Introduction to Integrability

Lecture Slides, Chapter 4

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

4. Spectral Curves

4.1 Spectral Curve

Start with some generic state $\vec{S}_j(t) \to T(u)$ Lax matrix
know spectrum of $T(u)$, in particular trace $F(u)$ is conserved

<u>Eigenvalues</u> spectrum of $T(u)$ is time-independent for all $u \in \bar{\mathbb{C}}$.

here trace $F(u)$ determines spectrum

recall $\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} \Rightarrow \det T(u) = \left(1 + 1/u^2\right)^L$

$F(u) = \operatorname{tr} T(u)$ is a polynomial of deg. L in $1/u$

$$\begin{aligned} \tau_1 \tau_2 &= \det T, \\ \tau_1 + \tau_2 &= \operatorname{tr} T = F \end{aligned} \Rightarrow \tau_{1,2}(u) = \frac{1}{2} F(u) \pm \sqrt{\frac{1}{4} F(u)^2 - \left(1 + 1/u^2\right)^L}$$

Singularities

$\tilde{u} = 0$

elements of $T(u)$ is polynomial in $1/u$ of degree L $\Rightarrow$ analytic except at L-fold pole at $u = \tilde{u} = 0$ $\Rightarrow$ $\tau_{1,2}(u)$ will have L-fold pole at $u = 0$.

nevertheless $\tau_{1,2}(u)$ do not need to be analytical at $u \neq 0$

Some exceptions to analyticity possible due to solving EV.

Namely where radicand of soln of $\tau_{1,2}$ equals zero.

$\Rightarrow$ square-root branch points $\hat{u}_j$ where $\frac{1}{4} F(\hat{u}_j)^2 = \left(1 + \frac{1}{\hat{u}_j^2}\right)^L$

algebraic eq. of deg $2L$ in $1/\hat{u}$. $\Rightarrow$ $2L$ solutions $\hat{u}_j$ $j = 1 .. 2L$.

These are where $\tau_1(\hat{u}_j) = \tau_2(\hat{u}_j)$

Note that $F(u)$ is special at $u=\infty$

$$F(u) = 2 + \frac{0}{u} - \frac{1}{u^2}\left(J^2 - L\right) + \ldots$$

leading two coefficients of alg. eq match $\Rightarrow$ 2 fixed solutions $\hat{u}_{2L-1} = \hat{u}_{2L} = \infty$

(related to $so(3)$ symmetry)

and $2L-2$ poles which are not universally fixed.

Simple Solutions

$L=2$ $\quad s_{1/2}(t) = \begin{pmatrix} 1 \\ I\tan(\vartheta/2)\, e^{-i\omega t} \end{pmatrix}$ $\quad \omega = \frac{2}{\cos\vartheta}$ pow.

$$T(u) = \mathrm{id} + \frac{2i}{u}\cos\vartheta\,\sigma^z - \frac{1}{u^2}\begin{pmatrix} \cos(2\vartheta) & e^{+i\omega t}\sin(2\vartheta) \\ -e^{-i\omega t}\sin(2\vartheta) & \cdot\cos(2\vartheta) \end{pmatrix}$$

$$F(u) = \mathrm{tr}\,(u) = 2 - \frac{2}{u^2}\cos(2\vartheta) \qquad H = -\log\frac{F(i)F(-i)}{16} = -4\log|\cos\vartheta|$$

$$\tilde{u}_{1,2}(u) = 1 + \frac{\cos(2\vartheta)}{u^2} \pm \frac{2i\cos\vartheta}{u}\sqrt{1 + \frac{\sin^2\vartheta}{u^2}}$$

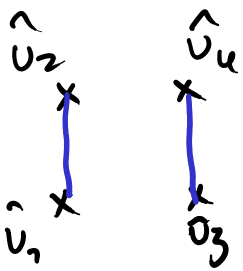
square-root pt at $\hat{u}_{1,2} = \mp i \sin\vartheta \leftarrow$ all information (ϑ) contained in $\hat{u}_j$

example at $L=3$ great circle $\quad F(u) = 2 + \dfrac{3 - J^2}{u^2}$

select case $1 < J \leq 3$ parametrise as $J^2 = 5 - 4\cos\mu$, $\mu \in]0,\pi]$

branch points at

$$\hat{u}_{1,4} = \pm \frac{e^{-i\mu}}{\sqrt{1 - 2e^{-i\mu}}} \qquad \hat{u}_{2,3} = \pm \frac{e^{i\mu}}{\sqrt{1 - 2e^{i\mu}}} = \hat{u}_{1,4}^*$$



Spectral Curve

investigate square root branch points $\hat{u}_n$ + neighbourhood

branch point $\hat{u}$ is where analyticity of $\tau_{1,2}(u)$ breaks

$$\tau_{1,2}(u) = \frac{1}{2} F(\hat{u}) \pm \hat{k}\sqrt{u-\hat{u}} + \mathcal{O}(u-\hat{u})$$

small circle

Follow function $\tau_1(v)$ around $u = \hat{u}$ $\quad u(\sigma) = \hat{u} + \epsilon e^{i\sigma}$

$$\tau_1(u(\sigma)) = \frac{1}{2} F(\hat{u}) + \hat{k}\sqrt{\epsilon}\, e^{i\sigma/2} + \mathcal{O}(\epsilon)$$

$\tau_1(u(\sigma))$ returns to initial value after rotation of σ by 4π.

rotation by 2π : interchanges eigenvalues $\tau_1 \leftrightarrow \tau_2$

$$\tau_1(u(\sigma+2\pi)) = \tau_2(u(\sigma))$$

$$\{\tau_j(u(\sigma+2\pi))\} = \{\tau_j(u(\sigma))\}.$$

2 eigenvalue functions $\tau_a(u)$ form a two-sheeted cover of $\bar{\mathbb{C}}$ (minus puncture at $u = \hat{u} = 0$)

branch points are connected in pairs by branch cuts.

eigenvalue functions $\tau_a(u)$ as single valued function $f(z)$

on a Rieman surface Γ ~~noth-triv. topology.~~ as follows

for every $z \in \Gamma$ associate a sheet $a(z) \in \{1,2\}$ and pt $u(z) \in \bar{\mathbb{C}}$

s.t. $f(z) = f_{a(z)}(u(z))$ and cuts are where $a(z)$ is discont.

Rieman surface is a complex curve (spectral curve)

1-d submfd Γ of 2-d complex space $(u, \tau) \in \mathbb{C}^2$

$$\Gamma = \{ (u, \tau) \in \bar{\mathbb{C}}^2 ;\ \det(T(u) - \tau) = 0 \}.$$

for every value of u there are two points $z \in \Gamma$

provide permutation map $z \to z^*$ st $u(z^*) = u(z)$

$$\tau(z^*) = \frac{\det T(u(z))}{\tau(z)} = F(u(z)) - \tau(z).$$

Example $L=2$

$$\tilde{\tau}_{1,2}(u) = 1 + \frac{\cos(2\vartheta)}{u^2} \pm \frac{2i\cos\vartheta}{u}\sqrt{1 + \frac{\sin^2\vartheta}{u^2}}$$

introduce $u(z) = \frac{1}{2}\sin\vartheta \cdot (z - 1/z)$

$$\tau(z) = \left(\frac{z + 1/z - 2i\cot\vartheta}{z - 1/z}\right)^2, \quad z \to z^* = -1/z$$

branch pt. $\hat{z}_{1,2} = \mp i$

4.2 Ground State and Excitations

Compare spectral curve to (perturbative) solutions:
Ferromagnetic ground state + excitations.

Ground State $\vec{S}_j = \vec{e}_z$

$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u}\sigma^z := \mathcal{L}$ eigenval $(u \pm i)/u$

$\Rightarrow T(u) = \mathcal{L}(u)^L$ has eigenvalues

$$T_{1,2}(u) = \frac{(u \pm i)^L}{u^L}.$$ two disconnected sheets.

have no square-root singularities.. $\Rightarrow g = -1$

but normal genus at L is $g = L - 2 \gg -1$

spectral curve is highly degenerate.

degeneracy of Γ. consider $F(u)$

$$F(u) = \tau_1(u) + \tau_2(u) = \frac{(u+i)^L + (u-i)^L}{u^L} \quad \text{pol. deg } L$$

$$\tau_{1,2}(u) = \frac{1}{2} F(u) \pm \sqrt{\frac{1}{4} F(u)^2 - \frac{(u^2+1)^L}{u^{2L}}}$$

potential branch points: $0 = \left(\frac{(u+i)^L - (u-i)^L}{2u^L}\right)^2$

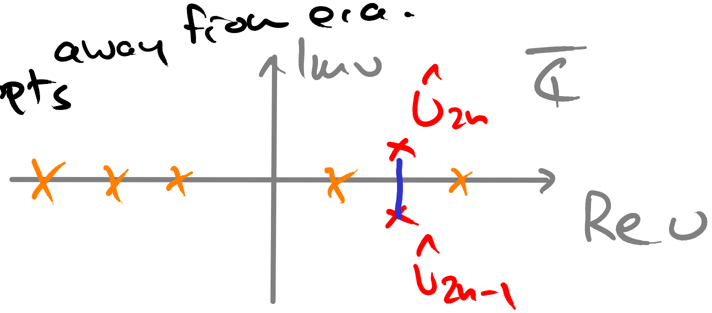
$2L-2$ double roots at $\hat{u}_{2k-1,2k} = \cot\frac{\pi k}{L} \quad k = 1 \ldots L-1$

no singular behaviour of $\tau_{1,2}(u)$ at $u = \hat{u}_k$

but this signals that a higher-genus curve has degenerated to this configuration by moving two nearby branch pts together.

Single Excitation

move two branchpts away from e.r.a.



how to change $F(u)$ to achieve this?

preserve polynomial nature of $F(u)$.

done by $F \to F + \delta F$ with

$$\delta F(u) = i\epsilon^2 \frac{(u+i)^L - (u-i)^L}{u^L (u - \hat{u}_{2n})}$$

- preserves polyn.
- zeros at $u = \hat{u}_{2n}$
- except at $u = \hat{u}_{2n}$

deformed eq.

$$F(\hat{u})^2 = 2F(\hat{u})\,\delta F(\hat{u}) + \ldots = \frac{4(\hat{u}^2+1)^L}{\hat{u}^{2L}}$$

solutions: $\hat{u} = \hat{u}_{2k}$ (twice) for $k \neq n$

$$\hat{u}_{2n-1,2n} = \hat{u}^{\circ}_{2n} \mp \frac{i\epsilon\sqrt{2/L}}{\sin(\pi n/L)}.$$

analyse charges of corresponding state through $F(u)$

$u=\infty$ $\quad \delta F(u) = \frac{2L\epsilon^2}{u^2} + \ldots \Rightarrow$ tot ang. mom. J $\quad \delta J^2 = -2L\epsilon^2$

$$\Rightarrow \delta \vec{J} = -\epsilon^2 \vec{e}_z$$

energy & momentum

$$\delta H = -\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = \frac{2\epsilon^2}{\hat{u}_{2n}^2+1} = 2\epsilon^2 \sin^2\frac{\pi n}{2}$$

similar to excitations of ferromag. vacuum.

combine J & H $\Rightarrow$ $\delta H = -2\delta J \sin^2\frac{\pi n}{L}$ matches prev.!

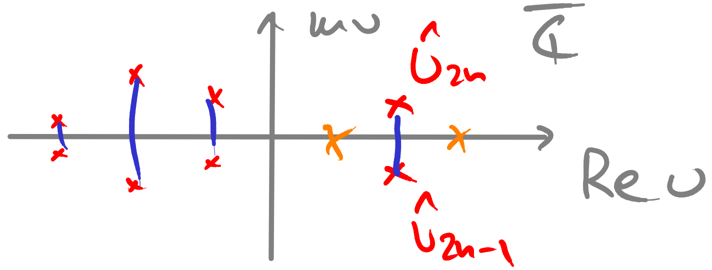
use action variables

$$\delta I_n = \pm\frac{1}{2\pi}\oint_{\hat{u}_{2n}} \frac{du\, \tau(u)}{\sqrt{\det \tau(u)}} = \epsilon^2 + \ldots$$

$$\left.\begin{aligned} \delta H &= 2\delta I_n \sin^2\frac{\pi n}{L} + \ldots \\ \delta\vec{J} &= -\delta I_n \vec{e}_z + \ldots \end{aligned}\right\} \text{agrees!}$$

$$\omega_n = \frac{\partial H}{\partial I_n} = 2\sin^2\frac{\pi n}{L}.$$

Multiple Excitations



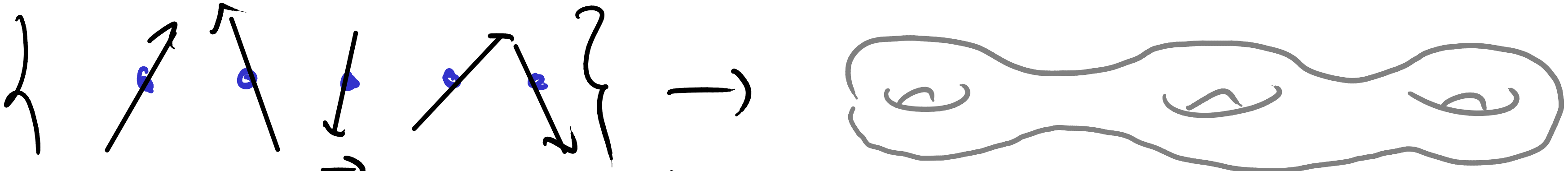
here to order ϵ^2 all deformations are independent $\Rightarrow$ qty add up $\qquad H = \sum_{n=1}^{L} I_n \cdot \omega_n$

leading order matches.

spectral curve provides an exact description beyond linear regime

e.g. take I_n larger, still obtain precise results including non-linear effects

L-1 excitation modes of f.m. vac



conserved charges $\vec{S}_j$ (half of phase space) $\qquad T$ L-1 moduli

4.3 Dynamical Divisor

Singularities

Eigenvectors determined by EV eq. τ_a eigenvalues $a=1,2$
ψ_a corr. eigenvectors

$$T(u)\,\psi_a(u) = \tau_a(u)\,\psi_a(u)$$

Eq. has a solution $\psi_a(u)$ for all $\tau_a(u)$ for all u
dependence on u is analytic almost everywhere
3 types...

1. monodromy $T(u)$ has a pole singularity
   $\Rightarrow \tau_a(u)$ has same singularity
   know $T(u)$ has L-fold pole at $u=\hat{u}=0$
   can remove singularity by rescaling by some pol. fn. u^L
   this does not affect eigenvectors
   So no particular singularity in $\psi_a(u)$ to be expected.

2. square-root singularities in $T_a(u)$ but not $T(u)$ (diagonalisation)
contradiction from assuming $\psi_a(u)$ to be analytic
$\Rightarrow$ $\psi_a(u)$ has a square-root singularity at branch pt.

3. normalisation of eigenvectors is undetermined by EV eq.
may renormalise $\psi_a(u)$ by $f(u)$; by this generate/remove pole sing.

Branch Points

- at square-root sing. both eigenvectors degenerate $\psi_1(\hat{u}) = \psi_2(\hat{u})$
- monodromy $T(\hat{u})$ is non-diagonalisable at these points.
  $\to$ single true eigenvector $\psi_1(\hat{u}^*) = \psi_2(\hat{u})$

non-diagonalisable $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

A, B, C, D are analytic at $u = \hat{u}$.
($T(\hat{u})$ is not)

eigenvalues

$$\tau_{1,2}(u) = \frac{1}{2}\left(A(u) + D(u)\right) \pm \sqrt{\frac{1}{4}\left(A(u) - D(u)\right)^2 + B(u)C(u)}$$

branch pt are where $\tau_1 = \tau_2$, radicand $= 0$

expand at $u = \hat{u}$ $\quad \tau(u) = \tau(\hat{u}) \pm \hat{k}\sqrt{u - \hat{u}} + \ldots$

$$\hat{k} = \sqrt{\frac{1}{2}(\hat{A} - \hat{D})(\hat{A}' - \hat{D}') + \hat{B}\hat{C}' + \hat{C}\hat{B}'}$$

assume $T(\hat{u})$ to be diagonalisable: two eigenval. $\tau_1 = \tau_2$

$\Rightarrow T(\hat{u}) = \tau_{1,2} \cdot id \quad \Rightarrow \hat{A} = \hat{D}, \quad \hat{B} = \hat{C} = 0$

$\Rightarrow \hat{k} = 0 \Rightarrow$ no square root branch point.

consider behaviour of eigenvectors at $u = \hat{u}$

$$\psi_a(u) = \begin{pmatrix} -B(u) \\ A(u) - \tau_a(u) \end{pmatrix}$$

Beneficial for formulating $\psi(z)$ as a function on Γ

$$\psi_1(\hat{u}) = \psi_2(\hat{u})$$

namely $\quad \psi(z) = \psi_{a(z)}(u(z))$ is analytic on Γ at $u = \hat{u}$

EV eq on Γ $\quad T(u(z))\,\psi(z) = \tau(z)\,\psi(z)$

$\tau(z)$, $\psi(z)$ are analytic on Γ

example chain with $L=2$

$$\psi(z) = \begin{pmatrix} 1 \\ i e^{-i\omega t} z \end{pmatrix}$$

Dynamical Divisor

scaling of $\psi(z)$ is not determined. where are singularities?

$$\psi(z) \equiv \lambda(z)\psi(z)$$

Therefore we normalise $\psi(z)$ is some particular way our choice

eg. $$v_r \cdot \psi(z) \stackrel{!}{=} 1 \quad \text{for some vector } v_r.$$

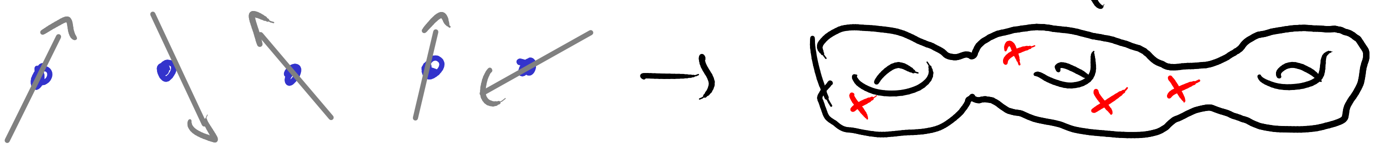
for choice $v_r = (1\ 0)$ $\Rightarrow$ $\psi(z) = \begin{pmatrix} 1 \\ \zeta(z) \end{pmatrix}$ stereographic projection.

reduces information in $\psi(z)$ to a function $\zeta(z)$

well-defined (but dependent on v_r) set of poles $\{\check{z}_k\}$

this set encodes all dynamical data of state

$\Rightarrow$ $\{\check{z}_k\}$ dynamical divisor for state (set of marked points on Γ)



Alternative picture for $\{\breve{z}_n\}$:

ψ is map $\Gamma \to \mathbb{CP}^1$ (rather than $\mathbb{C}^2$)

namely: ψ is defined up to scaling, ψ describes direction

$\breve{z}_n$ are poles of $\zeta(z)$ but these originate from normalisation

$V_r \cdot \psi(z) = 1$ $\quad$ $\breve{z}_n$ is where $\psi(z) \sim V_r^{\perp}$

Divisor consists of all points $\breve{z}_n$ where $\psi(\breve{z}_n)$ takes a specific direction.

claim: $\{\breve{z}_n\}$ consists of $g+1$ points on Γ $\quad$ $\underset{\downarrow}{\psi} \sim \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$

where g is genus of Γ.

Define function $f(u) := \left(\psi_1(u)^T \epsilon\, \psi_2(u)\right)^2 = \left(\zeta_1(u) - \zeta_2(u)\right)^2$

1. $f(u)$ is a meromorphic function of $u \in \bar{\mathbb{C}}$

- certainly of z! • interchange two eigenvalues/vectors $\psi_1 \leftrightarrow \psi_2$
  $f(u)$ remains the same $\Rightarrow$ also analytic here.

2. zeros of $f(u)$ are branch points.
- note $f(u) = 0$ if two vectors are collinear at branch pt.
- if $T(u)$ is diagonalisable (generic u) $\Rightarrow$ two eigenvectors span $\mathbb{C}^2$ $\Rightarrow f(u) \neq 0$

further each branch point contributes single zero for $f(u)$.

for a curve Γ of genus g two sheets are connected by
$g+1$ branch cuts $\Rightarrow$ $2g+2$ branch points.

3. meromorphic fn. $f(u)$ on compact $\bar{\mathbb{C}}$ has as many poles as zeros.
$2(g+1)$ poles. all poles are double by construction $f(u) = (\ldots)^2$
double pole due to either $\zeta_1(u)$ or $\zeta_2(u)$ (single) $\Rightarrow$ $g+1$ poles in $\zeta(z)$.

example $L=2$ state $\quad v_r = (1, -1/\varsigma_r) \quad \varsigma_r \in \overline{\mathbb{C}}$.

normalise ψ st. $\quad v_r \cdot \psi = \psi_1 - \psi_2/\varsigma_r = 1$

$$\psi(z) = \frac{1}{1 - i\,\varsigma_r^{-1} e^{-i\omega t} z} \begin{pmatrix} 1 \\ i\, e^{-i\omega t} z \end{pmatrix}$$

pole at $\quad \breve{z}(t) = -i\,\varsigma_r\, e^{i\omega t} \quad$ (rotates with ω on $\Gamma = \overline{\mathbb{C}}$)

Evolution

$\{\breve{z}_k\}$ describes truly dynamical data of state

set moves around on Γ in well-prescribed way

$$\frac{dT}{dt} = [M, T] \Rightarrow \frac{d\psi}{dt} = M\psi + \lambda\psi$$

normalisation as t progresses

keep $v_r \cdot \psi = 1$ solve for λ

$$\frac{d}{dt}\psi(z) = M(z)\psi(z) - \left(v_r \cdot M(z)\psi(z)\right) \cdot \psi(z)$$

non-linear, but nevertheless has solution.

consider eq. near a pole $\breve{z}$. double poles on both sides: cancel;

$$\frac{d\breve{z}}{dt} = -\operatorname*{res}_{z=\breve{z}}\left(v_r \cdot M(z)\psi(z)\right)$$

Example:

$$M(u) \equiv \frac{1}{u^2+1}\frac{1}{\cos\vartheta}\begin{pmatrix} i & u\, e^{+i\omega t}\sin\vartheta \\ -u\, e^{-i\omega t}\sin\vartheta & -i \end{pmatrix}$$

EV evolution

$$\frac{d}{dt}\psi + \lambda_1\psi = \frac{2}{\cos\vartheta}\begin{pmatrix} 0 \\ z\, e^{-i\omega t} \end{pmatrix} = M\psi + \lambda_2\psi$$

verify using solution $\breve{z} = -i\,\zeta_r\, e^{i\omega t}$

$$\operatorname*{res}_{z=\breve{z}} \psi(z) = \breve{z}\begin{pmatrix} 1 \\ \zeta_r \end{pmatrix}$$

$$(1-\zeta_r^{-1})\, M(u)\begin{pmatrix} 1 \\ \zeta_r \end{pmatrix} = \frac{2i}{\cos\vartheta}\,\frac{u\, u(\breve{z})+1}{u^2+1}$$

$$\Rightarrow \quad \frac{d\breve{z}}{dt} = i\omega\,\breve{z} = \frac{2i}{\cos\vartheta}\,\breve{z}$$ holds for actual ang. vel. $\omega = 2/\cos\vartheta$

Symmetry

system has $SO(3)$ rotation symmetry and cons. charge $\vec{J}$

- lowers the typical genus of curve from $g = L-1 \to g = L-2$

because pt $u = \infty$ related to symmetry is double pt of Γ

means that direction $\vec{J}$ is not encoded in Γ,

not in divisor

review expansion at $u = \infty$

$$T(u) = \mathrm{id} + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \ldots$$

at $u = \infty$ eigenvectors of $T(u)$ are not fixed by EV eq.

because $T(\infty) = \mathrm{id}$. nevertheless can consider $u \to \infty$

suppose $\upsilon(z) = \dfrac{c}{z - z_\infty} + \dots$ on Γ

$J = |\vec{J}|$

then $\tau(z) = 1 \pm \dfrac{iJ}{c}(z - z_\infty) + \dots$

Eigenvectors $\psi_{1,2}(z)$ as $z \to z_\infty / z_\infty^\times$

$$(\vec{J}\cdot\vec{\sigma})\,\psi(z_\infty) = \pm J\,\psi(z_\infty)$$

$$(\vec{J}\cdot\vec{\sigma})\,\psi(z_\infty^\times) = \mp J\,\psi(z_\infty^\times)$$

4.4 Construction of Solutions

Spectral Curve

construct $\tau(z)$ on Riemann surface Γ

$$\tau(z)^2 - F(u(z))\,\tau(z) + \det T(u(z)) = 0$$

$F(u)$ is a polynomial of deg. L in $1/u$

leading terms $F(u) = 2 + 0/u + \ldots$ $\quad L-1$ d.o.f.

$\det T(u) = (1 + 1/u^2)^L$

alg. eq. describes $2L-2$ branch pt $\Rightarrow$ $L-1$ cuts, genus $g = L-2$

has $L-1$ indep. moduli

correspond to $L-1$ action variables

Dynamical Divisor

assume normalisation $\Psi(z) = \begin{pmatrix} 1 \\ \zeta(z) \end{pmatrix}$

as a meromorphic function of degree $g+1$ (poles)

Riemann-Roch thm $\Rightarrow$ $3+g$ d.o.f. in choosing $\psi(z)$

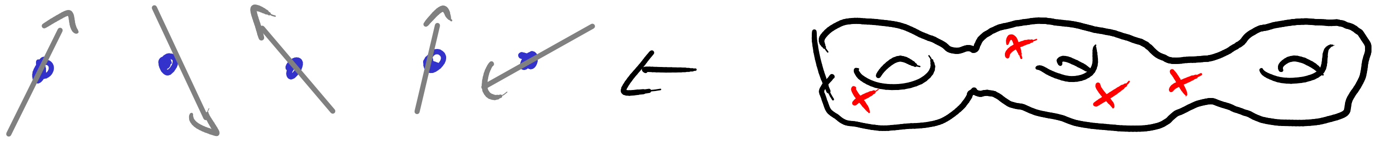
$(g+1$ poles, 1 $\underbrace{\text{scaling}, 1 \text{ shift}}_{\text{direction of } \vec{\zeta} / J})$

$\{\tilde{z}_k\}$

Reconstruct

$$T(u(z)) = \tau(z) \frac{\psi(z)\psi(z^*)^T \epsilon}{\psi(z^*)^T \epsilon \psi(z)} + \tau(z^*) \frac{\psi(z^*)\psi(z)^T \epsilon}{\psi(z)^T \epsilon \psi(z^*)}.$$

reconstruct state $\vec{S}_k$ from $T(u)$



consider d.o.f. curve generically has $g = L-2$

eigenvector has $g+3 = L+1$ d.o.f.

Γ has $L-1$ dof from $F(u)$

altogether: $2L$ d.o.f. $\simeq$ dim of phase space S^2 for each site.