Introduction to Integrability

Lecture Slides, Chapter 3

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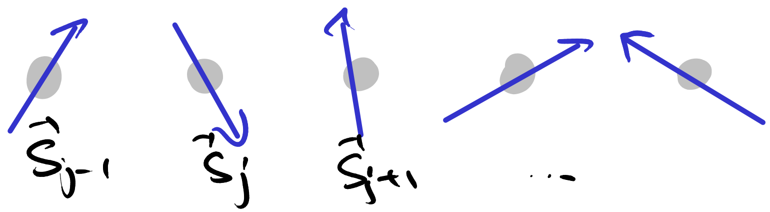
The current version of this work can be found at: `http://people.phys.ethz.ch/~nbeisert/lectures/`.

3 Classical Spin Chains

chain models

- simple systems on each site (integrable)
- interactions between neighbouring sites

3.1 Heisenberg Spin Chain



$$\|\vec{S}_j\| = 1 = \vec{S}_j^{\,2}$$

$$\{S_j^a, S_k^b\} = \delta_{j,k} \cdot \varepsilon^{abc} S_j^c$$

$$\{\vec{S}_j, \vec{S}_k^{\,2} - 1\} = 0$$

$$H = \Sigma_j H_j$$

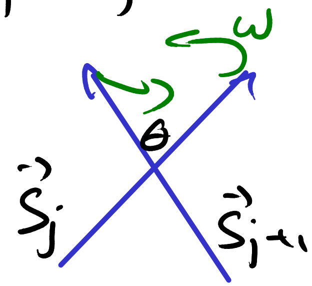
$$H_j = -\log \frac{1 + \vec{S}_j \cdot \vec{S}_{j+1}}{2}$$

- homogeneous along chain,
- rot inv, relative spin matters.

required for integrability

EOM. $$\frac{d\vec S_j}{dt} = -\{H,\vec S_j\} = -\frac{\vec S_{j-1}\times\vec S_j}{1+\vec S_{j-1}\cdot\vec S_j} + \frac{\vec S_j\times\vec S_{j+1}}{1+\vec S_j\times S_{j+1}}.$$

$$\omega \simeq \frac{1}{\cos\theta/2}$$



convert to stereograshic proj. / spinors

$$\frac{1+\vec S_j\cdot\vec S_k}{2} = \frac{(1+\zeta_j\zeta_k^*)(1+\zeta_k\zeta_j^*)}{(1+|\zeta_j|^2)(1+|\zeta_k|^2)} = \frac{(s_j^+ s_k)(s_k^+ s_j)}{(s_j^+ s_j)(s_k^+ s_k)}$$

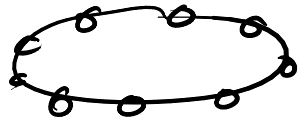
$$\frac{d\zeta_j}{dt} = \frac{i}{2}\sum_\pm \frac{1+|\zeta_j|^2}{1+\zeta_{j\pm1}\zeta_j^*}(\zeta_{j\pm1}-\zeta_j)$$

$$\frac{ds_j}{dt} = \frac{i}{2}\frac{s_j^+ s_j}{s_j^+ s_{j-1}} s_{j-1} + \frac{i}{2}\frac{s_j^+ s_j}{s_j^+ s_{j+1}} s_{j+1} + i\lambda_j s_j$$

Boundary Conditions

various choices which are integrable

- closed / periodic BC



L

$$\vec{S}_{j+L} = \vec{S}_j \qquad H = \sum_{j=1}^{L} H_j$$

1 … L

- open BC $H = \sum_{j=1}^{L-1} H_j$

- infinite BC $\vec{S}_j \to \vec{S}_{L/R}$ $j \to \mp\infty$ $H = \sum_{j=-\infty}^{+\infty} H_j$

other combinations of the above

- semi-infinite chains
- boundary interactions at ends
- twisted closed BC.

Global Symmetry

here SO(3) rotational symmetry, rotate all $\vec{S}_j$ simultaneously

$$\delta\vec{S}_i = -\{\delta\vec{X}\cdot\vec{S}_j, \vec{S}_i\}$$

total angular mom. vector (conserved) $\vec{J} = \sum_j \vec{S}_j$

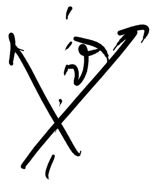
cons. $\{H, \vec{J}\} = 0$

discrete current $(\vec{Q}_j, \vec{k}_j)$ $\vec{Q}_j = \vec{S}_j$ $\vec{k}_j = \dfrac{\vec{S}_j \times \vec{S}_{j+1}}{1+\vec{S}_j \cdot \vec{S}_{j+1}}$

$$\frac{d}{dt}\vec{Q}_j = -\{H, \vec{Q}_j\} = \vec{k}_j - \vec{k}_{j-1}$$

$$\{J^a, J^b\} = \varepsilon^{abc} J^c \leftarrow \text{lie algebra of } SO(3)$$

Simple Solutions (closed)



$L=1$ single isotropic spin, no dynamics $H=0$

$L=2$ two spin vectors rotating around middle axis with constant $\omega = 2/\cos\theta/2 = 2/\cos\vartheta$ ← angle between axis and spin

$$H = -4\log|\cos\vartheta| \qquad \vec{J} = 2\cos\vartheta\,\vec{e}_z$$

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta\cos(2\pi nj/L - \omega t) \\ \sin\vartheta\sin(2\pi nj/L - \omega t) \\ \cos\vartheta \end{pmatrix}$$

$L=2$
$n=1$

↑ works for arbitrary L, n

$0 < n < L$

$$\omega = \frac{2\cos\vartheta\sin^2\pi n/L}{1-\sin^2\vartheta\sin^2\pi n/L} \qquad \vec{J} = L\cos\vartheta\,\vec{e}_z$$

$$H = -L\log\left(1-\sin^2\vartheta\sin^2(\pi n/L)\right)$$

L=3 solutions are more difficult (elliptic fn)
but some special cases (L=3, n=1,2 above):

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta_j \cos(-\omega t) \\ \sin\vartheta_i \sin(-\omega t) \\ \cos\vartheta_i \end{pmatrix}$$ all spins on a common plane

$$H = -2\log\frac{|J^2-1|}{8} \qquad \omega = \frac{4J}{J^2-1}$$

$$J^2 = 3 + 2\sum_j \cos(\vartheta_j - \vartheta_{j+1})$$

curious: two regimes of solutions (disconnected)
depending on $0<J<1$ $1<J<9$

Excitations of the Ferromagnetic Ground State

Ground state: all spins are aligned along z-axis

$$\vec{S}_k(t) = \vec{e}_z \quad \rightarrow H = 0 \qquad \vec{J} = L\vec{e}_z$$

Stereographic variables $\vec{S}_k \rightarrow \zeta_k \in \mathbb{C}$ $\qquad \zeta_k \sim \epsilon$

EoM $\quad \dfrac{d\zeta_j}{dt} = \dfrac{i}{2}\left(\zeta_{j-1} - 2\zeta_j + \zeta_{j+1}\right) + O(\epsilon^3)$

solve linear diff. eq. $\quad$ plane wave b/c homogeneous

$$\zeta_j(t) = \epsilon\, a_n \exp\frac{2\pi i n j}{L} \exp(-i\omega_n t) + O(\epsilon^2)$$

angular velocities $\quad \omega_n = 2\sin^2\dfrac{\pi n}{L}$

total ang. mom $\qquad\qquad$ energy

$$\vec{J} = \left(L - 2\epsilon^2|a_n|^2 L\right)\vec{e}_z + \ldots \qquad H = 4\epsilon^2|a_n|^2 L \sin^2\frac{\pi n}{L} + \ldots$$

more natural to express eqs. in terms of action variables

symplectic structure

$$\hat{\omega} = \sum_j 2i \, d\zeta_j \wedge d\zeta_j^*$$
$$= 2\epsilon |a_n|^2 L \omega_n \, dt \wedge d\epsilon + O(\epsilon^3)$$

$$dI_n = \frac{1}{2\pi} \oint \hat{\omega} = 4 \, d\epsilon \, \epsilon \, |a_n|^2 L + \ldots$$

$$I_n = 2|a_n|^2 \epsilon^2 L + \ldots$$

$$\vec{J} = (L - I_n) \vec{e}_z' + \ldots \qquad H = \omega_n I_n + \ldots$$

3.2 Integrable Structure

Express model in algebraic integrable framework

Lax pair (T, M) s.t. T encodes state and Lax eq.

$$\frac{d}{dt} T = [M, T]$$

Lax Transport

construct T recursively over the sites of chain.

introduce el. Lax transport $\mathcal{L}_j$, evol. M_j: transport eq.

$$\frac{d}{dt} \mathcal{L}_j = M_j \mathcal{L}_j - \mathcal{L}_j M_{j-1}.$$

construct composite Lax transport over sites $j+1 \ldots k$

$$W_{k,j} := \mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_{j+2} \mathcal{L}_{j+1}$$

Lax transport eq. holds for $W_{k,j}$ as:

$$\frac{d}{dt} W_{k,j} = M_k W_{k,j} - W_{k,j} M_j$$

For a closed chain of length L: Lax monodromy T

$T := W_{L,0} = \mathcal{L}_L \dots \mathcal{L}_1$ serves a Lax matrix T

$$\frac{d}{dt} T = [M, T]$$ with evol. $M = M_0 = M_L$

For Heisenberg spin chain

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u} \vec{S}_j \cdot \vec{\sigma}$$

$$M_j(u) = \frac{i}{u^2+1} \frac{(\vec{S}_j + \vec{S}_{j+1} + u\, \vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}} \qquad u \in \mathbb{C}$$

Elementary to show that Lax transp. eq. holds, use

$$(\vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma} = i(\vec{S}_{j+1} \cdot \vec{\sigma})(\vec{S}_j \cdot \vec{\sigma}) - i(\vec{S}_j \cdot \vec{S}_{j+1})\, \mathrm{id}$$

We have a Lax pair $\Rightarrow$ traces of powers of T are conserved:

$F = F_1 \qquad F_m(u) := \frac{1}{m} \operatorname{tr} T(u)^m$ need only $m = 1$

because $\det \mathcal{L}_j = 1 + \frac{1}{u^2} \Rightarrow \det T = \left(1 + \frac{1}{u^2}\right)^L$

r-matrix for Heisenberg-chain

classical RTT relation extends to $\mathcal{L}_j$ as follows

$$\{\mathcal{L}_j(u_1) \overset{\otimes}{,} \mathcal{L}_k(u_2)\} = \delta_{jk}\, r_j(u_1,u_2)\,(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2))$$
$$- \delta_{jk}\,(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2))\, r_{j-1}(u_1,u_2)$$

allows to combine Lax transport into Lax monodromy T

$$\{T(u_1) \overset{\otimes}{,} T(u_2)\} = [r_L(u_1,u_2), T(u_1)\otimes T(u_2)]$$

$$\Rightarrow \{F_m(u_1), F_n(u_2)\} = 0$$

- later show that all d.o.f. encoded into $T(u)$

$$r_j(u_1,u_2) = r(u_1,u_2) = -\frac{\sum_a \sigma^a\otimes\sigma^a}{2(u_1-u_2)}$$ std. solution to class. YBE.

3.3 Spectral Parameter

Lax Matrices are 2×2, but depend on $u \in \bar{\mathbb{C}}$

- can encode all $2n$ d.o.f of phase space of chain.
- can do complex analysis in u.

Hamiltonian

complication: $H = \sum H_j$ is "local" but $T(u)$ is non-local

question how to extract local information from non-local qty.?

hint: $\mathcal{L}_j(u)$ must become special for extraction of local data.

Heisenberg chain: $\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} = 0$ for $u = \pm i$

will arrive at $H = -\log \frac{F(+i)\,F(-i)}{4^L}$.

want to verify. use form of $\mathcal{L}_i$ at $t=\pm i$

$$\mathcal{L}_j(\pm i) = \mathrm{id} \pm \vec{S}_j \cdot \vec{\sigma}$$

Projector: EV are 2, 0. matrix has lower rank, rank 1

$$\mathrm{tr}\, \mathcal{L}_j(\pm i) = 2 \qquad \mathcal{L}_j(\pm i)^\dagger = \mathcal{L}_j(\pm i)$$

write $\mathcal{L}_j(\pm i)$ using spinors s_j

$$\mathcal{L}_j(+i) = \frac{2}{s_j^\dagger s_j}\, s_j s_j^\dagger$$

relate $\mathcal{L}_j(-i)$ to $\mathcal{L}_j(+i)$ by transposition

$$\mathcal{L}_j(-i) = \varepsilon\, \mathcal{L}_j(+i)^T \varepsilon^{-1} = \frac{2}{s_j^* s_j}\, \varepsilon s_j^* s_j^T \varepsilon^{-1}.$$

compute products of $\mathcal{L}_j$:

$$F(+i) = 2^L \prod_{j=1}^{L} \frac{S_{j+1}^{+} S_j}{S_j^{+} S_j} \qquad F(-i) = 2^L \prod_{j=1}^{L} \frac{S_j^{+} S_{j+1}}{S_j^{+} S_j}$$

$$F(+i)\,F(-i) = 4^L \prod_{j=1}^{L} \underbrace{\frac{\cdots}{\cdots}}_{\text{sites } j,\, j+1} = 2^L \prod_{j=1}^{L} \left(1 + \vec{S}_j \cdot \vec{S}_{j+1}\right)$$

$$\exp(-H) = \prod_{j=1}^{L} \frac{(S_{j+1}^{+} S_j)(S_j^{+} S_{j+1})}{(S_j^{+} S_j)(S_{j+1}^{+} S_{j+1})}$$

$$\frac{F(-i)}{F(+i)} = \exp(iP) = \prod_{j=1}^{L} \frac{S_j^{+} S_{j+1}}{S_{j+1}^{+} S_j}$$

also can generate further local qty from expansion of $F(u)$ around $u = \pm i$

Reconstruction

want to extract $\vec{S}_j$ from $T(u)$. via pt $u = \pm i$

consider Lax monodromy at $u = +i$ depends only on sites $L, 1$

$$T(+i) = 2^L \frac{s_L s_1^+}{s_L^+ s_L} \prod_{j=1}^{L-1} \frac{s_{j+1} s_j^+}{s_j^+ s_j} = F(+i) \frac{s_L s_1^+}{s_1^+ s_L}$$

consider EVect.

$EV=0$ the eigenvector of $T(u)$ is spinor εs_1^* ← determines s_1^*

likewise $T(-i) = F(-i)\left(\mathrm{id} - \frac{s_1 s_L^+}{s_L^+ s_1}\right)$; $EV=0 \to$ eigvect. s_1

compose spin vector $\vec{S}_j = \frac{s_j^+ \vec{\sigma} s_j}{s_j^+ s_j}$

how to obtain other sites $j \neq 1, L$?

recursion. consider shifted monodromy $T = T_L$

$$T_{j-1}(u) = \mathcal{L}_{j-1}(u) \dots \mathcal{L}_1(u)\, \mathcal{L}_L(u) \dots \mathcal{L}_j(u)$$

recursion relation

$$T_j(u) = \mathcal{L}_j(u)\, T_{j-1}(u)\, \mathcal{L}_j(u)^{-1}$$

Procedure:
- compute $\vec{S}_1$ from T_L
- compute $\mathcal{L}_1$ from $\vec{S}_1$
- compute T_1 from $\mathcal{L}_1$
- compute $\vec{S}_2$ from T_1

$\vdots$

obtain all $\vec{S}_k$ after $3L$ steps.

proves that $T(u)$ encodes all of phase space

Global Symmetry

$T(u)$ contains total ang. mom $\vec{J}$ at $u=\infty$

at $u=\infty$ we have expansion of $\mathcal{L}_j(u)$

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u}\vec{S}_j\cdot\vec{\sigma} + \dots$$

expand T around $u=\infty$

$$T(u) = \mathrm{id} + \frac{i}{u}\sum_{j=1}^{L}\vec{S}_j\cdot\vec{\sigma} + \dots = \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} + \dots$$

for monodromy trace

$$F(u) = 2 - \frac{1}{u^2}\left(J^2 - L\right) + \dots$$

Casimir for $\vec{J}$

further terms in expansion of $T(u)$ around $u=\infty$ are multi-local carges eg. $\vec{Y} := \sum_{k=1}^{L}\sum_{j=1}^{k-1}\vec{S}_j\times\vec{S}_k$