

Introduction to Integrability

Lecture Slides, Chapter 8

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8. Quantum Integrability

8.1 R-Matrix formalism

Recall scattering matrices encountered so far

$$S_{ab}^{cd}(v, \bar{v}) = \frac{(v - v) \delta_a^c \delta_b^d + i \delta_a^d \delta_b^c}{v - v - i}$$

for many magnon flavours & $SU(N)$ $N \geq 3$ chains, also spinors $\psi_{\pm}^{a,b,c,d} =$

Here introduce an operator R (R-matrix)

$$R_{ab}^{cd}(v, \bar{v}) = \frac{(v - v) \delta_a^c \delta_b^d + i \delta_a^d \delta_b^c}{v - v + i}$$

R as tensor op

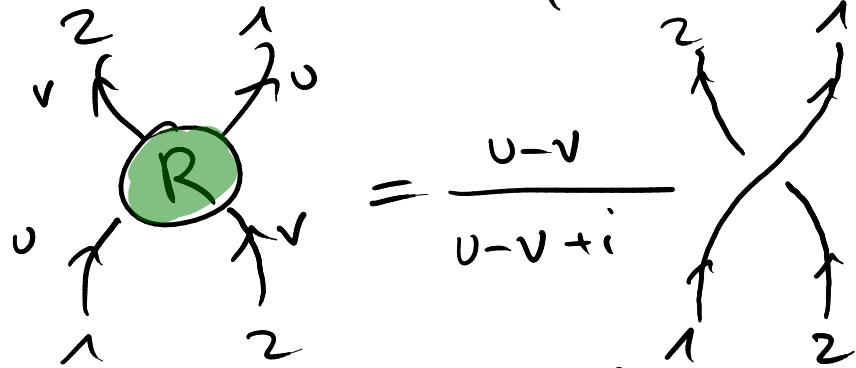
$$R: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$$

$$R = \frac{(v - v) \text{id} + i \epsilon_k}{v - v + i}$$

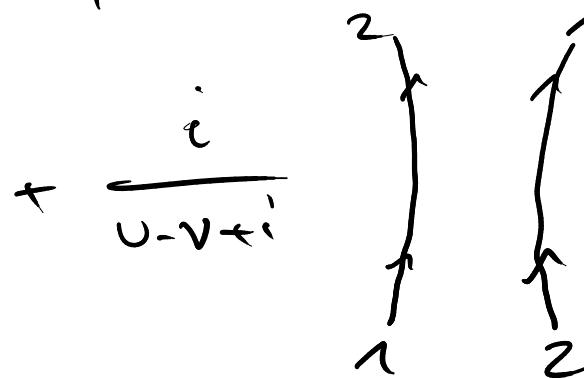
for many sites N_K could use
short cut notation $R_{K,k}$

Graphical Representation

want to represent op. R and composition of it in diagrams

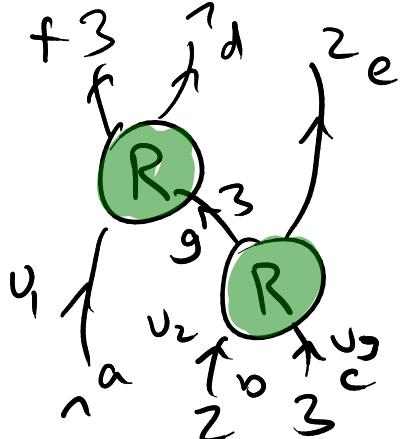


$$R_{12} \sim R(u, v) \quad \begin{matrix} u \rightarrow 1 \\ v \rightarrow 2 \end{matrix}$$



composition

$$R_{13} R_{23} =$$



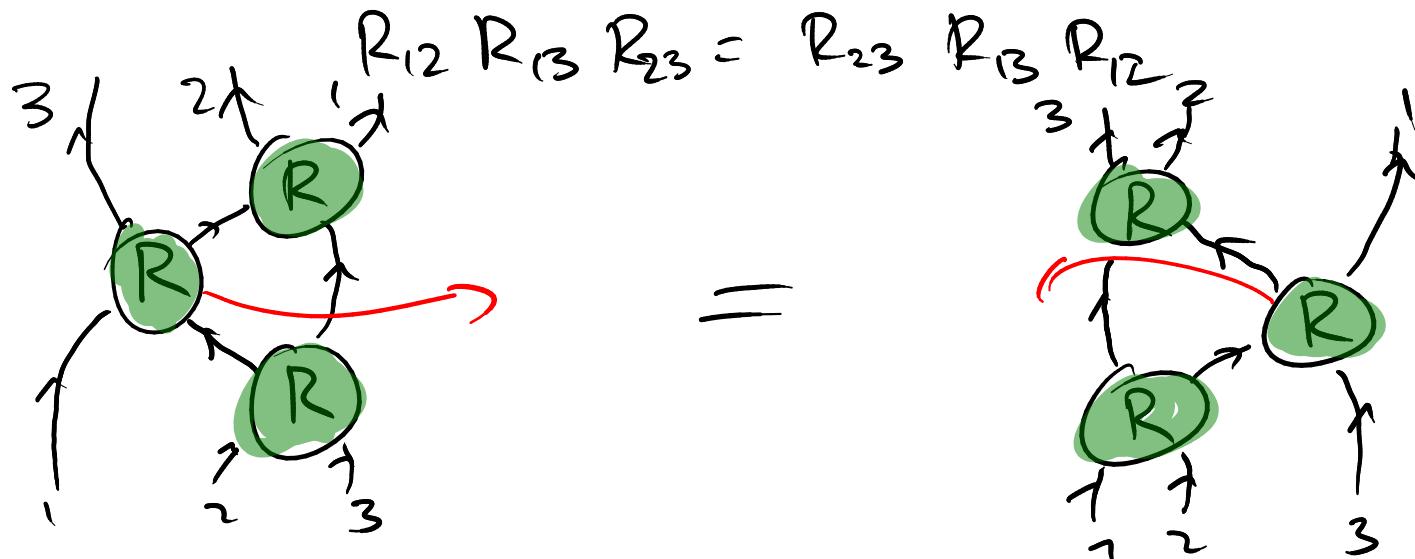
in components:

$$R_{ag}^{df}(u, v_3) \quad R_{bc}^{eg}(v_2, v_3)$$

Properties of R-Matrices

For fact. scattering : Yang-Baxter-Eq.

$$R_{12}(v_1, v_2) R_{13}(v_1, v_3) R_{23}(v_2, v_3) = R_{23}(v_2, v_3) R_{13}(v_1, v_3) R_{12}(v_1, v_2)$$



YBE allows to deform / shift interact. across lines

Similar property : $R_{21} = (R_{12})^{-1}$ or $R_{21} R_{12} = \text{id}_{12}$

$\stackrel{1 \rightarrow 2}{\text{R}}$ $=$ $\left. \begin{array}{c} \\ \end{array} \right\} \cong \text{id}$

note

$$\begin{aligned} R_{21} &:= R_{21}(v_2, v_1) \\ &= \text{ex}_{12} R_{12}(v_2, v_1) \text{ex}_{12} \\ &= \dots = (R_{12})^{-1} \end{aligned}$$

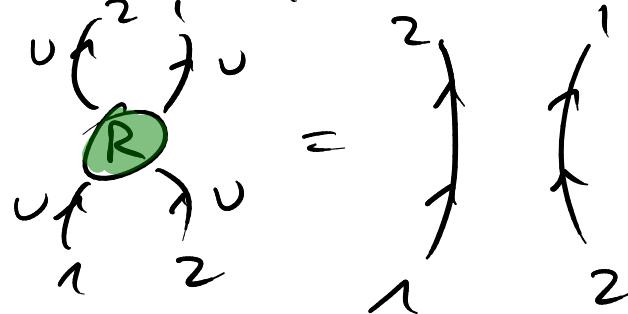
altogether:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \text{ and } R_{12} R_{21} = \text{id}.$$

equivalent to permutation group $S_N^{k^{\# \text{sites}}}$

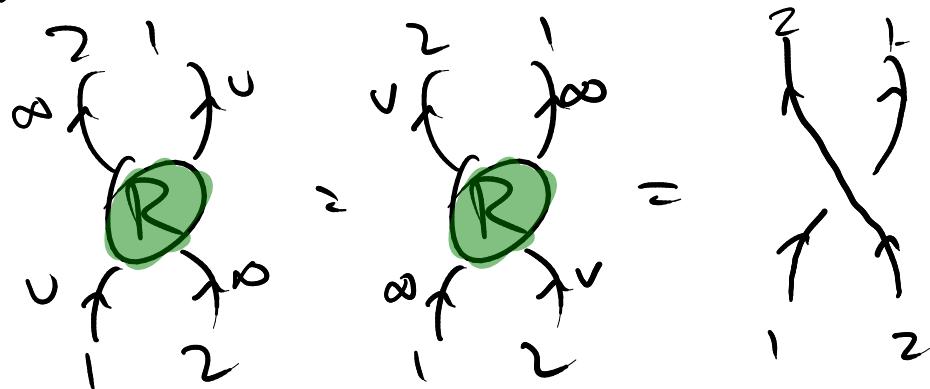
two aux properties related to physics.

$$R(v_1, v_2) = \text{ex}$$



for scattering: identical particles

for argument $v, v = \infty$ R trivializes $R(v, \infty) = R(\infty, v) = \text{id}$



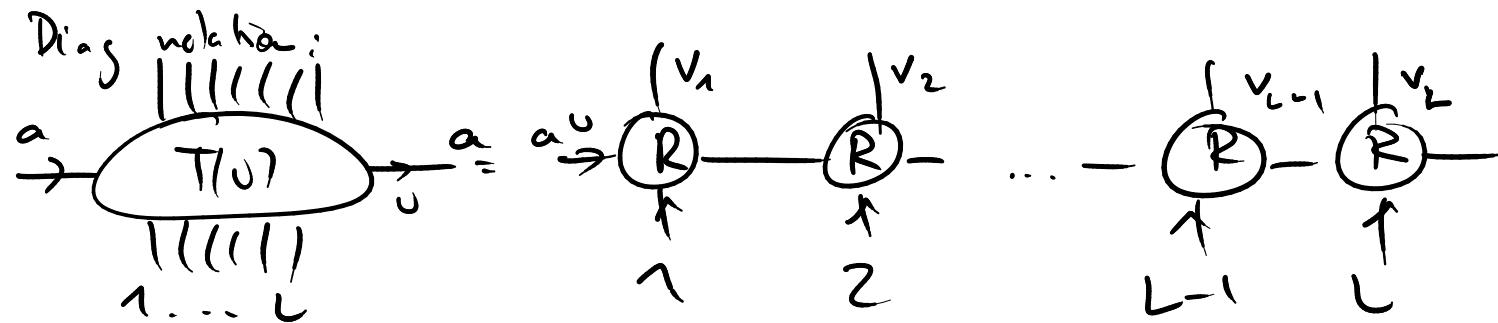
related to
 $SU(N)$ symmetry
of R .

8.2 Charges

Monodromy and Traces

Closed boundary monodromy matrix $T(u)$ defined as

$$T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdot \dots \cdot R_{a,2} \cdot R_{a,1}$$



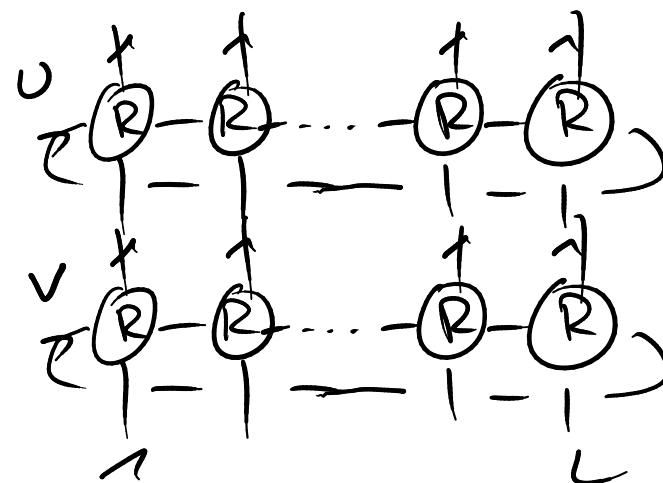
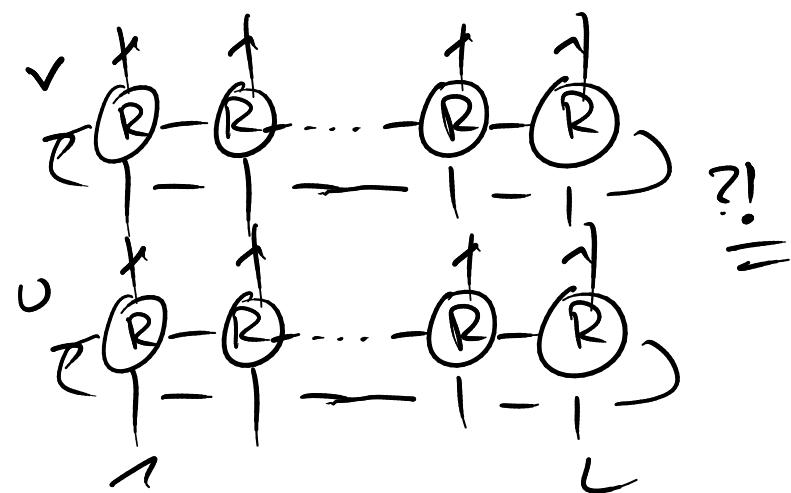
for a hor. spin chain all v_k equal $v_k = 0$.

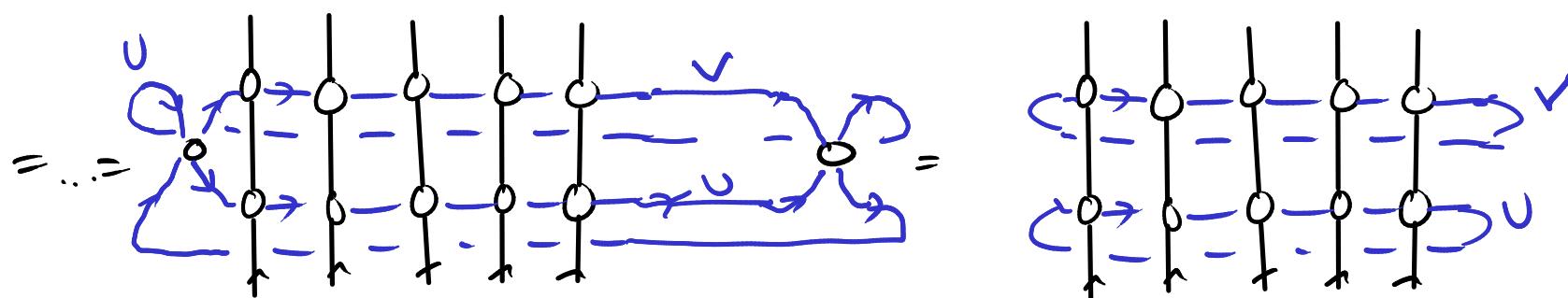
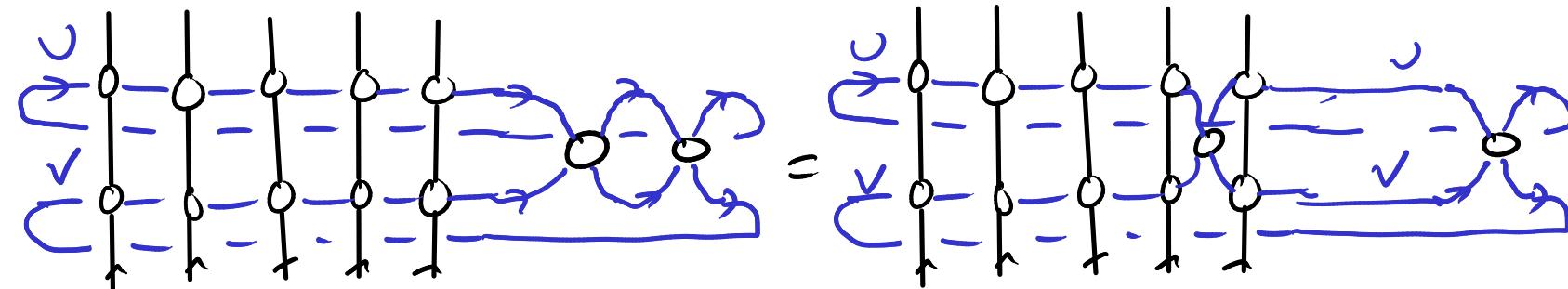
$$\text{trace } F(u) = \text{tr}_a T_a(u)$$



in class mech : $\{F(u), F(v)\} = 0$

$$\text{in DM: } [F(u), F(v)] = 0$$





Local Charges

$F(v)$ is non-local operator, contains some local information

here: at point $v=0=v_k$. expand

$$\text{use } R_{\alpha,j}(v,0) = ex_{\alpha,j} + iv ex_{\alpha,j} H_{\alpha,j} - \frac{1}{2} v^2 ex_{\alpha,j} H_{\alpha,j}^2 + \dots$$

$$H_{k,l} = id_{k,l} - ex_{k,l} \cdot \text{kernel of free ham. op.}$$

$$\begin{aligned}
 & \text{Diagram: } k \xrightarrow{v} \textcircled{R} \xrightarrow{k} e \xrightarrow{e} 0 \\
 & \quad = k \xrightarrow{} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} + iv \xrightarrow{k} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} - \frac{1}{2} v^2 \xrightarrow{k} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} + \dots \\
 & \text{Diagram: } k \xrightarrow{} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} l \xrightarrow{k} \textcircled{H} \xrightarrow{k} e \\
 & \quad = k \xrightarrow{} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} - k \xrightarrow{} \textcircled{H} \xrightarrow{k} e \xrightarrow{e} l
 \end{aligned}$$

expand $F(\omega)$

$$\begin{array}{c} \text{|||||} \\ | \\ F(\omega) = \langle \text{R} \text{--- R} \text{--- R} \text{--- R} \text{--- R} \rangle \end{array}$$

$$= \begin{array}{c} \overset{1}{\text{---}} \overset{2}{\text{---}} \overset{3}{\text{---}} \overset{L}{\text{---}} \\ | \quad | \quad | \quad | \quad | \\ -1 \quad 2 \quad \dots \quad L-1 \quad L \end{array} \quad \begin{array}{c} \overset{1}{\text{---}} \overset{2}{\text{---}} \overset{L}{\text{---}} \\ | \quad | \quad | \quad | \quad | \\ -1 \quad 1 \quad \dots \quad L-1 \quad L \end{array} \quad \begin{array}{l} \text{cyclic shift op.} \\ \exp(iP) \end{array}$$

$$+ i\omega \sum_{j=1}^L \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \quad | \\ -1 \quad j-1 \quad j \quad j+1 \quad L \end{array} \quad \leftarrow \exp(iP) \text{ w H}$$

$$+ \dots$$
$$\begin{array}{c} \text{|||||} \\ | \\ F = \sum_{j=1}^L \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ | \quad | \quad | \quad | \quad | \\ j \quad j+1 \quad j+1 \quad j+1 \quad j+1 \end{array} \end{array}$$

$$= \exp(iP) + i\omega \exp(iP) H + \dots = \exp(iP + i\omega H + \dots)$$

at order v^2

$$F(v) = \dots - v^2 \sum_{\substack{j < k=1 \\ |j-k|>1}}^L \quad \text{Diagram: Two loops with two vertices labeled H, connected by a horizontal line between them.}$$

$$- v^2 \sum_{j=1}^L \quad \text{Diagram: A loop with one vertex labeled H, connected to a horizontal line P.}$$

$$- \frac{1}{2} v^2 \sum_{j=1}^L \quad \text{Diagram: Three vertices labeled H in a triangle configuration, connected to a horizontal line P.}$$

[almost v^2 term of $\exp(iP + i\Omega t)$]

$$= \exp(iP + i\Omega t + iv^2 F_3 + \dots) = i[H_{j+1}, H_j]$$

$$\overline{F_3} = \sum_{j=1}^L \overline{|F_2|} \quad \text{Diagram: A circle labeled } F_3 \text{ with three vertical lines extending downwards.}$$

$$F_3 = \frac{i}{2} \left(\overline{H} \overset{\uparrow}{\text{---}} \overset{\uparrow}{H} - \overline{H} \overset{\downarrow}{\text{---}} \overset{\uparrow}{H} \right) \quad \text{Diagram: Two configurations of three vertices labeled H, each with a vertical line above it. The first has an upward arrow between the top two vertices, the second has a downward arrow between the bottom two vertices.}$$

altogether: $F_2 = H$, $F_1 \sim P$ $\sum [F_r, F_s] = 0$

Multi-local charges

Consider the (symmetric) part $v=\infty$

$$R_{\alpha ij}(v, 0) = i d_{\alpha ji} + i v^{-1} S_{\alpha ij} - \frac{1}{2} v^{-2} S_{\alpha ij}^2 + \dots$$

$$S_{\alpha ij} = ex_{\alpha ij} - id_{\alpha ij} = \begin{array}{c} \textcircled{S} \\ \downarrow \end{array} - \begin{array}{c} \int \\ \Gamma \end{array} - \frac{1}{i} \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$\begin{array}{c} k \\ \uparrow \\ \textcircled{R} \\ \downarrow \\ k \end{array} \rightarrow \begin{array}{c} l \\ \uparrow \\ \textcircled{l} \\ \downarrow \\ l \end{array} = \begin{array}{c} k \\ \uparrow \\ \textcircled{k} \\ \downarrow \\ k \end{array} + \begin{array}{c} i \\ \uparrow \\ \textcircled{i} \\ \downarrow \\ i \end{array} - \begin{array}{c} \textcircled{S} \\ \downarrow \end{array} - \frac{1}{2v^2} \begin{array}{c} \textcircled{S^2} \\ \downarrow \end{array}$

Expand $T(v)$ monodromy

$$a \xrightarrow{v} \textcircled{T(v)} = - \begin{array}{c} \textcircled{R} \\ \downarrow \end{array} \begin{array}{c} \textcircled{R} \\ \downarrow \end{array} \dots \begin{array}{c} \textcircled{R} \\ \downarrow \end{array} \begin{array}{c} \textcircled{R} \\ \downarrow \end{array} \dots$$

$$= - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| \leftarrow \text{id}$$

$$\rightarrow \frac{i}{\nu} \sum_{j=1}^L - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - a \leftarrow \frac{i}{\nu} J_a \stackrel{\text{J}_a \text{ total}}{\text{ang. mom.}}$$

$$- \frac{1}{\nu^2} \sum_{j=1}^L - \left| \begin{array}{c} \circled{S} \\ \downarrow \end{array} \right| \} \text{ almost } - \frac{1}{2\nu^2} (J_a)^2$$

$$- \frac{1}{2\nu^2} \sum_{j=1}^L - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \circled{S^2} \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \uparrow \\ \downarrow \end{array} \right| - a \} \quad Y_a = \frac{i}{2} \sum_{j=1}^L - \left| \begin{array}{c} \circled{S} \\ \downarrow \end{array} \right| - \left| \begin{array}{c} \circled{S} \\ \downarrow \end{array} \right| -$$

$$= \exp \left(\frac{i}{\nu} J_a + \frac{i}{\nu^2} Y_a + \dots \right)$$

bi-local operators

commutator of $\{ S_{a,ij}^\dagger, S_{a,kl} \}$

$$- \frac{i}{2} \sum_{j=1}^L \left[\left| \begin{array}{c} \circled{S} \\ \downarrow \end{array} \right|, \left| \begin{array}{c} \circled{S} \\ \downarrow \end{array} \right| \right]$$

8.3 Alternative Types of Bethe Ansätze

Algebraic Bethe Ansatz

use monodromy $T_a^{(v)}$ in aux space a acts as 2×2 matrix

$$T(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} \quad A, B, C, D \text{ are operators acting on chain (Hilbert space)}$$

obey an algebra: RTT-relations (Yang-Baxter eq)

$$R_{ab}(v, v) T_a(v) T_b(v) = T_b(v) T_a(v) R_{ab}(v, v)$$

in a basis $| \uparrow \rangle, | \downarrow \rangle$ and using that R is $su(2)$ invariant

A,D preserve # up/down spins, B flips \downarrow to \uparrow , C flips \uparrow to \downarrow

use as a framework of creation (T), annihilation (C) and charge (A,D) op
as in QM/QFT vacuum $| \downarrow \downarrow \dots \downarrow \rangle = | 0 \rangle$ ^{magnon states} $| v_1 \dots v_M \rangle = B(v_1) \dots B(v_M) | 0 \rangle$.

states $B(u_1) \dots B(u_m)$ are eigenstates of $F(u) = A(u) + D(u)$
eigenvalue is

$$F(u) = \prod_{k=1}^M \frac{u - u_k - i/2}{u - u_k + i/2} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^M \frac{u - u_k + 3i/2}{u - u_k + i/2}$$

$\xrightarrow{A(u)}$ $\xrightarrow{D(u)}$ + off-diagonal terms which cancel provided that

$$\left(\frac{u+i/2}{u-i/2}\right)^L = \prod_{\substack{k=1 \\ k \neq h}}^M \frac{u_h - u_k + i}{u_h - u_k - i} \quad \text{for all } k=1 \dots M \quad \text{Before eq!}$$

expand $F(u) = \exp(iP + iuE + iu^2 F_3 + \dots)$

$$\exp(iP) = \prod_{k=1}^M \frac{u_k + i/2}{u_k - i/2} \quad E = \sum_{k=1}^M \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2} \right) \quad F_3 = \sum_{k=1}^M \left(\frac{i}{2(u_k + i/2)^2} - \frac{i}{2(u_k - i/2)^2} \right)$$

Algebraic Bethe Ansatz for Higher-Rank Chains / Symmetries

Here $\text{SU}(2)$ $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$\text{SU}(N)$ $T = \begin{pmatrix} A^1 & B^1 & * & * & * \\ C^1 & A^2 & B^2 & * & * \\ * & C^2 & \ddots & B^{N-1} & \\ * & * & * & C^{N-1} & A^N \\ * & * & * & * & \end{pmatrix}$ $A^r \sim \text{Cartan subalg. el.}$
 B^r, C^r simple roots (± 1)

all other generators as products of A^r, B^r, C^r

Create meson quasi-particles from a vacuum $|N\ N\dots\ N\rangle = |0\rangle$
 by using $B^r |r+1\rangle \rightarrow |r\rangle$ populate full Hilbert space $(\mathbb{C}^N)^L$

eigenstates $|v_k^{(r)} v_e^{(s)} \dots\rangle = B^r (v_k^{(r)}) B^s (v_e^{(s)}) \dots |0\rangle$

Analytic Bethe Ansatz

start with expression

$$F(u) = \prod_{k=1}^M \frac{u - u_k - i/2}{u - u_k + i/2} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^M \frac{u - u_k + 3i/2}{u - u_k + i/2}$$

what does follow? recall $F(u) \sim (R)^L$ $R \sim \frac{u+i}{u-i}$

$$F(u) \sim \frac{P_L(u)}{(u+i)^L} \quad (\text{with q. op as coefficients})$$

compare this to above $F(u)$: mismatch, add. poles at $u = u_k - i/2$

$$\begin{aligned} \text{Residues } F(u_k - i/2 + \epsilon) &\sim -\frac{i}{\epsilon} \prod_{\substack{\ell=1 \\ \ell \neq k}}^M \frac{u_k - u_\ell - i}{u_k - u_\ell} + \frac{i}{\epsilon} \left(\frac{u_k - i/2}{u_k + i/2} \right)^L \prod_{\substack{\ell=1 \\ \ell \neq k}}^M \frac{u_k - u_\ell + i}{u_k - u_\ell} \\ &\sim -\frac{i}{\epsilon} \prod_{\substack{\ell=1 \\ \ell \neq k}}^M \frac{u_k - u_\ell - i}{u_k - u_\ell} \left(1 - \left(\frac{u_k - i/2}{u_k + i/2} \right)^L \prod_{\substack{\ell=1 \\ \ell \neq k}}^M \frac{u_k - u_\ell + i}{u_k - u_\ell - i} \right) \end{aligned}$$

Bethe Eq. hold $\Leftarrow = 0$ iff

Baxter Equation

$$\tilde{F}(v) := (v+i/2)^L F(v-i/2) \quad \text{Polynomial in } v$$

$$\tilde{F}(v) = (v+i/2)^L \prod_{k=1}^M \frac{v - v_k - i}{v - v_k} + (v-i/2)^L \prod_{k=1}^M \frac{v - v_k + i}{v - v_k}$$

introduce polynomial $Q(v) = \prod_{k=1}^M (v - v_k)$ poly. of Bethe roots

above eq. for $\tilde{F}(v)$ as

$$\tilde{F}(v) Q(v) = (v+i/2)^L Q(v-i) + (v-i/2)^L Q(v+i)$$

Difference eq.: Baxter eq. for $Q(v)$:

given some $\tilde{F}(v)$. defines 2-dim. space of solutions $Q(v)$

$Q(v)$ are polynomials in v ($\deg M$) only for specific $\tilde{F}(v)$.
↑ iff Bethe Eq. hold.