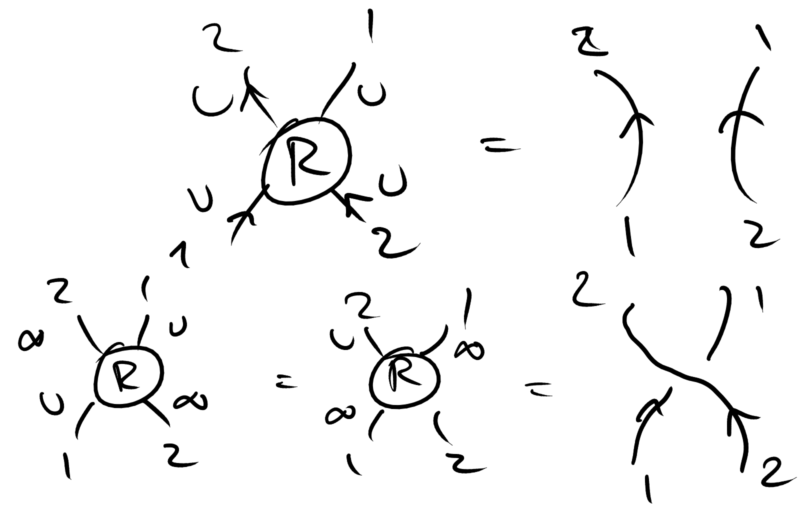
two properties $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, $R_{21}R_{12} = id$ realise perm. group S_*

further properties for specific R

$R(u,u) = ex$

$R(u,\infty) = R(\infty,u) = id$

$\sim SU(N)$ symmetry



8.2 Charges

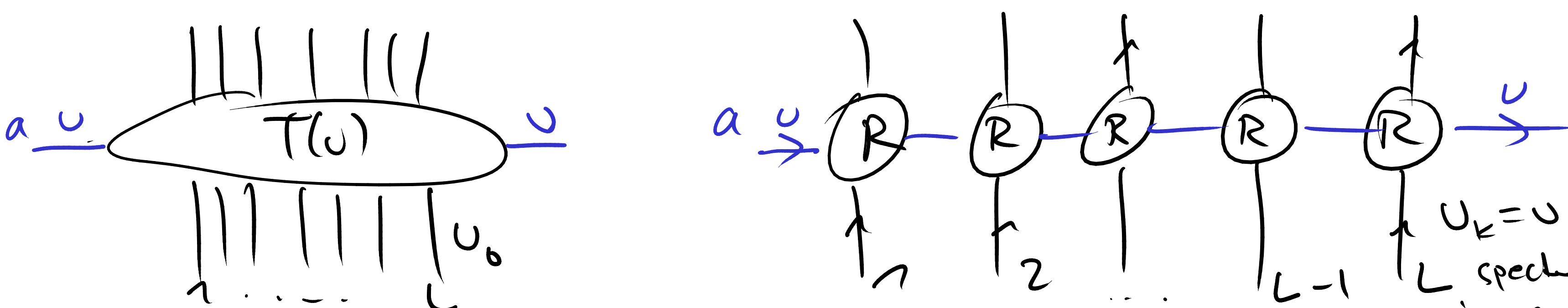
Monodromy and Traces

Lax transport → quantum operator $\quad \mathcal{L}(u) \to R_a(u-u_0)$

Monodromy $\quad T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdot \ldots \cdot R_{a,2} \cdot R_{a,1}$

$\mathbb{V}_a$ $\quad a, u \quad$ is auxiliary space

$\mathbb{V}_k$ $\quad u_k = u_0 \quad$ is quantum spin space at site k $\quad \mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_L$ is Hilb. spc.

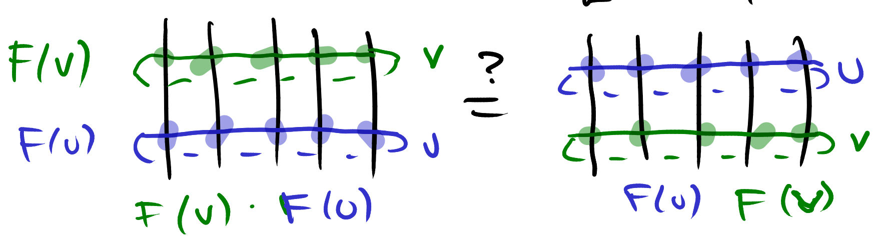


$u_k = u_0$ is spectral par in a hom. spin chain.

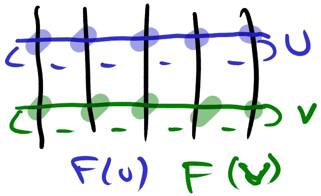
$u_0 = 0$

monodromy trace $F(u) = \mathrm{tr}_a \, T_a(u)$ for closed boundaries

commutation relation $[F(u), F(v)] = 0$ at all $u, v \in \mathbb{C}$



Proof:



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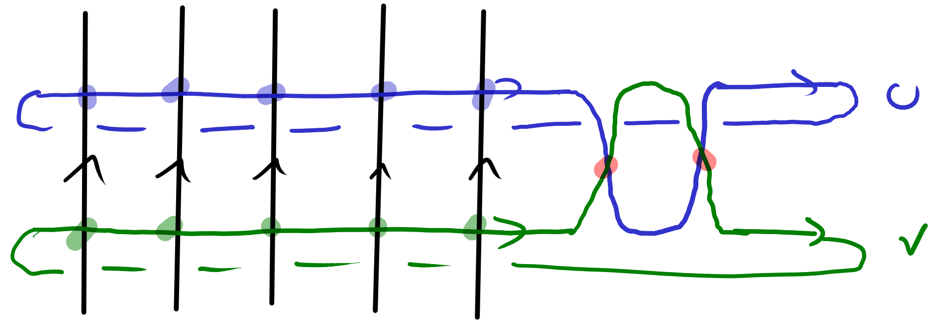

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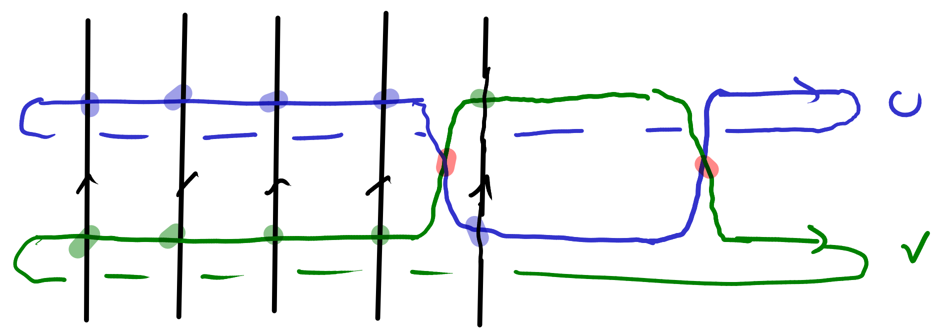

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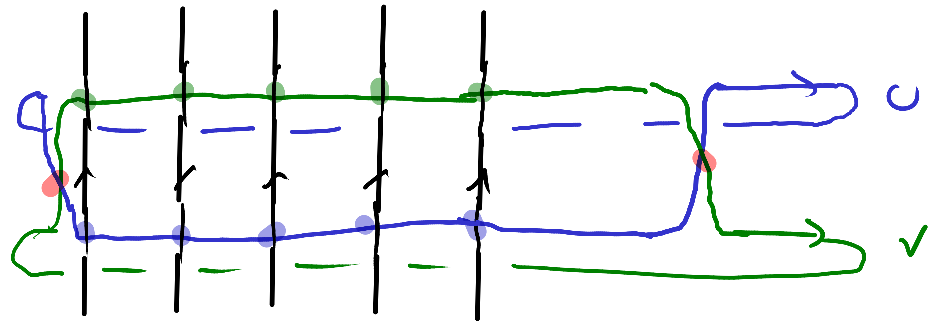

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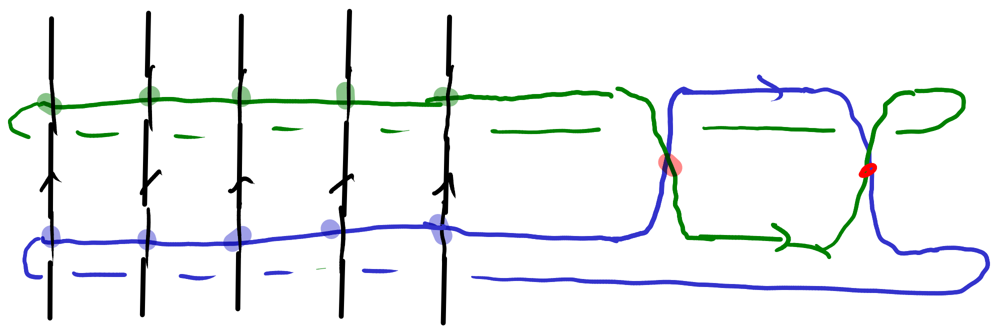

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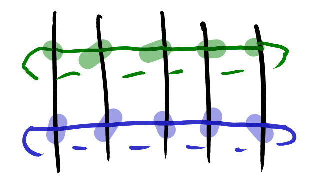


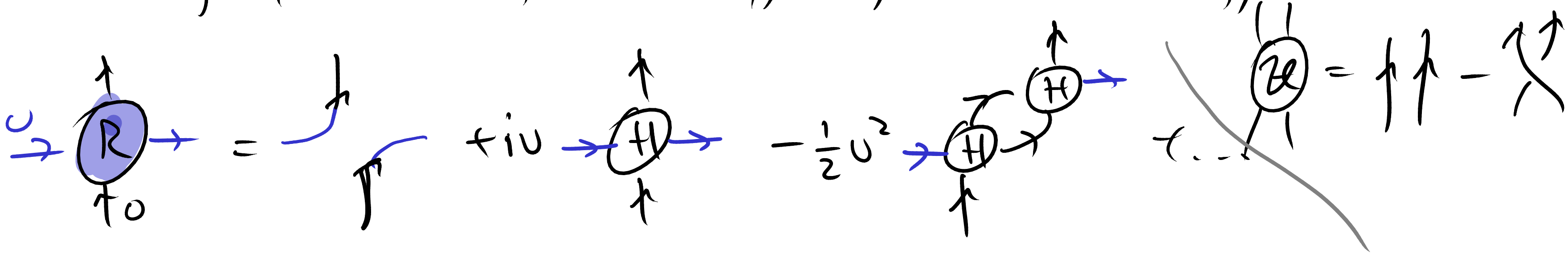
$F(v) \cdot F(u)$

Local Charges

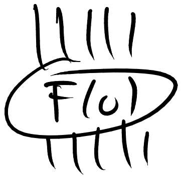
local charges are associated to point $u=0$

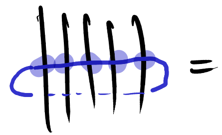
local Ham. density $\mathcal{H}_{k,l} = \mathrm{id}_{k,l} - \mathrm{ex}_{k,l}$

$$R_{a,j}(u,0) = \mathrm{ex}_{a,j} + iu\,\mathrm{ex}_{a,j}\,H_{a,j} - \tfrac{1}{2}u^2\,\mathrm{ex}_{a,j}\,H_{a,j}^2 + \dots$$



Expand monodromy trace $F(u)$ at $u=0$



$=$

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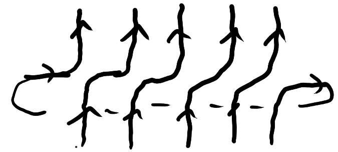
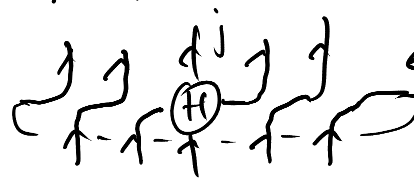
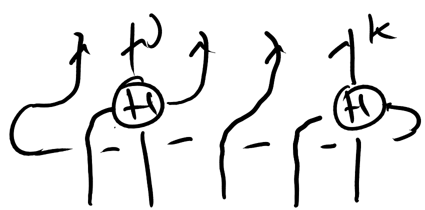
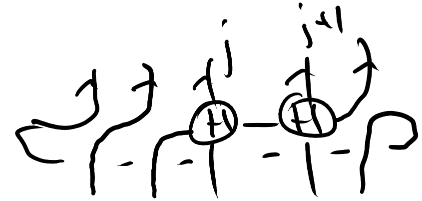

$\leftarrow \exp(iP)$

$+\, iu \sum_{j=1}^{L}$


$\leftarrow iu \exp(iP)\, H$

$-u^2 \sum_{\substack{j<k=1 \\ |j-k|>1}}^{L}$


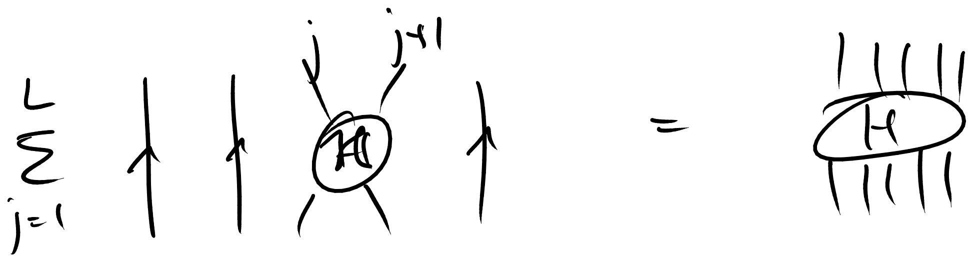
$-\, u^2 \sum_{j=1}^{L}$


$-\frac{1}{2} \sum_{j=1}^{L}$

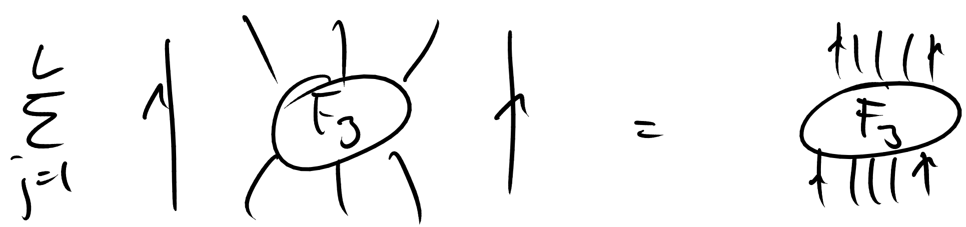

$\uparrow$ almost $\frac{1}{2}\exp(iP)\, H^2$

$-\frac{1}{2} u^2 \exp(iP)\, H^2 + i u^2 \exp(iP)\, F_3$

cyclic shift operator



local hom. N.N Hamiltonian



$$\mathcal{F}_{3,j} = \frac{i}{2}\left[\mathcal{H}_{j+1}, \mathcal{H}_j\right]$$

altogether expansion as: $F(u) = \exp(iP)\exp(iuH + iu^2 F_3 + \ldots)$

$$= \exp(iP + iuH + iu^2 F_3 + \ldots)$$

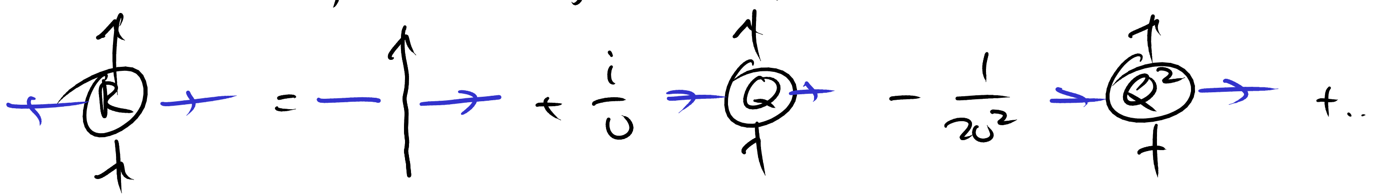
local operators

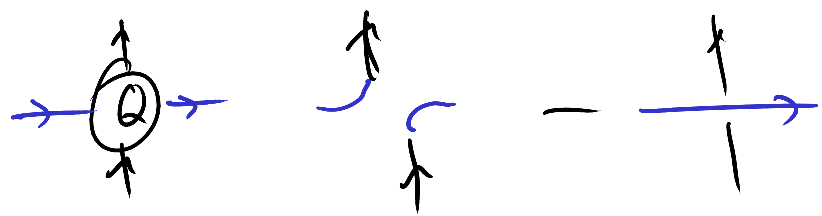
found/constructed local conserved commuting charges F_s $[F_s, F_r] = 0$

Multi-local Charges   expansion at $u = \infty$

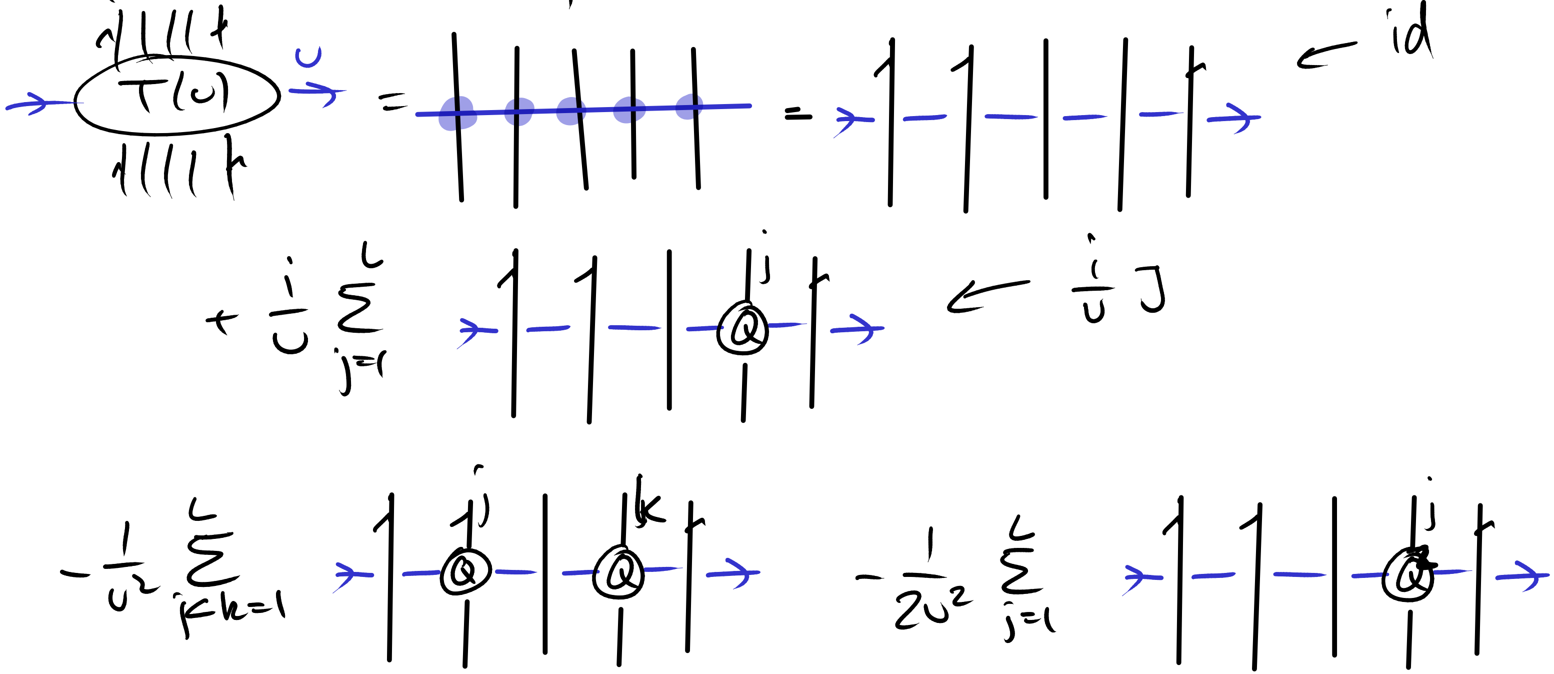
$$R_{a,j}(u,0) = id_{a,j} + \frac{i}{u} Q_{a,j} - \frac{1}{2u^2} Q_{a,j}^2 + \dots$$

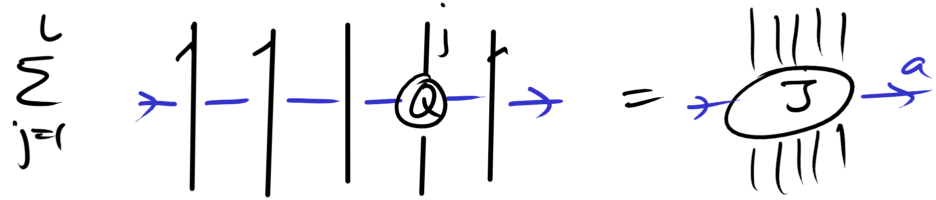
$$Q_{a,j} := ex_{a,j} - id_{a,j}$$





expand monodromy matrix $T_a(u)$ at $u=\infty$



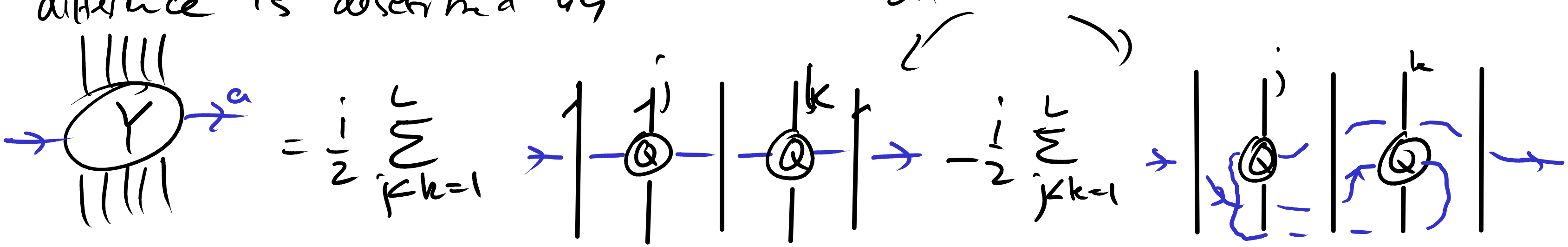


SU(N) symmetry generator

remaining term at $O(1/u^2)$ is almost $\left(\frac{i}{u}\right)^2 J^2$

difference is described by

bilocal



$$T_a(u) = \exp\Big(\underbrace{\frac{i}{u} J_a}_{\text{local}} + \underbrace{\frac{i}{u^2} Y_a}_{\text{bilocal}} + \dots\Big)$$

extends SU(N) symmetry to ∞/L "copies"

⇝ Yangian algebra Y(SU(N))