Introduction to Integrability

Lecture Slides, Chapter 6

ETH Zurich, 2023 HS

PROF. N. BEISERT

The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

6. Quantum Spin Chains

Focus on Spin Chain Models:

- they form a large class of int. QM models
- they can be treated with some uniform framework
- they can have several parameters to tune
- short chains are genuine QM models
- long chains approximate (1+1) D QFT models
- for large quantum numbers approach classical mechanics.
- they model magnetic materials

QM magnetism: ↑↓ energy

			ferromagnetism	anti-ferrom.
nearby spins	opp. aligned	↑↓	high energy	low energy
	equal aligned	↑↑	low energy	high energy

- Ising model: class. stat mech model. lattice of spins, alignment det eng.
- Heisenberg quantum spin chain: QM model ↑↓. 1D model. Ham acts on NN.

6.1. Heisenberg Spin Chain

Setup Spin state $|\uparrow\rangle$, $|\downarrow\rangle$, or any complex lin. comb.

$\Rightarrow$ spin site described by vector space $\mathbb{V} = \mathbb{C}^2$.

Spin chain of length L is L-fold tensor product

$$\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_L$$

Hilbert space $\mathbb{V}^{\otimes L}$ has dimension 2^L. Basis from "pure" states

$$|\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\rangle$$

Hamiltonian operator $H : \mathbb{V}^{\otimes L} \to \mathbb{V}^{\otimes L}$, acts locally homogeneously

$$H = \sum_j H_j \qquad H_j = \mathbb{V}_j \otimes \mathbb{V}_{j+1} \to \mathbb{V}_j \otimes \mathbb{V}_{j+1}$$

Heisenberg Hamiltonian density

$$H_{12} = \lambda_0 (1 \otimes 1) + \lambda_x (\sigma^x \otimes \sigma^x) + \lambda_y (\sigma^y \otimes \sigma^y) + \lambda_z (\sigma^z \otimes \sigma^z)$$

all combinations of parameters $\lambda_{0,x,y,z} \in \mathbb{R}$ is integrable:

- general $\lambda_x \neq \lambda_y \neq \lambda_z \neq \lambda_x \to$ "XYZ" modell
- two λ's equal $\lambda_x = \lambda_y \neq \lambda_z \to$ "XXZ" model $SO(2)$
- all λ_{xyz} equal $\lambda_x = \lambda_y = \lambda_z \to$ "XXX" model $SO(3)$

- λ_0 has trivial effect: shifts all energies equally (by $L \cdot \lambda_0$)

Focus on XXX: $\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z =: \frac{1}{2}\lambda$ $\quad$ ferromag. $\lambda > 0$

$$H_j \in \mathrm{End}(\mathbb{V}_j \otimes \mathbb{V}_{j+1}): \quad H_j = \lambda (id_{j,j+1} - ex_{j,j+1})$$
$$= id - ex \quad (\lambda = 1)$$

$ex|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$ $\quad ex|ab\rangle = |ba\rangle$

$ex|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$

Boundary Conditions

specify boundary conditions

- finite closed, periodic boundaries : $\mathbb{V}_{j+L} \equiv \mathbb{V}_j$
- finite open chains
- infinite chains, asymptotic boundaries

Choice has impact on spectrum

- finite chains, finite spectrum => discrete
- infinite chains, continuous spectrum

Symmetry

XXX model has $SO(3)$ / $SU(2)$ symmetry H_j obey sym.

$SU(2)$ defines spin-1/2 irrep $|\uparrow\rangle$ $|\downarrow\rangle$, acts by Pauli matrices

$$\vec{S}_j = \frac{1}{2}\hbar\vec{\sigma}_j$$

Commutation rel.: $$[S_j^a, S_k^b] = i\hbar\,\delta_{j,k}\,\varepsilon^{abc} S_j^c$$

$$\vec{S}_j^{\,2} = \frac{3}{4}\hbar^2\,\mathrm{id}_j$$

Overall $SU(2)$ generated by angular mom. $\vec{J}$

$$\vec{J} = \sum_{j=1}^{L}\vec{S}_j = \sum_{j=1}^{L}\frac{1}{2}\hbar\vec{\sigma}_j$$

tensor product representation on L-fold tensor prod. of 1/2

Symmetry generator commutes with Hamiltonian

$$[J, H] = 0$$

spin j modules

$\Rightarrow$ spectrum has many degeneracies, multiplets of $su(2)$

tensor product decomposition of $(1/2)^{\otimes L}$ into $su(2)$ irreps

$L=2$: $(1) \oplus (0)$

$L=3$: $(3/2) \oplus 2\,(1/2)$

$L=4$: $(2) \oplus 3\,(1) \oplus 2\,(0)$

multiplicity of such multiplets.

spin j of multiplet $\Rightarrow$ $2j+1$ states of equal energy

Classical limit and Higher Spin Repr

for classical limit first generalise Heisenberg chain from spin 1/2 "$XXX_{1/2}$" to arbitrary spin $s \in \mathbb{Z}_0^+$: "XXX_s"

elementary vector space $\mathbb{V} = \mathbb{C}^{2s+1}$. introduce spin op $\vec{S}_j$

$$[S^a_j, S^b_k] = i\hbar\, \delta_{j,k} S^c_j, \quad \vec{S}_j^2 = \hbar^2 s(s+1)$$

eigenvalues of spin comp. $\vec{e}\cdot\vec{S}_j$ range $-\hbar s$ to $+\hbar s$ steps of $\hbar$

generalise Heisenberg NN Ham. dens H_j respecting $SU(2)$

introduce two-site total spin op.

$$J_{j,k} := \sqrt{(\vec{S}_j + \vec{S}_k)^2 + \tfrac{1}{2}\hbar^2} - \tfrac{1}{2}\hbar$$

specify Γ st eigenvalues of J are non-neg. int.

A unique Ham. dens. that respects SU(2) and is integrable

$$H_j = 2\psi(2s+1) - 2\psi\left(\frac{1}{\hbar} J_{j,j+1} + 1\right)$$

where digamma $\psi(z) := d \log \Gamma(z)/dz$ $\quad \Gamma(z) = (z-1)!$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \qquad \psi(n+1) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k}$$

- show for $s = 1/2$ get above Ham. dens.

spec. of $J_{j,k}$ is $\{0, \hbar\}$

$$J_{j,k} = \frac{3}{4}\hbar\, id + \frac{1}{4}\hbar\, \vec{\sigma}_j \cdot \vec{\sigma}_k = \frac{1}{2}\hbar\, id_{j,k} + \frac{1}{2}\hbar\, ex_{j,k}$$

using $\psi(2) = \psi(1) + 1$ $\quad H_j = 2 - 2\frac{1}{\hbar} J_{j,j+1} = \frac{id_{j,k} - ex_{j,k}}{2}$

- for $s \to \infty$ obtain classical chain $\quad \hbar = \frac{1}{s}$, assumpt. $\psi(x) \sim \log x$

$$H_j^{qu} \to -\log \frac{J_{j,j+1}}{4} \to = -\log \frac{1 + \vec{S}_j \cdot \vec{S}_{j+1}}{4} = H^{cl}$$

6.2 Spectrum of the Closed Chain

Conventional Strategy

How to obtain spectrum of Heisenberg chain by conv. methods:

- Enumerate basis of $V^{\otimes L}$ $|\downarrow\ldots\downarrow\rangle$, $|\downarrow\ldots\downarrow\uparrow\rangle$, $|\downarrow\ldots\downarrow\uparrow\downarrow\rangle$, ...
- Evaluate Ham H as a $2^L \times 2^L$ matrix in this basis
  combinatorial problem (id − ex). sparse matrix
- solve eigenvalue problem of $2^L \times 2^L$ matrix → alg. eq
- method can be used by hand for $L \simeq 6$
- computer algebra can addr. problem $L \simeq 20$
- method does not help for long chains

Short Chains

$L=2$
$(1) \times E=0$
$(0) \times E=4$

$L=3$
$(3/2) \times E=0$
$2(1/2) \times E=3$

$L=4$
$(2) \times E=0$
$2(1) \times E=2$
$(1) \times E=4$
$(0) \times E=6$
$(0) \times E=2$

...

$L=6$
$(0) \times$
$E=5+\sqrt{13}$

Bethe Equations

We can set up a sys. of alg. eq. for M variables $u_k \in \mathbb{C}$ (Bethe roots)

$$\left(\frac{u_k + \overset{\color{red}{is}}{i/2}}{u_k - \overset{\color{red}{is}}{i/2}}\right)^L = \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad k = 1 \dots M$$

M indep. eq. for M unknowns $u_k \Rightarrow$ sols to be discrete

Claim: for every eigenstate (multiplet) with tot. ang. mom. $J = \frac{L}{2} - M$ there is one soln. to eq. with $M \leq L/2$ distinct Bethe roots u_k.

and energy eigenvalue $E = \sum_{k=1}^{M} \left(\frac{i}{u_k + \overset{\color{red}{is}}{i/2}} - \frac{i}{u_k - \overset{\color{red}{is}}{i/2}} \right)$.

example $L = 6, M = 3$ su(2) singlet

$$u_{1,2} = \pm \sqrt{-\frac{5}{12} + \frac{\sqrt{13}}{6}}, \quad u_3 = 0$$

$$\Rightarrow E = 5 + \sqrt{13}$$

6.3 Coordinate Bethe Ansatz

Solution of Heisenberg XXX by Hans Bethe

Based on a quasiparticle picture on an infinite chain.

Start with ferromagnetic vacuum, put M spin flips acting as particles.

Vacuum State

Ferromagnetic vacuum simple: $|0\rangle := |\downarrow\downarrow \ldots \downarrow\rangle$.

Ham density acts trivially $H_j|0\rangle = \mathrm{id}_{j,j+1}|0\rangle - \mathrm{ex}_{j,j+1}|0\rangle = \underbrace{|0\rangle - |0\rangle}_{=0}$

$\Rightarrow H|0\rangle = E|0\rangle = 0 \quad \Rightarrow \quad E = 0$

solves the problem for $M=0$ spin flips ($L=\infty$, $L=$ finite)

Magnon States $M=1$

elem. state $|j\rangle = |\downarrow\dots\downarrow\overset{j}{\uparrow}\downarrow\dots\downarrow\rangle$ $L=\infty$ $L=$ finite

Ham closes on such states because of J^z conservation.

Eigenstates in $M=1$ sector? use H is homogeneous

$\to$ eigenstates have def. mom., are plane waves

$$|p\rangle := \sum_j e^{ijp}|j\rangle \qquad \text{magnon state}$$

magnon is a (quasi) particle with one d.o.f. p.

$$H|p\rangle = \sum_j e^{ipj}\left(H_{j-1}|j\rangle + H_j|j\rangle\right)$$

$$\overset{L=\infty}{=} \sum_j e^{ipj}\left(|j\rangle - |j-1\rangle + |j\rangle - |j+1\rangle\right)$$

$$= \sum_j e^{ipj}\left(1-e^{ip}+1-e^{-ip}\right)|j\rangle = e(p)|p\rangle$$

magnon disp. rel. $e(p) = 2(1-\cos p) = 4\sin^2(p/2)$

so far $L = \infty$ for evaluating sum leading $e(p)$.

want L = finite. need to set $p = \frac{2\pi n}{L}$ $n = 0, \ldots, L-1$

for proper periodicity $|j\rangle$ has same coeff as $|j+L\rangle$

closed boundary condition quantise p to above values.

solved sector with $M = 1$ for both $L = \infty$, L = finite

Scattering Factor

States with two spin flips

$$|j<k\rangle := |\overset{}{\downarrow}\dots\downarrow\overset{j}{\uparrow}\downarrow\dots\downarrow\overset{k}{\uparrow}\downarrow\dots\downarrow\rangle$$

form a closed sector under H. H acts locally $\to |p\rangle \otimes |q\rangle$

eigenstate ansatz $$|p<q\rangle = \sum_{j<k=-\infty}^{+\infty} e^{ipj+iqk} |j<k\rangle$$

energy eigenvalues $$E = e(p) + e(q)$$

now act with $H - E = H - e(p) - e(q)$ on $|p<q\rangle$

$$\dots = (e^{ip+iq} - 2e^{iq} + 1) \underbrace{\sum_{j=-\infty}^{+\infty} e^{i(p+q)} |j<j+1\rangle}_{\text{symmetric in } p,q}$$ contact term

act instead on $|q<p\rangle$

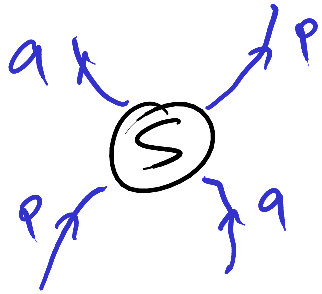
$$\ldots = \left(e^{iq+ip} - 2e^{ip} + 1\right) \sum_{j=-\infty}^{+\infty} e^{i(p+q)} \, |j<j+1\rangle$$

compose true eigenstate

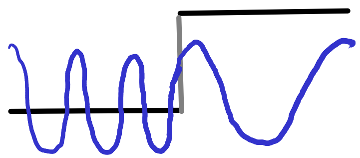
$$|p,q\rangle := |p<q\rangle + S(p,q)\,|q<p\rangle$$

Scattering factor S

$$S(p,q) := -\frac{e^{ip+iq} - 2e^{iq} + 1}{e^{ip+iq} - 2e^{ip} + 1}$$



similar to QM potential barrier problems



Factorised Scattering

$M=3$ magnons. there are $6=3!$ asymptotic regions

each magnon carries index. mom p_k. Bethe Ansatz for eigenstate

$$|p_1,p_2,p_3\rangle = |p_1<p_2<p_3\rangle + S_{12}S_{13}S_{23}\,|p_3<p_2<p_1\rangle$$
$$+ S_{12}\,|p_2<p_1<p_3\rangle + S_{13}S_{23}\,|p_3<p_1<p_2\rangle$$
$$+ S_{23}\,|p_1<p_3<p_2\rangle + S_{12}S_{13}\,|p_2<p_3<p_1\rangle$$

note $S_{12}=S_{21}^{-1}$. all pairwise contact terms are cancelled.

typically expect: $E=e(p_1)+e(p_2)+e(p_3)$

$$(H-E)|p_1,p_2,p_3\rangle = \sum_j e^{iPj}\,|j<j+1<j+2\rangle$$

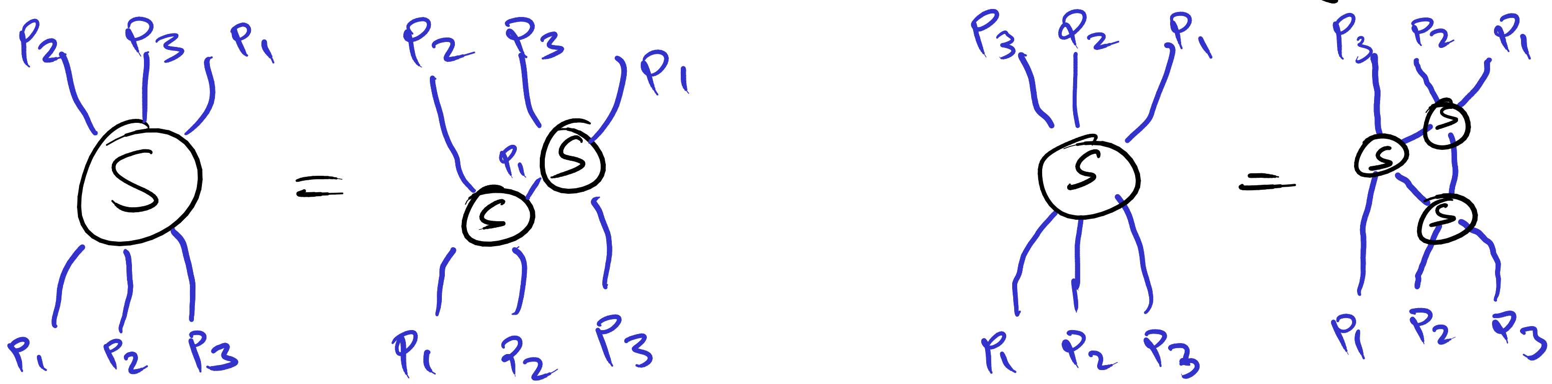
would have to cancel by different comb. of $p_1'\,p_2'\,p_3'$

with equal E,P. Here Miracle: no 3-magnon contact term

$|p_1,p_2,p_3\rangle$ exact eigst.

Miracle is integrability. No contact terms for $M \geq 3$.
H has range 2 => expected.

Scattering of $M \geq 3$ magnons is factorised
(into a sequence of two-magnon scattering events).



Solution on Infinite Chain

$$|0\rangle = |\downarrow \ldots \downarrow\rangle \qquad E = 0$$

$$|p\rangle = \sum_j e^{ipj} |\ldots \overset{j}{\uparrow} \ldots\rangle \qquad E = e(p)$$

$$|p,q\rangle = |p<q\rangle + S(p,q)\,|q<p\rangle \qquad E = e(p) + e(q)$$

$$|\{p_k\}\rangle = \sum_{\pi \in S_M} S_\pi \,|p_{\pi(1)} < \ldots < p_{\pi(M)}\rangle \qquad E = \sum_k e(p_k)$$

Note: p_k defined mod 2π (lattice)

ordering of p_k matters only for prefactor of $|\{p_m\}\rangle$

No identical momenta: $S(p,p) = -1 \Rightarrow$ Fermions!

su(2) symmetry related to $p=0$: $\quad S(p,0) = 1 \quad e(0) = 0$

$\uparrow$ ladder op. for su(2) multiplets.

Bound States

want states to be normalisable (difficult on ∞ chain with def. mom)

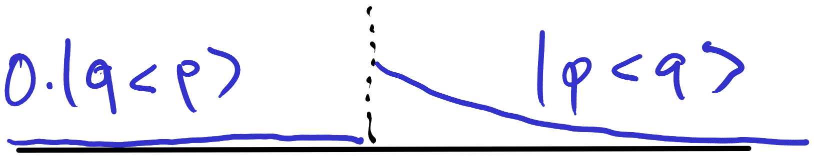
don't want exponential growth as $j \to \pm\infty$

demand that all p_k are real.

but can also allow complex p_k if exponential growth is excluded

- exponential growth for each plane wave factor happens only on one side
- make sure coefficient of this asymptotical partial contribution is zero.

achieved by $S(p,q) = 0, \infty$ for corresponding momenta.



bound states of magnons.

$$e_2(p) = 2\sin^2(p/2) \qquad e_n(p) = \frac{4}{n}\sin^2(p/2)$$

p overall mom. of n bound magnons.

6.4 Bethe Equations

Focus on finite, periodic chains / states.

Closed Chains

roughly: periodic wave functions: $\langle j_1 \ldots | \psi \rangle = \langle j+L, \ldots | \psi \rangle$

construction: pick a range of L sites on ∞ chain.

consider contrib. to $|\psi\rangle$ where all spin flips are in range

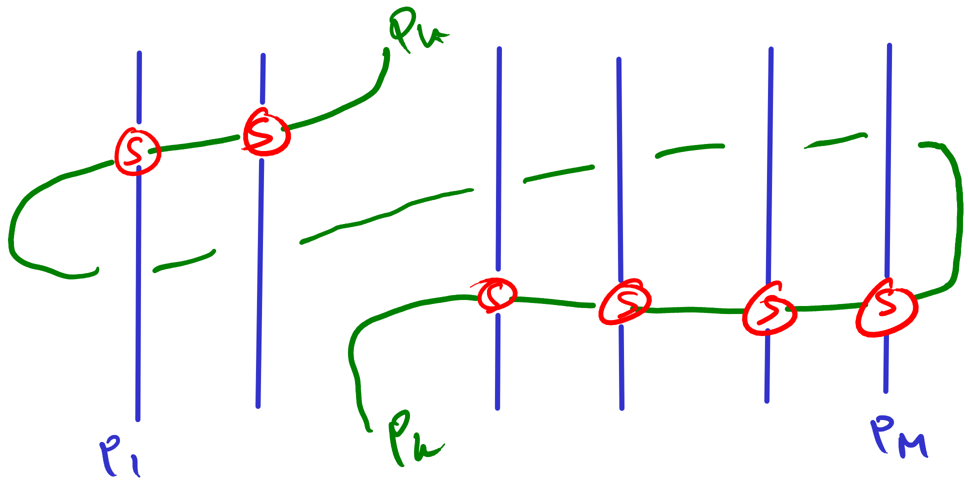
to match boundaries to be periodic:

- pick leftmost magnon with mom p_k
- shift it by L sites to right $\quad e^{ip_k L}$
- scatter with all $M-1$ magnons $\quad \prod_{j \neq k} S(p_k, p_j)$

want $\langle j_1, j_2, \ldots j_M | \psi \rangle \overset{!}{=} 1 \cdot \langle j_2, j_3, \ldots, j_M, j_1 + L | \psi \rangle$

Obtain Bethe Equations

$$e^{ip_k L} \prod_{\substack{l=1 \\ l\neq k}}^{M} S(p_k, p_l) = 1 \quad \text{for } k = 1, \dots, M$$



$\doteq 1$

Product of all B.E.

$\downarrow$

$$E = \sum_{k=1}^{M} e(p_k) \qquad P = \sum_{k=1}^{M} p_k \pmod{2\pi} \quad \text{note } e^{iPL} = 1$$

P is quantised as $\frac{2\pi n}{L}$.

Rapidity Variables

BE are in trigonometric form.

introduce new set of Bethe roots $\{u_k\}$

$$p_k = 2\,\mathrm{arccot}(2u_k) \qquad u_k = \frac{1}{2}\cot(p_k/2) \qquad e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}$$

$$S(u,v) = \frac{u-v-i}{u-v+i} \qquad e(u) = \frac{i}{u+i/2} - \frac{i}{u-i/2}$$

Bethe equations

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad k = 1, \ldots, M$$

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} \qquad E = \sum_{k=1}^{M}\left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right) = \sum \frac{1}{u_k^2 + 1/4}$$

6.5 Generalisations

Open Chains

$$H = \sum_{j=1}^{L-1} H_n$$ finite open chain, extend range to $j = 1, \ldots \infty$ semi-infinite

· Act with $H - e(p)$ on a one-magnon state $|+p\rangle = \sum_{j=1}^{+\infty} e^{ipj} |j\rangle$

$$(H - e(p))|+p\rangle = (1 - e^{+ip}) \, |1\rangle$$

contact term at boundary

Need to cancel with another state of same $E = e(+p)$

$\Rightarrow +p \to -p \qquad e(-p) = e(+p) \qquad \bar{p} = -p$

$$(H - e(p))|-p\rangle = (1 - e^{-ip}) \, |1\rangle$$

exact eigenstate at left boundary

$$||p|\rangle_L = e^{-ip} |+p\rangle + e^{+ip} k_L(+p) \, |-p\rangle$$

with boundary scattering factor k_L

$$k_L(+p) = -e^{-2ip}\frac{1-e^{+ip}}{1-e^{-ip}} = e^{-ip}$$

Analogously for right boundary at site $j=L$

$$||p|\rangle_R = e^{-ipL}|+p\rangle + e^{+ipL}k_R(+p)\,|-p\rangle$$

$$k_R(+p) = e^{+ip}$$

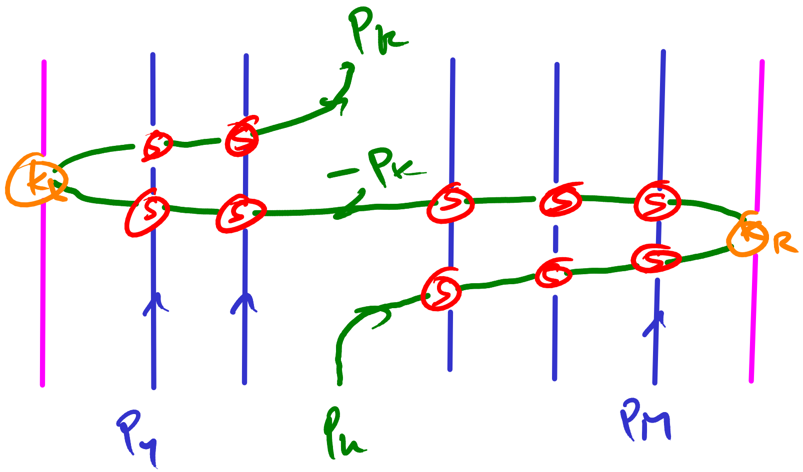
for consistency we need to satisfy $k_{L/R}(-p) = k_{L/R}(+p)^{-1}$.

as well as $$\frac{S(+p,+q)}{S(-p,+q)} = \frac{S(+p,-q)}{S(-p,-q)}$$

For multi-magnon states: Bethe Equations

$$\frac{e^{i(L-1)(+p_k)}\, K_R(+p_k)}{e^{i(L-1)(-p_k)}\, K_L(+p_k)} \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{S(+p_k,p_l)}{S(-p_k,p_l)} = 1$$

$$\frac{e^{+ip_k(L-1)}}{e^{-ip_k(L-1)}}$$



$= 1$

rational form $\left(\frac{u_k+i/2}{u_k-i/2}\right)^{2L} = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i}\, \frac{u_k+u_l+i}{u_k+u_l-i}$

XXZ model

Extend Ham slightly to local terms

$$H = \alpha_1 (1 \otimes 1) + \alpha_2 (\sigma^z \otimes 1) + \alpha_3 (1 \otimes \sigma^z) + \alpha_4 (\sigma^z \otimes \sigma^z)$$
$$+ \alpha_5 (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + i\alpha_6 (\sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x)$$

Six parameters related to:

- one overall energy shift ($\sim L$) $\to \delta\alpha_1$
- one triv. deformation for closed chains $\to \delta\alpha_2 = -\delta\alpha_3$
- one shift prop to J^z: $\to \delta\alpha_2 = +\delta\alpha_3$
- one overall scaling " $\to$ $\delta\alpha_k = \alpha_k \cdot \delta\beta$
- one def. parameter h, $q = e^{ih}$ anisotr. $\Delta = \frac{1}{2}(q + 1/q)$
- one magnetic flux parameter ρ

Bethe eq. (trigonometric)

$$\frac{\sinh(u_k + i/2)}{\sinh(u_k - i/2)} e^{ipL} = \prod_{\substack{l=1 \\ l\neq k}}^{M} \frac{\sinh(u_k - u_l + i)}{\sinh(u_k - u_l - i)}$$

$$e^{ip(u)} = \frac{\sinh(u + i/2)}{\sinh(u - i/2)} \qquad e(u) = -p'(u)$$

obtain XXX model for $h=0$

Higher Spin XXX_s concretely $s=1$ $|0\rangle, |1\rangle, |2\rangle$

Preserves J^z for two-site contrib. to H find block-diag.

$$H = \left(\begin{array}{c|cc|ccc|cc|c}
* & & & & & & & & \\ \hline
 & * & * & & & & & & \\
 & * & * & & & & & & \\ \hline
 & & & * & * & * & & & \\
 & & & * & * & * & & & \\
 & & & * & * & * & & & \\ \hline
 & & & & & & * & * & \\
 & & & & & & * & * & \\ \hline
 & & & & & & & & *
\end{array}\right)
\qquad
E = \left(\begin{array}{c}
|00\rangle \\ \hline
|10\rangle \\
|01\rangle \\ \hline
|20\rangle \\
|11\rangle \\
|02\rangle \\ \hline
|21\rangle \\
|12\rangle \\ \hline
|22\rangle
\end{array}\right)$$

carry out Bethe ansatz according to J^z

vacuum state $|0\rangle = |0\ldots 0\rangle$

magnon state $|p\rangle = \sum e^{ipj}\, |0\ldots 0 \overset{j}{1} 0 \ldots 0\rangle$

two magnons: new contact term

$$|p<q\rangle = \sum_{j<k} e^{ipj+iqk} |\ldots \overset{j}{1} \ldots \overset{k}{1} \ldots\rangle$$

$$|p;2\rangle = \sum_{j} e^{ipj} |\ldots \overset{j}{2} \ldots\rangle$$

$$(H-E)|p<q\rangle = \sum_j e^{i(p+q)j}\left(*|\ldots \overset{j\,j+1}{11} \ldots\rangle + *|\ldots \overset{j}{2} \ldots\rangle\right)$$

$$(H-E)\,|p;2\rangle = \sum_j e^{ipj}\left(*|\ldots \overset{j\,j+1}{11} \ldots\rangle + *|\ldots \overset{j}{2} \ldots\rangle\right)$$

total exact eigenstate

$$|p,q\rangle = |p<q\rangle + S|q<p\rangle + C|p+q;2\rangle$$

note S: scattering factor → IR → relevant to BE
C: contact term → UV → irrelevant.

resulting Bethe equations

$$\left(\frac{u_k+i}{u_k-i}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip}=\frac{u+i}{u-i} \qquad e(u)=-p'(u)$$

$$XXX_{1/2} \to XXX_1 \to XXX_s \sim \quad {}^{i}/_{2} \to i \to is$$

$$\left(\frac{u_k+is}{u_k-is}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip}=\frac{u+is}{u-is} \qquad e(u)=-p'(u)$$