Introduction to Integrability

Lecture Slides, Chapter 2

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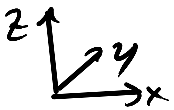
The current version of this work can be found at:
`http://people.phys.ethz.ch/~nbeisert/lectures/`.

2 Algebraic Integrability

2.1 Spin Models

elementary classical spin d.o.f.: phase space $M = S^2$

Spinning Top / Rigid body fixed at c.o.m., no gravity



co-moving frame, axes aligned with princ. mom. inert.
$\Omega_x, \Omega_y, \Omega_z$

Euler angles $\vartheta, \varphi, \psi \leftrightarrow$ ang. mom. vect. $\vec{S}$ in co-moving frame

$$
\begin{aligned}
S_x &= -\Omega_x (\dot\varphi \sin\vartheta \sin\psi + \dot\vartheta \cos\psi) \\
S_y &= -\Omega_y (\dot\varphi \sin\vartheta \cos\psi - \dot\vartheta \sin\psi) \\
S_z &= -\Omega_z (\dot\varphi \cos\vartheta + \dot\psi)
\end{aligned}
$$

Lagrangian $$L = \frac{S_x^2}{2\Omega_x} + \frac{S_y^2}{2\Omega_y} + \frac{S_z^2}{2\Omega_z} = L(\varphi, \vartheta, \psi, \dot\varphi, \dot\vartheta, \dot\psi)$$

E.O.M → Euler Eq.

$$\frac{d}{dt} S_x = \left(\frac{1}{\Omega_y} - \frac{1}{\Omega_z}\right) S_y S_z$$

$$\frac{d}{dt} S_y = \left(\frac{1}{\Omega_z} - \frac{1}{\Omega_x}\right) S_z S_x$$

$$\frac{d}{dt} S_z = \left(\frac{1}{\Omega_x} - \frac{1}{\Omega_y}\right) S_x S_y$$

conserved charges : H, $\vec{J}$ in inertial frame (4/6)

3 Poisson commute: H, $\vec{J}^2$, J_z

integrable (super-int; SO(3) symmetry)

Focus on $\vec{S}$ subspace $\mathbb{R}^3$ (3/6)

$|\vec{S}| = J$ fixed $\Rightarrow$ Phase space $\Rightarrow$ $M = S^2 \subset \mathbb{R}^3$

$\to$ elementary spin d.o.f. / spin model

$\vec{S}$ spin vector

generate $SO(3)$ $\Leftrightarrow$ Lie brackets

Poisson brackets for $\vec{S}$ $\quad \{S_j, S_k\} = \varepsilon_{jkl} S_l$. $\quad$ tot antisym 3-tensor

Hamiltonian for $\vec{S}$ $\quad H = \frac{1}{2} \vec{S}^T \Omega^{-1} \vec{S} \quad \Omega = \mathrm{diag}(\Omega_x, \Omega_y, \Omega_z)$

Ham E.o.M.

$$\frac{d}{dt}\vec{S} = -\{H, \vec{S}\} = (\Omega^{-1}\vec{S}) \times \vec{S}.$$

Phase space reduces to $M = S^2$ $\quad F_1, F_2 : S^2 \to \mathbb{R}$

$$\{F_1, F_2\} = \varepsilon_{jkl} S_l \frac{\partial F_1}{\partial S_j} \frac{\partial F_2}{\partial S_k}.$$

note $\{|\vec{S}|, F\} \overset{\text{always}}{=} 0$

fix $|\vec{S}| = J \doteq$ const.

Spin Parametrisations

different phase space coordinates useful.

- $\vec{S}$ is a vector. but $|\vec{S}| = J$ $\vec{S}^2 = J^2$ non-lin const.

- spherical coordinates $\vec{S} = J \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$
  manifestly 2D coord.

$$\{\vartheta, \varphi\} = \frac{1}{J \sin\vartheta}.$$

Drawback: periodically identified.
two singular pt: N, S

- stereographic proj $S^2 \to \bar{\mathbb{C}}$: complex number $\zeta \in \bar{\mathbb{C}}$

$$\vec{S} = \frac{J}{1+|\zeta|^2} \begin{pmatrix} 2\,\mathrm{Re}\,\zeta \\ 2\,\mathrm{Im}\,\zeta \\ 1-|\zeta|^2 \end{pmatrix} \qquad \zeta = \tan(\vartheta/2)\, e^{i\varphi} = \frac{S_x + i S_y}{J + S_z}$$

just one complex coord. (ζ, ζ^*)

$$\{\zeta, \zeta^*\} = -\frac{i}{2J}(1+|\zeta|^2)^2.$$

• spinor repr. $SO(3) \simeq SU(2)$ express $\vec{S}$ as 2×2 matrix

$\vec{S} \to \vec{S}\cdot\vec{\sigma}$ $\quad \vec{\sigma}$ Pauli matrices

hermitian, traceless, eigenvalues $\pm J$.

eigenvector relation $\quad (\vec{S}\cdot\vec{\sigma})s = +Js \qquad s \in \mathbb{C}^2$

$$\vec{S} = J\frac{s^+\vec{\sigma}s}{s^+s} \qquad (\vec{S}\cdot\vec{\sigma})\epsilon s^* = -J\epsilon s^* \qquad \epsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$s \in \mathbb{C}^2$ is a spinor repr. of pt. on S^2

Caveat: projective space $\quad s \in \mathbb{C}P^1 \quad s \equiv \lambda s \quad \lambda \in \mathbb{C}^*$

$s = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$ stereographic ζ $\qquad$ resolves $\zeta = \infty$: $s \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \infty \end{pmatrix}$

Poisson brackets $\quad \{F_1, F_2\} = -\frac{i}{2J} s^+s\left(\frac{\partial F_1}{\partial s}\cdot\frac{\partial F_2}{\partial s^*} - \frac{\partial F_1}{\partial s^*}\cdot\frac{\partial F_2}{\partial s}\right)$

$F(s) \overset{!}{=} F(\lambda s)$

altogether 4 reprs of M: $\mathbb{R}^3 \supset S^2 = \bar{\mathbb{C}} = \mathbb{C}P^1$

Classes of solutions

- explicit solutions, general case $\Omega_x \neq \Omega_y \neq \Omega_z \neq \Omega_x$
  solution in terms of Jacobi elliptic functions sn, cn, dn

elliptic

$$S_x = c_x \,\mathrm{cn}(\lambda t + \varphi ; k)$$
$$S_y = c_y \,\mathrm{sn}(\lambda t + \varphi ; k)$$
$$S_z = c_z \,\mathrm{dn}(\lambda t + \varphi ; k)$$

sine, cosine, delta elliptic functions

c_k are functions of λ, k, depend on E, J, Ω_k; φ initial pos.

- for $k=0$ special case where e.g. $\Omega_x = \Omega_y \neq \Omega_z$

trigonometric

$$S_x = c \cos(\lambda t + \varphi)$$
$$S_y = c \sin(\lambda t + \varphi)$$
$$S_z = \text{const.}$$

$\leftarrow$ SO(2) res. rot sym in co-moving frame.

- most symmetric case $\Omega_x = \Omega_y = \Omega_z$: SO(3) rot. sym, no dynamics.

"rational"

classification of integrable systems

type	rational	trigonometric	elliptic
symbols	XXX	XXZ	XYZ
$\Omega_x \Omega_y \Omega_z$	$\Omega_x \Omega_x \Omega_x$	$\Omega_x \Omega_x \Omega_z$	$\Omega_x \Omega_y \Omega_z$
symmetry	SO(3)	SO(2) (Cartan subalg)	—

equal values of Ω_k

2.2 Lax Pair

Formulate Integrability using algebraic methods.

Spin Model $\quad SO(3) = SU(2)$

$\{S_j, S_k\} = \varepsilon_{jkl} S_l$ represent $\vec{S}$ as a matrix using Pauli matrix generator of su(2)

$$\vec{S}\cdot\vec{\sigma} = \begin{pmatrix} +S_z & S_x - iS_y \\ S_x + iS_y & -S_z \end{pmatrix}$$

note $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c \;//\; [\vec{v}\cdot\vec{\sigma}, \vec{w}\cdot\vec{\sigma}] = 2i(\vec{v}\times\vec{w})\cdot\vec{\sigma}$

E.o.M. $\dfrac{d}{dt}\vec{S}\cdot\vec{\sigma} = ((\Omega^{-1}\vec{S})\times\vec{S})\cdot\vec{\sigma} = -\dfrac{i}{2}[(\Omega^{-1}\vec{S})\cdot\vec{\sigma}, \vec{S}\cdot\vec{\sigma}]$

Def $\quad T := \vec{S}\cdot\vec{\sigma}, \quad M := -\dfrac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$

EoM: $\quad \dfrac{d}{dt}T = [M, T] \implies$ spec T is conserved

spec $T = \{\pm J\}$ $\quad J$ const indep of M.

Lax Pair

E.o.M can be formulated in terms of a Lax Pair (T, M), two square matr. ($\mathrm{End}(V)$) and whose elem. are phase space functions.

T is Lax matrix, M is evolution matrix

Lax eq. $\frac{d}{dt} T = -\{H, T\} = [M, T]$ statement in phase space

Ham form.

holds by virtue of EoM / is equiv. to Eo.M.

conseq. EV spectrum of T is conserved sim trans (t)

$$T(t) = g(t)\, T(t_0)\, g(t)^{-1} \quad t_0 \text{ refr. time}$$

characteristic pol. $\det(\tau\, \mathrm{id} - T)$ is indep of time.

If Lax pair (satisfying Lax eq) exists, generate conserved charges F_k as traces of powers of T

$$F_k := \frac{1}{k} \operatorname{tr} T^k \qquad 1/k \text{ sym factor.}$$

$$\frac{d}{dt} F_k = \operatorname{tr}\left([M, T]\, T^{k-1} \right) \overset{\text{cycl. of trace}}{=} 0 \qquad \text{conservation!}$$

Note: • not all F_k are necess. indep.

• there may be additional cons. charges.

Complete Lax Pairs

Lax formulation is nice and useful for integrability: but

- Lax pair is never unique
- not every Lax pair is useful
- no recipe for construction of Lax pair (general)
- abstract, not necessarily related to physics
- size of T, M is not related to features of system.

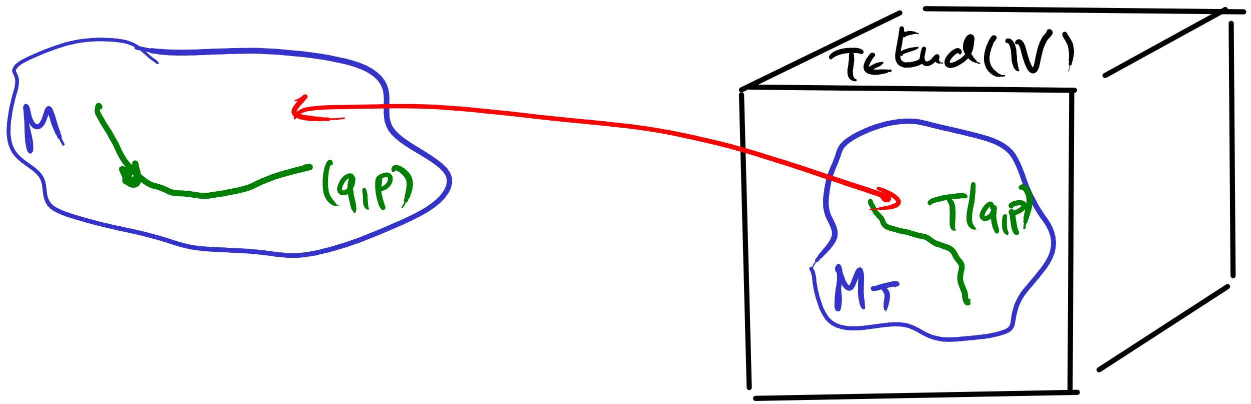
How to formulate a useful Lax pair for integrable system.

Generalisations of Lax pair for spin system:

- other representations $2\times 2 \to (2s+1)\times(2s+1)$
- add unit matrix to T $\quad T = \vec{S}\cdot\vec{\sigma} + uH\,\mathrm{id}\ ;\ M = -\frac{i}{2}(\Omega^{-1}\vec{S}^{\,h})\cdot\vec{\sigma}$

changes spectrum of T in a useful way : $\operatorname{spec} T = \{uH \pm J\}$ not contain of M

Lax formulation as algebraic fcn. of phase space



Establish a 1:1 map between M and $M_T \subset \mathrm{End}(\mathbb{V})$

Point in $M_T \subset \mathrm{End}(\mathbb{V})$ specifies state uniquely.

· furthermore eigenvalues of T represent conserved charges

Desirable Properties for a complete Lax Pair formulation.

i) (T, M) obeys Lax eq. $dT/dt = [M, T]$

ii) T encodes all $2n$ phase space coordinates

iii) T is diagonalisable almost everywhere on M

iv) spectrum must encode n indep. variables

v) these variables are in involution.

iv) + v) $\Rightarrow$ Liouville integrable system.

2.3. Lax-Poisson Structure

Lax-Poisson Equation

Phase space variables encoded into matrix T, elements T_{jk}

$$\{T_{jk}, T_{\ell m}\} = \sum_n R_{(j\ell)(nm)} T_{nk} - \sum_n T_{jn} R_{(nj)(km)}$$
$$- \sum_n R_{(\ell j)(nk)} T_{nm} + \sum_n T_{\ell n} R_{(nj)(mk)}$$

$R_{(j\ell)(km)}$ are elements of rank-2 tensor operator

combination of terms guarantees that eigenvalues of T Poisson commute

Tensor Notation

Matrix A in components $\sum_{jk} A_{jk} E_{jk}$

E_{jk} matrix with all elements 0 except for 1 in row j, col k.

Poisson brackets of matrices

$$\{A \overset{\otimes}{,} B\} := \sum_{jklm} \{A_{jk}, B_{lm}\}\, E_{jk} \otimes E_{lm}$$

tensor operator R

$$R := \sum_{jklm} R_{\overset{\text{out}}{(jl)}\overset{\text{in}}{(km)}}\, E_{jk} \otimes E_{lm}$$

$$P(R) := \sum_{jklm} R_{(jl)(km)}\, E_{lm} \otimes E_{jk}$$

P is tensor product perm.

$$\Rightarrow \{T \overset{\otimes}{,} T\} = [R, T \otimes \mathrm{id}] - [P(R), \mathrm{id} \otimes T]$$

short hand notation for tensor operators / sites:

index denotes site on which tensor acts, no label means: id

$$R \to R_{12} \qquad T_1 := T \otimes \mathrm{id} \qquad T_2 := \mathrm{id} \otimes T \qquad P(R_{12}) \to R_{21}$$

$$\{T_1, T_2\} := \{T \otimes T\} = [R_{12}, T_1] - [R_{21}, T_2]$$

Properties and Applications

Consider Poisson brackets of conserved charges $F_k := \frac{1}{k} \operatorname{tr}(T^k)$

$$\{F_j, F_k\} = \frac{1}{jk} \{\operatorname{tr} T^j, \operatorname{tr} T^k\} = \frac{1}{jk} \operatorname{tr}_{1,2} \{T_1^j, T_2^k\}$$

$$= \frac{1}{jk} \sum_{l=1}^{j} \sum_{m=1}^{k} \operatorname{tr}_{1,2} \left(T_1^{l-1} T_2^{m-1} \{T_1, T_2\} T_1^{j-l} T_2^{k-m} \right)$$

$$= \operatorname{tr}_{1,2} \left(T_1^{j-1} T_2^{k-1} \{T_1, T_2\} \right)$$

$$= \operatorname{tr}_{1,2} \left(T_1^{j-1} T_2^{k-1} [R_{12}, T_1] - T_1^{j-1} T_2^{k-1} [R_{21}, T_2] \right)$$

$$= 0 \qquad \text{(due to cyclicity)}$$

Jacobi identity?

$$0 = [T_1, [\![R,R]\!]_{123} + \{T_2, R_3\} - \{T_3, R_{12}\}] + 2 \text{ cycl.} \qquad 123 \to 231, 312$$

symbol $[\![\cdot,\cdot]\!]$ defined $\qquad [\![X,Y]\!]_{123} = - [\![X,Y]\!]_{132}$

$$[\![X,Y]\!]_{123} := [Y_{12}, Y_{13}] + [Y_{12}, X_{23}] + [X_{32}, Y_{13}]$$

Example: el. spin model $\vec{S}$, $\quad T := \vec{S}\cdot\vec{\sigma} + u\,\mathrm{H}\,\mathrm{id}$

$$\{T_1, T_2\} = (\vec{\sigma}_1 \times \vec{\sigma}_2)\cdot\vec{S}$$
$$+ u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_1 - u\left((\Omega^{-1}\vec{S})\times\vec{S}\right)\cdot\vec{\sigma}_2$$

Lax Poisson Eq solved by Lax Poisson ch. for T:

$$R_{12} = -\frac{i}{4}\,\vec{\sigma}_1\cdot\vec{\sigma}_2 - \frac{i}{2}u\,(\Omega^{-1}\vec{S})\cdot\vec{\sigma}_1$$

use Lax Poisson Structure for improved def. of complete Lax Pair

→ complete Lax-Poisson structure (T, M, R)

i) pair L, M obeys Lax eq. $dT/dt = [M, T]$

ii) Lax matrix T encodes all $2n$ phase space d.o.f.

iii) T diagonalisable almost everywhere

iv) spectrum of T encodes n indep. var.

v) Lax Poisson stru. R obeys Lax Poisson equation.

Evolution from Lax-Poisson Structure

H is conserved $\Rightarrow$ $H = h(T)$ ← spectrum of T

show that Lax eq. holds $\frac{d}{dt} T = -\{H, T\} = [M, T]$

with evolution matrix M given by $\quad h = h(F_k) \quad F_k = \frac{1}{k} \operatorname{tr} T^k$

$$M_1 = \sum_k \frac{\partial h}{\partial F_k} \operatorname{tr}\left(T_2^{k-1} R_{12} \right) \qquad dh = \sum_k \frac{\partial h}{\partial F_k} dF_k$$

in def of compl. Lax Poisson struct: i) Ham H is given by $h(T)$

Ex: $T = \vec{S} \cdot \vec{\sigma} + u \,\mathrm{id} \quad R_{12} = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u \left(\Omega^{-1} \vec{S} \right) \cdot \vec{\sigma}_1 \, \mathrm{id}_2$

$H = \operatorname{tr} T / 2u \quad \ldots \quad M_1 = \frac{1}{2u} \operatorname{tr}_2 R_{12} = -\frac{i}{2} \left(\Omega^{-1} \vec{S} \right) \cdot \vec{\sigma}_1$

Parametric Lax Pairs

we can have Lax Pair that depend on a (complex) par. u.

→ by expanding in u can package many qty into small matrices

→ perform complex analysis on u-dependence.

$$\frac{d}{dt} T(u) = [M(u), T(u)] \qquad \text{spectrum } F_k(u)$$

$$F_1(u) = 2uH, \quad F_2(u) = J^2 + u^2 H^2, \quad F_3(u) = 2uH\left(J^2 + \tfrac{1}{3}u^2 H^2\right) \quad \ldots$$

Extend to R Lax Poisson Str.

$$\{T_1(u_1), T_2(u_2)\} = [R_{12}(u_1, u_2), T_1(u_1)] - [R_{21}(u_2, u_1), T_2(u_2)]$$

short

$$T_1 := T_1(u_1) \xrightarrow{} \{T_1, T_2\} = [R_{12}, T_1] - [R_{21}, T_2]$$

$$F_j(u) = \frac{1}{j} \operatorname{tr}(T(u)^j)$$

implies

$$\{F_j(u_1), F_k(u_2)\} = 0 \quad \text{for all } j, k, u_1, u_2$$

For sph model $\quad R_{12}(u_1, u_2) = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u_2 (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_1$

Classical r-Matrix

alternative to describe the Poisson structure of T

relevant relation : RTT relation

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T] \quad // \quad \{T_1, T_2\} = [r_{12}, T_1 T_2]$$

typically have $r_{12} = -r_{21}$ (antisymmetry of $\{\}$)

can construct R from r, Jacobi id

$$0 = [[[r,r]], T_1 T_2 T_3] + [\{r_{12}, T_3\}, T_1 T_2] + \text{2 cyclic}$$

often r is indep of phase space

classical Yang-Baxter eq. $[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$