Introduction to Integrability

Lecture Slides, Chapter 1

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

1 Integrable Mechanics

1.1. Hamiltonian Mechanics

Phase space M: dim $2n$ coordinates (q_k, p_k) $k = 1 \dots n$

Hamiltonian function $H: M \to \mathbb{R}$

Solution of system is a curve $(q_k(t), p_k(t))$ obeying Hamiltonian equations of motion

$$\dot{q}_k := \frac{\partial q_k}{\partial t} = + \frac{\partial H}{\partial p_k} \qquad \dot{p}_k := \frac{\partial p_k}{\partial t} = - \frac{\partial H}{\partial q_k}$$

Poisson brackets. Functions $F_1, F_2 : M \to \mathbb{R}$

$$\{F_1, F_2\} = \sum_k \left(\frac{\partial F_1}{\partial q_k} \frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k} \frac{\partial F_2}{\partial q_k} \right)$$

- bi-linear, • derivations (Leibniz), • anti-sym, • Jacobi-Id.

Formulate Ham. E.O.M.

$$\dot{q}_n = -\{H, q_n\} \qquad \dot{p}_n = -\{H, p_n\}$$

Generalised to arb. phase space functions $F(q, p, t)$

evaluated on (any) solution: $\hat{F}(t) := F(q(t), p(t), t)$

$$\frac{d\overset{(\wedge)}{F}}{dt} := \frac{\partial F}{\partial t} - \{H, F\} \quad \text{total time der}$$

here: consider time-indep function $F = F(q, p)$

$$\frac{dF}{dt} = -\{H, F\} \qquad \frac{dF}{dt} = 0 : F \text{ is conserved}$$

Phase space is a symplectic space

symplectic structure : 2-form on phase space

$$\omega = \sum_k dq_k \wedge dp_k \qquad \text{(non-degenerate)}$$

ω is inverse of Poisson brackets

canonical structure (1-form) $\sum_k p_k \, dq_k$

Canonical transformations $(q,p) \to (\tilde{q}, \tilde{p})$

phase space diffeomorphisms

use them to trivialise dynamics in integrable models

req for can. transf. $\{\tilde{q}_k, \tilde{p}_l\} = \delta_{k,l}$; $\{\tilde{q}_k, \tilde{q}_l\} = \{\tilde{p}_k, \tilde{p}_l\} = 0$

$$\tilde{\omega} = \sum_k d\tilde{q}_k \wedge d\tilde{p}_k = \sum_k dq_k \wedge dp_k = \omega$$

1.2. Integrals of Motion

time-indep Ham. $H = H(q,p,\cancel{t})$

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} - \{H,H\} = 0$$

$E = H(q,p)$

Energy (as value of H on a solution) is constant

⟺ Dynamics is constrained to hypersurfaces M_E of const energy E

hypersurf constrained by cond $E = H(q,p)$

Can be further (time-independent) phase space functions $F_k(q,p)$

constant on (all) solutions

$$\frac{d}{dt}F_k = -\{H,F_k\} \stackrel{!}{=} 0$$

→ integral of motion, a conserved qty, conserved charge

$$\Rightarrow \text{charge} \quad F_k(q,p) = f_k = \text{const}$$

level set M_f

$$M_f := \{ (q,p) \in M ;\ F_k(q,p) = f_k \text{ for all } k \}$$

motion/dynamics takes place within a common M_f

Noether's theorem: conserved charge $\leftrightarrow$ cont. symmetry transf.

transformation generates new solutions from existing ones

flow $-\partial F_{k'} \cdot \}$ generates new solutions

$$(\tilde{q}(t), \tilde{p}(t)) = (q(t), p(t)) + \delta (q(t), p(t))$$ (inf. def. of solution)

with $\delta q(t) = -\epsilon \{ F_k, q(t) \}$ $\delta p(t) = -\epsilon \{ F_k, p(t) \}$

deformed solution has same energy E and same charge under f_k (not necessarily same f_ℓ)

More simplifications if charges F_k Poisson commute

$$\{F_k, F_l\} = 0 \quad \text{for all } k, l$$

all solutions generated by flows have same charges (on same M_f)

→ commuting charges F_k

by constr. Ham H is among charges

- $H = F_1$ eg.
- $H = H(F_k)$

Finding such charges is not easy, not straight-forward

- trial and error
- Noether's theorem, symmetry
- add. charges generate novel/hidden symmetries.

2D Central Potetial

particle, mass m, symm. potential $V(r)$, $r := \sqrt{x^2+y^2}$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - V\left(\sqrt{x^2+y^2}\right)$$

convenient to use radial coord. $x = r\cos\varphi$

$y = r\sin\varphi$

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - V(r).$$

angle appears only through $\dot{\varphi}$, shift of φ: symm. of L

Ham form: Legendre transf. $p = m\dot{r}$, $\psi = m r^2 \dot{\varphi}$

$$H = \frac{p^2}{2m} + \frac{\psi^2}{2mr^2} + V(r) \quad \dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m}; \ \dot{p} = -\frac{\partial H}{\partial r} = \frac{\psi^2}{mr^3} - V'(r)$$

$$\dot{\varphi} = \frac{\partial H}{\partial \psi} = \frac{\psi}{mr^2}; \ \dot{\psi} = -\frac{\partial H}{\partial \varphi} = 0$$

conserved charge $F = \psi =: J$ angular mom. in 2D

and H is conserved, use F, H to express φ, ψ

$$P(r, E, J) = \sqrt{2m\,(E - V(r)) - J^2/r^2}$$

$$\Psi(J) = J \qquad r, \varphi, E, J \text{ new coord.}$$

can exp. r, ψ

know $\frac{dr}{dt} = \frac{P}{m}$ sep. of. var.

$$\int_{r_0}^{r} \frac{m\,dr'}{P(r', E, J)} = t - t_0 \quad \Rightarrow \text{ find sol. } r(t)$$

substitute

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{J\,dt}{m\,r(t)^2} = \varphi_0 + \int_{r_0}^{r(t)} \frac{J\,dr'}{r'^2\,P(r', E, J)}$$

1.3 Liouville Integrability

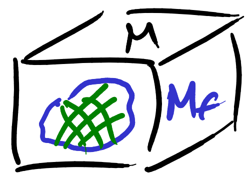
Mech. sys 2n-dim phase space M is Liouville integrable

- n independent
- everywhere differentiable
- conserved charges F_k (integrals of motion)
- in involution: Poisson commute $\{F_k, F_l\} = 0$.

Solve it by so-called quadratures. finite sequence of

- resolve relations among coordinates
  (non-linear eq, non-integral, non-differential eq.)
- calculate ordinary integrals (multi-dim)

not necessarily easy or doable in well-est. functions.



Phase Space Structure

For int. sys level sets M_f have a very nice structure

- M_f has dimension $n = \frac{1}{2} \dim M$
- there are n indep. commuting flows acting on it
- can specify coordinates on M using the F_k

n coordinates f_k (as values of $F_k(q,p)$)

n coord. functions $G_k(q,p)$ flow functions defined by

$$\{F_k, G_\ell\} = -\delta_{k\ell} \quad \text{with pt. } q_0(f) \text{ as origin.}$$

As such: coordinates per level set. Want to extend to all M.

add. rel $\{G_k, G_\ell\} = 0 \Rightarrow G_k$ can be constructed consistently

$(q,p) \to (G,F)$ canonical transf. $\{G_k, F_\ell\} = \delta_{k\ell}$, $\{F_k, F_\ell\} = \{G_k, G_\ell\} = 0$.

one very useful corollary: time evolution is linear in (g, f) coordinates:

$$H = H(q, p, \cancel{x}) = H(\cancel{g}, f).$$

$$\frac{d}{dt} F_k = -\{H, F_k\} = 0$$

$$\frac{d}{dt} G_k = -\{H, G_k\} = -\sum_\ell \frac{\partial H}{\partial f_\ell}\{F_\ell, G_k\} = \frac{\partial H}{\partial f_k} =: v_k(f) = \text{const}.$$

$$f_k(t) = f_{0,k} = \text{const} \qquad g_k(t) = g_{0,k} + v_k (t - t_0).$$

Note: Flow coordinates defined by diff eq.
if level set has non-trivial cycles, flow coord. not necessarily globally defined.

Follow G_ℓ around a non-triv. cycle $C_k(f)$ way obtain shift:

$$\Omega_{k\ell}(f) := \oint_{C_k(f)} dG_\ell \leftarrow \text{def from } \{F_k, G_\ell\} = -\delta_{k\ell}$$

note $\Omega_{k\ell}(f)$ is inv under smooth def of $C_k(f)$

$\Omega_{k\ell}(f)$ is called period matrix for level set M_f

Charges

Mech. sys., Phase spc (q_k, p_k).

Need to establish cons. charges $F_k(q,p)$.

No recipe that works in many systems.

However given some set of n ind. charges F_k can establish integrability by straight-forward verification.

Invert momentum coordinates in terms of F_k and q_k fixed

$$f_k = F_k(q,p) \iff p_k = P_k(q,f)$$

$$P_k(q, F_k(q,p)) = p_k$$

these are non-lin. (non-int; non-diff) eq. to be solved.

$$(q,p) \to (q,f)$$

Generating Functions

$\{G_k, G_l\} = 0$

we want to obtain flow functions $\{G_k, F_l\} = \delta_{kl}$

use tech. Gen. Funct.

$$S(q,f) := \int_{\gamma(q,f)} \sum_{k=1}^{n} p_k \, dq_k'$$

$\gamma(q,f)$ path on M_f connecting $q_0(f)$ to q.

$\int$ is inv. under cont. def b/c $p_k dq_k$ is closed 1-form.

$$dp_k \wedge dq_k = \underbrace{df_l}_{=0 \text{ on level set } M_f} \wedge dq_k \frac{\partial P_k}{\partial f_l} + dq_l \wedge dq_k \underbrace{\frac{\partial P_k}{\partial q_l}}_{\text{symmetric in } k,l}$$

symmetry: $f_j = F_j(q,p)$

diff. wrt. q_l

$$0 = \frac{\partial F_m}{\partial p_l} \frac{\partial F_j}{\partial q_l} + \frac{\partial F_m}{\partial p_l} \frac{\partial F_j}{\partial p_k} \frac{\partial P_k}{\partial q_l}$$

contract same rel. with $m \leftrightarrow j$ exch.

$$\Rightarrow \quad 0 = \underbrace{\{F_m, F_j\}}_{=0 \text{ by integr.}} + \underbrace{\frac{\partial F_m}{\partial p_l} \frac{\partial F_j}{\partial p_k}}_{\text{invertible}} \underbrace{\left(\frac{\partial P_k}{\partial q_l} - \frac{\partial P_l}{\partial q_k}\right)}_{\overset{!}{=} 0}$$

$\Rightarrow$ S is invariant under path deformations

computing S amounts to computing an ord. integral.

Flow Functions

Note that S reproduces $P_k(q,f)$ $\quad \dfrac{\partial S}{\partial q_k}(q,f) = P_k(q,f)$.

but also $\quad G_k(q,p) := \dfrac{\partial S}{\partial f_k}(q, F(q,p))$

$(q,p) \to (g,f)$ is canonical transf. of M.

2D Central Potential

radial pot $V(r)$, $H = F_1 = E$, $F_2 = \varphi = J$

solve for p $\quad P(r,\cancel{\varphi},E,J) = \sqrt{2m(E-V(r)) - J^2/r^2}$

solve for φ $\quad \Psi(\cancel{r},\cancel{\varphi},\cancel{E},J) = J$

Generating function

$$S(r,\varphi,E,J) = \int_{(r_0,\varphi_0)}^{(r,\varphi)} \left(P(r',E,J)\,dr' + J\,d\varphi' \right)$$

$$= \int_{r_0}^{r} P(r',E,J)\,dr' + (\varphi-\varphi_0)\,J$$

measure & initial angle

flow fct. T, Φ

$$T = \frac{\partial S}{\partial E} = \int_{r_0}^{r} \frac{m\,dr'}{P(r',E,J)} = t - t_0$$

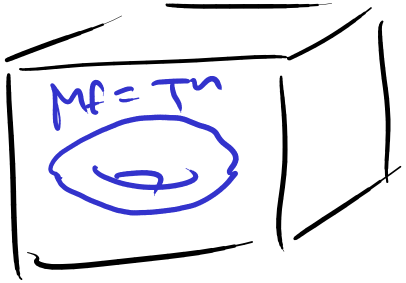
$$\Phi = \frac{\partial S}{\partial J} = \varphi - \varphi_0 - \int \frac{J\,dr'}{r'^2 P(r',E,J)} = \varphi(t_0) - \varphi_0$$

Compact Level Set and Action-Angle variables

M_f is compact:
- standardised set of coordinates
- quasi-periodic motion

Liouville Thm.

if M_f is compact: $\underline{M_f}$ is diffeomorphic to n-torus T^n



Liouville Torus

follow from existence of commuting flows

compact mfd of dim n with n com. vector fields $\Rightarrow T^n$

Can define convenient coordinates on compact level set T^n

n non-contr. cycles C_k. Use cycles to define alt repr. of charge coordinates f_k

$$I_k(f) := \frac{1}{2\pi} \oint_{C_k(f)} \sum_j p_j \, dq_j \quad \text{action variables}$$

$\{I_k\}$ replaces $\{F_k\}$.

construct flow functions dual to action variables

$$\theta_k := \frac{\partial S/\partial f_j}{\partial I_k/\partial f_j} \qquad S(q,f) = \int_{q_0(f)}^{q} \sum_j p_j(q',f) \, dq'_j$$

S defined up to shifts by non-triv. cycles

$$S \to S + \oint_{C_k(f)} \sum_j p_j \, dq_j = S + 2\pi I_k$$

Period matrix has very nice form

$$\Omega_{kl}(f) = \oint_{C_k(f)} d\theta_l = 2\pi\,\delta_{kl}$$

Angle variables θ_k increase by 2π over assoc. cycle $C_k(f)$

all θ_a are well-def mod 2π.

$$\{\theta_j, I_k\} = \delta_{jk} \qquad \{I_j, I_k\} = \{\theta_j, \theta_k\} = 0$$

$$I_k(t) = I_{0,k} = \text{const}$$

$$\theta_k(t) = \theta_{0,k} + (t - t_0)\,\omega_k(I_0) \qquad \omega_k(I_0) := \frac{\partial H}{\partial I_k}(I_0)$$

Motion along each cycle C_k is periodic $\Rightarrow$ altogether is quasi-periodic

1.4. Variation of Integrability

Darboux Theorem

cons. gen. mech. sys 2n-dim phase space, Ham H, sym $\hat{\omega}$

Darboux thm.: can choose coord. F_k, G_k for phase space st.

$$\hat{\omega} = \sum_{k=1}^{n} dG_k \wedge dF_k \qquad G_k, F_k \text{ diff. fct} \quad F_1 = H$$

looks like conditions for integrability. but:

F_k, G_k are defined only locally

Proper integrability requires F_k to be globally defined.

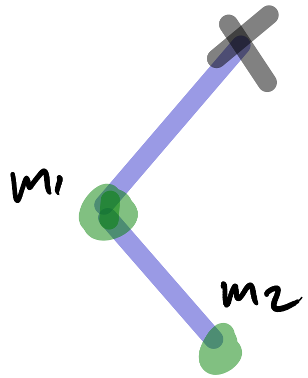
$\Rightarrow$ integrability is a global property of phase space not a local one.

Insufficient Charges

Systems with less than n charges or less than n charges in involution.

most systems with phase space dim ≥ 4 are non-int.

famous seemingly simple system: double pendulum



irregular motion

Chaos chaotic motion

integrability is absence of chaotic motion.

Broken Integrability

non-integrable deformation of an integrable system.

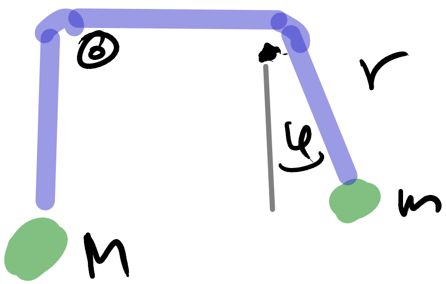
KAM theorem: Kolmogorov - Arnold - Moser

quasi-periodic motion of an integrable system extends to quasi-per. motion in dof. sys almost always for small def.

Ex: Swinging Atwood machine

dof. r, φ (+ can. conj)

constants: M, m, ℓ



$$L = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - g r (M - m\cos\varphi)$$

$p = (M+m)\dot{r}$

$\psi = m r^2 \dot{\varphi}$

$$H = \frac{p^2}{2(M+m)} + \frac{\psi^2}{2mr^2} + g r (M - m\cos\varphi)$$

Integrability for $\mu = M/m = 3$

$$F = \frac{p\psi}{4m^2}\cos(\varphi/2) - \frac{\psi^2}{2m^2 r}\sin(\varphi/2) + g r^2 \sin(\varphi/2)\cos^2(\varphi/2)$$

Super-Integrability

if in addition to F_k $k=1\dots n$ there are extra conserved charges $\Rightarrow$ super-integrable.

Note: cannot be in involution.

$\Rightarrow$ some angular velocities ω_k, ω_ℓ are rationally compatible

Extreme is $n-1$ further comm. charges: max. super-intg.

all ang. vel. ω_a are rat. comp. $\Rightarrow$ truly periodic.

- 2d phase space
- planetary motion / hydrogen atom
- multi-dim. HO with compatible freq.

Non-abelian symmetries

integrable system with some non-abelian sym

$\Rightarrow$ super-integrable $\qquad J_k$ form non-abelian Lie gr.

$\{H, J_k\} = 0$

$\{J_k, J_\ell\} = f_{k\ell m} J_m$ Lie brackets.

imply extra non-comm. cons. charges.

$SO(3)/SU(2)$ rot. symmetry $\quad J_x, J_y, J_z$

$$F_k \begin{cases} J^2 = J_x^2 + J_y^2 + J_z^2 & \text{commutes with all } J_k \\ J_z \end{cases}$$

extra: J_x s.t. $\{J_z, J_x\} \neq 0$