Introduction to Integrability

Lecture Slides, Chapter 9

ETH Zurich, 2023 HS

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The current version of this work can be found at: `http://people.phys.ethz.ch/~nbeisert/lectures/`.

9. Quantum Algebra

9.1 Lie Algebra

Lie algebra g is vector space with Lie brackets $[,]$ as prod

$$[\cdot,\cdot]: g \times g \to g$$

• bilinear, • anti-symmetric • satisfy Jacobi-id

Repr of g: $\rho: g \to \mathrm{End}(V)$ (linear map on V)

such that Lie algebra is respected as a commutator of maps

$$[\rho(a), \rho(b)] = \rho([a,b]) \qquad a,b \in g.$$

Introduce a (imaginary) basis $J^a \in ig$:

$$[J^a, J^b] = i f^{ab}{}_c J^c \qquad f^{ab}{}_c \text{ structure const. for } g.$$

Invariant quad. form

$$M = c_{ab} J^a \otimes J^b$$

inverse of Cartan-Killing form

$$k(a,b) = \mathrm{tr}\, \rho_{ad}(a)\, \rho_{ad}(b) \qquad c^{ab} \sim k(J^a, J^b)$$

For $g = su(2) = so(3)$

$$c_{ab} = c^{ab} = \delta_{ab} \qquad f^{ab}{}_c = \varepsilon^{abc}$$

Loop Algebras

Can encode dep. on spectral par u into alg:

loop algebra $g[z, z^{-1}]$ is spanned by elements

$$J^a_n := z^n J^a \quad \text{where } n \in \mathbb{Z} \quad J^a \in g \text{ span } g.$$

n: loop level of a generator J^a_n.

loop alg is a Lie alg with

$$[J^a_m, J^b_n] = i f^{ab}{}_c J^c_{m+n}$$

$$[z^m J^a, z^n J^b] = z^{n+m} i f^{ab}{}_c J^c$$

Invariant quad form(s)

$$M_m = \sum_{k=-\infty}^{+\infty} c_{ab} J^a_k \otimes J^b_{m-k}$$

half loop algebras: only pos / non-neg levels

$$z\,g[z] \;/\; g[z]$$

evaluation repr.: given a rep ρ of g on $\mathbb{V}$ def

$$\rho_z : g[z, z^{-1}] \to \mathrm{End}(\mathbb{V})$$

$$\rho_z(J^a_n) := z^n \rho(J^a) \qquad z \in \mathbb{C} \text{ eval. par.}$$

eval. rep are useful for integrability due to enhanced irreducibility

two eval. rep $\rho'_{z'}, \rho''_{z''}$

$$\rho_{z',z''} = \rho'_{z'} \otimes 1 + 1 \otimes \rho''_{z''} \quad \text{is irreducible if } \rho'_{z'}, \rho''_{z''} \text{ are and } z' \neq z''$$

9.2 Classical Integrability

classical r-matrix r fits well into framework of Lie bialgebras.

Lie Bialgebra

Lie algebra g whose dual g^* is also a Lie algebra such that the two Lie algebra structures are compatible

dual of Lie brackets: $\mu^*: g^* \to g^* \otimes g^*$
such that for all $a, b \in g$, $c^* \in g^*$

$$c^*([a,b]) = \mu^*(c^*)(a \otimes b)$$

dual of dual Lie bracket is called Lie cobracket δ

$$\delta: g \to g \otimes g$$

- must be antisymmetric
  $\delta(a) \in g \wedge g$

dual Jacobi id $(1 + P_{12}P_{23} + P_{23}P_{12})(\delta \otimes 1)\delta(c) = 0$

Compatibility between $\mathfrak{g}, \mathfrak{g}^*$ / $[\,,\,]$, δ

$$\delta([a,b]) = [a, \delta(b)] + [\delta(a), b]$$

$$[a, b\otimes c] := [a,b]\otimes c + b \otimes [a,c].$$ as used for inv quad form M

Classical r-Matrix

A class r-matrix $r \in \mathfrak{g}\otimes\mathfrak{g}$ such that

$$\delta(a) = [r, a].$$

- anti-sym of δ implies that $r + P(r)$ is inv. elem. for $\mathfrak{g}$
- dual Jacobi id. requires that

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] \in \mathfrak{g}^{\otimes 3}$$ is inv. cubic elem. for $\mathfrak{g}$

$\Rightarrow$ $\mathfrak{g}$ is called coboundary ($[[r,r]] = 0 \Rightarrow \mathfrak{g}$ quasi-triangular)

Example

$$r(u,v) = \frac{-2\, c_{ab}\, J^a \otimes J^b}{u-v} = \frac{-2M}{u-v}$$

compare to loop algebra basis with $J^a_n = u^n J^a$

Expand for $|u| \gg |v|$

want to express $r \in g[u,u^{-1}] \otimes g[v,v^{-1}]$

$$r = -2 \sum_{n=0}^{\infty} \frac{v^n}{u^{n+1}} M = -2 \sum_{n=0}^{\infty} c_{ab}\, J^a_{-n-1} \otimes J^b_n$$

r-matrix satisfies classical Yang-Baxter eq $[[r,r]]=0$

symmetric part of expanded r:

$$r + P(r) = -2 \sum_{n=-\infty}^{+\infty} c_{ab}\, J^a_{-n-1} \otimes J^b_n = -2M_{-1}$$

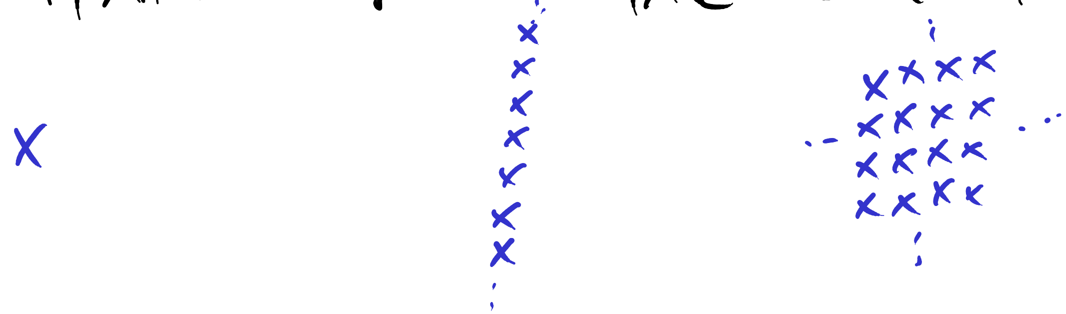
quad inv. form of loop alg. at level -1

Classification and construction

Parametric solutions to class. YBE (difference form) have been classified by Belavin + Drinfeld:

Three distinct types (related to the pattern of poles in $\mathbb{C}$)

rational / XXX   trigonometric / XXZ   elliptic / XYZ



Towards construction useful to make sample $r \in u^{-1} g[u^{-1}] \otimes g[v]$

half loop algebras $u^{-1} g[u^{-1}]$ and $g[u]$ are related by conjugation w.r.t. quad form Ψ_{-1}:

full $g[u,u^{-1}]$ is classical double $dg[u] \to$ bialgebra structure on $g[u,u^{-1}]$ + classical r-matrix

9.3 Quantum Algebras

Enveloping Algebra

Put together Lie algebra g, corresponding Lie group G as well as products of all of their elements. $\to U(g)$

Define first tensor algebra $T(g)$: arbitrary polynomials in elements of g with respecting order of letters in words.

$$J^a J^b J^c \neq J^a J^c J^b \quad \text{two indep. monomials}$$

Env. algebra $U(g)$ is obtained by identifying Lie brackets with commutators:
$$J^a J^b - J^b J^a = [J^a, J^b] = i f^{ab}{}_c J^c$$
(should hold with any polynomial $X \dots Y = X \dots Y$)

$$U(g) = T(g) / \mathrm{span}(J^a J^b - J^b J^a - i f^{ab}{}_c J^c).$$

Why $U(g)$ in quantum physics?

- Lie algebra $g \subset U(g)$ as $g = \mathrm{span}(J^a)$
- can express products of Lie generators $J^a J^b$ or $J^b J^a$ etc. while respecting Lie alg structure
- Lie group $G \subset U(g)$ as $\{\exp(a) ; a \in g\}$
- Tensor products are naturally defined
- We can do non-trivial deformations of $U(g)$ for integr. syst.

Hopf Algebra

$U(g)$ has a natural Hopf algebra struct.

Hopf alg: biunital, biassociative, bialgebra with antipode consider some Hopf alg A over field $\mathbb{k}$

- Product μ, coproduct Δ are $\mathbb{k}$-linear (co)associative maps

$$\mu : A \otimes A \to A \qquad \Delta : A \to A \otimes A$$

bialgebra: compatibility between $\mu \Leftrightarrow \Delta$

$$\Delta(\mu(X \otimes Y)) = (\mu_{13} \otimes \mu_{24})(\Delta(X) \otimes \Delta(Y)).$$

Note: needed for consistency of tensor prod. representations.

$$\rho_{12}(X) := (\rho_1 \otimes \rho_2)(\Delta(X)).$$

- unit ϵ and counit η

$$\epsilon : \mathbb{k} \to A \qquad \eta : A \to \mathbb{k}$$

compatibility: $\mu(\epsilon(a) \otimes X) = aX$, $\eta_1(\Delta(X)) = X$

- antipode $\Sigma : A \to A$ satisfying

$$\mu(\Sigma, 1(\Delta(X))) = \epsilon(\eta(X))$$

if Σ exists it is unique.

Σ is anti-homomorphism of alg / coalg.

$$\mu(\Sigma(X) \otimes \Sigma(Y)) = \Sigma(\mu(Y \otimes X))$$

$$\Delta(\Sigma(X)) = (\Sigma \otimes \Sigma)(\tilde{\Delta}(X))$$

opposite coproduct $\tilde{\Delta}(X) = P \circ \Delta(X)$

Σ incorporates negative / inverse of elements of A

<u>Example</u> for $A = U(g)$

$\mu(X \otimes Y) = XY$ (modulo Lie brackets identification)

coproduct:

$$\Delta(1) = 1 \otimes 1 \qquad \Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a \quad J^a \in g$$

$$\Delta(\exp(a)) = \exp(a) \otimes \exp(a) \qquad \exp(a) \in G$$

(L-1)-fold coproduct act on $A^{\otimes L}$

$$\Delta^{L-1}(1) = 1 \qquad \Delta^{L-1}(J^a) = \sum_{k=1}^{L} J^a_k$$

unit, counit $\quad \epsilon(1) = 1, \quad \eta(1) = 1, \quad \eta(J^a) = 0 \quad a \in g$

antipode: $\quad \Sigma(1) = 1 \qquad \Sigma(J^a) = -J^a, \; \Sigma(\exp(a)) = \exp(-a)$

Universal R-Matrix

introduce univ R-Matrix $\quad R \in A \otimes A$

note R matrices of integr sys. are typically repr. $(\rho_1 \otimes \rho_2) R$ rank-2 tensor operators on $V_1 \otimes V_2$

R relates $\Delta(x)$ with $\tilde{\Delta}(x)$ for any x

$$R \Delta(x) = \tilde{\Delta}(x) R \qquad \tilde{\Delta}(x) = R \Delta(x) R^{-1}$$

coproduct and opp. coproduct are not (necessarily) the same (no cocommutativity) but they are related by similarity transformation R

$\Rightarrow$ quasi-cocommutativity

ordering of factors in a tensor product matters only in terms of basis

Quasi-triangularity

$$\Delta_1(R) = R_{13} R_{23}$$
$$\Delta_2(R) = R_{13} R_{12}$$

imply the Yang-Baxter equation

$$R_{12}(R_{13} \; R_{23}) = R_{12} \Delta_1(R) = \tilde{\Delta}_1(R) R_{12}$$
$$= (R_{23} R_{13}) R_{12}$$

ultimately QT incorporates fusion
can interchangably ~~treat~~ treat 2 particles as 1 composite obj.

9.4 Yangian Algebra

Quantum algebra framework for XXX Heisenberg spin chain

Algebra

Yangian $Y(g)$ is def. of $U(g[u])$.

Generated by (products/polynomials in) level-zero gen $J^a \simeq J^a_0$
and level-one generator $Y^a \simeq J^a_1$, $a = 1 \dots \dim(g)$

$$[J^a, J^b] = i f^{ab}{}_c J^c \quad \leftarrow \text{level-zero} \equiv g$$

$$[J^a, Y^b] = i f^{ab}{}_c Y^c \quad \leftarrow Y^a \text{ transforms in adj of } g.$$

Plus Serre relation

$$[[J^a, Y^b], Y^c] + 2 \text{ cyclic} = \hbar^2 \cdot \text{"} J^3 \text{"}$$

Note higher levels J^a_n, $n > 1$ are expressed as commutators of Y's

Hopf algebra

coproduct $\Delta(1) = 1 \otimes 1$, $\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a$,

$$\Delta(Y^a) = Y^a \otimes 1 + 1 \otimes Y^a + i\hbar f^a{}_{bc} J^b \otimes J^c.$$

antipode $\Sigma(1) = 1$ $\Sigma(J^a) = -J^a$

$$\Sigma(Y^a) = -Y^a + \frac{i}{2}\hbar f^a{}_{bc} f^{bc}{}_d J^d$$

$$= -Y^a + i\hbar J^a \quad \text{for } g = su(2)$$

$\Sigma^2(J^a) = J^a$ $\Sigma^2(Y^a) = Y^a - 2i\hbar J^a$ (not involution)

Evaluation Representation

as for $g[u]$

$\rho_u(1) = 1$ $\rho_u(J^a) = \rho(J^a)$ $\rho_u(Y^a) = u \cdot \rho(J^a)$

(r.h.s. of Serre relation $\stackrel{!}{=} 0$ for consistency)

Spin Chains

Homogeneous chain of L sites, use $u_j = 0$

$$\rho_0(1) = 1, \qquad \rho_0(J^a) = \rho(J^a), \quad \rho_0(Y^a) = 0.$$

Repr. ρ_{ch} on a chain of L sites

$$\rho_{ch} = (\rho_0 \otimes \dots \otimes \rho_0) \circ \Delta^{L-1}$$

note:

$$\Delta^{L-1}(J^a) = \sum_{j=1}^{L} J_j^a \quad , \quad \Delta^{L-1}(Y^a) = \sum_{j=1}^{L} Y_j^a + \hbar f^a{}_{bc} \sum_{j<k=1}^{L} J_j^b \, J_k^c$$

repr.

$$\rho_{ch}(J^a) = \sum_{j=1}^{L} \rho_j(J^a) \qquad \rho_{ch}(Y^a) = \hbar f^a{}_{bc} \sum_{j<k=1}^{L} \rho_j(J^b)\, \rho_k(J^c)$$

matches with monodromy $T(u)$ at $u = \infty$: $T(u) = \exp\left(\frac{i}{u} J + \frac{i}{u^2} Y + \dots\right)$

Symmetry

Let us consider Ham $H = \sum_k H_k$ $\quad H_k = id_{k,k+1} - ex_{k,k+1}$

$[e_{ch}(J^a), H] = 0$ $\quad$ g is a symmetry of chain

$[e_{ch}(Y^a), H] \neq 0$ (by terms at boundary $j = 1, L$)

$Y(g)$ is not a symmetry of chain

is broken by periodic boundary conditions

is symmetry of bulk

is provides useful quantum operators (creation/annihilation)

if $Y(g)$ were symmetry $\Rightarrow$ large/full degeneracy of spectrum.

Magnon States

How does $Y(g)$ act on magnon states $(L=\infty)$ $|p_1, \dots, p_M\rangle$

need to regularise J^z

$$\rho(J^z)_{reg} = \frac{1}{2}\sum(\sigma_j^z + id_j)$$

eigenvalue of $\rho(J^z)_{reg}$.

$$\rho(J^z)_{reg}|p_1 \dots p_M\rangle = M \cdot |p_1 \dots p_M\rangle$$

$$\rho(Y^z)_{ch} = \frac{i}{2}\hbar \sum_{j<k}(\sigma_j^- \sigma_k^+ - \sigma_j^+ \sigma_k^-)$$

$$\rho(Y^z)_{ch}|p\rangle = \frac{i}{2}\hbar \sum_{j<k}\left(e^{ipj}|k\rangle - e^{ipk}|j\rangle\right)$$

$$= \frac{i}{2}\hbar \sum_{k=1}^{\infty}\left(e^{-ipk} - e^{ipk}\right) \underbrace{\sum_j e^{ipj}|j\rangle}_{|p\rangle} = \frac{1}{2}\hbar \cot(p/2)|p\rangle = u|p\rangle$$

rap.

$$u = \frac{1}{2}\hbar \cot(p/2)$$

← mag. mom.

$\hbar = 1$

R-Matrix

S matrix of (multiple flavours of) magnons is R

Symmetry of S extends to $Y(g)$: quasi-cocommutativity

$$R\,\Delta(X) = \tilde{\Delta}(X)\,R \qquad R \sim \frac{1}{u-v+i}\left((u-v)\,\mathrm{id} + i\,\mathrm{ex}\right)$$

can do for fund. eval. rep. with $X = J^a, Y^a$

$$\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a \qquad g = SU(N)$$

$$\overset{(\sim)}{\Delta}(Y^a) = u\,(J^a \otimes 1) + v\,(1 \otimes J^a) \pm \hbar f^a{}_{bc}\, J^b \otimes J^c$$

Q. cocomm. implies for $X = J^a$ $\quad R = R_1^{(u,v)}\,\mathrm{id} + R_2^{(u,v)}\,\mathrm{ex}$

for $X = Y^a$ implies $i\hbar R_1 = (u-v) R_2$

$$R \sim (u-v)\,\mathrm{id} + i\hbar\,\mathrm{ex}$$

Tensor Products

R/S matrix acts on tensor product of two sites/particles

suppose $\mathfrak{g} = SU(N)$, site/part. repr. is fund.

from Lie repr $\quad \square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$

$\qquad$ fund $\otimes$ fund = sym $\oplus$ anti-sym.

$\qquad (\tfrac{1}{2}) \otimes (\tfrac{1}{2}) = (1) \oplus (0) \qquad$ for $SU(2)$

What changes in $Y(\mathfrak{g})$?

consider 3 states:

$|0\rangle = |\downarrow\downarrow\rangle \in \square\square$

$|s\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \in \square\square$

$|a\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \in \begin{array}{c}\square\\[-4pt]\square\end{array}$

can act with raising/lowering level-zero/level-one gen:

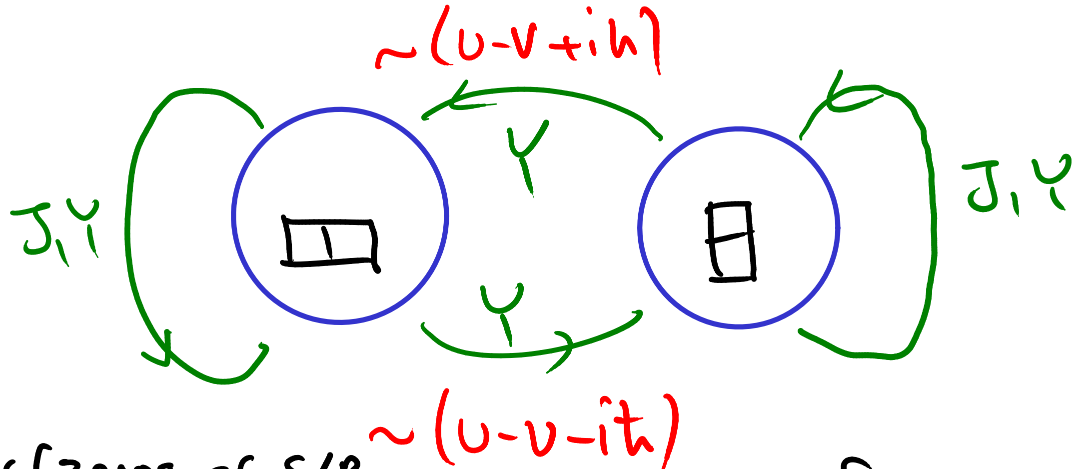
$$\Delta(J^+)|0\rangle = \tfrac{1}{2}|\uparrow\downarrow\rangle + \tfrac{1}{2}|\downarrow\uparrow\rangle = \tfrac{1}{2}|s\rangle$$

$$\Delta(Y^+)|0\rangle = \tfrac{1}{2}u\,|\uparrow\downarrow\rangle + \tfrac{1}{2}v\,|\downarrow\uparrow\rangle - \tfrac{i}{4}\hbar\,|\uparrow\downarrow\rangle + \tfrac{i}{4}\hbar\,|\downarrow\uparrow\rangle$$

$$= \tfrac{1}{4}(u+v)\,|s\rangle + \tfrac{1}{4}(u-v-i\hbar)\,|a\rangle$$

$$\Delta(J^-)\,|a\rangle = 0$$

$$\Delta(Y^-)\,|a\rangle = \tfrac{1}{2}(u-v+i\hbar)\,|0\rangle$$



$\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are unrelated by g

$\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are generically related by $Y(g)$

poles/zeros of S/R

bound states of single-particle

→ tensor prod. is indecomposable but reducible.

for $u-v=\pm i\hbar$ one direction is forbidden.