Introduction to Integrability

Lecture Slides, Chapter 7

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Prof. N. Beisert

The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

7. Long Quantum Chains

7.1. Magnon Spectrum

Ferromagnetic Vacuum $|0\rangle$ (note: su(2) descendants)
States with M magnons, consider lowest energies

Mode Numbers

consider BE in log. form

mode numbers

$$iL \log \frac{u_k + i/2}{u_k - i/2} - i \sum_{\substack{l=1 \\ l \neq k}}^{M} \log \frac{u_k - u_l + i}{u_k - u_l - i} + 2\pi n_k = 0$$

n_k depend on branch cut of log (default): imag $-\pi \to +\pi$
mode numbers range between $-L/2$ and $+L/2$

Single Magnons

$$iL \log \frac{u+i/2}{u-i/2} + 2\pi n = 0$$

$$u = \frac{1}{2}\cot\frac{\pi n}{L} \qquad p = \frac{2\pi n}{L} \qquad e = 4\sin^2\frac{\pi n}{L}$$

low energies for small $|n| \ll L$ $\quad$ n finite fixed as $L \to \infty$

$$u = \frac{L}{2\pi n} \qquad p = \frac{2\pi n}{L} \qquad e = \frac{4\pi^2 n^2}{L^2}$$

total mom, eng: $\quad P \sim 1/L \qquad E \sim 1/L^2$

Several Magnons

M magnons with distinct mode numbers n_k

interactions to be small b/c gas of magnons (assumption)

$u_k = \frac{L}{2\pi n_k}$ at L.O. then scattering term

$$-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} \simeq -i \log \frac{L/2\pi n_k - L/2\pi n_\ell + i}{L/2\pi n_k - L/2\pi n_\ell - i} \simeq -i \log 1 = 0$$

compute P, E, simple $\Rightarrow$ sum of single magnon terms.

consider several magnons at coincident mode number

ansatz $$u_k = \frac{L}{2\pi n} + \delta u_k$$

momentum term (l.h.s) $$iL \log \frac{u_k + i/2}{u_k - i/2} = -2\pi n + \frac{4\pi^2 n^2}{L} \delta u_k + O(\delta u_k^2 / L^2)$$

scattering term (r.h.s) $-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} = \frac{2}{\delta u_k - \delta u_\ell} + O(1/\delta u_k^2)$

together: $\frac{4\pi^2 n^2}{L} \delta u_k + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{M} \frac{2}{\delta u_k - \delta u_\ell} = 0$

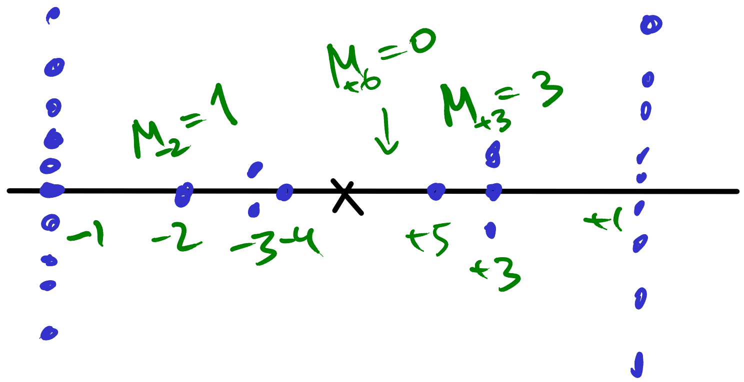
can assume $\delta u_k \sim \frac{1}{n} \sqrt{\frac{L}{M}}$

leads to some alg. eq. for δu_k with a good solution (purely imag. δu_k at L.O.)

vertically stacked Bethe Roots in $\mathbb{C}$

contrib. to P, E is M times single magnon + corrections

Magnon Spectrum



$$M = \sum_n M_n \qquad P = \sum_n M_n \cdot \frac{2\pi n}{L} \qquad E = \sum_n M_n \frac{4\pi^2 n^2}{L^2}$$

Essentially magnons are all bosons.

+ finite size corrections $O(1/L)$ relative to L.O.

7.2 Ferromagnetic Continuum

may as well take M to be large as $L \to \infty$

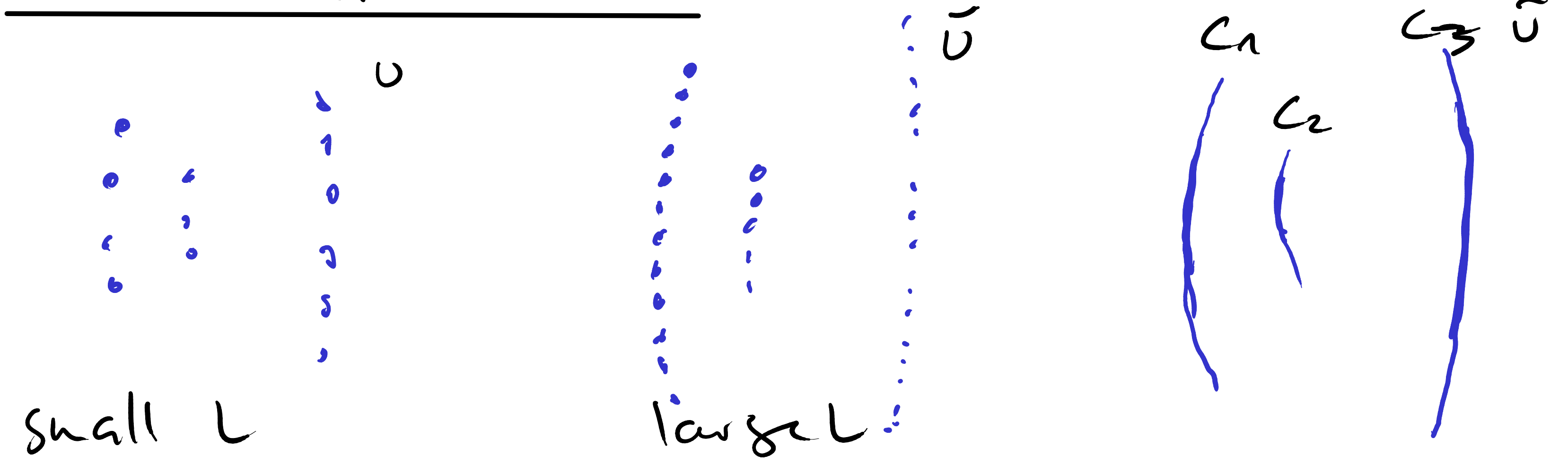
- how to distribute mode numbers n_k? note $E \sim M \cdot n^2$

assume that mode numbers remain bounded, but heavily pop.

convince oneself that $1 \ll M \ll L \Rightarrow$ no corrections to before

new behaviour at $M \sim L$ new analysis.

Distributions of Bethe Roots



To address $L \to \infty$, $M \sim L$ rescale $u = L\tilde{u}$

assume Bethe Roots reside on contours C_k, $C = \bigcup_k C_k$

density $\rho(\tilde{u})$ defined on contours C

$$u \to \tilde{u} L \qquad \sum_k \to L \sum_k \int_{C_k} d\tilde{u}\, \rho(\tilde{u})$$

Bethe Equation have following limit

mode number for C_k

$$P \int_C \frac{2 d\tilde{v}\, \rho(\tilde{v})}{\tilde{u} - \tilde{v}} - \frac{1}{\tilde{u}} + 2\pi n_k = 0 \quad \text{for } \tilde{u} \in C$$

Principal value prescription for $\int$ is due to $\sum_{\ell \neq k}$

For a sol to integral eq. finds the charges (Riemann-Hilbert prob)

$$M_k = L \int_{C_k} d\tilde{u}\, \rho(\tilde{u}) \qquad P = \underbrace{\int_C \frac{d\tilde{u}\, \rho(\tilde{u})}{\tilde{u}}}_{\sim O(1)} \qquad E = \underbrace{\frac{1}{L} \int_C \frac{d\hat{u}\, \rho(\tilde{u})}{\tilde{u}^2}}_{\sim \frac{1}{L}}$$

multiplicity for contour C_k

Spectral Curve

introduce quasi-momentum function $q(\hat{v})$ on $\mathbb{C}$

$$q(\hat{v}) := \int_C \frac{d\tilde{v}\, \rho(\tilde{v})}{\tilde{v}-\hat{v}} + \frac{1}{2\hat{v}}.$$

analyse $q(\tilde{v})$ on $\mathbb{C}$: pole at $\hat{v}=0$ $\quad q(\hat{v}) \sim 1/2\hat{v}$

furthermore branch cut at C_n (discontinuities)

Bethe eq: $\lim_{\epsilon \to 0} \left(q(\tilde{v}+\epsilon) + q(\tilde{v}-\epsilon) \right) = 2\pi n_k$ for $\hat{v} \in C_k$

discont $q(\tilde{v}+\epsilon)$ to $2\pi n_k - q(\tilde{v}-\epsilon)$ going through branch cut at C_n

derivative $q'(\tilde{v}+\epsilon) \to -q'(\tilde{v}-\epsilon)$ Heisenb.

q' describes 2-sheeted cover of $\mathbb{C}$ $\Rightarrow$ Large L limit of Chain
cont Heisenb. model
spectral curve offsets

Heisenberg Framework

get class. cont. Heis. model from quantum chain $L \to \infty$, $\mu \sim L$

use coherent states of q. model, exp. values.

Spin 1/2 state $|S\rangle$ prepared as

$$\langle S | \vec{\sigma} | S \rangle = \vec{S}$$

Operator X exp. val $\langle X \rangle_S = \mathrm{tr}\left(\frac{1}{2} (1 + \vec{S} \cdot \vec{\sigma}) X \right)$

apply to H_j:

$$\langle H_j \rangle_S = \mathrm{tr}_{j,j+1} \left(\frac{1}{4} (1 + \vec{S}_j \cdot \vec{\sigma}_j)(1 + \vec{S}_{j+1} \cdot \vec{\sigma}_{j+1}) (\mathrm{id} - \mathrm{ex})_{j,j+1} \right)$$

$$= \dots = \frac{1}{2} - \frac{1}{2} \vec{S}_j \cdot \vec{S}_{j+1}$$

$$H = \frac{1}{2} \sum_j \left(1 - \vec{S}_j \cdot \vec{S}_{j+1} \right) \quad \text{only from 2 sites}$$

also take $L \to \infty$: $\vec{S}_j = \vec{S}(j\epsilon)$

compute H in $L \to \infty$ limit.

$$H = \frac{1}{\epsilon} \int dx \, \frac{1}{2} \left(1 - \vec{S} \cdot \left(\vec{S} + \epsilon \vec{S}' + \frac{1}{2} \epsilon^2 \vec{S}'' + \dots \right) \right)$$

$$= \frac{1}{4} \epsilon \int dx \, \vec{S}'^2$$ Ham of cont Heisenberg model scaled by $\epsilon/2$

7.3. AntiFerromagnetic Ground State

Consider highest states in spectrum at $L \to \infty$ (aka. low. eng. of antif.)

Entanglement

lowest energy is obtained by aligning spins, $L=2$, $L>2$

highest energy is obtain by opposite alignment and spin 0 for $L=2$

$$|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \quad \text{vs} \quad |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$E=0 \quad S=1 \qquad\qquad E=2 \quad S=0$

cannot be extrapolated to $L>2$, not exactly

- want pairs of neighbours to be in $S=0$ config.
- not possible exactly

note $*|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle \pm *|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ + all other config.

difficult combinatorics $+ |\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\ldots\rangle$

Bethe Equations

ranges betw. $-i\pi$ and $+i\pi$ with 0 at $u=\infty$

Assume all $u_k \in \mathbb{R}$ $\qquad \log \frac{u+i}{u-i} = i\pi\, \mathrm{sign}(u) - 2i \arctan u$

write Bethe eq. as

$$2 \arctan(2u_k) - \frac{2}{L} \sum_{l=1}^{M} \arctan(u_k - u_l) + \frac{2\pi \tilde{n}_k}{L} = 0$$

shifted mode no.: $\tilde{n}_k = n_k + k - \frac{1}{2} M \underbrace{- \frac{1}{2} - \frac{L}{2}\, \mathrm{sign}(u_k)}_{\in \frac{1}{2}\mathbb{Z}}$

permissible mode numbers for $L \to \infty$, high energy:

$-\frac{L}{2} \leq n_k \leq +\frac{L}{2}$, $\quad n_k = 0$ special (exact $U(2)$ sym) $\leftarrow u_k = \infty$

- only single occupation
- neighbouring mode numbers $n_k \pm 1$ should be unoccupied.

assume L = even $\quad M = L/2$

$$\begin{array}{ccccccccccc} -1 & & -3 & & -5 & & +5 & & +3 & & +1 \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} \qquad L=12$$

$$\begin{array}{ccccccccccccc} -1 & & -3 & & -5 & & \pm 7 & & +5 & & +3 & & +1 \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} \qquad L=14$$

anti-ferromagnetic ground state

$$M = L/2 \qquad n_k = L\,\Theta_{2k>M} - 2k+1$$

Integral Equations

distribution of Bethe roots described by density on $\mathbb{R}$

$$\rho(u) = \frac{1}{L}\frac{dk}{du} \qquad k(u) = L\int_{-\infty}^{u} dv\, \rho(v)$$

density of Bethe roots $\qquad$ index of Bethe root

Bethe eq.

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v) \arctan(u-v) - 2\pi \int_{-\infty}^{u} dv\, \rho(v) + \frac{1}{2}\pi$$

differentiate

kernel of difference form

$$\frac{4}{1+4u^2} - \int \frac{2dv\, \rho(v)}{1+(u-v)^2} - 2\pi \rho(u) = 0$$

solve int. eq. by Fourier transform

$$\rho(u) = \int \frac{d\theta}{2\pi} e^{iu\theta} R(\theta) \qquad R(\theta) = \int dv\, e^{-iu\theta} \rho(v)$$

Note Fourier integral

$$\int \frac{dv}{2\pi} \frac{2e^{-iv\theta}}{1+v^2} = e^{-|\theta|}$$

transformed eq

$$e^{-|\theta|/2} - e^{-|\theta|} R(\theta) - R(\theta) = 0 \Rightarrow R(\theta) = \frac{1}{2\cosh(\theta/2)}$$

transform back

$$\rho(v) = \frac{1}{2\cosh(\pi v)} \qquad k(v) = \frac{L}{4} + \frac{L}{\pi} \arctan \tanh\left(\tfrac{1}{2}\pi v\right)$$

Ground State Properties

$$E = L\int \frac{4\,du\,\rho(u)}{1+4u^2} = L\int d\theta\, e^{-|\theta|/2} R(\theta) = 2L \overset{0.69}{\log 2} < 2L$$

$P = 0$ or $P = \pi$ consider exact mode numbers n_i

$$P = \begin{cases} 0 & M = L/2 \text{ even} \\ \pi & M = L/2 \text{ odd} \end{cases}$$

$J = L/2 - M = 0$ b/c half-filling

7.4 Spinons

Bethe Equations

Excitation by inserting a gap of 2 unoccupied modes at mode $k \to v_0$

Integral equation for this config is:

$$0 = 2\arctan(2v) - 2\int_{-\infty}^{+\infty} dv \, \rho(v) \arctan(v-v) - 2\pi \int_{-\infty}^{v} dv \, \rho(v) + \frac{1}{4}\pi - \frac{\pi}{2L}\operatorname{sign}(v-v_0).$$

modification $O(1/L)$: consider variation $\delta\rho$ of density. After diff.:

$$-\int_{-\infty}^{+\infty} \frac{2\, dv\, \delta\rho(v)}{1+(v-v)^2} - 2\pi\,\delta\rho(v) - \frac{2\pi}{L}\delta(v-v_0) = 0$$

Fourier trans, solve to obtain $\delta R(\theta) = -\frac{1}{L}\frac{e^{|\theta|/2 - iv_0\theta}}{2\cosh(\theta/2)}$.

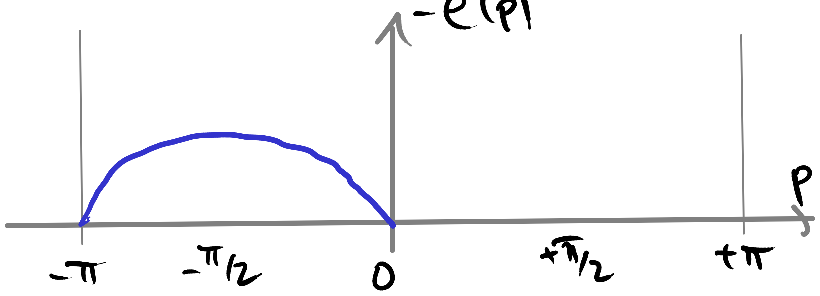
Spinon Properties

energy shift: $e(v_0) = -\int \frac{d\theta\, e^{-iv_0\theta}}{2\cosh(\theta/2)} = -\frac{\pi}{\cosh(\pi v_0)}$.

momentum shift: $p(v_0) = L \int dv\, \delta\rho(v) \left(\pi - 2\arctan(2v)\right)$

$= 2\arctan\tanh(\tfrac{1}{2}\pi v_0) - \tfrac{1}{2}\pi$.

dispersion relation e vs p: $e(p) = -\pi \sin(-p)$



dispersion relation only for

$-\pi < p(v_0) < 0$

only half of Brioullin zone is occup.

furthermore spin $\delta J^z = L(\delta R(0) - \tfrac{1}{2}) = -\tfrac{1}{2}$ How? all Bethe roots carry spin 1

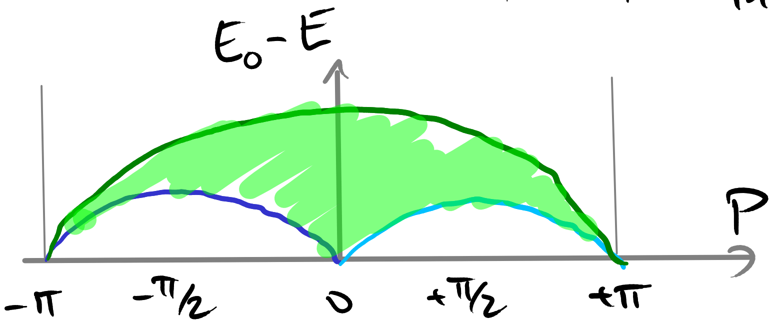
Physical Spinon States

Spinon as described above is <u>not</u> elem. spin flip (like magnon) but a collective excitation of all Bethe roots of afm vac.

It carries spin 1/2 indeed → doublet.

Important point: spinons (on own length L) can exist in <u>pairs</u> only!
resolves δJ^z issue ⇒ $\delta J^z \in \mathbb{Z}$. two spinons in $J=0$, $J=1$ state.

momentum and energy $P = p_1 + p_2 + P_0$ $\quad E = e(p_1) + e(p_2) + E_0$ $\quad L \equiv 2 \bmod 4$, $P_0 = \pi$



$p_2 = -\pi$ $\quad e_2 = 0$ $\qquad p_2 = 0$ $\qquad e_2 = 0$

$p_1 = p_2$

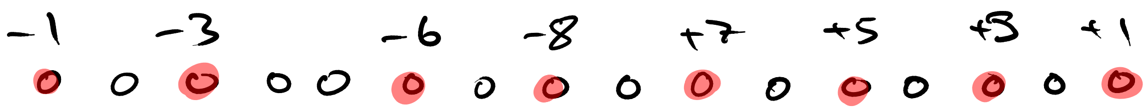
$E = E_0 + e_1 + e_2 = E_0 + 2e_1$

-1 $\quad -3$ $\quad -6$ $\quad \pm 8$ $\quad +5$ $\quad +3$ $\quad +1$

two d.o.f. to eat gaps ⇒ two quasi-particles ⇒ spinons.

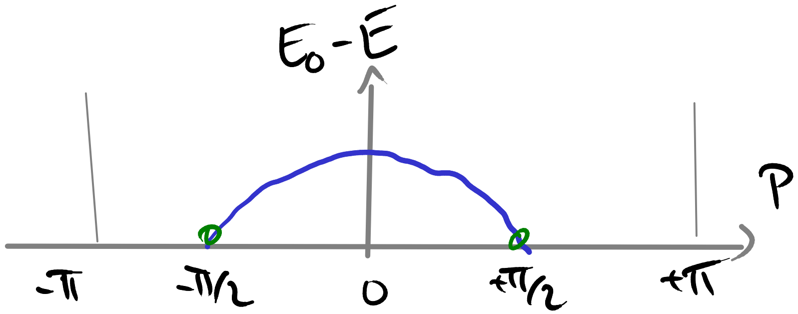
Odd Length

Slightly different: no perfect pattern of alternating occupation from $-1, -3, \ldots, +3, +1$. Typically:



single spinon ~ 4.5

on odd-length chains, only odd number of spinons is permitted.



$L \equiv 1 \mod 4$

For ground state (lowest energy): 2 doublets near $P = \pm\pi/2$

$$E = 2L \log 2 + O(1)$$

Spinon Scattering

Spinons are particle excitations of a ground state.

Scatter on an infinite line (discrete nature of chain is preserved, see Brioullin zones)

spinons are spin-1/2: doublets → scattering matrix $\curvearrowright$ tensor op rank 2

$$S(u,v) = \frac{\Gamma\left(1-\frac{i}{2}(u-v)\right)\Gamma\left(\frac{1}{2}+\frac{i}{2}(u-v)\right)}{\Gamma\left(1+\frac{i}{2}(u-v)\right)\Gamma\left(\frac{1}{2}-\frac{i}{2}(u-v)\right)}\left(\frac{u-v}{u-v+i}\,\mathrm{id}+\frac{i}{u-v+i}\,\mathrm{ex}\right)$$

difference form (difference of resp. u,v): $S(u,v) = S(u-v)$

rapidities $u = \frac{2}{\pi}\,\mathrm{artanh}\,\tan\left(\frac{1}{2}p+\frac{1}{4}\pi\right)$

up to prefactor same S-matrix as for magnons in SU(3) chain.

7.5 Spectrum Overview

- Ferromagnetic vacuum: $E=0$ $P=0$ $J=L/2$
- Magnon excitations (finitely many of finite mode number)

$$E=\sum_n M_n \frac{4\pi^2 n^2}{L^2} \qquad P=\sum_n M_n \frac{2\pi n}{L} \qquad J^3 = L/2 - \sum_n M_n$$

- large number of magnons ($M\sim L$) at finite mode number → non-linear terms

$$E\sim \frac{1}{L} \qquad -\pi < P < +\pi \qquad J\sim L$$

← described by continuous Heisenberg model (field th.)

Bethe Eq.

- Spinon excitations of anti-ferro-magnetic vacuum (come in pairs)

dispersion relation $e \sim -\sin(-p)$ $\quad p, e \sim O(L^0)$

- anti-ferromagn. vacuum $E=2L\log 2$ $\quad P=\frac{1}{2}\pi L \pmod 4$ $\quad J=0$

$$E_0 - E = \sum_n \frac{2\pi^2 (n_n)}{L} \qquad P=\pi\mathbb{Z}+\sum \frac{2\pi n_n}{L} \qquad J\leq \sum_n \tfrac{1}{2}$$

lattice system