Introduction to Integrability

Lecture Slides, Chapter 8

ETH Zurich, 2023 HS

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8. Quantum Integrability

8.1 R-Matrix Formalism

Recall scattering matrices encountered so far

$$S^{cd}_{ab}(u,v) = \frac{(u-v)\,\delta^c_a\delta^d_b + i\,\delta^d_a\delta^c_b}{u-v-i}$$

for many magnon flavours in $SU(N)$ $N\geq 3$ chains, also spinors $a,b,c,d = \uparrow,\downarrow$

Here introduce an operator R (R-matrix)

$$R^{cd}_{ab}(u,v) = \frac{(u-v)\,\delta^c_a\delta^d_b + i\,\delta^d_a\delta^c_b}{u-v+i}$$

R as tensor op

$R: \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}$

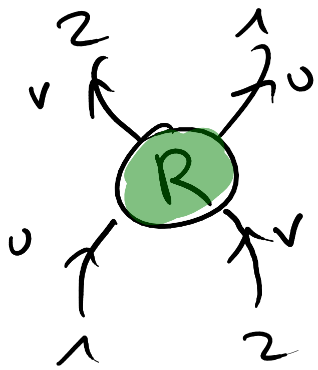
$$R = \frac{(u-v)\,\mathrm{id} + i\,\mathrm{ex}}{u-v+i}$$

for many sites $\mathbb{V}_k$ could use short cut notation R_{kl}

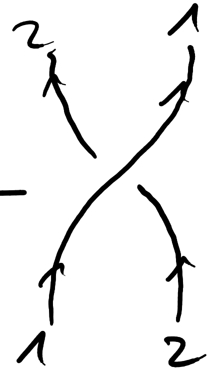
Graphical Representation

$R_{12} \sim R(u,v)$ $\quad u \to 1,\; v \to 2$

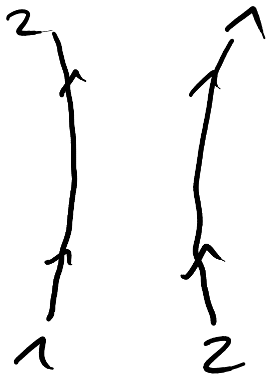
want to represent op. R and composition of it in diagrams



$= \dfrac{u-v}{u-v+i}$

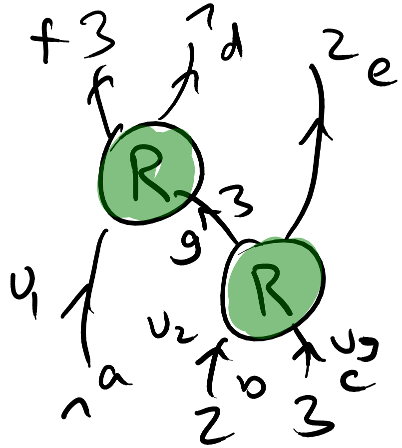


$+ \dfrac{i}{u-v+i}$



composition

$R_{13} R_{23} =$



in components:

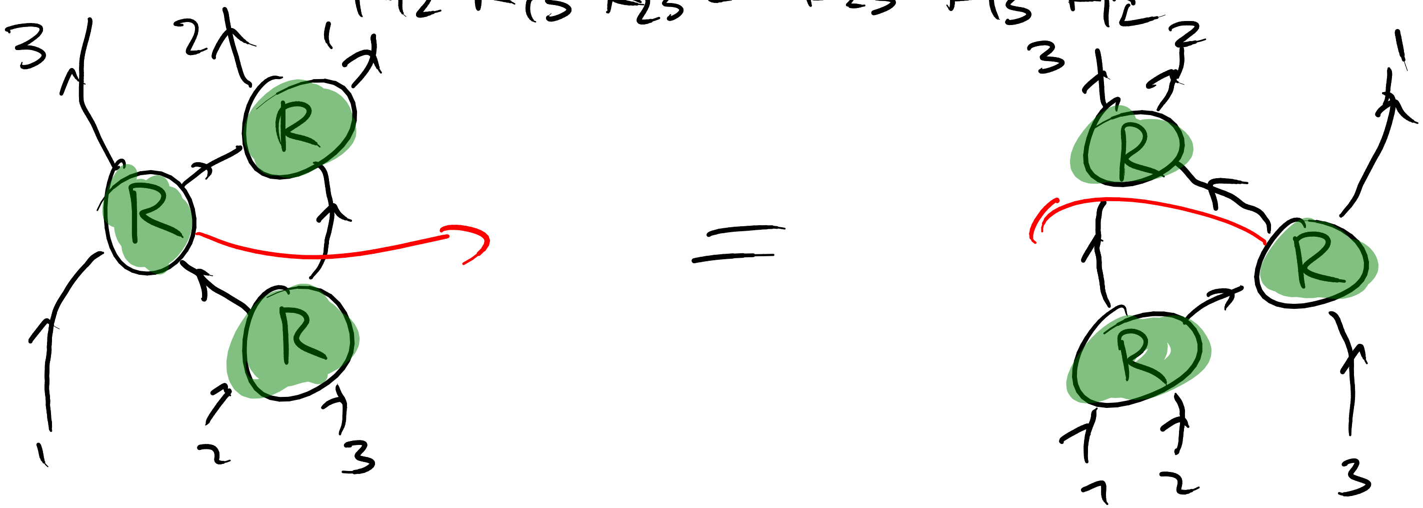
$R^{df}_{ag}(u_1, u_3)\, R^{eg}_{bc}(u_2, u_3)$

Properties of R-Matrices

For fact. scattering: Yang-Baxter-Eq.

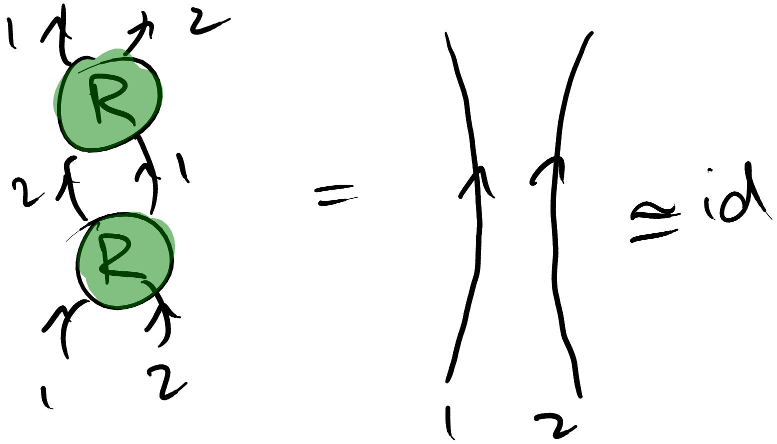
$$R_{12}(u_1 u_2)\, R_{13}(u_1 u_3)\, R_{23}(u_2 u_3) = R_{23}(u_2 u_3)\, R_{13}(u_1 u_3)\, R_{12}(u_1 u_2)$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$



YBE allows to deform/shift intersect. across lines

Similar property: $R_{21} = (R_{12})^{-1}$ or $R_{21}\, R_{12} = \mathrm{id}_{12}$



note

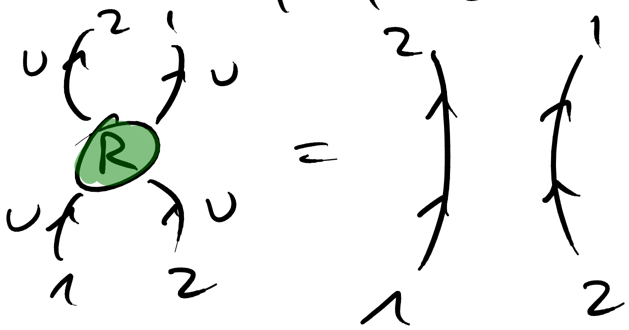
$$R_{21} := R_{21}(u_2, u_1)$$
$$= ex_{12}\, R_{12}(u_2, u_1)\, ex_{12}$$
$$= \dots = (R_{12})^{-1}$$

altogether:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad \text{and} \quad R_{12} R_{21} = \mathrm{id}.$$

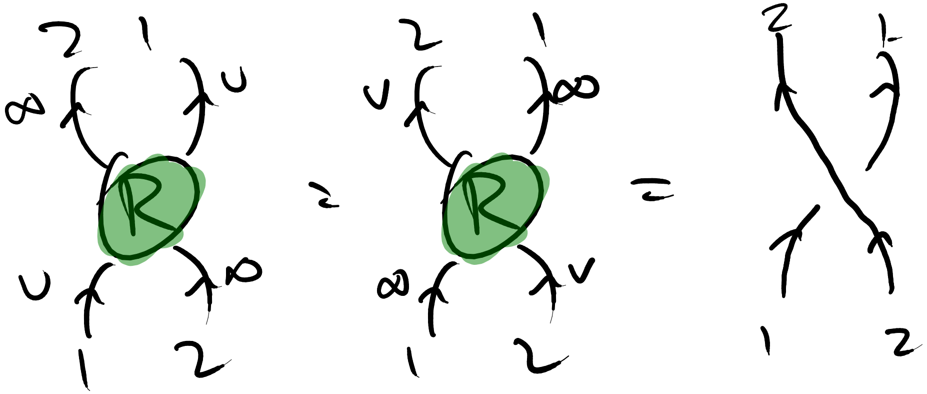
equivalent to permutation group $S_N \leftarrow$ #sites two aux properties related to physics.

$R(u,u) = $ ex



for scattering: identical particles

for argument $u, v = \infty$ R trivializes $R(u,\infty) = R(\infty,v) = \mathrm{id}$



related to $SU(N)$ symmetry of R.

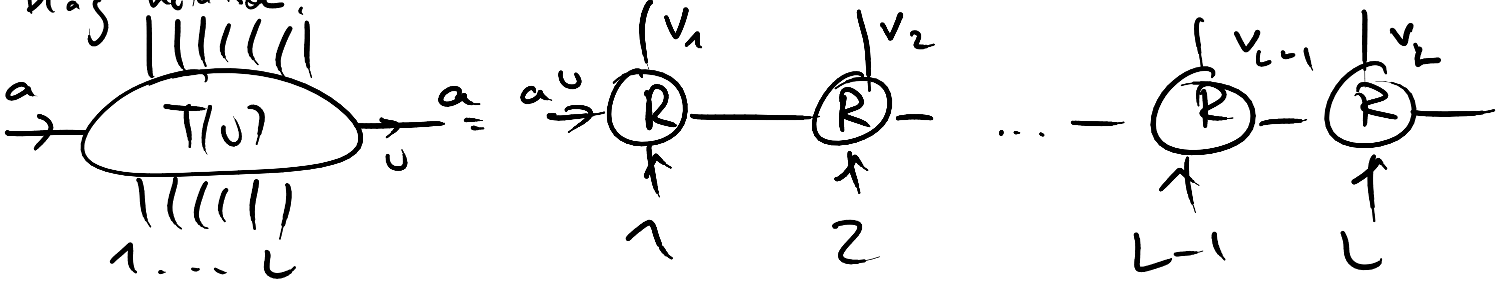
8.2 Charges

Monodromy and Traces

Closed boundary monodromy matrix $T(u)$ defined as

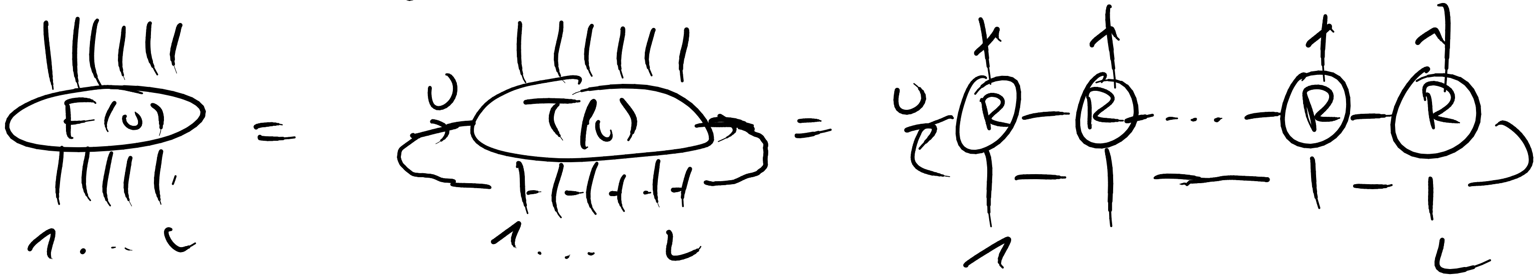
$$T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdot \ldots \cdot R_{a,2} \cdot R_{a,1}$$

Diag notation:



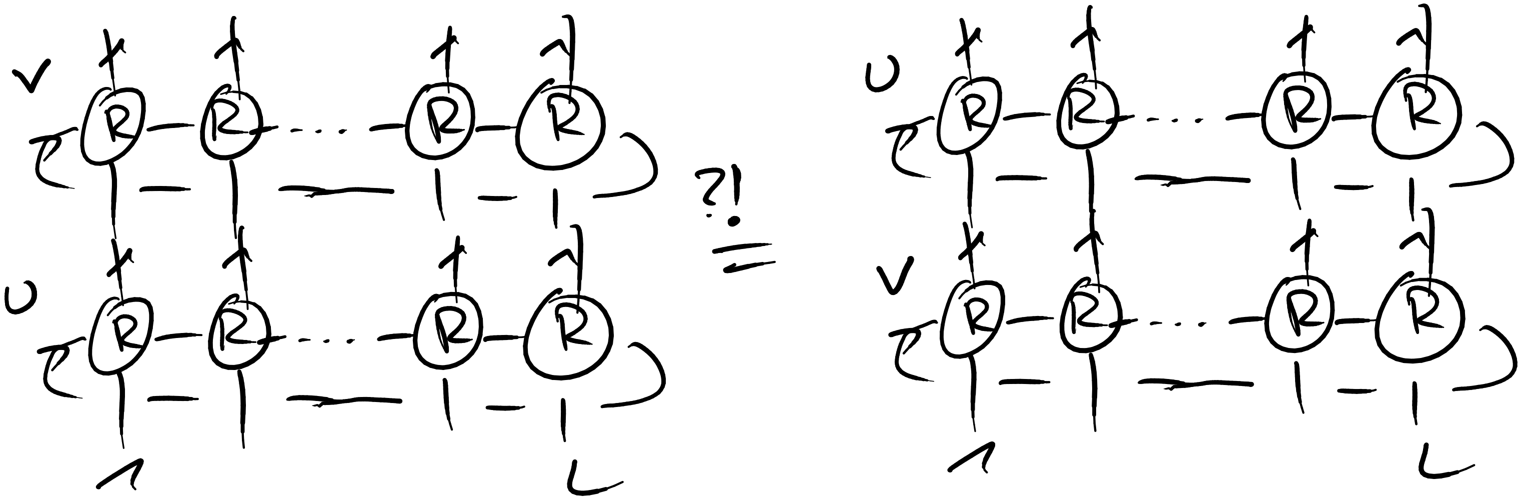
for a hom. spin chain all v_k equal $\quad v_k = 0$.

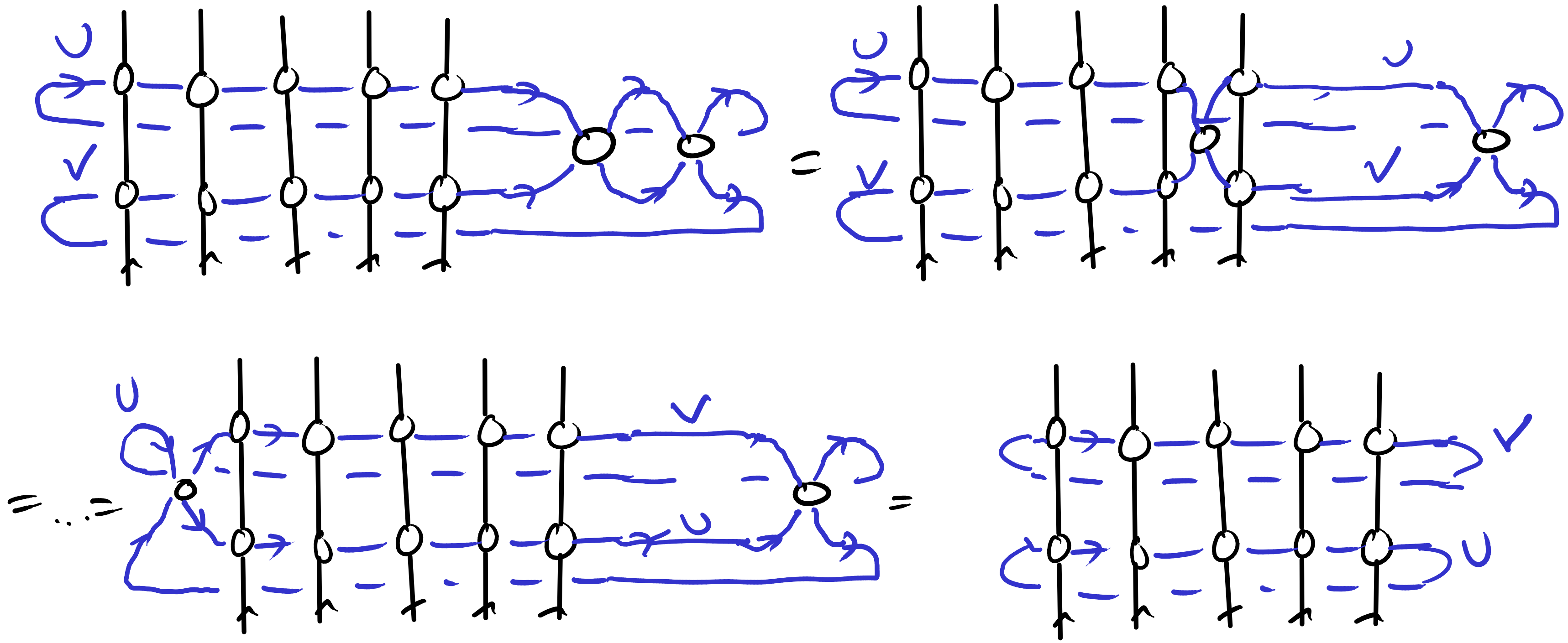
Trace $F(u) = \mathrm{tr}_a\, T_a(u)$



in class mech: $\{F(u), F(v)\} = 0$

in QM: $[F(u), F(v)] \overset{?}{=} 0$




U
V
=
=
= ... =
=

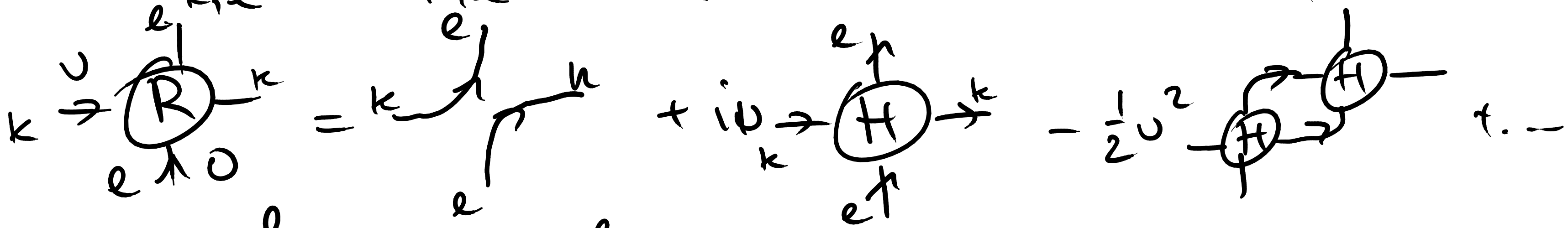
Local Charges

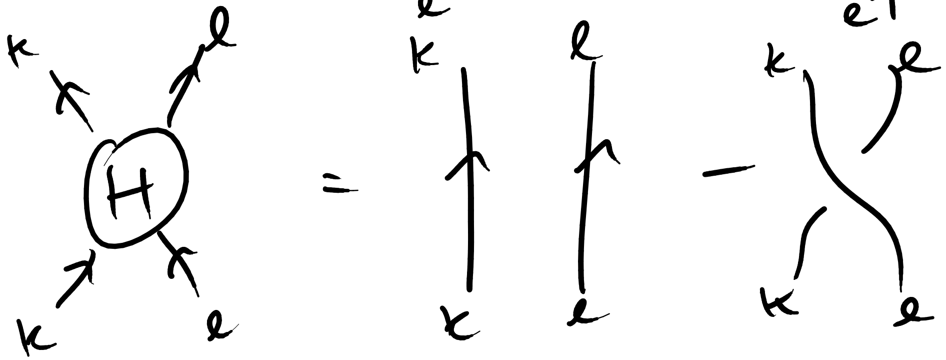
F(u) is non-local operator, contains some local information

here: at point $u = 0 = v_k$. expand

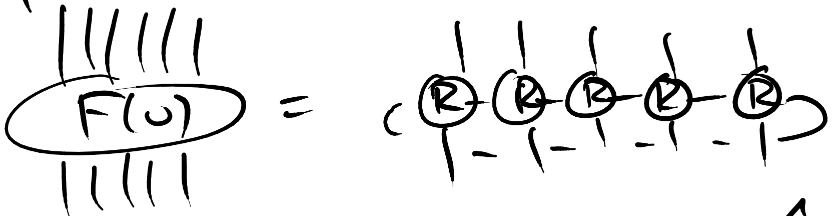
use $R_{a,j}(u,0) = ex_{a,j} + iu\, ex_{a,j} H_{a,j} - \frac{1}{2} u^2 ex_{a,j} H_{a,j}^2 + \dots$

$H_{k,l} = id_{k,l} - ex_{k,l}$. kernel of Ham. op.

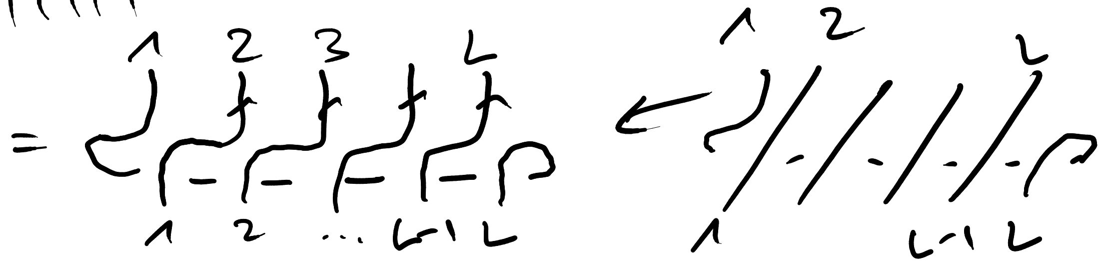




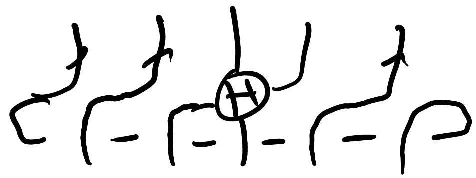
Expand F(u)



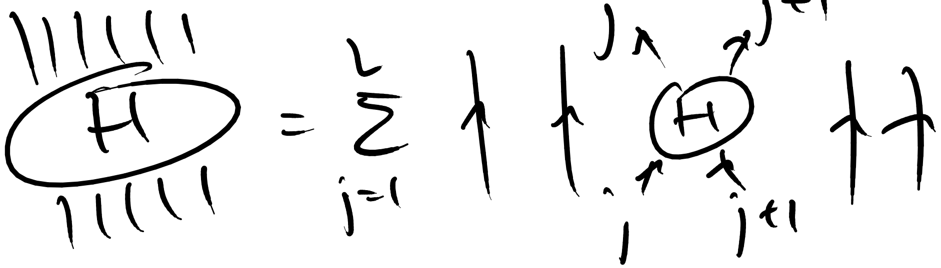
cyclic shift op.



$\exp(iP)$

$+\, iu \sum_{j=1}^{L}$  $\leftarrow \exp(iP)\, iu\, H$

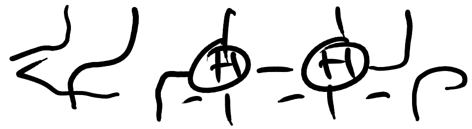
$+ \ldots$



$$= \exp(iP) + iu \exp(iP) H + \ldots = \exp(iP + iu H + \ldots)$$

at order u^2

$F(u) = \dots \quad - u^2 \sum_{\substack{j<k=1 \\ |j-k|>1}}^{L}$ 

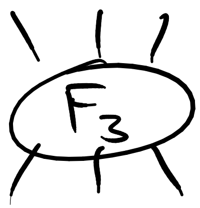
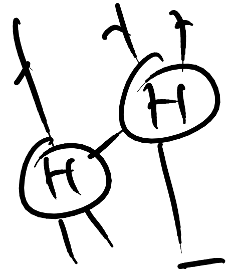
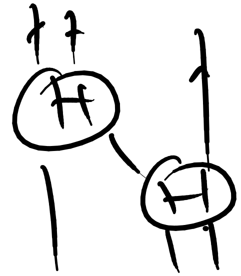
$- u^2 \sum_{j=1}^{L}$ 

$- \frac{1}{2} u^2 \sum_{j=1}^{L}$ 

[almost u^2 term of $\exp(iP + iuH)$]

$= \exp(iP + iuH + iu^2 F_3 + \dots)$ $\qquad = \frac{i}{2}[H_{j+1}, H_j]$

 $- \sum_{j=1}^{L}$ 

 $= \frac{i}{2}$  $- \frac{i}{2}$ 

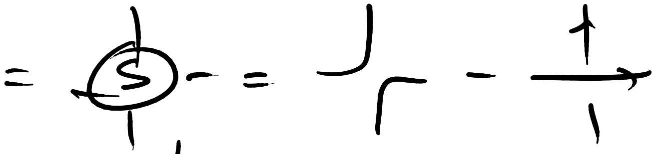
altogether: $F_2 = H$, $F_1 \sim P$ $\qquad [F_r, F_s] = 0$

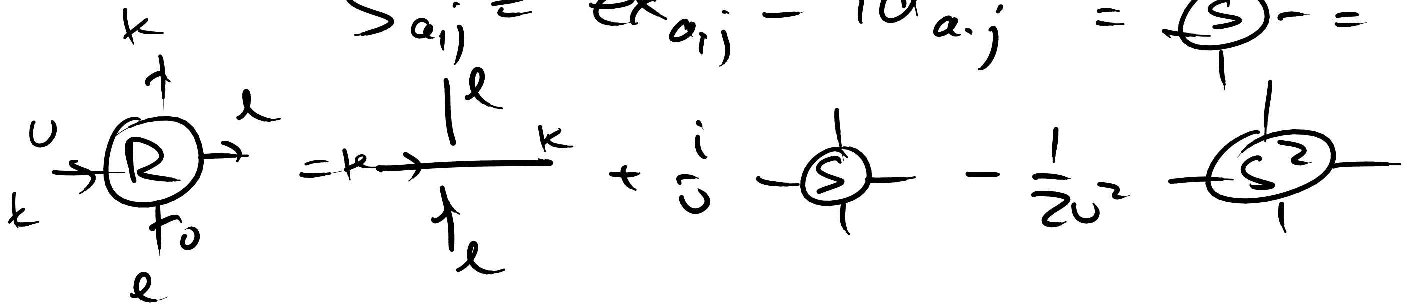
Multi-local Charges

consider the (symmetry) point $u=\infty$

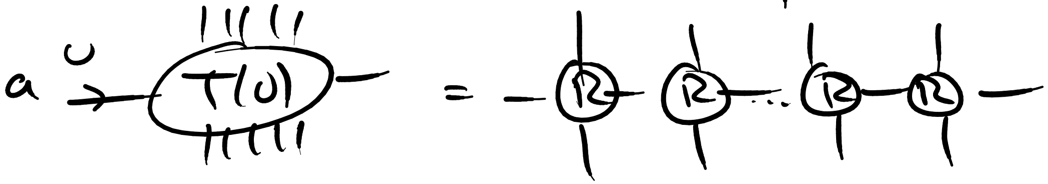
$$R_{a,j}(u,0) = id_{a,j} + iu^{-1} S_{a,j} - \tfrac{1}{2}u^{-2} S_{a,j}^2 + \dots$$

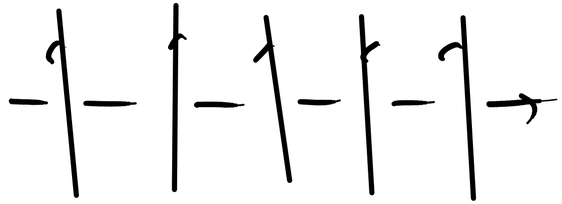
$$S_{a,j} = ex_{a,j} - id_{a,j}$$

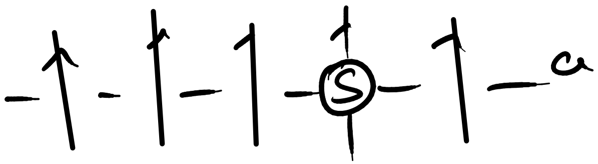




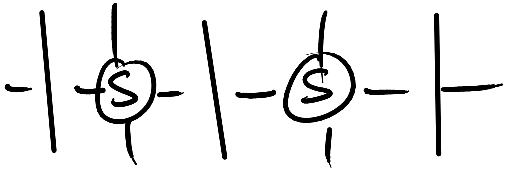
Expand $T(u)$ monodromy

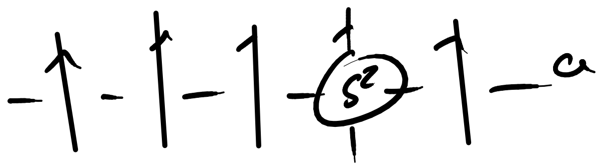


$=$  $\leftarrow$ id

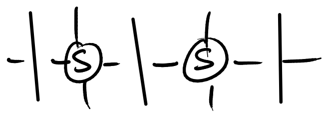
$+\frac{i}{U}\sum_{j=1}^{L}$  $\leftarrow \frac{i}{U} J_a$

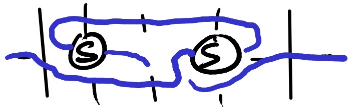
J_a total ang. mom.

$-\frac{1}{U^2}\sum_{j>k=1}^{L}$ 

$-\frac{1}{2U^2}\sum_{j=1}^{L}$ 

almost $-\frac{1}{2U^2}(J_a)^2$

$Y_a = \frac{i}{2}\sum_{j>k=1}^{L}$ 

$-\frac{i}{2}\sum_{j>k=1}^{L}$ 

$= \exp\left(\frac{i}{U} J_a + \frac{i}{U^2} Y_a + \dots\right)$

bi-local operator

commutator of $[S_{a,j}, S_{a,k}]$

8.3 Alternative Types of Bethe Ansätze

Algebraic Bethe Ansatz

use monodromy $T_a(u)$ in aux space a acts as 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$ A, B, C, D are operators acting on chain / Hilbert space

obey an algebra: RTT-relations (Yang-Baxter eq)

$$R_{ab}(u,v)\, T_a(u)\, T_b(v) = T_b(v)\, T_a(u)\, R_{ab}(u,v)$$

in a basis $|\uparrow\rangle$, $|\downarrow\rangle$ and using that R is so(2) invariant

A, D preserve # up/down spins, B flips one $\downarrow$ to $\uparrow$, C flips $\uparrow$ to $\downarrow$

use as a framework of creation (B), annihilation (C) and charge (A, D) op

as in QM II / QFT vacuum $|\downarrow\downarrow\ldots\downarrow\rangle = |0\rangle$ magnon states $|u_1 \ldots u_M\rangle = B(u_1)\ldots B(u_M)|0\rangle$.

states $B(u_1)\dots B(u_M)|0\rangle$ are eigenstates of $F(u) = A(u) + D(u)$

eigenvalue is

$$F(u) = \underbrace{\prod_{k=1}^{M} \frac{u-u_k-i/2}{u-u_k+i/2}}_{A(u)} + \underbrace{\left(\frac{u}{u+i}\right)^{L} \prod_{k=1}^{M} \frac{u-u_k+3i/2}{u-u_k+i/2}}_{D(u)}$$

+ off-diagonal terms which cancel provided that

$$\left(\frac{u+i/2}{u-i/2}\right)^{L} = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i}$$

for all $k=1\dots M$ Bethe eq!

expand $F(u) = \exp(iP + iuE + iu^2 F_3 + \dots)$

$$\exp(iP) = \prod_{k=1}^{M} \frac{u_k+i/2}{u_k-i/2} \qquad E = \sum_{k=1}^{M}\left(\frac{i}{u_k+i/2} - \frac{i}{u_k-i/2}\right) \qquad F_3 = \sum_{k=1}^{M}\left(\frac{i}{2(u_k+i/2)^2} - \frac{i}{2(u_k-i/2)^2}\right)$$

Algebraic Bethe Ansatz for Higher-Rank Chains / Symmetries

Here su(2) $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

SU(N) $T = \begin{pmatrix} A^1 & B^1 & * & * & * \\ C^1 & A^2 & B^2 & * & * \\ * & C^2 & \ddots & \vdots & B^{N-1} \\ * & * & & C^{N-1} & A^N \\ * & * & & & \end{pmatrix}$ $A^r \sim$ Cartan subalg. el.

B^r, C^r single roots ($\pm$)

all other generators as products of A^r, B^r, C^r.

create magnon quasi particles from a vacuum $|N\,N \ldots N\rangle = |0\rangle$

by using $B^r\,|r+1\rangle \to |r\rangle$ populate all Hilbert space $(\mathbb{C}^N)^L$

eigenstates $|u_k^{(r)}\, u_e^{(s)} \ldots\rangle = B^r(u_k^{(r)})\, B^s(u_e^{(s)}) \ldots |0\rangle$

Analytic Bethe Ansatz

start with expression

$$F(u) = \prod_{k=1}^{M} \frac{u-u_k-i/2}{u-u_k+i/2} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^{M} \frac{u-u_k+3i/2}{u-u_k+i/2}$$

what does follow? recall $F(u) \sim (R)^L$ $\quad R \sim \frac{*u+*1}{u+i}$

$$F(u) \sim \frac{P_L(u)}{(u+i)^L} \quad \text{(with q. op as coefficients)}$$

compare this to above $F(u)$: mismatch, add. poles at $u = u_k - i/2$

Residues $F(u_k - i/2 + \epsilon) \sim -\frac{i}{\epsilon} \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l-i}{u_k-u_l} + \frac{i}{\epsilon}\left(\frac{u_k-i/2}{u_k+i/2}\right)^L \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l}$

$$\sim -\frac{i}{\epsilon} \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l-i}{u_k-u_l} \left(1 - \left(\frac{u_k-i/2}{u_k+i/2}\right)^L \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \right) \leftarrow = 0 \text{ iff Bethe Eq. hold}$$

Baxter Equation

$\tilde{F}(u) := (u+i/2)^L F(u-i/2)$ Polynomial in u

$$\tilde{F}(u) = (u+i/2)^L \prod_{k=1}^{M} \frac{u-u_k-i}{u-u_k} + (u-i/2)^L \prod_{k=1}^{M} \frac{u-u_k+i}{u-u_k}$$

introduce polynomial $Q(u) = \prod_{k=1}^{M} (u-u_k)$ ← poly. of Bethe roots

above eq. for $\tilde{F}(u)$ as

$$\tilde{F}(u)\,Q(u) = (u+i/2)^L\, Q(u-i) + (u-i/2)^L\, Q(u+i)$$

Difference eq.: Baxter eq. for $Q(u)$:

given some $\tilde{F}(u)$: defines 2-dim. space of solutions $Q(u)$

$Q(u)$ are polynomials in u (deg M) only for specific $\tilde{F}(u)$.

↑ iff Bethe Eq. hold.