Introduction to Integrability

Lecture Slides

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The current version of this work can be found at:
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Chapter 0

Overview

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Introduction to Integrability

Overview

- Peculiar feature of some theoretical physics models.
- Makes calculations feasible → (complete) solvability
- map models to problems in complex funct. analysis
- hidden symmetry enhancement
- absence of chaotic motion
- colourful mixture of theoretical physics & maths.
- a lot of fun

Integrable Models

- Many of the simple models of classical mechanics.
  - free parti., HO, spinning top, Kepler problem / hydrogen.
- 1+1 dimension (1 space, 1 time)

  discrete space: lattice / continuous space: field
  - Korteweg-de Vries (KdV) eq.
  - sine Gordon
  - Einstein Gravity (2D)
  - sigma models on coset spaces
  - classical magnets (1D)
  - string theory
- Quantum Mechanical models, QFT (1+1 dim)
- Statistical Mechanics (Vertex models)
- AdS/CFT correspondence, higher dim large-N YM.

Prerequisites

- Classical analytical Mechanics
- (classical) Fields
- Algebra, Groups (QM)
- Complex Functional Analysis

References

- many books on integrable models

Chapter 1

Integrable Mechanics

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1 Integrable Mechanics

1.1. Hamiltonian Mechanics

Phase space M: dim $2n$ coordinates (q_k, p_k) $k = 1 \dots n$

Hamiltonian function $H : M \to \mathbb{R}$

Solution of system is a curve $(q_k(t), p_k(t))$ obeying Hamiltonian equations of motion

$$\dot{q}_k := \frac{\partial q_k}{\partial t} = + \frac{\partial H}{\partial p_k} \qquad \dot{p}_k := \frac{\partial p_k}{\partial t} = - \frac{\partial H}{\partial q_k}$$

Poisson brackets. Functions $F_1, F_2 : M \to \mathbb{R}$

$$\{F_1, F_2\} = \sum_k \left(\frac{\partial F_1}{\partial q_k} \frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k} \frac{\partial F_2}{\partial q_k} \right)$$

• bi-linear, • derivations (Leibniz), • anti-sym, • Jacobi-Id.

Formulate Ham. E.O.M.

$$\dot{q}_u = -\{H, q_u\} \qquad \dot{p}_u = -\{H, p_u\}$$

Generalised to arb. phase space functions $F(q, p, t)$

evaluated on (any) solution: $\hat{F}(t) := F(q(t), p(t), t)$

$$\frac{d\overset{(\wedge)}{F}}{dt} := \frac{\partial F}{\partial t} - \{H, F\} \qquad \text{total time der}$$

here: consider time-indep function $F = F(q, p)$

$$\frac{dF}{dt} = -\{H, F\} \qquad \frac{dF}{dt} = 0 \;:\; F \text{ is conserved}$$

Phase space is a symplectic space

symplectic structure : 2-form on phase space

$$\omega = \sum_k dq_k \wedge dp_k \qquad \text{(non-degenerate)}$$

ω is inverse of Poisson brackets

canonical structure (1-form) $\quad \sum_k p_k \, dq_k$

Canonical transformations $(q,p) \to (\tilde{q}, \tilde{p})$

phase space diffeomorphisms

use them to trivialise dynamics in integrable models

req for can. transf. $\{\tilde{q}_k, \tilde{p}_l\} = \delta_{kl}$; $\{\tilde{q}_k, \tilde{q}_l\} = \{\tilde{p}_k, \tilde{p}_l\} = 0$

$$\tilde{\omega} = \sum_k d\tilde{q}_k \wedge d\tilde{p}_k = \sum_k dq_k \wedge dp_k = \omega$$

1.2. Integrals of Motion

time-indep Ham. $H = H(q,p,\cancel{t})$

$$\frac{d}{dt} H = \frac{\partial H}{\partial t} - \{H, H\} = 0$$

$E = H(q,p)$

Energy (as value of H on a solution) is constant

$\Leftrightarrow$ Dynamics is constrained to hypersurfaces M_E of const energy E

hypersurf constrained by cond $E = H(q,p)$

Can be further (time-independent) phase space functions $F_k(q,p)$

constant on (all) solutions

$$\frac{d}{dt} F_k = -\{H, F_k\} \stackrel{!}{=} 0$$

$\rightarrow$ integral of motion, a conserved qty, conserved charge

$$\Rightarrow \text{charge} \quad F_k(q,p) = f_k = \text{const}$$

level set M_f

$$M_f := \{ (q,p) \in M ; \; F_k(q,p) = f_k \text{ for all } k \}$$

motion/dynamics takes place within a common M_f

Noether's theorem: conserved charge $\leftrightarrow$ cont. symmetry transf.

transformation generates new solutions from existing ones

flow $-\{F_{k'}, \cdot\}$ generates new solutions

$$(\tilde{q}(t), \tilde{p}(t)) = (q(t), p(t)) + \delta(q(t), p(t))$$ (inf. def. of solution)

with $\delta q(t) = -\epsilon \{F_k, q(t)\}$ $\quad \delta p(t) = -\epsilon \{F_k, p(t)\}$

deformed solution has same energy E and same charge under F_k (not necessarily same f_ℓ)

More simplifications if charges F_k Poisson commute

$$\{F_k, F_l\} = 0 \quad \text{for all } k, l$$

all solutions generated by flows have same charges (on same M_f)

→ commuting charges F_k

by constr. Ham H is among charges

- $H = F_1$ eg.
- $H = H(F_k)$

Finding such charges is not easy, not straight-forward

- trial and error
- Noether's theorem, symmetry
- add. charges generate novel/hidden symmetries.

2D Central Potetial

particle, mass m, symm. potential $V(r)$, $r := \sqrt{x^2+y^2}$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - V\left(\sqrt{x^2+y^2}\right)$$

convenient to use radial coord. $x = r\cos\varphi$

$y = r\sin\varphi$

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - V(r).$$

angle appears only through $\dot{\varphi}$, shift of φ: symm. of L

Ham form: Legendre transf. $p = m\dot{r}$, $\psi = m r^2 \dot{\varphi}$

$$H = \frac{p^2}{2m} + \frac{\psi^2}{2mr^2} + V(r) \qquad \dot{r} = \frac{\partial H}{\partial p} = \frac{p}{m};\ \dot{p} = -\frac{\partial H}{\partial r} = \frac{\psi^2}{mr^3} - V'(r)$$

$$\dot{\varphi} = \frac{\partial H}{\partial \psi} = \frac{\psi}{mr^2};\ \dot{\psi} = -\frac{\partial H}{\partial \varphi} = 0$$

conserved charge $F = \psi =: J$ angular mom. in 2D
and H is conserved, use F, H to express φ, ψ

$$P(r, E, J) = \sqrt{2m\left(E - V(r)\right) - J^2/r^2}$$

$$\Psi(J) = J \qquad r, \varphi, E, J \text{ new coord.}$$

can exp. r, ψ

know $\dfrac{dr}{dt} = \dfrac{P}{m}$ sep. of. var.

$$\int_{r_0}^{r} \frac{m\,dr'}{P(r', E, J)} = t - t_0 \quad \Rightarrow \text{find sol. } r(t)$$

substitute

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{J\,dt}{m\,r(t)^2} = \varphi_0 + \int_{r_0}^{r(t)} \frac{J\,dr'}{r'^2 P(r', E, J)}$$

1.3 Liouville Integrability

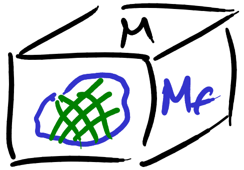
Mech. sys 2n-dim phase space M is Liouville integrable

- n independent
- everywhere differentiable
- conserved charges F_k (integrals of motion)
- in involution: Poisson commute $\{F_k, F_l\} = 0$.

Solve it by so-called quadratures. finite sequence of

- resolve relations among coordinates
  (non-linear eq, non-integral, non-differential eq.)
- calculate ordinary integrals (multi-dim)

not necessarily easy or doable in well-est. functions.



Phase Space Structure

For int. sys level sets M_f have a very nice structure

- M_f has dimension $n = \frac{1}{2} \dim M$
- there are n indep. commuting flows acting on it
- can specify coordinates on M using the F_k

n coordinates f_k (as values of $F_k(q,p)$)

n coord. functions $G_k(q,p)$ flow functions defined by

$$\{F_k, G_l\} = -\delta_{kl} \quad \text{with pt. } q_0(f) \text{ as origin.}$$

As such: coordinates per level set. Want to extend to all M.

add. rel $\{G_k, G_l\} = 0 \Rightarrow G_k$ can be constructed consistently

$(q,p) \to (G,F)$ canonical transf. $\{G_k, F_l\} = \delta_{kl}$, $\{F_k, F_l\} = \{G_k, G_l\} = 0$.

one very useful corollary: time evolution is linear in (g,f) coordinates:

$$H = H(q,p,\cancel{x}) = H(\cancel{g},f).$$

$$\frac{d}{dt} F_k = -\{H, F_k\} = 0$$

$$\frac{d}{dt} G_k = -\{H, G_k\} = -\sum_l \frac{\partial H}{\partial f_l}\{F_l, G_k\} = \frac{\partial H}{\partial f_k} =: v_k(f) = \text{const}.$$

$$f_k(t) = f_{0,k} = \text{const} \qquad g_k(t) = g_{0,k} + v_k\,(t-t_0).$$

Note: Flow coordinates defined by diff eq.
if level set has non-trivial cycles, flow coord.
not necessarily globally defined.
Follow G_ℓ around a non-triv. cycle $C_k(f)$ may obtain shift:

$$\Omega_{k\ell}(f) := \oint_{C_k(f)} d G_\ell \leftarrow \text{def from } \{F_k, G_\ell\} = -\delta_{k\ell}$$

note $\Omega_{k\ell}(f)$ is inv under smooth def of $C_k(f)$

$\Omega_{k\ell}(f)$ is called period matrix for level set M_f

Charges

Mech. sys., Phase spc (q_k, p_k).

Need to establish cons. charges $F_k(q,p)$.

No recipe that works in many systems.

However given some set of n ind. charges F_n can establish integrability by straight-forward verification.

Invert momentum coordinates in terms of F_k and q_k fixed

$$f_k = F_k(q,p) \quad \Leftrightarrow \quad p_k = P_k(q,f)$$

$$P_k(q, F_k(q,p)) = p_k$$

these are non-lin. (non-int; non-diff) eq. to be solved.

$$(q,p) \to (q,f)$$

Generating Functions

we want to obtain flow functions $\{G_k, G_l\} = 0$, $\{G_k, F_l\} = \delta_{kl}$

use tech. Gen. Funct.

$$S(q,f) := \int_{\gamma(q,f)} \sum_{k=1}^{n} p_k \, dq_k'$$

$\gamma(q,f)$ path on M_f connecting $q_0(f)$ to q.

$\int$ is inv. under cont. def b/c $p_k dq_k$ is closed 1-form.

$$dp_k \wedge dq_k = \underbrace{df_l}_{=0 \text{ on level set } M_f} \wedge dq_k \frac{\partial P_k}{\partial f_l} + dq_l \wedge dq_k \underbrace{\frac{\partial P_k}{\partial q_l}}_{\text{symmetric in } k,l}$$

symmetry: $f_j = F_j(q,p)$
diff. w.r.t. q_l

$$0 = \frac{\partial F_m}{\partial p_l}\frac{\partial F_j}{\partial q_l} + \frac{\partial F_m}{\partial p_l}\frac{\partial F_j}{\partial p_k}\frac{\partial P_k}{\partial q_l}$$

contract same rel. with $m \leftrightarrow j$ exch.

$$\Rightarrow \quad 0 = \underbrace{\{F_m, F_j\}}_{=0 \text{ by integr.}} + \underbrace{\frac{\partial F_m}{\partial q_l}}_{\text{invertible}} \underbrace{\frac{\partial F_j}{\partial p_k}}\underbrace{\left(\frac{\partial p_k}{\partial q_l} - \frac{\partial p_l}{\partial q_k}\right)}_{\overset{!}{=}0}$$

$\Rightarrow$ S is invariant under path deformations

computing S amounts to computing an ord. integral.

Flow Functions

Note that S reproduces $p_k(q,f)$ $\dfrac{\partial S}{\partial q_k}(q,f) = p_k(q,f)$.

but also $G_k(q,p) := \dfrac{\partial S}{\partial f_k}(q, F(q,p))$

$(q,p) \to (g,f)$ is canonical transf. of M.

2D Central Potential

radial pot $V(r)$, $H = F_1 = E$, $F_2 = \varphi = J$

solve for p $\quad P(r, \cancel{\varphi}, E, J) = \sqrt{2m\left(E - V(r)\right) - J^2/r^2}$

solve for φ $\quad \Psi(\cancel{r}, \cancel{\varphi}, \cancel{E}, J) = J$

Generating function

$$S(r, \varphi, E, J) = \int_{(r_0,\varphi_0)}^{(r,\varphi)} \left(P(r', E, J)\, dr' + J\, d\varphi' \right)$$

$$= \int_{r_0}^{r} P(r', E, J)\, dr' + (\varphi - \varphi_0) J$$

measure & initial angle

flow fct. T, Φ

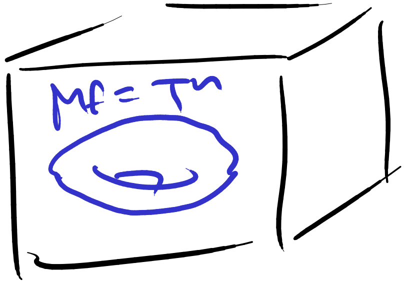
$$T = \frac{\partial S}{\partial E} = \int_{r_0}^{r} \underbrace{\frac{m\, dr'}{P(r', E, J)}}_{= t - t_0} \qquad \Phi = \frac{\partial S}{\partial J} = \varphi - \varphi_0 - \underbrace{\int \frac{J\, dr'}{r'^2 P(r', E, J)}}_{= \varphi(t_0) - \varphi_0}$$

Compact Level Set and Action-Angle variables

M_f is compact: • standardised set of coordinates
• quasi-periodic motion

Liouville Thm.

if M_f is compact: $\underline{M_f}$ is diffeomorphic to n-torus T^n



Liouville Torus

follow from existence of commuting flows

compact mfd of dim n with n com. vector fields $\Rightarrow T^n$

Can define convenient coordinates on compact level set T^n

n non-contr. cycles C_k. Use cycles to define alt repr. of charge coordinates f_k

$$I_k(f) := \frac{1}{2\pi} \oint_{C_k(f)} \sum_j p_j \, dq_j$$ action variables

$\{I_k\}$ replaces $\{F_k\}$.

Construct flow functions dual to action variables

$$\theta_k := \frac{\partial S/\partial f_j}{\partial I_k/\partial f_j} \qquad S(q,f) = \int_{q_0(f)}^{q} \sum_j p_j(q',f) \, dq'_j$$

S defined up to shifts by non-triv. cycles $\quad S \to S + \oint_{C_k(f)} \sum_j p_j \, dq_j = S + 2\pi I_k$

Period matrix has very nice form

$$\Omega_{k\ell}(f) = \oint_{C_k(f)} d\theta_\ell = 2\pi\,\delta_{k\ell}$$

Angle variables θ_k increase by 2π over assoc. cycle $C_k(f)$

all θ_a are well-def mod 2π.

$$\{\theta_j, I_k\} = \delta_{jk} \qquad \{I_j, I_k\} = \{\theta_j, \theta_k\} = 0$$

$$I_k(t) = I_{0,k} = \text{const}$$

$$\theta_k(t) = \theta_{0,k} + (t-t_0)\,\omega_k(I_0) \qquad \omega_k(I_0) := \frac{\partial H}{\partial I_k}(I_0)$$

Motion along each cycle C_k is periodic $\Rightarrow$ altogether is quasi-periodic

1.4. Variation of Integrability

Darboux Theorem

cons. gen. mech. sys 2n-dim phase space, Ham H, sym $\hat{\omega}$

Darboux thm.: can choose coord. F_k, G_k for phase space st.

$$\hat{\omega} = \sum_{k=1}^{n} dG_k \wedge dF_k$$

G_k, F_k diff. fct.

$F_1 = H$

looks like conditions for integrability. but:

F_k, G_k are defined only locally

Proper integrability requires F_k to be globally defined.

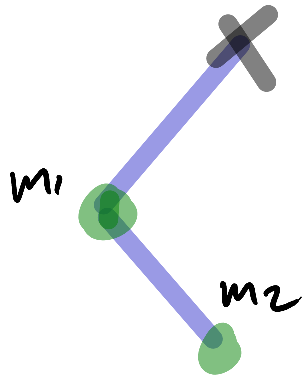
$\Rightarrow$ integrability is a global property of phase space not a local one.

Insufficient Charges

Systems with less than n charges or less than n charges in involution.

most systems with phase space dim ≥ 4 are non-int.

famous seemingly simple system: double pendulum



irregular motion

Chaos chaotic motion

integrability is absence of chaotic motion.

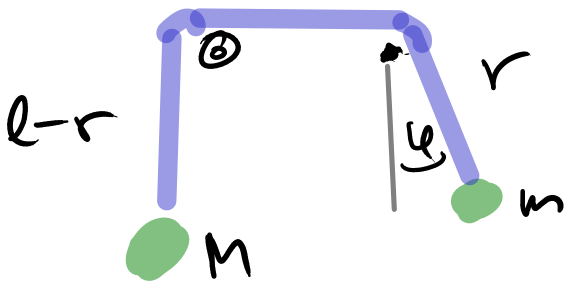
Broken Integrability

non-integrable deformation of an integrable system.

KAM theorem: Kolmogorov-Arnold-Moser

quasi-periodic motion of an integrable system extends to quasi-per. motion in dof. sys almost always for small def.

Ex: Swinging Atwood machine



dof. r, φ (+ can. conj)

constants: M, m, l

$$L = \tfrac{1}{2}(M+m)\dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\varphi}^2 - g r (M - m\cos\varphi)$$

$$p = (M+m)\dot{r} \qquad \psi = m r^2 \dot{\varphi}$$

$$H = \frac{p^2}{2(M+m)} + \frac{\psi^2}{2mr^2} + g r (M - m\cos\varphi)$$

Integrability for $\mu = M/m = 3$

$$F = \frac{p\psi}{4m^2}\cos(\varphi/2) - \frac{\psi^2}{2m^2 r}\sin(\varphi/2) + g r^2 \sin(\varphi/2)\cos^2(\varphi/2)$$

Super-Integrability

if in addition to F_k $k = 1 \ldots n$ there are extra conserved charges $\Rightarrow$ super-integrable.

Note: cannot be in involution.

$\Rightarrow$ some angular velocities ω_k, ω_ℓ are rationally compatible

Extreme is $n-1$ further comm. charges: max. super-intg.

all ang. vel. ω_a are rath. comp. $\Rightarrow$ truly periodic.

- 2d phase space
- planetary motion / hydrogen atom
- multi-dim. HO with compatible freq.

Non-abelian symmetries

integrable system with some non-abelian sym

$\Rightarrow$ super-integrable $\qquad$ J_k form non-abelian Lie gr.

$\{H, J_k\} = 0$

$\{J_k, J_\ell\} = f_{k\ell m} J_m$ $\qquad$ Lie brackets.

imply extra non-comm. cons. charges.

$SO(3)/SU(2)$ rot. symmetry $\quad J_x, J_y, J_z$

$$F_k \begin{cases} J^2 = J_x^2 + J_y^2 + J_z^2 & \text{commutes with all } J_k \\ J_z & \end{cases}$$

extra. J_x $\quad$ st. $\quad \{J_z, J_x\} \neq 0$

Chapter 2

Algebraic Integrability

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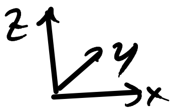
2 Algebraic Integrability

2.1 Spin Models

elementary classical spin d.o.f.: phase space $M = S^2$

<u>Spinning Top</u> / Rigid body fixed at c.o.m., no gravity



co-moving frame, axes aligned with princ. mom. inert. $\Omega_x, \Omega_y, \Omega_z$

Euler angles $\vartheta, \varphi, \psi \leftrightarrow$ ang. mom. vect. $\vec{S}$ in co-moving frame

$$
\begin{aligned}
S_x &= -\Omega_x (\dot\varphi \sin\vartheta \sin\psi + \dot\vartheta \cos\psi) \\
S_y &= -\Omega_y (\dot\varphi \sin\vartheta \cos\psi - \dot\vartheta \sin\psi) \\
S_z &= -\Omega_z (\dot\varphi \cos\vartheta + \dot\psi)
\end{aligned}
$$

Lagrangian $L = \frac{S_x^2}{2\Omega_x} + \frac{S_y^2}{2\Omega_y} + \frac{S_z^2}{2\Omega_z} = L(\varphi, \vartheta, \psi, \dot\varphi, \dot\vartheta, \dot\psi)$

E.o.M → Euler Eq.

$$\frac{d}{dt} S_x = \left(\frac{1}{\Omega_y} - \frac{1}{\Omega_z}\right) S_y S_z$$

$$\frac{d}{dt} S_y = \left(\frac{1}{\Omega_z} - \frac{1}{\Omega_x}\right) S_z S_x$$

$$\frac{d}{dt} S_z = \left(\frac{1}{\Omega_x} - \frac{1}{\Omega_y}\right) S_x S_y$$

conserved charges : $H, \vec{J}$ in inertial frame (4/6)

3 Poisson commute: $H, \vec{J}^2, J_z$

integrable (super-int; SO(3) symmetry)

Focus on $\vec{S}$ subspace $\mathbb{R}^3$ (3/6)

$|\vec{S}| = J$ fixed $\Rightarrow$ Phase space $\Rightarrow$ $M = S^2 \subset \mathbb{R}^3$

$\to$ elementary spin d.o.f. / spin model

$\vec{S}$ spin vector

generate $SO(3)$

$\Leftrightarrow$ Lie brackets

Poisson brackets for $\vec{S}$ $\quad \{S_j, S_k\} = \varepsilon_{jkl} S_l$. tot antisym 3-tensor

Hamiltonian for $\vec{S}$ $\quad H = \frac{1}{2} \vec{S}^T \Omega^{-1} \vec{S} \quad \Omega = \mathrm{diag}(\Omega_x, \Omega_y, \Omega_z)$

Ham E.o.M. $\quad \frac{d}{dt} \vec{S} = -\{H, \vec{S}\} = (\Omega^{-1} \vec{S}) \times \vec{S}$.

Phase space reduces to $M = S^2$ $\quad F_1, F_2 : S^2 \to \mathbb{R}$

$$\{F_1, F_2\} = \varepsilon_{jkl} S_l \frac{\partial F_1}{\partial S_j} \frac{\partial F_2}{\partial S_k}.$$

note $\{|\vec{S}|, F\} \overset{\text{always}}{=} 0$

fix $|\vec{S}| = J \doteq \text{const}$,

Spin Parametrisations

different phase space coordinates useful.

- $\vec{S}$ is a vector. but $|\vec{S}| = J$ $\vec{S}^2 = J^2$ non-lin const.
- spherical coordinates $\vec{S} = J \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$

  manifestly 2D coord.

  $\{\vartheta, \varphi\} = \dfrac{1}{J\sin\vartheta}$. Drawback: periodically identified. two singular pt: N, S

- stereographic proj $S^2 \to \bar{\mathbb{C}}$: complex number $\zeta \in \bar{\mathbb{C}}$

  $$\vec{S} = \frac{J}{1+|\zeta|^2}\begin{pmatrix} 2\,\mathrm{Re}\,\zeta \\ 2\,\mathrm{Im}\,\zeta \\ 1-|\zeta|^2 \end{pmatrix} \qquad \zeta = \tan(\vartheta/2)\, e^{i\varphi} = \frac{S_x + i S_y}{J + S_z}$$

  just one complex coord. (ζ, ζ^*)

  $$\{\zeta, \zeta^*\} = -\frac{i}{2J}(1+|\zeta|^2)^2.$$

• spinor repr. $SO(3) \simeq SU(2)$ express $\vec{S}$ as 2×2 matrix

$\vec{S} \to \vec{S}\cdot\vec{\sigma}$ $\quad \vec{\sigma}$ Pauli matrices

hermitian, traceless, eigenvalues $\pm J$.

eigenvector relations $(\vec{S}\cdot\vec{\sigma})s = +Js$ $\quad s \in \mathbb{C}^2$

$$\vec{S} = J\frac{s^\dagger\vec{\sigma}s}{s^\dagger s} \qquad (\vec{S}\cdot\vec{\sigma})\epsilon s^* = -J\epsilon s^* \qquad \epsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$s \in \mathbb{C}^2$ is a spinor repr. of pt. on S^2

Caveat: projective space $s \in \mathbb{C}P^1$ $\quad s \equiv \lambda s \quad \lambda \in \mathbb{C}^*$

$s = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$ stereographic ζ $\quad$ resolves $\zeta = \infty$: $s \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \infty \end{pmatrix}$

Poisson brackets $\{F_1, F_2\} = -\frac{i}{2J} s^\dagger s \left(\frac{\partial F_1}{\partial s} \cdot \frac{\partial F_2}{\partial s^*} - \frac{\partial F_1}{\partial s^*} \cdot \frac{\partial F_2}{\partial s} \right)$

$F(s) \overset{!}{=} F(\lambda s)$

altogether 4 reprs of M: $\mathbb{R}^3 \supset S^2 = \bar{\mathbb{C}} = \mathbb{C}P^1$

Classes of solutions

- explicit solutions, general case $\Omega_x \neq \Omega_y \neq \Omega_z \neq \Omega_x$

  solution in terms of Jacobi elliptic functions sn, cn, dn

elliptic

$$S_x = c_x \, \mathrm{cn}(\lambda t + \varphi ; k)$$
$$S_y = c_y \, \mathrm{sn}(\lambda t + \varphi ; k)$$
$$S_z = c_z \, \mathrm{dn}(\lambda t + \varphi ; k)$$

sine, cosine, delta elliptic functions

c_k are functions of λ, k, depend on E, J, Ω_k; φ initial pos.

- for $k=0$ special case where e.g. $\Omega_x = \Omega_y \neq \Omega_z$

trigonometric

$$S_x = c \cos(\lambda t + \varphi)$$
$$S_y = c \sin(\lambda t + \varphi)$$
$$S_z = \text{const.}$$

$\Leftarrow$ SO(2) res. rot sym in co-moving frame.

- most symmetric case $\Omega_x = \Omega_y = \Omega_z$: SO(3) rot. sym, no dynamics.

"rational"

classification of integrable systems

type	rational	trigonometric	elliptic
symbols	XXX	XXZ	XYZ
$\Omega_x \Omega_y \Omega_z$	$\Omega_x \Omega_x \Omega_x$	$\Omega_x \Omega_x \Omega_z$	$\Omega_x \Omega_y \Omega_z$
symmetry	$SO(3)$	$SO(2)$ (Cartan subalg)	—

equal values of Ω_k

2.2 Lax Pair

Formulate Integrability using algebraic methods.

Spin Model $\quad SO(3) = SU(2)$

$\{S_j, S_k\} = \varepsilon_{jkl} S_l$ $\quad$ represent $\vec{S}$ as a matrix using Pauli matrix generators of $su(2)$

$$\vec{S}\cdot\vec{\sigma} = \begin{pmatrix} +S_z & S_x - iS_y \\ S_x + iS_y & -S_z \end{pmatrix}$$

note $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c \;//\; [\vec{v}\cdot\vec{\sigma}, \vec{w}\cdot\vec{\sigma}] = 2i(\vec{v}\times\vec{w})\cdot\vec{\sigma}$

E.o.M. $\quad \dfrac{d}{dt}\vec{S}\cdot\vec{\sigma} = ((\Omega^{-1}\vec{S})\times\vec{S})\cdot\vec{\sigma} = -\dfrac{i}{2}[(\Omega^{-1}\vec{S})\cdot\vec{\sigma}, \vec{S}\cdot\vec{\sigma}]$

Def $\quad T := \vec{S}\cdot\vec{\sigma}, \quad M := -\frac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$

EoM: $\quad \dfrac{d}{dt}T = [M, T] \implies$ spec T is conserved

spec $T = \{\pm J\}$ $\quad$ J const indep of M.

Lax Pair

E.o.M can be formulated in terms of a Lax Pair (T, M), two square matr. $(\mathrm{End}(V))$ and whose elem. are phase space functions.

T is Lax matrix, M is evolution matrix

Lax eq. $\frac{d}{dt} T = -\{H, T\} = [M, T]$ statement in phase space

Ham form.

holds by virtue of EoM / is equiv. to Eo.M.

conseq. EV spectrum of T is conserved sim trans (t)

$$T(t) = g(t)\, T(t_0)\, g(t)^{-1} \quad t_0 \text{ refr. time}$$

characteristic pol. $\det(\tau\, \mathrm{id} - T)$ is indep of time.

If Lax pair (satisfying Lax eq) exists, generate conserved charges F_k as traces of powers of T

$$F_k := \frac{1}{k} \operatorname{tr} T^k \qquad \text{1/k sym factor.}$$

$$\frac{d}{dt} F_k = \operatorname{tr}\left([M,T]\, T^{k-1}\right) \overset{\text{cycl. of trace}}{=} 0 \qquad \text{conservation!}$$

Note: • not all F_k are necess. indep.
• there may be additional cons. charges.

Complete Lax Pairs

Lax formulation is nice and useful for integrability: but

- Lax pair is never unique
- not every Lax pair is useful
- no recipe for construction of Lax pair (general)
- abstract, not necessarily related to physics
- size of T, M is not related to features of system.

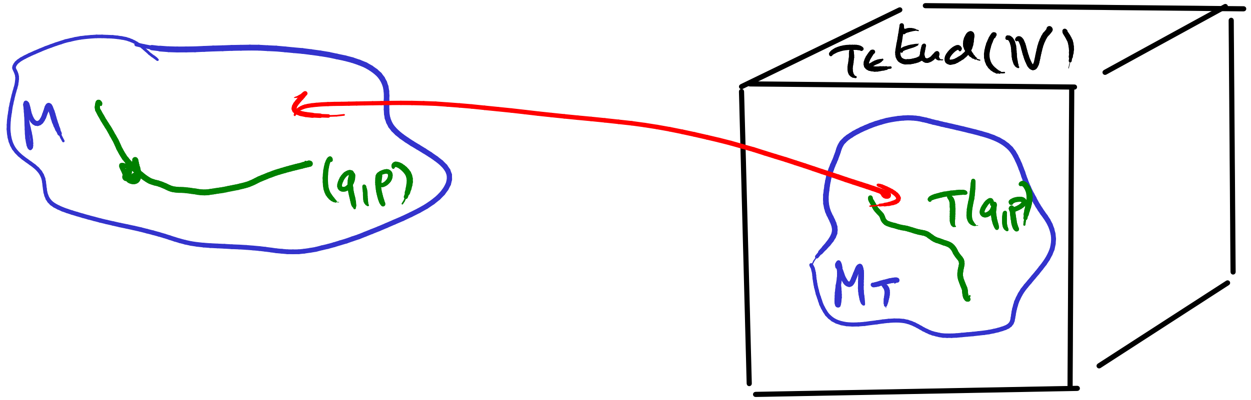
How to formulate a useful Lax pair for integrable system.

Generalisations of Lax pair for spin system:

- other representations $2\times 2 \to (2s+1)\times(2s+1)$
- add unit matrix to T $\quad T = \vec{S}\cdot\vec{\sigma} + uH\,\mathrm{id}\ ;\ M = -\frac{i}{2}(\Omega^{-1}\vec{S})\cdot\vec{\sigma}$

changes spectrum of T in a useful way : $\operatorname{spec} T = \{uH \pm J\}$ not const of M

Lax formulation as algebraic fcn. of phase space



Establish a 1:1 map between M and $M_T \subset \mathrm{End}(V)$

Point in $M_T \subset \mathrm{End}(V)$ specifies state uniquely.

· furthermore eigenvalues of T represent conserved charges

Desirable Properties for a complete Lax Pair formulation:

i) (T, M) obeys Lax eq. $dT/dt = [M, T]$

ii) T encodes all $2n$ phase space coordinates

iii) T is diagonalisable almost everywhere on M

iv) spectrum must encode n indep. variables

v) these variables are in involution.

iv) + v) $\Rightarrow$ Liouville integrable system.

2.3. Lax-Poisson Structure

Lax-Poisson Equation

Phase space variables encoded into matrix T, elements T_{jk}

$$\{T_{jk}, T_{lm}\} = \sum_n R_{(jl)(nm)} T_{nk} - \sum_n T_{jn} R_{(nl)(km)} - \sum_n R_{(lj)(nk)} T_{nm} + \sum_n T_{ln} R_{(nj)(mk)}$$

$R_{(jl)(km)}$ are elements of rank-2 tensor operator

combination of terms guarantees that eigenvalues of T Poisson commute

Tensor Notation

Matrix A in components $\sum_{jk} A_{jk} E_{jk}$

E_{jk} matrix with all elements 0 except for 1 in row j, col k.

Poisson brackets of matrices

$$\{A \overset{\otimes}{,} B\} := \sum_{jklm} \{A_{jk}, B_{lm}\}\, E_{jk} \otimes E_{lm}$$

tensor operator R out in

$$R := \sum_{jklm} R_{(jl)(km)}\, E_{jk} \otimes E_{lm}$$

$$P(R) := \sum_{jklm} R_{(jl)(km)}\, E_{lm} \otimes E_{jk}$$

P is tensor product perm.

$$\Rightarrow \{T \overset{\otimes}{,} T\} = [R, T \otimes id] - [P(R), id \otimes T]$$

short hand notation for tensor operators / sites:

index denotes site on which tensor acts, no label means: id

$$R \to R_{12} \qquad T_1 := T \otimes \mathrm{id} \qquad T_2 := \mathrm{id} \otimes T \qquad P(R_{12}) \to R_{21}$$

$$\{T_1, T_2\} := \{T \otimes T\} = [R_{12}, T_1] - [R_{21}, T_2]$$

Properties and Applications

Consider Poisson brackets of conserved charges $F_k := \frac{1}{k} \mathrm{tr}(T^k)$

$$\{F_j, F_k\} = \frac{1}{jk}\{\mathrm{tr}\, T^j, \mathrm{tr}\, T^k\} = \frac{1}{jk} \mathrm{tr}_{1,2} \{T_1^j, T_2^k\}$$

$$= \frac{1}{jk} \sum_{l=1}^{j} \sum_{m=1}^{k} \mathrm{tr}_{1,2} \left(T_1^{l-1} T_2^{m-1} \{T_1, T_2\} T_1^{j-l} T_2^{k-m} \right)$$

$$= \mathrm{tr}_{1,2} \left(T_1^{j-1} T_2^{k-1} \{T_1, T_2\} \right)$$

$$= \mathrm{tr}_{1,2} \left(T_1^{j-1} T_2^{k-1} [R_{12}, T_1] - T_1^{j-1} T_2^{k-1} [R_{21}, T_2] \right)$$

$$= 0 \qquad \text{(due to cyclicity)}$$

Jacobi identity?

$$123 \to 231, 312$$

$$0 = [T_1, [\![R, R]\!]_{123} + \{T_2, R_3\} - \{T_3, R_{12}\}] + 2 \text{ cycl.}$$

symbol $[\![\cdot, \cdot]\!]$ defined $\qquad [\![X, Y]\!]_{123} = -[\![X, Y]\!]_{132}$

$$[\![X, Y]\!]_{123} := [Y_{12}, Y_{13}] + [Y_{12}, X_{23}] + [X_{32}, Y_{13}]$$

Example: el. spin model $\vec{S}$, $T := \vec{S} \cdot \vec{\sigma} + u H \, \mathrm{id}$

$$\{T_1, T_2\} = (\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot \vec{S}$$
$$+ u((\Omega^{-1} \vec{S}) \times \vec{S}) \cdot \vec{\sigma}_1 - u((\Omega^{-1} \vec{S}) \times \vec{S}) \cdot \vec{\sigma}_2$$

Lax Poisson Eq solved by Lax Poisson ch. for T:

$$R_{12} = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u (\Omega^{-1} \vec{S}) \cdot \vec{\sigma}_1$$

use Lax Poisson Structure for improved def. of complete Lax Pair

→ complete Lax-Poisson structure (T, M, R)

i) pair L, M obeys Lax eq. $dT/dt = [M, T]$

ii) Lax matrix T encodes all 2n phase space d.o.f.

iii) T diagonalisable almost everywhere

iv) spectrum of T encodes n indep. var.

v) Lax Poisson stru. R obeys Lax Poisson equation.

Evolution from Lax-Poisson Structure

H is conserved $\Rightarrow$ $H = h(T)$ ← spectrum of T

show that Lax eq. holds $\frac{d}{dt} T = -\{H, T\} = [M, T]$

with evolution matrix M given by $\qquad h = h(F_k)$ $\quad F_k = \frac{1}{k} \operatorname{tr} T^k$

$$M_1 = \sum_k \frac{\partial h}{\partial F_k} \operatorname{tr}\left(T_2^{k-1} R_{12}\right) \qquad dh = \sum_k \frac{\partial h}{\partial F_k} dF_k$$

in def of compl. Lax Poisson struct: i) Ham H is given by $h(T)$

Ex: $T = \vec{S} \cdot \vec{\sigma} + u \mathbb{1} \operatorname{id}$ $\quad R_{12} = -\frac{i}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u \left(\Omega^{-1} \vec{S}\right) \cdot \vec{\sigma}_1 \operatorname{id}_2$

$H = \operatorname{tr} T/2u$ $\;\cdots\;$ $M_1 = \frac{1}{2u} \operatorname{tr}_2 R_{12} = -\frac{i}{2} \left(\Omega^{-1} \vec{S}\right) \cdot \vec{\sigma}_1$

Parametric Lax Pairs

we can have Lax Pair that depend on a (complex) par. u.

→ by expanding in u can package many qty into small matrices

→ perform complex analysis on u-dependence.

$$\frac{d}{dt} T(u) = [M(u), T(u)] \qquad \text{spectrum } F_k(u)$$

$$F_1(u) = 2uH, \quad F_2(u) = J^2 + u^2 H^2, \quad F_3(u) = 2uH\left(J^2 + \tfrac{1}{3}u^2 H^2\right) \quad \ldots$$

Extend to R Lax Poisson Str.

$$\{T_1(u_1), T_2(u_2)\} = [R_{12}(u_1, u_2), T_1(u_1)] - [R_{21}(u_2, u_1), T_2(u_2)]$$

$$T_1 := T_1(u_1) \quad \xrightarrow{\text{short}} \quad \{T_1, T_2\} = [R_{12}, T_1] - [R_{21}, T_2]$$

$$F_j(u) = \frac{1}{j}\,\mathrm{tr}\left(T(u)^j\right)$$

implies

$$\{F_j(u_1),\, F_k(u_2)\} = 0 \quad \text{for all } j, k,\ u_1, u_2$$

For sph model

$$R_{12}(u_1, u_2) = -\frac{i}{4}\,\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{i}{2} u_2 \left(\Omega^{-1}\vec{S}\right) \cdot \vec{\sigma}_1$$

Classical r-Matrix

alternative to describe the Poisson structure of T

relevant relation : RTT relation

$$\{T \overset{\otimes}{,} T\} = [r, T \otimes T] \quad // \quad \{T_1, T_2\} = [r_{12}, T_1 T_2]$$

typically have $r_{12} = -r_{21}$ (antisymmetry of $\{\}$)

can construct R from r, Jacobi id

$$0 = [[[r, r]], T_1 T_2 T_3] + [\{r_{12}, T_3\}, T_1 T_2] + \text{2 cyclic}$$

often r is indep of phase space

classical Yang-Baxter eq. $[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0$

Chapter 3

Classical Spin Chains

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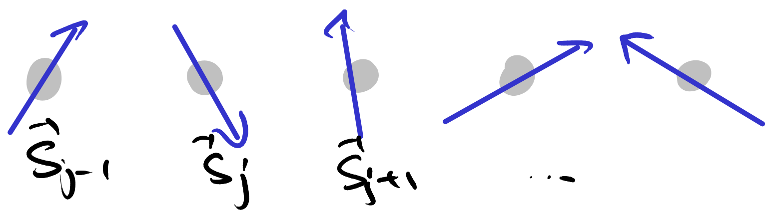
3 Classical Spin Chains

chain models

- simple systems on each site (integrable)
- interactions between neighbouring sites

3.1 Heisenberg Spin Chain



$$\|\vec S_j\| = 1 = \vec S_j^{\,2}$$

$$\{S^a_j, S^b_k\} = \delta_{j,k}\, \varepsilon^{abc} S^c_j$$

$$\{\vec S_j, \vec S_k^{\,2} - 1\} = 0$$

- homogeneous along chain
- rot inv, relative spin matters.

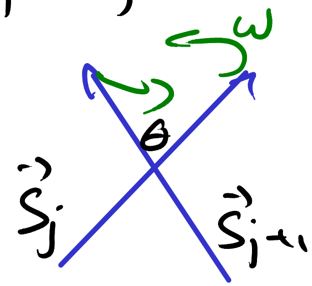
$$H = \sum_j H_j$$

$$H_j = -\log \frac{1 + \vec S_j \cdot \vec S_{j+1}}{2}$$

↖ required for integrability

EOM. $$\frac{d\vec{S}_j}{dt} = -\{H, \vec{S}_j\} = -\frac{\vec{S}_{j-1} \times \vec{S}_j}{1+\vec{S}_{j-1}\cdot\vec{S}_j} + \frac{\vec{S}_j \times \vec{S}_{j+1}}{1+\vec{S}_j \cdot \vec{S}_{j+1}} .$$

$$\omega \simeq \frac{1}{\cos\theta/2}$$



convert to stereographic proj. / spinors

$$\frac{1+\vec{S}_j\cdot\vec{S}_k}{2} = \frac{(1+\zeta_j\zeta_k^*)(1+\zeta_k\zeta_j^*)}{(1+|\zeta_j|^2)(1+|\zeta_k|^2)} = \frac{(s_j^\dagger s_k)(s_k^\dagger s_j)}{(s_j^\dagger s_j)(s_k^\dagger s_k)}$$

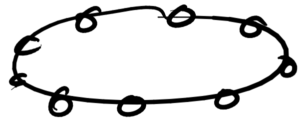
$$\frac{d\zeta_j}{dt} = \frac{i}{2}\sum_{\pm} \frac{1+|\zeta_j|^2}{1+\zeta_{j\pm1}\zeta_j^*}(\zeta_{j\pm1}-\zeta_j)$$

$$\frac{ds_j}{dt} = \frac{i}{2}\frac{s_j^\dagger s_j}{s_j^\dagger s_{j-1}} s_{j-1} + \frac{i}{2}\frac{s_j^\dagger s_j}{s_j^\dagger s_{j+1}} s_{j+1} + i\lambda_j s_j$$

Boundary Conditions

various choices which are integrable

- closed / periodic BC



$$\vec{S}_{j+L} = \vec{S}_j \qquad H = \sum_{j=1}^{L} H_j$$

- open BC $\quad H = \sum_{j=1}^{L-1} H_j$

- infinite BC $\quad \vec{S}_j \to \vec{S}_{L/R} \quad j \to \mp\infty$

$$H = \sum_{j=-\infty}^{+\infty} H_j$$

other combinations of the above

- semi-infinite chains
- boundary interactions at ends
- twisted closed BC.

Global Symmetry

here $SO(3)$ rotational symmetry, rotate all $\vec{S}_j$ simultaneously

$$\delta \vec{S}_i = -\{\delta\vec{X}\cdot\vec{S}_j, \vec{S}_i\}$$

total angular mom. vector (conserved) $\vec{J} = \sum_j \vec{S}_j$

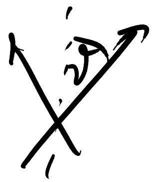
cons: $\{H, \vec{J}\} = 0$

discrete current $(\vec{Q}_j, \vec{k}_j)$ $\vec{Q}_j = \vec{S}_j$ $\vec{k}_j = \dfrac{\vec{S}_j \times \vec{S}_{j+1}}{1+\vec{S}_j \cdot \vec{S}_{j+1}}$

$$\frac{d}{dt}\vec{Q}_j = -\{H, \vec{Q}_j\} = \vec{k}_j - \vec{k}_{j-1}$$

$$\{J^a, J^b\} = \varepsilon^{abc} J^c \quad \leftarrow \text{ lie algebra of } SO(3)$$

Simple Solutions (closed)



$L=1$ single isotropic spin, no dynamics $H=0$

$L=2$ two spin vectors rotating around middle axis with constant $\omega = 2/\cos\theta/2 = 2/\cos\vartheta$ ← angle between axis and spin

$$H = -4\log|\cos\vartheta| \qquad \vec{J} = 2\cos\vartheta\,\vec{e}_z$$

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta\cos(2\pi nj/L - \omega t) \\ \sin\vartheta\sin(2\pi nj/L - \omega t) \\ \cos\vartheta \end{pmatrix} \qquad \begin{matrix} L=2 \\ n=1 \end{matrix}$$

↑ works for arbitrary L, n, $0<n<L$

$$\omega = \frac{2\cos\vartheta\sin^2\pi n/L}{1-\sin^2\vartheta\sin^2\pi n/L} \qquad \vec{J} = L\cos\vartheta\,\vec{e}_z$$

$$H = -L\log\left(1-\sin^2\vartheta\sin^2(\pi n/L)\right)$$

$L=3$ solutions are more difficult (elliptic fn)
but some special cases ($L=3$, $n=1,2$ above):

$$\vec{S}_j(t) = \begin{pmatrix} \sin\vartheta_j & \cos(-\omega t) \\ \sin\vartheta_j & \sin(-\omega t) \\ \cos\vartheta_j & \end{pmatrix}$$

all spins on a common plane

$$H = -2\log\frac{|J^2-1|}{8} \qquad \omega = \frac{4J}{J^2-1}$$

$$J^2 = 3 + 2\sum_j \cos(\vartheta_j - \vartheta_{j+1})$$

curious: two regimes of solutions (disconnected)
depending on $0<J<1$ $\quad 1<J<9$

Excitations of the Ferromagnetic Ground State

Ground state: all spins are aligned along z-axis

$$\vec{S}_k(t) = \vec{e}_z \quad \rightarrow H = 0 \qquad \vec{J} = L\vec{e}_z$$

Stereographic variables $\vec{S}_k \rightarrow \zeta_k \in \mathbb{C}$ $\qquad \zeta_k \sim \epsilon$

EOM $\quad \dfrac{d\zeta_j}{dt} = \dfrac{i}{2}\left(\zeta_{j-1} - 2\zeta_j + \zeta_{j+1}\right) + O(\epsilon^3)$

solve linear diff. eq. $\quad$ planewave b/c homogeneous

$$\zeta_j(t) = \epsilon a_n \exp\frac{2\pi i n j}{L} \exp(-i\omega_n t) + O(\epsilon^2)$$

angular velocity $\qquad \omega_n = 2\sin^2\dfrac{\pi n}{L}$

total ang. mom $\qquad\qquad$ energy

$$\vec{J} = \left(L - 2\epsilon^2 |a_n|^2 L\right)\vec{e}_z + \dots \qquad H = 4\epsilon^2 |a_n|^2 L \sin^2\frac{\pi n}{L} + \dots$$

more natural to express eqs. in terms of action variables

symplectic structure

$$\hat{\omega} = \sum_j 2i \, d\zeta_j \wedge d\zeta_j^*$$
$$= 2\epsilon |a_n|^2 L \omega_n \, dt \wedge d\epsilon + O(\epsilon^3)$$

$$dI_n = \frac{1}{2\pi} \oint \hat{\omega} = 4 \, d\epsilon \, \epsilon \, |a_n|^2 L + \ldots$$

$$I_n = 2|a_n|^2 \epsilon^2 L + \ldots$$

$$\vec{J} = (L - I_n) \vec{e}_z' + \ldots \qquad H = \omega_n I_n + \ldots$$

3.2 Integrable Structure

Express model in algebraic integrable framework

Lax pair (T, M) s.t. T encodes state and Lax eq.

$$\frac{d}{dt} T = [M, T]$$

Lax Transport

construct T recursively over the sites of chain.

introduce el. Lax transport $\mathcal{L}_j$, evol. M_j: transport eq.

$$\frac{d}{dt} \mathcal{L}_j = M_j \mathcal{L}_j - \mathcal{L}_j M_{j-1}.$$

construct composite Lax transport over sites $j+1 \ldots k$

$$W_{k,j} := \mathcal{L}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_{j+2} \mathcal{L}_{j+1}$$

Lax transport eq. holds for $W_{k,j}$ as:

$$\frac{d}{dt} W_{k,j} = M_k W_{k,j} - W_{k,j} M_j$$

For a closed chain of length L: Lax monodromy T

$$T := W_{L,0} = \mathcal{L}_L \ldots \mathcal{L}_1 \quad \text{serves a Lax matrix } T$$

$$\frac{d}{dt} T = [M, T] \quad \text{with evol. } M = M_0 = M_L$$

For Heisenberg spin chain

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u} \vec{S}_j \cdot \vec{\sigma}$$

$$M_j(u) = \frac{i}{u^2+1} \frac{(\vec{S}_j + \vec{S}_{j+1} + u\, \vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma}}{1 + \vec{S}_j \cdot \vec{S}_{j+1}} \qquad u \in \mathbb{C}$$

Elementary to show that Lax transp. eq. holds, use

$$(\vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{\sigma} = i(\vec{S}_{j+1} \cdot \vec{\sigma})(\vec{S}_j \cdot \vec{\sigma}) - i(\vec{S}_j \cdot \vec{S}_{j+1})\, \mathrm{id}$$

We have a Lax pair $\Rightarrow$ traces of powers of T are conserved:

$F = F_1 \qquad F_m(u) := \frac{1}{m} \operatorname{tr} T(u)^m$ need only $m = 1$

because $\det \mathcal{L}_j = 1 + \frac{1}{u^2} \Rightarrow \det T = \left(1 + \frac{1}{u^2}\right)^L$

r-matrix for Heisenberg-chain

classical RTT relation extends to $\mathcal{L}_j$ as follows

$$\{\mathcal{L}_j(u_1) \overset{\otimes}{,} \mathcal{L}_k(u_2)\} = \delta_{jk}\, r_j(u_1,u_2)\,(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2))$$
$$- \delta_{jk}\,(\mathcal{L}_j(u_1)\otimes\mathcal{L}_j(u_2))\, r_{j-1}(u_1,u_2)$$

allows to combine Lax transport into Lax monodromy T

$$\{T(u_1) \overset{\otimes}{,} T(u_2)\} = [r_L(u_1,u_2), T(u_1)\otimes T(u_2)]$$

.. $\Rightarrow$ $\{F_m(u_1), F_n(u_2)\} = 0$

- later show that all d.o.f. encoded into $T(u)$

$$r_j(u_1,u_2) = r(u_1,u_2) = -\frac{\sum_a \sigma^a\otimes\sigma^a}{2(u_1-u_2)}$$ std. solution to class. YBE.

3.3 Spectral Parameter

Lax Matrices are 2×2, but depend on $u \in \bar{\mathbb{C}}$

- can encode all $2n$ d.o.f of phase space of chain.
- can do complex analysis in u.

<u>Hamiltonian</u>

complication: $H = \sum H_j$ is "local" but $T(u)$ is non-local

question how to extract local information from non-local qty.?

hint: $\mathcal{L}_j(u)$ must become special for extraction of local data.

Heisenberg chain: $\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} = 0$ for $u = \pm i$

will arrive at $H = -\log \frac{F(+i)\, F(-i)}{4^L}$.

want to verify. use form of $\mathcal{L}_i$ at $t = \pm i$

$$\mathcal{L}_j(\pm i) = \mathrm{id} \pm \vec{S}_j \cdot \vec{\sigma}$$

Projector: EV are 2, 0. matrix has lower rank, rank 1

$$\mathrm{tr}\, \mathcal{L}_j(\pm i) = 2 \qquad \mathcal{L}_j(\pm i)^+ = \mathcal{L}_j(\pm i)$$

write $\mathcal{L}_j(\pm i)$ using spinors s_j

$$\mathcal{L}_j(+i) = \frac{2}{s_j^+ s_j}\, s_j s_j^+$$

relate $\mathcal{L}_j(-i)$ to $\mathcal{L}_j(+i)$ by transposition

$$\mathcal{L}_j(-i) = \varepsilon \mathcal{L}_j(+i)^T \varepsilon^{-1} = \frac{2}{s_j^+ s_j}\, \varepsilon s_j^* s_j^T \varepsilon^{-1}.$$

compute products of $\mathcal{L}_j$:

$$F(+i) = 2^L \prod_{j=1}^{L} \frac{s_{j+1}^+ s_j}{s_j^+ s_j} \qquad F(-i) = 2^L \prod_{j=1}^{L} \frac{s_j^+ s_{j+1}}{s_j^+ s_j}$$

$$F(+i)F(-i) = 4^L \prod_{j=1}^{L} \frac{\overset{\text{sites } j,j+1}{\cdots}}{\cdots} = 2^L \prod_{j=1}^{L} \left(1 + \vec{S}_j \cdot \vec{S}_{j+1}\right)$$

$$\exp(-H) = \prod_{j=1}^{L} \frac{(s_{j+1}^+ s_j)(s_j^+ s_{j+1})}{(s_j^+ s_j)(s_{j+1}^+ s_{j+1})}$$

$$\frac{F(-i)}{F(+i)} = \exp(iP) = \prod_{j=1}^{L} \frac{s_j^+ s_{j+1}}{s_{j+1}^+ s_j}$$

also can generate further local qty from expansion of $F(u)$ around $u = \pm i$

Reconstruction

want to extract $\vec{S}_j$ from $T(u)$. via pt $u = \pm i$

consider Lax monodromy at $u = +i$ depends only on sites $L, 1$

$$T(+i) = 2^L \frac{s_L s_1^+}{s_L^+ s_L} \prod_{j=1}^{L-1} \frac{s_{j+1} s_j^+}{s_j^+ s_j} = F(+i) \frac{s_L s_1^+}{s_1^+ s_L}$$

consider EVect.

EV $= 0$ the eigenvector of $T(u)$ is spinor εs_1^* ← determines s_1^*

likewise $T(-i) = F(-i) \left(\mathrm{id} - \frac{s_1 s_L^+}{s_L^+ s_1} \right)$; EV$=0 \to$ eigvect. s_1

compose spin vector $\vec{S}_j = \frac{s_j^+ \vec{\sigma} s_j}{s_j^+ s_j}$

how to obtain other sites $j \neq 1, L$?

recursion. consider shifted monodromy $T = T_L$

$$T_{j-1}(u) = \mathcal{L}_{j-1}(u) \dots \mathcal{L}_1(u)\, \mathcal{L}_L(u) \dots \mathcal{L}_j(u)$$

recursion relation

$$T_j(u) = \mathcal{L}_j(u)\, T_{j-1}(u)\, \mathcal{L}_j(u)^{-1}$$

Procedure:
- compute $\vec{S}_1$ from T_L
- compute $\mathcal{L}_1$ from $\vec{S}_1$
- compute T_1 from $\mathcal{L}_1$
- compute $\vec{S}_2$ from T_1

$\vdots$

Proves that $T(u)$ encodes all of phase space

obtain all $\vec{S}_k$ after $3L$ steps.

Global Symmetry

$T(u)$ contains total ang. mom $\vec{J}$ at $u = \infty$

at $u = \infty$ we have expansion of $\mathcal{L}_j(u)$

$$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u} \vec{S}_j \circ \vec{\sigma} + \ldots$$

expand T around $u = \infty$

$$T(u) = \mathrm{id} + \frac{i}{u} \sum_{j=1}^{L} \vec{S}_j \cdot \vec{\sigma} + \ldots = \mathrm{id} + \frac{i}{u} \vec{J} \cdot \vec{\sigma} + \ldots$$

Casimir for $\vec{J}$

for monodromy trace $\quad F(u) = 2 - \frac{1}{u^2} (J^2 - L) + \ldots$

further terms in expansion of $T(u)$ around $u = \infty$
are multi-local charges eg. $\vec{Y} := \sum_{k=1}^{L} \sum_{j=1}^{k-1} \vec{S}_j \times \vec{S}_k$

Chapter 4

Spectral Curves

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4. Spectral Curves

4.1 Spectral Curve

Start with some generic state $\vec{S}_j(t) \to T(u)$ Lax matrix

know spectrum of $T(u)$, in particular trace $F(u)$ is conserved

Eigenvalues spectrum of $T(u)$ is time-independent for all $u \in \overline{\mathbb{C}}$.

here trace $F(u)$ determines spectrum

recall $\det \mathcal{L}_j(u) = 1 + \frac{1}{u^2} \Rightarrow \det T(u) = \left(1 + 1/u^2\right)^L$

$F(u) = \operatorname{tr} T(u)$ is a polynomial of deg. L in $1/u$

$$\left.\begin{array}{l} \tau_1 \tau_2 = \det T \\ \tau_1 + \tau_2 = \operatorname{tr} T = F \end{array}\right\} \Rightarrow \tau_{1,2}(u) = \tfrac{1}{2} F(u) \pm \sqrt{\tfrac{1}{4} F(u)^2 - \left(1 + 1/u^2\right)^L}$$

Singularities

elements of $T(u)$ is polynomial in $1/u$ of degree L $\Rightarrow$ analytic except at $\tilde{u} = 0$

L-fold pole at $u = \tilde{u} = 0$ $\Rightarrow$ $\tau_{1,2}(u)$ will have L-fold pole at $u = 0$.

nevertheless $\tau_{1,2}(u)$ do not need to be analytical at $u \neq 0$

Some exceptions to analyticity possible due to solving EV.

Name where radicand of soln of $\tau_{1,2}$ equals zero.

$\Rightarrow$ square-root branch points $\hat{u}_j$ where $\frac{1}{4} F(\hat{u}_j)^2 = \left(1 + \frac{1}{\hat{u}_j^2}\right)^L$

algebraic eq. of deg $2L$ in $1/\hat{u}$. $\Rightarrow$ $2L$ solutions $\hat{u}_j$ $j = 1 .. 2L$.

These are where $\tau_1(\hat{u}_j) = \tau_2(\hat{u}_j)$

Note that $F(u)$ is special at $u = \infty$

$$F(u) = 2 + \frac{D}{u} - \frac{1}{u^2}(J^2 - L) + \ldots$$

leading two coefficients of alg. eq match $\Rightarrow$ 2 fixed solutions $\hat{u}_{2L-1} = \hat{u}_{2L} = \infty$

(related to so(3) symmetry)

and $2L-2$ poles which are not universally fixed.

<u>Simple Solutions</u>

$L=2$ $\quad s_{1,2}(t) = \begin{pmatrix} 1 \\ I \tan(\vartheta/2)\, e^{-i\omega t} \end{pmatrix}$ $\quad \omega = \frac{2}{\cos\vartheta}$ pow.

$$T(u) = \mathrm{id} + \frac{2i}{u}\cos\vartheta\, \sigma^z - \frac{1}{u^2}\begin{pmatrix} \cos(2\vartheta) & e^{+i\omega t}\sin(2\vartheta) \\ -e^{-i\omega t}\sin(2\vartheta) & \cdot\cos(2\vartheta) \end{pmatrix}$$

$$F(u) = \mathrm{tr}(u) = 2 - \frac{2}{u^2}\cos(2\vartheta) \qquad H = -\log\frac{F(i)F(-i)}{16} = -4\log|\cos\vartheta|$$

$$\tilde{v}_{1,2}(v) = 1 + \frac{\cos(2\vartheta)}{v^2} \pm \frac{2i\cos\vartheta}{v}\sqrt{1+\frac{\sin^2\vartheta}{v^2}}$$

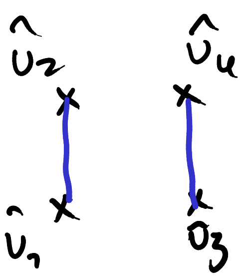
square-root pt at $\hat{v}_{1,2} = \mp i \sin\vartheta \leftarrow$ all information (ϑ) contained in $\hat{v}_j$

example at $L=3$ great circle $\quad F(v) = 2 + \dfrac{3-J^2}{v^2}$

select case $1 < J \leq 3$ parametrise as $J^2 = 5 - 4\cos\mu \quad \mu \in]0,\pi]$

branch points at

$$\hat{v}_{1,4} = \pm \frac{e^{-i\mu}}{\sqrt{1-2e^{-i\mu}}} \qquad \hat{v}_{2,3} = \pm \frac{e^{i\mu}}{\sqrt{1-2e^{i\mu}}} = \hat{v}^*_{1,4}$$



Spectral Curve

investigate square root branch points $\hat{u}_n$ + neighbourhood

branch point $\hat{u}$ is where analyticity of $\tau_{1,2}(u)$ breaks

$$\tau_{1,2}(u) = \frac{1}{2} F(\hat{u}) \pm \hat{k}\sqrt{u-\hat{u}} + \mathcal{O}(u-\hat{u})$$

small circle

Follow function $\tau_1(u)$ around $u = \hat{u}$ $\qquad u(\sigma) = \hat{u} + \epsilon e^{i\sigma}$

$$\tau_1(u(\sigma)) = \frac{1}{2} F(\hat{u}) + \hat{k}\sqrt{\epsilon}\, e^{i\sigma/2} + \mathcal{O}(\epsilon)$$

$\tau_1(u(\sigma))$ returns to initial value after rotation of σ by 4π.

rotation by 2π : interchanges eigenvalues $\tau_1 \leftrightarrow \tau_2$

$$\tau_1(u(\sigma+2\pi)) = \tau_2(u(\sigma))$$

$$\{\tau_j(u(\sigma+2\pi))\} = \{\tau_j(u(\sigma))\}.$$

2 eigenvalue functions $\tau_a(u)$ form a two-sheeted cover of $\bar{\mathbb{C}}$ (minus puncture at $u = \hat{u} = 0$)

branch points are connected in pairs by branch cuts.

eigenvalue functions $\tau_a(u)$ as single valued function $f(z)$ on a Rieman surface Γ non-triv. topology. as follows

for every $z \in \Gamma$ associate a sheet $\underset{\{1,2\}}{a(z)}$ and pt $u(z) \in \bar{\mathbb{C}}$

s.t. $f(z) = f_{a(z)}(u(z))$ and cuts are where $a(z)$ is discont.

Rieman surface is a complex curve (spectral curve)

1-d submfd Γ of 2-d complex space $(u, \tau) \in \mathbb{C}^2$

$$\Gamma = \{ (u, \tau) \in \bar{\mathbb{C}}^2 ; \det(T(u) - \tau) = 0 \}.$$

for every value of u there are two points $z \in \Gamma$

provide permutation map $z \to z^*$ st $u(z^*) = u(z)$

$$\tau(z^*) = \frac{\det T(u(z))}{\tau(z)} = F(u(z)) - \tau(z).$$

Example $L=2$

$$\tilde{\tau}_{1,2}(u) = 1 + \frac{\cos(2\vartheta)}{u^2} \pm \frac{2i\cos\vartheta}{u}\sqrt{1+\frac{\sin^2\vartheta}{u^2}}$$

introduce $u(z) = \frac{1}{2}\sin\vartheta \cdot (z - 1/z)$

$$\tau(z) = \left(\frac{z + 1/z - 2i\cot\vartheta}{z - 1/z}\right)^2, \quad z \to z^* = -1/z$$

branch pt. $\hat{z}_{1,2} = \mp i$

4.2 Ground State and Excitations

Compare spectral curve to (perturbative) solutions:
Ferromagnetic ground state + excitations.

Ground State $\vec{S}_j = \vec{e}_z$

$\mathcal{L}_j(u) = \mathrm{id} + \frac{i}{u}\sigma^z := \mathcal{L}$ eigenval $(u \pm i)/u$

$\Rightarrow T(u) = \mathcal{L}(u)^L$ has eigenvalues

$$T_{1,2}(u) = \frac{(u \pm i)^L}{u^L}.$$ two disconnected sheets.

have no square-root singularities.. $\Rightarrow g = -1$

but normal genus at L is $g = L-2 \gg -1$

spectral curve is highly degenerate.

degeneracy of Γ. consider $F(u)$

$$F(u) = \tau_1(u) + \tau_2(u) = \frac{(u+i)^L + (u-i)^L}{u^L} \quad \text{pol. deg } L$$

$$\tau_{1,2}(u) = \frac{1}{2} F(u) \pm \sqrt{\frac{1}{4} F(u)^2 - \frac{(u^2+1)^L}{u^{2L}}}$$

potential branch points: $0 = \left(\frac{(u+i)^L - (u-i)^L}{2u^L} \right)^2$

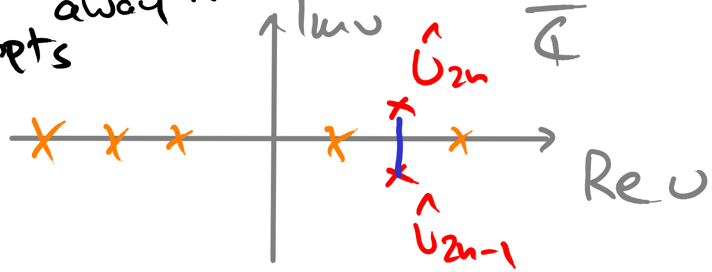
$2L-2$ double roots at $\hat{u}_{2k-1,2k} = \cot \frac{\pi k}{L}$ $k = 1 \ldots L-1$

no singular behaviour of $\tau_{1,2}(u)$ at $u = \hat{u}_k$

but this signals that a higher-genus curve has degenerated to this configuration by moving two nearby branch pts together.

Single Excitation

move two branchpts away from e.a.



how to change $F(u)$ to achieve this?

preserve polynomial nature of $F(u)$.

done by $F \to F + \delta F$ with

$$\delta F(u) = i\epsilon^2 \frac{(u+i)^L - (u-i)^L}{u^L (u - \hat{u}_{2n})}$$

← • preserves polyn.
• zeros at $u = \hat{u}_{2n}$
• except at $u = \hat{u}_{2n}$

deformed eq.

$$F(\hat{u})^2 = 2F(\hat{u})\,\delta F(\hat{u}) + \ldots = \frac{4(\hat{u}^2+1)^L}{\hat{u}^{2L}}$$

solutions: $\hat{u} = \hat{u}_{2k}$ (twice) for $k \neq n$

$$\hat{u}_{2n-1,2n} = \hat{u}^0_{2n} \mp \frac{i\epsilon\sqrt{2/L}}{\sin(\pi n/L)}.$$

analyse charges of corresponding state through $F(u)$

$u = \infty$ $\qquad \delta F(u) = \dfrac{2L\epsilon^2}{u^2} + \ldots \Rightarrow$ tot ang. mom J $\quad \delta J^2 = -2L\epsilon^2$

$$\Rightarrow \delta \vec{J} = -\epsilon^2 \vec{e}_z$$

energy & momentum

$$\delta H = -\frac{\delta F(+i)}{F(+i)} - \frac{\delta F(-i)}{F(-i)} = \frac{2\epsilon^2}{\hat{u}_{2n}^2+1} = 2\epsilon^2 \sin^2\frac{\pi n}{2}$$

similar to excitations of ferromag. vacuum.

combine $J+H \Rightarrow \delta H = -2\,\delta J \sin^2\frac{\pi n}{L}$ matches prev!

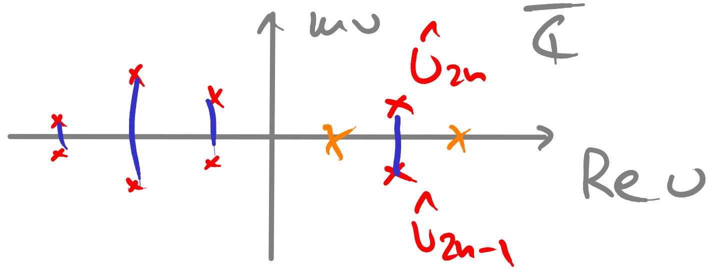
use action variables

$$\delta I_n = \pm \frac{1}{2\pi} \oint_{\hat{u}_{2n}} \frac{du\, \tau(u)}{\sqrt{\det \tau(u)}} = \epsilon^2 + \ldots$$

$$\left.\begin{aligned} \delta H &= 2\,\delta I_n \sin^2\frac{\pi n}{L} + \ldots \\ \delta \vec{J} &= -\delta I_n \vec{e}_z + \ldots \end{aligned}\right\} \text{agrees!}$$

$$\omega_n = \frac{\partial H}{\partial I_n} = 2\sin^2\frac{\pi n}{L}.$$

Multiple Excitations



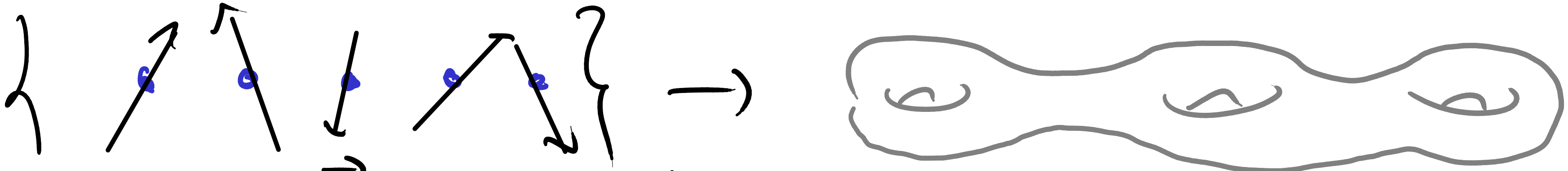
here to order ϵ^2 all deformations are independent $\Rightarrow$ qty add up $\quad H = \sum_{n=1}^{L} I_n \cdot \omega_n$

leading order matches.

spectral curve provides an exact description beyond linear regime

eg. take I_n larger, still obtain precise results including non-linear effects.

$L-1$ excitation modes of f.m. vac



conserved charges $\vec{S}_j$ (half of phase space) $\quad \Gamma \quad L-1$ moduli

4.3 Dynamical Divisor

Singularities

Eigenvectors determined by EV eq.

τ_a eigenvalues $a = 1, 2$
ψ_a corr. eigenvectors

$$T(u)\,\psi_a(u) = \tau_a(u)\,\psi_a(u)$$

Eq. has a solution $\psi_a(u)$ for all $\tau_a(u)$ for all u
dependence on u is analytic almost everywhere
3 types...

1. monodromy $T(u)$ has a pole singularity
   $\Rightarrow \tau_a(u)$ has same singularity
   know $T(u)$ has L-fold pole at $u = \hat{u} = 0$
   can remove singularity by rescaling by some pol. fn. u^L
   this does not affect eigenvectors
   So no particular singularity in $\psi_a(u)$ to be expected.

2. square-root singularities in $T_a(u)$ but not $T(u)$ (diagonalisation) contradiction from assuming $\psi_a(u)$ to be analytic
$\Rightarrow$ $\psi_a(u)$ has a square-root singularity at branch pt.

3. normalisation of eigenvectors is undetermined by EV eq.
may renormalise $\psi_a(u)$ by $f(u)$; by this generate/remove pole sing.

Branch Points

- at square-root sing. both eigenvectors degenerate $\psi_1(\hat{u}) = \psi_2(\hat{u})$
- monodromy $T(\hat{u})$ is non-diagonalisable at these points.
  → single true eigenvector $\psi_1(\hat{u}^*) = \psi_2(\hat{u})$

non-diagonalisable $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

A, B, C, D are analytic at $u = \hat{u}$.
($T(\hat{u})$ is not)

eigenvalues

$$\tau_{1,2}(u) = \frac{1}{2}\left(A(u)+D(u)\right) \pm \sqrt{\frac{1}{4}\left(A(u)-D(u)\right)^2 + B(u)C(u)}$$

branch pt are where $\tau_1 = \tau_2$, radicand $= 0$

expand at $u = \hat{u}$ $\quad \tau(u) = \tau(\hat{u}) \pm \hat{k}\sqrt{u-\hat{u}} + \dots$

$$\hat{k} = \sqrt{\frac{1}{2}(\hat{A}-\hat{D})(\hat{A}'-\hat{D}') + \hat{B}\hat{C}' + \hat{C}\hat{B}'}$$

assume $T(\hat{u})$ to be diagonalisable: two eigenval. $\tau_1 = \tau_2$

$\Rightarrow T(\hat{u}) = \tau_{1,2}\cdot \mathrm{id} \Rightarrow \hat{A} = \hat{D}, \quad \hat{B} = \hat{C} = 0$

$\Rightarrow \hat{k} = 0 \Rightarrow$ no square root branch point.

consider behaviour of eigenvectors at $u = \hat{u}$

$$\psi_a(u) = \begin{pmatrix} -B(u) \\ A(u) - \tau_a(u) \end{pmatrix}$$

Beneficial for formulating $\psi(z)$ as a function on Γ

$$\psi_1(\hat{u}) = \psi_2(\hat{u})$$

namely $\quad \psi(z) = \psi_{a(z)}(u(z))$ is analytic on Γ at $u = \hat{u}$

EV eq on Γ $\quad T(u(z))\,\psi(z) = \tau(z)\,\psi(z)$

$\tau(z)$, $\psi(z)$ are analytic on Γ

example chain with $L=2$

$$\psi(z) = \begin{pmatrix} 1 \\ ie^{-i\omega t} z \end{pmatrix}$$

Dynamical Divisor

scaling of $\psi(z)$ is not determined. where are singularities?

$$\psi(z) \equiv \lambda(z)\psi(z)$$

Therefore we normalise $\psi(z)$ in some particular way, our choice

eg. $$v_r \cdot \psi(z) \overset{!}{=} 1 \quad \text{for some vector } v_r.$$

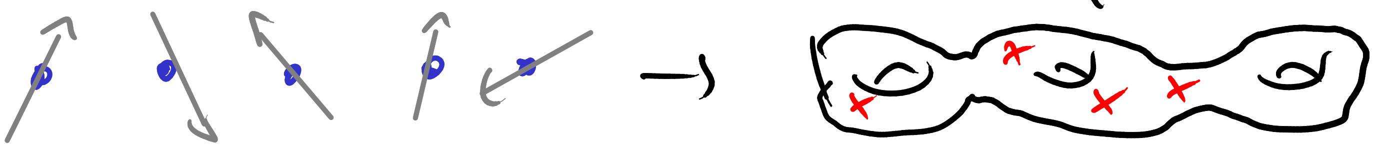
for choice $v_r = (1\ 0)$ $\Rightarrow$ $\psi(z) = \begin{pmatrix} 1 \\ \zeta(z) \end{pmatrix}$ stereographic projection.

reduces information in $\psi(z)$ to a function $\zeta(z)$

well-defined (but dependent on v_r) set of poles $\{\breve{z}_k\}$

this set encodes all dynamical data of state

$\Rightarrow$ $\{\breve{z}_k\}$ dynamical divisor for state (set of marked points on Γ)



Alternative picture for $\{\breve{z}_n\}$:

ψ is map $\Gamma \to \mathbb{C}P^1$ (rather than $\mathbb{C}^2$)

namely: ψ is defined up to scaling, ψ describes direction

$\breve{z}_n$ are poles of $\zeta(z)$ but these originate from normalisation

$V_r \cdot \psi(z) = 1$ $\breve{z}_n$ is where $\psi(z) \sim V_r^{\perp}$

Divisor consists of all points $\breve{z}_n$ where $\psi(\breve{z}_n)$ takes a specific direction.

Claim: $\{\breve{z}_n\}$ consists of $g+1$ points on Γ $\psi \sim \begin{pmatrix} 1 \\ \zeta \end{pmatrix}$
where g is genus of Γ. $\downarrow$

Define function $f(u) := \left(\psi_1(u)^T \epsilon \psi_2(u)\right)^2 = \left(\zeta_1(u) - \zeta_2(u)\right)^2$

1. $f(u)$ is a meromorphic function of $u \in \bar{\mathbb{C}}$

- certainly of z!
- interchange two eigenvalues/vectors $\psi_1 \leftrightarrow \psi_2$
  $f(u)$ remains the same $\Rightarrow$ also analytic here.

2. zeros of $f(u)$ are branch points.

- note $f(u) = 0$ if two vectors are collinear at branch pt.
- if $T(u)$ is diagonalisable (generic u) $\Rightarrow$ two eigenvectors span $\mathbb{C}^2$ $\Rightarrow f(u) \neq 0$

further each branch point contributes single zero for $f(u)$.

for a curve Γ of genus g two sheets are connected by $g+1$ branch cuts $\Rightarrow$ $2g+2$ branch points.

3. meromorphic fn. $f(u)$ on compact $\bar{\mathbb{C}}$ has as many poles as zeros.
$2(g+1)$ poles. all poles are double by construction $f(u) = (\ldots)^2$
double pole due to either $\zeta_1(u)$ or $\zeta_2(u)$ (single) $\Rightarrow$ $g+1$ poles in $\zeta(z)$.

example $L=2$ state $\quad v_r = (1, -1/\varsigma_r) \quad \varsigma_r \in \bar{\mathbb{C}}$.

normalise ψ st. $\quad v_r \cdot \psi = \psi_1 - \psi_2 / \varsigma_r = 1$

$$\psi(z) = \frac{1}{1 - i\,\varsigma_r^{-1} e^{-i\omega t} z} \begin{pmatrix} 1 \\ i\,e^{-i\omega t} z \end{pmatrix}$$

pole at $\quad \check{z}(t) = -i\,\varsigma_r\, e^{i\omega t} \quad$ (rotates with ω on $\Gamma = \bar{\mathbb{C}}$)

Evolution

$\{\breve{z}_k\}$ describes truly dynamical data of state

set moves around on Γ in well-prescribed way

$$\frac{dT}{dt} = [M, T] \Rightarrow \frac{d\psi}{dt} = M\psi + \lambda\psi$$

normalisation ab + progresses

keep $v_r \cdot \psi = 1$ solve for λ

$$\frac{d}{dt}\psi(z) = M(z)\psi(z) - \left(v_r \cdot M(z)\psi(z)\right) \cdot \psi(z)$$

non-linear, but nevertheless has solution.

consider eq. near a pole $\breve{z}$. double poles on both sides: cancel;

$$\frac{d\breve{z}}{dt} = -\operatorname*{res}_{z=\breve{z}} \left(v_r \cdot M(z)\psi(z)\right)$$

Example:

$$M(u) \equiv \frac{1}{u^2+1}\frac{1}{\cos\vartheta}\begin{pmatrix} i & u\, e^{+i\omega t}\sin\vartheta \\ -u\, e^{-i\omega t}\sin\vartheta & -i \end{pmatrix}$$

EV evolution

$$\frac{d}{dt}\psi + \lambda_1 \psi = \frac{2}{\cos\vartheta}\begin{pmatrix} 0 \\ z\, e^{-i\omega t}\end{pmatrix} = M\psi + \lambda_2 \psi$$

verify using solution $\breve{z} = -i\,\zeta_r\, e^{i\omega t}$

$$\operatorname*{res}_{z=\breve{z}} \psi(z) = \breve{z}\begin{pmatrix} 1 \\ \zeta_r \end{pmatrix}$$

$$(1-\zeta_r^{-1})\, M(u)\begin{pmatrix} 1 \\ \zeta_r\end{pmatrix} = \frac{2i}{\cos\vartheta}\,\frac{u\, u(\breve{z})+1}{u^2+1}$$

$$\Rightarrow \quad \frac{d\breve{z}}{dt} = i\omega\,\breve{z} = \frac{2i}{\cos\vartheta}\,\breve{z}$$ holds for actual ang. vel. $\omega = 2/\cos\vartheta$

Symmetry

system has SO(3) rotation symmetry and cons. charge $\vec{J}$

∘ lowers the typical genus of curve from $g = L-1 \to g = L-2$

because pt $u=\infty$ related to symmetry is double pt of Γ

means that direction $\vec{J}$ is not encoded in Γ,

not in divisor

review expansion at $u=\infty$

$$T(u) = \mathrm{id} + \frac{i}{u}\vec{J}\cdot\vec{\sigma} + \ldots$$

at $u=\infty$ eigenvectors of $T(u)$ are not fixed by EV eq.

because $T(\infty) = \mathrm{id}$. nevertheless can consider $u \to \infty$

suppose $u(z) = \dfrac{c}{z - z_\infty} + \dots$ on Γ

$J = |\vec{J}|$

then $\tau(z) = 1 \pm \dfrac{iJ}{c}(z - z_\infty) + \dots$

Eigenvectors $\psi_{1,2}(z)$ as $z \to z_\infty / z_\infty^\times$

$$(\vec{J} \cdot \vec{\sigma})\,\psi(z_\infty) = \pm J \psi(z_\infty)$$

$$(\vec{J} \cdot \vec{\sigma})\,\psi(z_\infty^\times) = \mp J \psi(z_\infty^\times)$$

4.4 Construction of Solutions

Spectral Curve

construct $\tau(z)$ on Riemann surface Γ

$$\tau(z)^2 - F(u(z))\tau(z) + \det T(u(z)) = 0$$

$F(u)$ is a polynomial of deg. L in $1/u$

leading terms $F(u) = 2 + 0/u + \dots$ $\quad L-1$ d.o.f.

$\det T(u) = (1 + 1/u^2)^L$

alg. eq. describes $2L-2$ branch pt $\Rightarrow$ $L-1$ cuts, genus $g = L-2$

has $L-1$ indep. moduli

correspond to $L-1$ action variables

Dynamical Divisor

assume normalisation $\psi(z) = \begin{pmatrix} 1 \\ \zeta(z) \end{pmatrix}$

as a meromorphic function of degree $g+1$ (poles)

Riemann-Roch thm $\Rightarrow$ $3+g$ d.o.f. in choosing $\psi(z)$

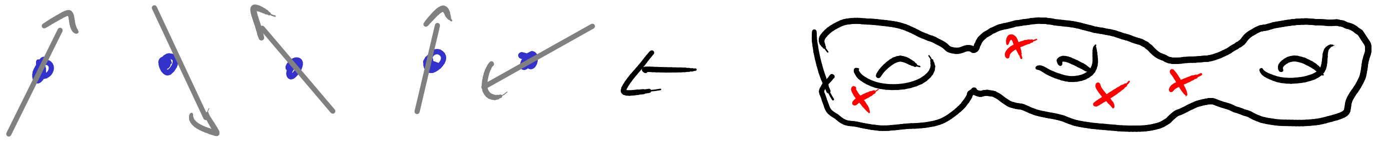
$(g+1$ poles, 1 $\underbrace{\text{scaling}, 1 \text{ shift}}_{\text{direction of } \vec{\zeta}/J})$

$\{\tilde{z}_k\}$

Reconstruct

$$T(u(z)) = \tau(z) \frac{\psi(z)\psi(z^\times)^T \epsilon}{\psi(z^\times)^T \epsilon \, \psi(z)} + \tau(z^\times) \frac{\psi(z^\times)\psi(z)^T \epsilon}{\psi(z)^T \epsilon \, \psi(z^\times)}.$$

reconstruct state $\vec{S}_k$ from $T(u)$



consider d.o.f. curve generically has $g = L-2$

eigenvector has $g+3 = L+1$ d.o.f.

Γ has $L-1$ dof from $F(u)$

altogether: $2L$ d.o.f. $\simeq$ dim of phase space S^2 for each site.

Chapter 6

Quantum Spin Chains

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6. Quantum Spin Chains

Focus on Spin Chain Models:

- they form a large class of int. QM models
- they can be treated with same uniform framework
- they can have several parameters to tune
- short chains are genuine QM models
- long chains approximate (1+1) D QFT models
- for large quantum numbers approach classical mechanics.
- they model magnetic materials

QM magnetism:	$\uparrow\downarrow$	energy	ferromagnetism	anti-ferrom.
nearby spins	opp. aligned	$\uparrow\downarrow$	high energy	low energy
	equal aligned	$\uparrow\uparrow$	low energy	high energy

- Ising model: class. stat mech model. lattice of spins, alignment det eny.

- Heisenberg quantum spin chain: QM model $\uparrow\downarrow$. 1D model. Ham acts on NN.

6.1. Heisenberg Spin Chain

Setup Spin state $|\uparrow\rangle$, $|\downarrow\rangle$, or any complex lin. comb.

$\Rightarrow$ spin site described by vector space $\mathbb{V} = \mathbb{C}^2$.

Spin chain of length L is L-fold tensor product

$$\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_L$$

Hilbert space $\mathbb{V}^{\otimes L}$ has dimension 2^L. Basis from "pure" states

$$|\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\rangle$$

Hamiltonian operator $H: \mathbb{V}^{\otimes L} \to \mathbb{V}^{\otimes L}$, acts locally homogeneously

$$H = \sum_j H_j \qquad H_j = \mathbb{V}_j \otimes \mathbb{V}_{j+1} \to \mathbb{V}_j \otimes \mathbb{V}_{j+1}$$

Heisenberg Hamiltonian density

$$H_{12} = \lambda_0 (1 \otimes 1) + \lambda_x (\sigma^x \otimes \sigma^x) + \lambda_y (\sigma^y \otimes \sigma^y) + \lambda_z (\sigma^z \otimes \sigma^z)$$

all combinations of parameters $\lambda_{0,x,y,z} \in \mathbb{R}$ is integrable:

- general $\lambda_x \neq \lambda_y \neq \lambda_z \neq \lambda_x \rightarrow$ "XYZ" modell
- two λ's equal $\lambda_x = \lambda_y \neq \lambda_z \rightarrow$ "XXZ" model $SO(2)$
- all λ_{xyz} equal $\lambda_x = \lambda_y = \lambda_z \rightarrow$ "XXX" model $SO(3)$

- λ_0 has trivial effect: shifts all energies equally (by $L \cdot \lambda_0$)

Focus on XXX: $\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z =: \frac{1}{2}\lambda$ ferromag. $\lambda > 0$

$$H_j \in \mathrm{End}(\mathbb{V}_j \otimes \mathbb{V}_{j+1}): \quad H_j = \lambda\left(id_{j,j+1} - ex_{j,j+1}\right)$$
$$= id - ex \quad (\lambda = 1)$$

$ex|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$ $\quad ex|ab\rangle = |ba\rangle$

$ex|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$

Boundary Conditions

specify boundary conditions

- finite closed, periodic boundaries : $\mathbb{V}_{j+L} \equiv \mathbb{V}_j$
- finite open chains
- infinite chains, asymptotic boundaries

Choice has impact on spectrum

- finite chains, finite spectrum => discrete
- infinite chains, continuous spectrum

Symmetry

XXX model has $SO(3)$ / $SU(2)$ symmetry H_j obey sym.

$SU(2)$ defines spin-½ irrep $|\uparrow\rangle$ $|\downarrow\rangle$, acts by Pauli matrices

$$\vec{S}_j = \frac{1}{2}\hbar\vec{\sigma}_j$$

Commutation rel.: $$[S_j^a, S_k^b] = i\hbar\,\delta_{j,k}\,\varepsilon^{abc} S_j^c$$

$$\vec{S}_j^2 = \frac{3}{4}\hbar^2\, \mathrm{id}_j$$

Overall $SU(2)$ generated by angular mom. $\vec{J}$

$$\vec{J} = \sum_{j=1}^{L} \vec{S}_j = \sum_{j=1}^{L} \frac{1}{2}\hbar\vec{\sigma}_j$$

tensor product representation on L-fold tensor prod. of ½

Symmetry generator commutes with Hamiltonian

$$[J, H] = 0$$

spin j modules

=> spectrum has many degeneracies, multiplets of $su(2)$

tensor product decomposition of $(1/2)^{\otimes L}$ into $su(2)$ irreps

$$L=2: \quad (1) \oplus (0)$$

$$L=3: \quad (3/2) \oplus 2(1/2)$$

$$L=4: \quad (2) \oplus 3(1) \oplus 2(0)$$

multiplicity of such multiplets.

spin j of multiplet
=> 2j+1 states of equal energy

Classical limit and Higher Spin Repr

for classical limit first generalise Heisenberg chain

from spin $1/2$ "$XXX_{1/2}$" to arbitrary spin $s \in \mathbb{Z}_0^+$: "XXX_s"

elementary vector space $V = \mathbb{C}^{2s+1}$. introduce spin op $\vec{S}_j$

$$[S^a_j, S^b_k] = i\hbar\,\delta_{j,k} S^c_j\,, \quad \vec{S}_j^{\,2} = \hbar^2 s(s+1)$$

eigenvalues of spin comp. $\vec{e}\cdot\vec{S}_j$ range $-\hbar s$ to $+\hbar s$ steps of $\hbar$

generalise Heisenberg NN Ham. dens H_j respecting $SU(2)$

introduce two-site total spin op.

$$J_{j,k} := \sqrt{(\vec{S}_j + \vec{S}_k)^2 + \tfrac{1}{2}\hbar^2} - \tfrac{1}{2}\hbar$$

specify Γ st eigenvalues of J are non-neg. int.

A unique Ham. dens. that respects $SU(2)$ and is integrable

$$H_j = 2\psi(2s+1) - 2\psi\left(\frac{1}{\hbar} J_{j,j+1} + 1\right)$$

where digamma $\psi(z) := d \log \Gamma(z) / dz$ $\quad \Gamma(z) = (z-1)!$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \qquad \psi(n+1) = \psi(1) + \sum_{k=1}^{n} \frac{1}{k}$$

- show for $s = 1/2$ get above Ham. dens.

spec. of $J_{j,k}$ is $\{0, \hbar\}$

$$J_{j,k} = \frac{3}{4}\hbar\, id + \frac{1}{4}\hbar\, \vec{\sigma}_j \cdot \vec{\sigma}_k = \frac{1}{2}\hbar\, id_{j,k} + \frac{1}{2}\hbar\, ex_{j,k}$$

using $\psi(2) = \psi(1) + 1$ $\quad H_j = 2 - 2\frac{1}{\hbar} J_{j,j+1} = \dfrac{id_{j,k} - ex_{j,k}}{2}$

- for $s \to \infty$ obtain classical chain $\quad \hbar = \frac{1}{s}$, assump. $\psi(x) \sim \log x$

$$H_j^{qu} \to - \log \frac{J_{j,j+1}}{4} = -\log \frac{1 + \vec{S}_j \cdot \vec{S}_{j+1}}{4} = H^{cl}$$

6.2 Spectrum of the Closed Chain

Conventional Strategy

How to obtain spectrum of Heisenberg chain by conv. methods:

- Enumerate basis of $V^{\otimes L}$ $|\downarrow\ldots\downarrow\rangle$, $|\downarrow\ldots\downarrow\uparrow\rangle$, $|\downarrow\ldots\downarrow\uparrow\downarrow\rangle$, ...
- Evaluate Ham $\mathcal{H}$ as a $2^L \times 2^L$ matrix in this basis
  combinatorial problem (id − ex). sparse matrix
- solve eigenvalue problem of $2^L \times 2^L$ matrix → alg. eq
- method can be used by hand for $L \simeq 6$
- computer algebra can addr. problem $L \simeq 20$
- method does not help for long chains

Short Chains

$L=2$
$(1) \times E=0$
$(0) \times E=4$

$L=3$
$(3/2) \times E=0$
$2(1/2) \times E=3$

$L=4$
$(2) \times E=0$
$2(1) \times E=2$
$(1) \times E=4$
$(0) \times E=6$
$(0) \times E=2$

...

$L=6$
$(0) \times$
$E=5+\sqrt{13}$

Bethe Equations

We can set up a sys. of alg. eq. for M variables $u_k \in \mathbb{C}$ (Bethe roots)

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad k = 1 \dots M$$

M indep. eq. for M unknows $u_k \Rightarrow$ sols to be discrete

Claim: for every eigenstate (multiplet) with tot. ang mom $J = \frac{L}{2} - M$ there is one soln. to eq. with $M \leq L/2$ distinct Bethe roots u_k.

and energy eigenvalue $E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2} \right)$.

example $L = 6,\ M = 3$ $u_{1,2} = \pm \sqrt{-\frac{5}{12} + \frac{\sqrt{13}}{6}}, \quad u_3 = 0$

su(2) singlet

$$\Rightarrow E = 5 + \sqrt{13}$$

6.3 Coordinate Bethe Ansatz

Solution of Heisenberg XXX by Hans Bethe

Based on a quasiparticle picture on an infinite chain.

Start with ferromagnetic vacuum, put M spin flips acting as particles.

Vacuum State

Ferromagnetic vacuum simple: $|0\rangle := |\downarrow\downarrow \ldots \downarrow\rangle$.

Ham density acts trivially $H_j|0\rangle = id_{j,j+1}|0\rangle - ex_{j,j+1}|0\rangle = \underbrace{|0\rangle - |0\rangle}_{=0}$

$\Rightarrow H|0\rangle = E|0\rangle = 0 \quad \Rightarrow \quad E = 0$

solves the problem for $M=0$ spin flips ($L=\infty$, $L=$ finite)

Magnon States $M=1$

elem. state $|j\rangle = |\downarrow \ldots \downarrow \overset{j}{\uparrow} \downarrow \ldots \downarrow\rangle$

$L=\infty$

$L=$ finite

Ham closes on such states because of J^z conservation.

Eigenstates in $M=1$ sector? use H is homogeneous

$\to$ eigenstates have def. mom., are plane waves

$$|p\rangle := \sum_j e^{ijp} |j\rangle \quad \text{magnon state}$$

magnon is a (quasi) particle with one d.o.f. p.

$$H|p\rangle = \sum_j e^{ipj} \left(H_{j-1} |j\rangle + H_j |j\rangle \right)$$

$$\overset{L=\infty}{=} \sum_j e^{ipj} \left(|j\rangle - |j-1\rangle + |j\rangle - |j+1\rangle \right)$$

$$= \sum_j e^{ipj} \left(1 - e^{ip} + 1 - e^{-ip} \right) |j\rangle = e(p) |p\rangle$$

magnon disp. rel. $e(p) = 2(1-\cos p) = 4 \sin^2 (p/2)$

so far $L=\infty$ for evaluating sum leading $e(p)$.

want $L=$ finite. need to set $p = \frac{2\pi n}{L}$ $n = 0, \dots, L-1$

for proper periodicity $|j\rangle$ has same coeff as $|j+L\rangle$

closed boundary condition quantise p to above values.

solved sector with $M=1$ for both $L=\infty$, $L=$ finite

Scattering Factor

States with two spin flips

$$|j<k\rangle := |\downarrow\ldots\downarrow\overset{j}{\uparrow}\downarrow\ldots\downarrow\overset{k}{\uparrow}\downarrow\ldots\downarrow\rangle$$

form a closed sector under H. H acts locally $\to |p\rangle \otimes |q\rangle$

eigenstate ansatz $$|p<q\rangle = \sum_{j<k=-\infty}^{+\infty} e^{ipj+iqk} |j<k\rangle$$

energy eigenvalues $E = e(p) + e(q)$

now act with $H - E = H - e(p) - e(q)$ on $|p<q\rangle$

$$\ldots = (e^{ip+iq} - 2e^{iq} + 1) \underbrace{\sum_{j=-\infty}^{+\infty} e^{i(p+q)} |j<j+1\rangle}_{\text{symmetric in } p,q}$$

contact term

act instead on $|q<p\rangle$

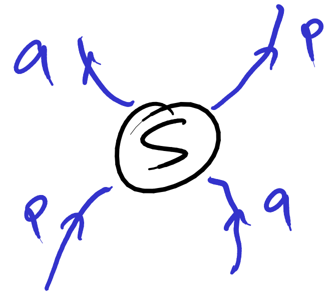
$$\ldots = \left(e^{iq+ip} - 2e^{ip} + 1\right) \sum_{j=-\infty}^{+\infty} e^{i(p+q)} \, |j<j+1\rangle$$

compose true eigenstate

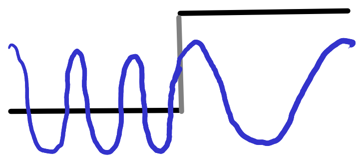
$$|p,q\rangle := |p<q\rangle + S(p,q)\,|q<p\rangle$$

Scattering factor S

$$S(p,q) := -\frac{e^{ip+iq} - 2e^{iq} + 1}{e^{ip+iq} - 2e^{ip} + 1}$$



similar to QM potential barrier problems



Factorised Scattering

$M=3$ magnons. there are $6=3!$ asymptotic regions
each magnon carries index. mom p_k. Bethe Ansatz for eigenstate

$$|p_1,p_2,p_3\rangle = |p_1<p_2<p_3\rangle + S_{12}S_{13}S_{23}\,|p_3<p_2<p_1\rangle$$
$$+ S_{12}\,|p_2<p_1<p_3\rangle + S_{13}S_{23}\,|p_3<p_1<p_2\rangle$$
$$+ S_{23}\,|p_1<p_3<p_2\rangle + S_{12}S_{13}\,|p_2<p_3<p_1\rangle$$

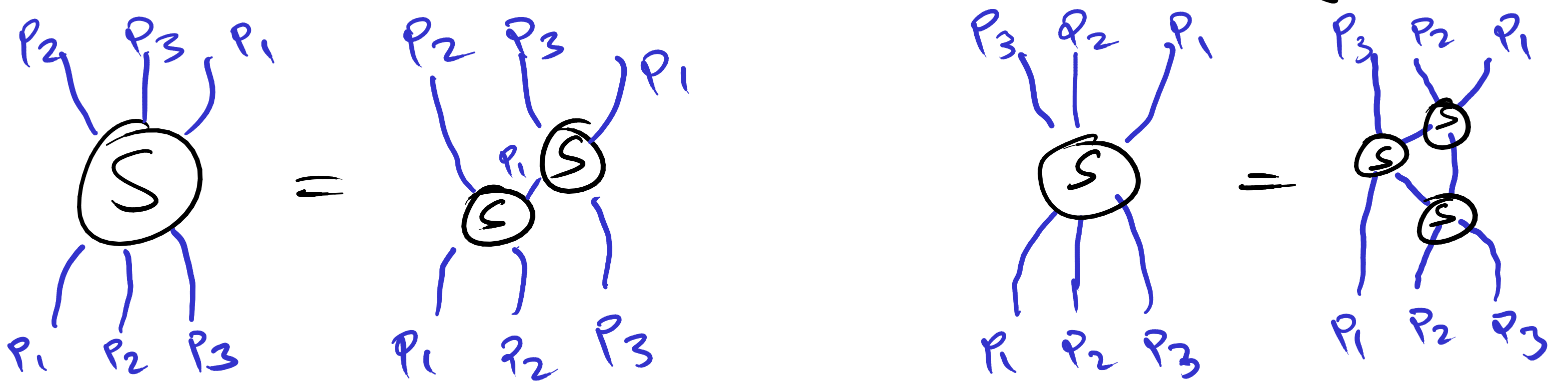
note $S_{12}=S_{21}^{-1}$. all pairwise contact terms are cancelled.
typically expect: $E=e(p_1)+e(p_2)+e(p_3)$

$$(H-E)|p_1,p_2,p_3\rangle = \sum_j e^{iPj}\,|j<j+1<j+2\rangle$$

would have to cancel by different comb. of $p_1'\,p_2'\,p_3'$
with equal E,P. Here Miracle: no 3-magnon contact term
$|p_1,p_2,p_3\rangle$ exact eigst.

Miracle is integrability. No contact terms for $M \geq 3$.
H has range 2 => expected.

Scattering of $M \geq 3$ magnons is factorised
(into a sequence of two-magnon scattering events).



Solution on Infinite Chain

$|0\rangle = |\downarrow \ldots \downarrow\rangle \qquad E = 0$

$|p\rangle = \sum_j e^{ipj} |\ldots \overset{j}{\uparrow} \ldots\rangle \qquad E = e(p)$

$|p,q\rangle = |p<q\rangle + S(p,q)\,|q<p\rangle \qquad E = e(p) + e(q)$

$|\{p_n\}\rangle = \sum_{\pi \in S_M} S_\pi\, |p_{\pi(1)} < \ldots < p_{\pi(M)}\rangle \qquad E = \sum_n e(p_n)$

Note: p_n defined mod 2π (lattice)

ordering of p_n matters only for prefactor of $|\{p_n\}\rangle$

No identical momenta: $S(p,p) = -1 \Rightarrow$ Fermions!

su(2) symmetry related to $p=0$: $\quad S(p,0) = 1 \qquad e(0) = 0$

ladder op. for su(2) multiplets.

Bound States

want states to be normalisable (difficult on ∞ chain with def. mom)

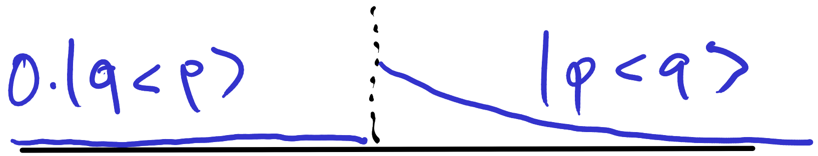
don't want exponential growth as $j \to \pm\infty$

demand that all p_k are real.

but can also allow complex p_k if exponential growth is excluded

- exponential growth for each plane wave factor happens only on one side
- make sure coefficient of this asymptotical partial contribution is zero.

achieved by $S(p,q) = 0, \infty$ for corresponding momenta.



bound states of magnons.

$$e_2(p) = 2\sin^2(p/2) \qquad e_n(p) = \frac{4}{n}\sin^2(p/2)$$

p overall mom. of n bound magnons.

6.4 Bethe Equations

Focus on finite, periodic chains / states.

Closed Chains

roughly: periodic wave functions: $\langle j_1 \ldots | \psi \rangle = \langle j_1 + L, \ldots | \psi \rangle$

construction: pick a range of L sites on ∞ chain.

consider contrib. to $|\psi\rangle$ whose all spin flips are in range

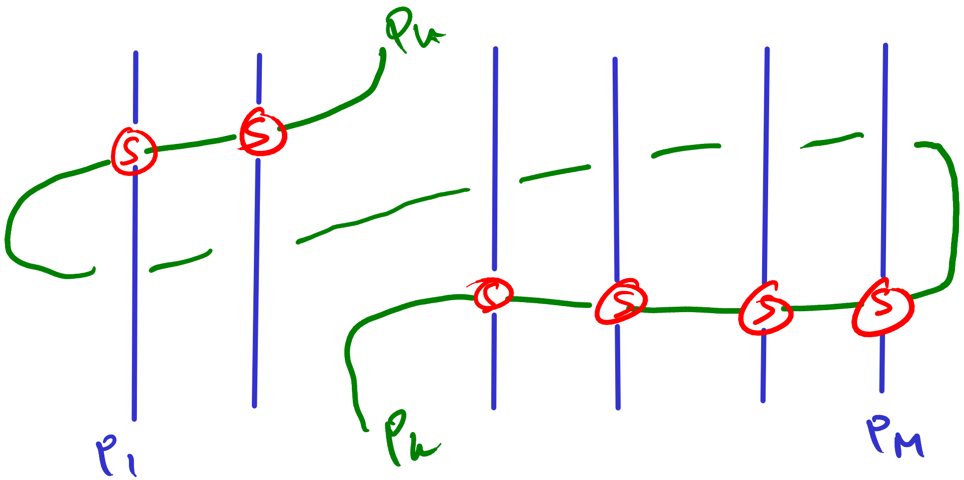
to match boundaries to be periodic:

- pick leftmost magnon with mom p_k
- shift it by L sites to right $\quad e^{i p_k L}$
- scatter with all $M-1$ magnons $\quad \prod_{j \neq k} S(p_k, p_j)$

want $\langle j_1, j_2, \ldots j_M | \psi \rangle \overset{!}{=} 1 \cdot \langle j_2, j_3, \ldots, j_M, j_1 + L | \psi \rangle$

Obtain Bethe Equations

$$e^{ip_k L} \prod_{\substack{l=1 \\ l \neq k}}^{M} S(p_k, p_l) = 1 \quad \text{for } k = 1, \dots, M$$



$\doteq 1$

Product of all B.E.

$\downarrow$

$$E = \sum_{k=1}^{M} e(p_k) \qquad P = \sum_{k=1}^{M} p_k \ (\mathrm{mod}\ 2\pi) \quad \text{note } e^{iPL} = 1$$

P is quantised as $\frac{2\pi m}{L}$.

Rapidity variables

BE are in trigonometric form.

introduce new set of Bethe roots $\{u_k\}$

$$p_k = 2\,\mathrm{arccot}(2u_k) \qquad u_k = \frac{1}{2}\cot(p_k/2) \qquad e^{ip_k} = \frac{u_k + i/2}{u_k - i/2}$$

$$S(u,v) = \frac{u-v-i}{u-v+i} \qquad e(u) = \frac{i}{u+i/2} - \frac{i}{u-i/2}$$

Bethe equations

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{l=1 \\ l\neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \qquad k = 1, \ldots, M$$

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} \qquad E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2} \right) = \sum \frac{1}{u_k^2 + 1/4}$$

6.5 Generalisations

Open Chains

$H = \sum_{j=1}^{L-1} H_n$ finite open chain, extend range to $j = 1, \ldots \infty$ semi-infinite

· Act with $H - e(p)$ on a one-magnon state $|+p\rangle = \sum_{j=1}^{+\infty} e^{ipj} |j\rangle$

$$(H - e(p))|+p\rangle = (1 - e^{+ip})\, |1\rangle$$

contact term at boundary

Need to cancel with another state of same $E = e(+p)$

$\Rightarrow +p \to -p \qquad e(-p) = e(+p) \qquad \bar{p} = -p$

$$(H - e(p))|-p\rangle = (1 - e^{-ip})\, |1\rangle$$

exact eigenstate at left boundary

$$||p|\rangle_L = e^{-ip} |+p\rangle + e^{+ip} K_L(+p) |-p\rangle$$

with boundary scattering factor k_L

$$k_L(+p) = -e^{-2ip}\frac{1-e^{+ip}}{1-e^{-ip}} = e^{-ip}$$

Analogously for right boundary at site $j=L$

$$||p|\rangle_R = e^{-ipL}|+p\rangle + e^{+ipL}k_R(+p)|-p\rangle$$

$$k_R(+p) = e^{+ip}$$

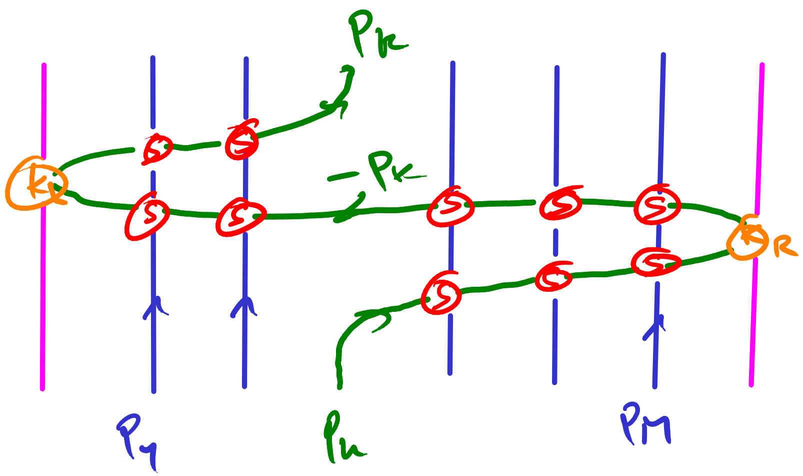
for consistency we need to satisfy $k_{L/R}(-p) = k_{L/R}(+p)^{-1}$.

as well as $$\frac{S(+p,+q)}{S(-p,+q)} = \frac{S(+p,-q)}{S(-p,-q)}$$

For multi-magnon states: Bethe Equations

$$\frac{e^{i(L-1)(+p_k)}\,K_R(+p_k)}{e^{i(L-1)(-p_k)}\,K_L(+p_k)}\prod_{\substack{l=1\\ l\neq k}}^{M}\frac{S(+p_k,p_l)}{S(-p_k,p_l)}=1$$

$$\frac{e^{+ip_k(L-1)}}{e^{-ip_k(L-1)}}$$



$$=1$$

rational form $$\left(\frac{u_k+i/2}{u_k-i/2}\right)^{2L}=\prod_{\substack{l=1\\ l\neq k}}^{M}\frac{u_k-u_l+i}{u_k-u_l-i}\,\frac{u_k+u_l+i}{u_k+u_l-i}$$

XXZ model

Extend Ham slightly to local terms

$$H = \alpha_1 (1 \otimes 1) + \alpha_2 (\sigma^z \otimes 1) + \alpha_3 (1 \otimes \sigma^z) + \alpha_4 (\sigma^z \otimes \sigma^z)$$
$$+ \alpha_5 (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) + i\alpha_6 (\sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x)$$

Six parameters related to:

- one overall energy shift ($\sim L$) $\to$ $\delta\alpha_1$
- one triv. deformation for closed chains $\to \delta\alpha_2 = -\delta\alpha_3$
- one shift prop to J^z: $\to \delta\alpha_2 = +\delta\alpha_3$
- one overall scaling " $\to$ $\delta\alpha_k = \alpha_k \cdot \delta\beta$
- one def. parameter h, $q = e^{ih}$ anisotr. $\Delta = \frac{1}{2}(q + 1/q)$
- one magnetic flux parameter ρ

Bethe eq. (trigonometric)

$$\frac{\sinh(u_k + i/2)}{\sinh(u_k - i/2)} e^{ipL} = \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{\sinh(u_k - u_l + i)}{\sinh(u_k - u_l - i)}$$

$$e^{ip(u)} = \frac{\sinh(u + i/2)}{\sinh(u - i/2)} \qquad e(u) = -p'(u)$$

obtain XXX model for $h = 0$

Higher Spin XXX_s concretely $s=1$ $|0\rangle, |1\rangle, |2\rangle$

Preserves J^z for two-site contrib. to H find block-diag.

$$H = \left(\begin{array}{c|cc|ccc|cc|c}
* & & & & & & & & \\ \hline
 & * & * & & & & & & \\
 & * & * & & & & & & \\ \hline
 & & & * & * & * & & & \\
 & & & * & * & * & & & \\
 & & & * & * & * & & & \\ \hline
 & & & & & & * & * & \\
 & & & & & & * & * & \\ \hline
 & & & & & & & & *
\end{array}\right)
\qquad
E = \left(\begin{array}{c}
|00\rangle \\ \hline
|10\rangle \\
|01\rangle \\ \hline
|20\rangle \\
|11\rangle \\
|02\rangle \\ \hline
|21\rangle \\
|12\rangle \\ \hline
|22\rangle
\end{array}\right)$$

carry out Bethe ansatz according to J^z

vacuum state $|0\rangle = |0\ldots0\rangle$

magnon state $|p\rangle = \sum e^{ipj} |0\ldots0 \overset{j}{1} 0\ldots0\rangle$

two magnons: new contact term

$$|p<q\rangle = \sum_{j<k} e^{ipj+iqk} |\ldots \overset{j}{1} \ldots \overset{k}{1} \ldots\rangle$$

$$|p;2\rangle = \sum_j e^{ipj} |\ldots \overset{j}{2} \ldots\rangle$$

$$(H-E)|p<q\rangle = \sum_j e^{i(p+q)j} \left(* |\ldots \overset{j}{1}\overset{j+1}{1} \ldots\rangle + * |\ldots \overset{j}{2} \ldots\rangle \right)$$

$$(H-E)\,|p;2\rangle = \sum_j e^{ipj} \left(* |\ldots \overset{j}{1}\overset{j+1}{1} \ldots\rangle + * |\ldots \overset{j}{2} \ldots\rangle \right)$$

total exact eigenstate

$$|p,q\rangle = |p<q\rangle + S|q<p\rangle + C|p+q;2\rangle$$

note S: scattering factor → IR → relevant to BE
C: contact term → UV → irrelevant.

resulting Bethe equations

$$\left(\frac{u_k+i}{u_k-i}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip}=\frac{u+i}{u-i} \qquad e(u)=-p'(u)$$

$$XXX_{1/2} \to XXX_1 \to XXX_s \sim \tfrac{i}{2} \to i \to is$$

$$\left(\frac{u_k+is}{u_k-is}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M} \frac{u_k-u_l+i}{u_k-u_l-i} \qquad e^{ip}=\frac{u+is}{u-is} \qquad e(u)=-p'(u)$$

Chapter 7

Long Spin Chains

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7. Long Quantum Chains

7.1. Magnon Spectrum

Ferromagnetic Vacuum $|0\rangle$ (note: su(2) descendants)
States with M magnons, consider lowest energies

Mode Numbers

consider BE in log. form

$$i L \log \frac{u_k + i/2}{u_k - i/2} - i \sum_{\substack{l=1 \\ l \neq k}}^{M} \log \frac{u_k - u_l + i}{u_k - u_l - i} + 2\pi \underset{\text{mode numbers}}{n_k} = 0$$

n_k depend on branch cut of log (default): imag $-\pi \to +\pi$
mode numbers range between $-L/2$ and $+L/2$

Single Magnons

$$iL \log \frac{u+i/2}{u-i/2} + 2\pi n = 0$$

$$u = \frac{1}{2}\cot\frac{\pi n}{L} \qquad p = \frac{2\pi n}{L} \qquad e = 4\sin^2\frac{\pi n}{L}$$

low energies for small $|n| \ll L$ $\quad$ n finite fixed as $L\to\infty$

$$u = \frac{L}{2\pi n} \qquad p = \frac{2\pi n}{L} \qquad e = \frac{4\pi^2 n^2}{L^2}$$

total mom, eng: $\quad P \sim 1/L \qquad E \sim 1/L^2$

Several Magnons

M magnons with distinct mode numbers n_k

interactions to be small b/c gas of magnons (assumption)

$u_k = \frac{L}{2\pi n_k}$ at L.O. then scattering term

$$-i \log \frac{u_k - u_\ell + i}{u_k - u_\ell - i} \simeq -i \log \frac{L/2\pi n_k - L/2\pi n_\ell + i}{L/2\pi n_k - L/2\pi n_\ell - i} \simeq -i \log 1 = 0$$

compute P, E, simple $\Rightarrow$ sum of single magnon terms.

consider several magnons at coincident mode number

ansatz
$$u_k = \frac{L}{2\pi n} + \delta u_k$$

momentum term (l.h.s)
$$iL \log \frac{u_k + i/2}{u_k - i/2} = -2\pi n + \frac{4\pi^2 n^2}{L} \delta u_k + O\left(\delta u_k^2 / L^2\right)$$

scattering term (r.h.s) $-i\log\frac{u_k-u_\ell+i}{u_k-u_\ell-i}=\frac{2}{\delta u_k-\delta u_\ell}+O(1/\delta u_k^2)$

together: $\frac{4\pi^2 n^2}{L}\delta u_k+\sum_{\substack{\ell=1\\ \ell\neq k}}^{M}\frac{2}{\delta u_k-\delta u_\ell}=0$

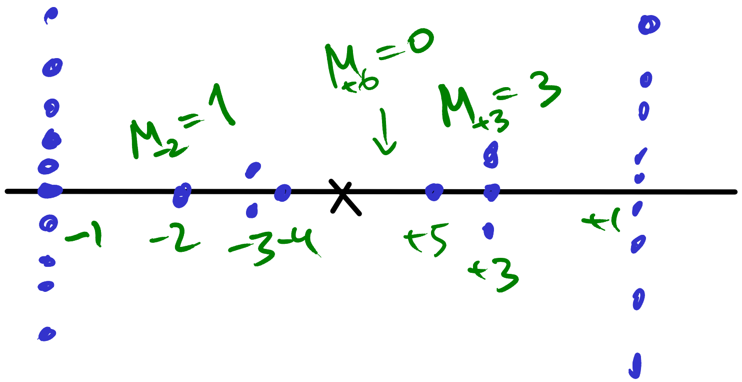
can assume $\delta u_k\sim\frac{1}{n}\sqrt{\frac{L}{M}}$

leads to some alg. eq. for δu_k with a good solution (purely imag. δu_k at L.O.)

vertically stacked Bethe Roots in $\mathbb{C}$

contrib. to P, E is M times single magnon + corrections

Magnon Spectrum



$$M = \sum_n M_n \qquad P = \sum_n M_n \cdot \frac{2\pi n}{L} \qquad E = \sum_n M_n \frac{4\pi^2 n^2}{L^2}$$

Essentially magnons are all bosons.

+ finite size corrections $O(1/L)$ relative to L.O.

7.2 Ferromagnetic Continuum

may as well take M to be large as $L \to \infty$

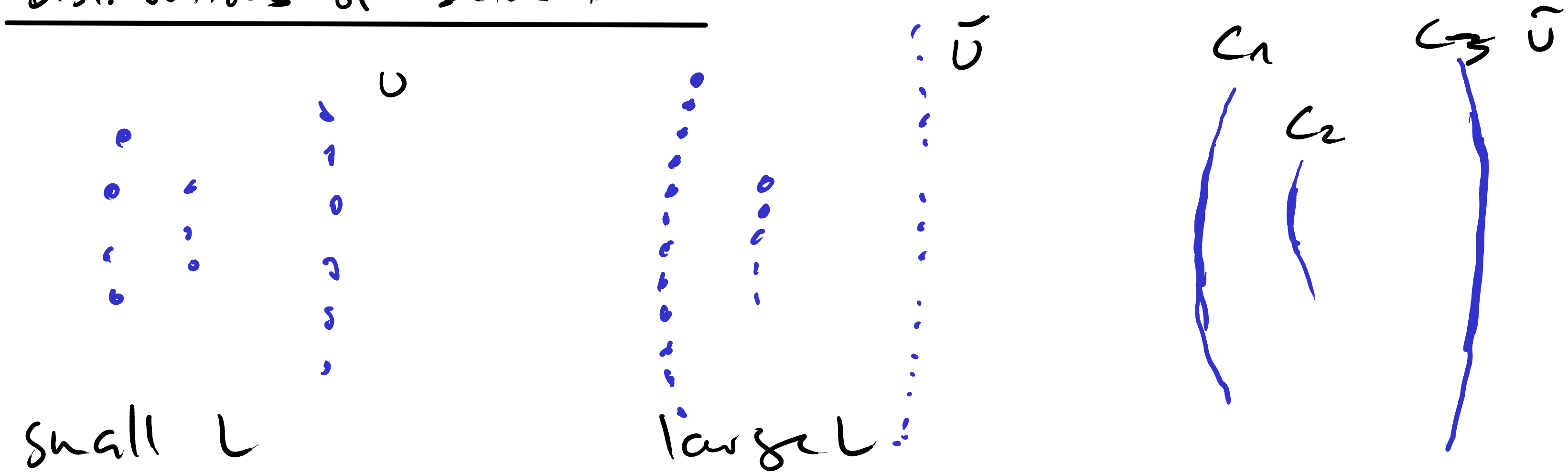
∘ how to distribute mode numbers n_k? note $E \sim M \cdot n^2$

assume that mode numbers remain bounded, but heavily pop.

convince oneself that $1 \ll M \ll L \Rightarrow$ no corrections to before

new behaviour at $M \sim L$ new analysis.

Distributions of Bethe Roots



To address $L\to\infty$, $M\sim L$ rescale $u = L\tilde{u}$

assume Bethe Roots reside on contours C_k, $C = \bigcup_k C_k$

density $\rho(\tilde{u})$ defined on contours C

$$u \to \tilde{u}L \qquad \sum_k \to L\sum_k \int_{C_k} d\tilde{u}\, \rho(\tilde{u})$$

Bethe Equation have following limit

mode number for C_k

$$P\int_C \frac{2d\tilde{v}\,\rho(\tilde{v})}{\tilde{u}-\tilde{v}} - \frac{1}{\tilde{u}} + 2\pi n_k = 0 \quad \text{for } \tilde{u}\in C$$

Principal value prescription for $\int$ is due to $\sum_{l\neq k}$

For a sol to integral eq. finds the charges (Riemann-Hilbert prob)

$$M_k = L\int_{C_k} d\tilde{u}\,\rho(\tilde{u}) \qquad P = \int_C \frac{d\tilde{u}\,\rho(\tilde{u})}{\tilde{u}} \qquad E = \frac{1}{L}\int_C \frac{d\hat{u}\,\rho(\tilde{u})}{\tilde{u}^2}$$

$$\sim O(1) \qquad \sim \frac{1}{L}$$

multiplicity for contour C_k

Spectral Curve

introduce quasi-momentum function $q(\tilde{u})$ on $\mathbb{C}$

$$q(\hat{u}) := \int_C \frac{d\tilde{v}\,\rho(\tilde{v})}{\tilde{v}-\tilde{u}} + \frac{1}{2\tilde{u}}.$$

analyse $q(\tilde{u})$ on $\mathbb{C}$: pole at $\tilde{u}=0$ $\quad q(\tilde{u}) \sim 1/2\tilde{u}$

furthermore branch cut at C_k (discontinuities)

Bethe eq: $\lim_{\epsilon\to 0}\left(q(\tilde{u}+\epsilon)+q(\tilde{u}-\epsilon)\right) = 2\pi n_k$ for $\hat{u}\in C_k$

discont $q(\tilde{u}+\epsilon)$ to $2\pi n_k - q(\tilde{u}-\epsilon)$ going through branch cut at C_k

derivative $q'(\tilde{u}+\epsilon) \to -q'(\tilde{u}-\epsilon)$

q' describes 2-sheeted cover of $\mathbb{C}$ $\Rightarrow$ Large L limit of Chain cont Heisenb. model spectral curve of Heisenb.

Heisenberg Framework

get class. cont. Heis. model from quantum chain $L \to \infty$, $\mu \sim L$

use coherent states of q. model, exp. values.

Spin $1/2$ state $|S\rangle$ prepared as

$$\langle S | \vec{\sigma} | S \rangle = \vec{S}$$

Operator X exp. val $\langle X \rangle_S = \mathrm{tr}\left(\frac{1}{2}(1 + \vec{S}\cdot\vec{\sigma}) X\right)$

apply to H_j:

$$\langle H_j \rangle_S = \mathrm{tr}_{j,j+1}\left(\frac{1}{4}(1 + \vec{S}_j \cdot \vec{\sigma}_j)(1 + \vec{S}_{j+1}\cdot\vec{\sigma}_{j+1})(\mathrm{id} - \mathrm{ex})_{j,j+1}\right)$$

$$= \dots = \frac{1}{2} - \frac{1}{2}\vec{S}_j \cdot \vec{S}_{j+1}$$

$$H = \frac{1}{2}\sum_j \left(1 - \vec{S}_j \cdot \vec{S}_{j+1}\right)$$ only from 2 sites

also take $L \to \infty$: $\qquad \vec{S}_j = \vec{S}(j\epsilon)$

compute H in $L \to \infty$ limit.

$$H = \frac{1}{\epsilon} \int dx \, \frac{1}{2} \left(1 - \vec{S} \cdot \left(\vec{S} + \epsilon \vec{S}' + \frac{1}{2} \epsilon^2 \vec{S}'' + \dots \right) \right)$$

$$= \frac{1}{4} \epsilon \int dx \, \vec{S}'^2$$

Ham of cont Heisenberg model scaled by $\epsilon/2$

7.3. Antiferromagnetic Ground State

Consider highest states in spectrum at $L \to \infty$ (aka. low. eng. of antif.)

Entanglement

lowest energy is obtained by aligning spins, $L=2$, $L>2$

highest energy is obtain by opposite alignment and spin 0 for $L=2$

$$|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \quad \text{vs} \quad |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$E=0 \quad S=1 \qquad\qquad E=2 \quad S=0$$

cannot be extrapolated to $L>2$, not exactly

- want pairs of neighbours to be in $S=0$ config.
- not possible exactly

note $*|\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle \pm *|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ + all other config.

difficult combinatorics $\quad + |\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\ldots\rangle$

Bethe Equations

ranges betw. $-i\pi$ and $+i\pi$ with 0 at $u=\infty$

Assume all $u_k \in \mathbb{R}$ $\quad \log \frac{u+i}{u-i} = i\pi \,\mathrm{sign}(u) - 2i \arctan u$

write Bethe eq. as

$$2\arctan(2u_k) - \frac{2}{L}\sum_{l=1}^{M} \arctan(u_k - u_l) + \frac{2\pi \tilde{n}_k}{L} = 0$$

shifted mode no.: $\tilde{n}_k = n_k + K - \frac{1}{2}M - \frac{1}{2} - \frac{L}{2}\,\mathrm{sign}(u_k)$ $\quad \in \frac{1}{2}\mathbb{Z}$

permissible mode numbers for $L \to \infty$, high energy:

$-\frac{L}{2} \le n_k \le +\frac{L}{2}$, $n_k = 0$ special (exact $u(?)$ sym) $\leftarrow u_k = \infty$

- only single occupation
- neighbouring mode numbers $n_k \pm 1$ should be unoccupied.

assume $L =$ even $\quad M = L/2$

$$\begin{array}{ccccccccccc} -1 & & -3 & & -5 & & +5 & & +3 & & +1 \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} \qquad L = 12$$

$$\begin{array}{ccccccccccccc} -1 & & -3 & & -5 & & \pm 7 & & +5 & & +3 & & +1 \\ \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} \qquad L = 14$$

anti-ferromagnetic ground state

$$M = L/2 \qquad n_k = L\,\Theta_{2k>M} - 2k + 1$$

Integral Equations

distribution of Bethe roots described by density on $\mathbb{R}$

$$\rho(u) = \frac{1}{L}\frac{dk}{du} \qquad k(u) = L\int_{-\infty}^{u} dv\, \rho(v)$$

density of Bethe roots $\qquad$ index of Bethe root

Bethe eq.

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v) \arctan(u-v)$$

$$- 2\pi \int_{-\infty}^{u} dv\, \rho(v) + \frac{1}{2}\pi$$

differentiate

kernel of difference form

$$\frac{4}{1+4u^2} - \int \frac{2dv\, \rho(v)}{1+(u-v)^2} - 2\pi\rho(u) = 0$$

solve int. eq. by Fourier transform

$$\rho(u) = \int \frac{d\theta}{2\pi} e^{iu\theta} R(\theta) \qquad R(\theta) = \int dv\, e^{-iu\theta} \rho(v)$$

Note Fourier integral

$$\int \frac{d\upsilon}{2\pi} \, \frac{2e^{-i\upsilon\theta}}{1+\upsilon^2} = e^{-|\theta|}$$

transformed eq

$$e^{-|\theta|/2} - e^{-|\theta|} R(\theta) - R(\theta) = 0 \Rightarrow R(\theta) = \frac{1}{2\cosh(\theta/2)}$$

transform back

$$\rho(\upsilon) = \frac{1}{2\cosh(\pi\upsilon)} \qquad k(\upsilon) = \frac{L}{4} + \frac{L}{\pi} \arctan \tanh\left(\tfrac{1}{2}\pi\upsilon\right)$$

Ground State Properties

$$E = L \int \frac{4\, du\, \rho(u)}{1+4u^2} = L \int d\theta\, e^{-|\theta|/2} R(\theta) = 2L \overset{0.69}{\log 2} < 2L$$

$P = 0$ or $P = \pi$ consider exact mode numbers m

$$P = \begin{cases} 0 & M = L/2 \text{ even} \\ \pi & M = L/2 \text{ odd} \end{cases}$$

$J = L/2 - M = 0$ b/c half-filling

7.4 Spinons

Bethe Equations

Excitation by inserting a gap of 2 unoccupied modes at mode $k \to u_0$

Integral equation for this config is:

$$0 = 2\arctan(2u) - 2\int_{-\infty}^{+\infty} dv\, \rho(v)\arctan(u-v) - 2\pi\int_{-\infty}^{u} dv\, \rho(v) + \frac{1}{4}\pi - \frac{\pi}{2L}\operatorname{sign}(u-u_0).$$

modification $O(1/L)$: consider variation $\delta\rho$ of density. After diff.:

$$-\int_{-\infty}^{+\infty} \frac{2\,dv\,\delta\rho(v)}{1+(u-v)^2} - 2\pi\,\delta\rho(u) - \frac{2\pi}{L}\delta(u-u_0) = 0$$

Fourier trans, solve to obtain $\delta R(\theta) = -\frac{1}{L}\frac{e^{|\theta|/2 - iu_0\theta}}{2\cosh(\theta/2)}.$

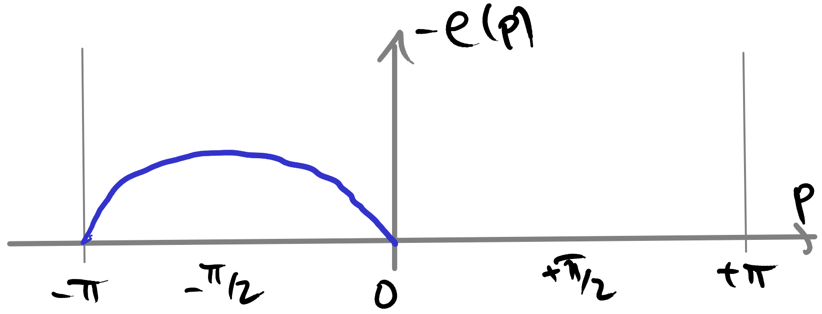
Spinon Properties

energy shift: $e(v_0) = -\int d\theta \, \frac{e^{-iv_0\theta}}{2\cosh(\theta/2)} = -\frac{\pi}{\cosh(\pi v_0)}$.

momentum shift: $p(v_0) = L\int dv \, \delta\rho(v)\left(\pi - 2\arctan(2v)\right)$

$$= 2\arctan \tanh\left(\tfrac{1}{2}\pi v_0\right) - \tfrac{1}{2}\pi .$$

dispersion relation e vs p: $e(p) = -\pi \sin(-p)$



dispersion relation only for

$$-\pi < p(v_0) < 0$$

only half of Brillouin zone is occup.

furthermore spin $\delta J^z = L\left(\delta R(0) - \frac{1}{2}\right) = -\frac{1}{2}$ How? all Bethe roots carry spin 1

Physical Spinon States

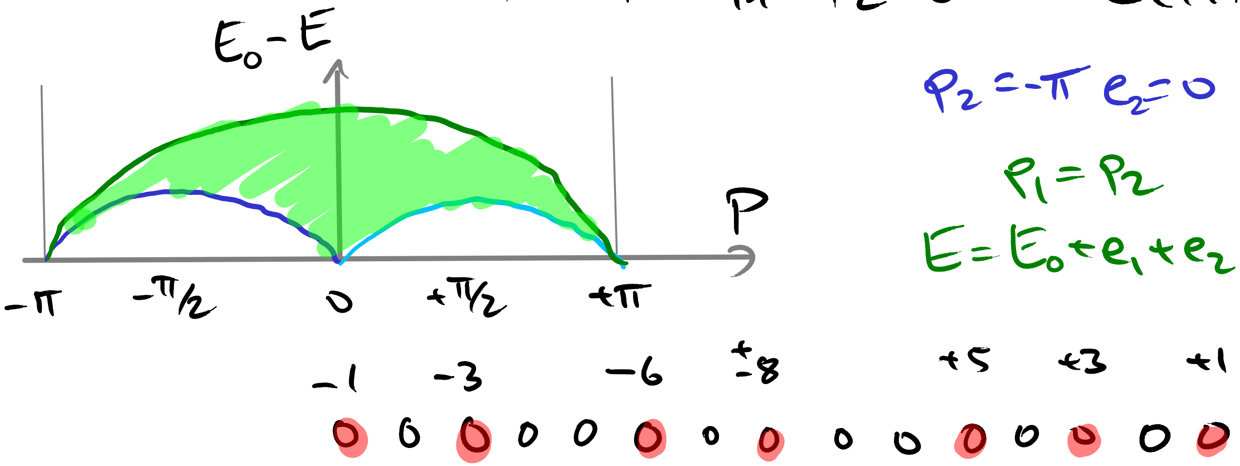
Spinon as described above is <u>not</u> elem. spin flip (like magnon) but a collective excitation of all Bethe roots of .af vac.

It carries spin 1/2 indeed → doublet.

Important point: spinons (on own length L) can exist in <u>pairs</u> only!

resolves δJ^z issue $\Rightarrow$ $\delta J^z \in \mathbb{Z}$. two spinons in $J=0$, $J=1$ state.

momentum and energy $\quad P = p_1 + p_2 + P_0 \quad E = e(p_1) + e(p_2) + E_0 \quad L \equiv 2 \bmod 4$, $P_0 = \pi$



$p_2 = -\pi \quad e_2 = 0 \qquad p_2 = 0 \qquad e_2 = 0$

$p_1 = p_2$

$E = E_0 + e_1 + e_2 = E_0 + 2e_1$

two d.o.f. to set gaps ⇒ two quasi-particles ⇒ spinons.

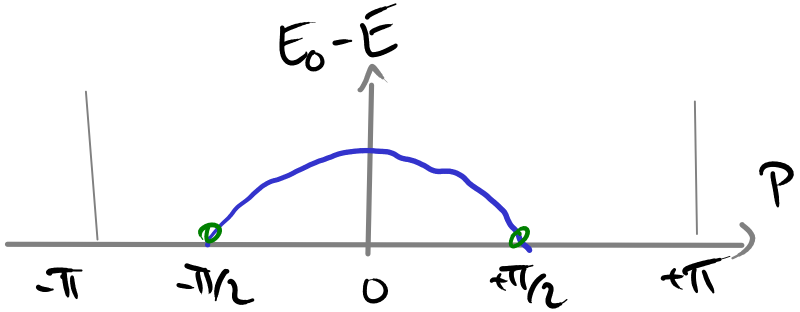
Odd Length

Slightly different: no perfect pattern of alternating occupation from $-1, -3, \dots, +3, +1$. Typically:

$$\begin{array}{ccccccccccccccc} -1 & & -3 & & & -6 & & -8 & & +7 & & +5 & & +3 & +1 \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \; \circ & \circ \end{array}$$

single sinon ~ 4.5

on odd-length chains, only odd number of spinons is permitted.



$L \equiv 1 \mod 4$

For ground state (lowest energy): 2 doublets near $P = \pm \pi/2$

$E = 2L \log 2 + O(1)$

Spinon Scattering

Spinons are particle excitations of a ground state.

Scatter on an infinite line (discrete nature of chain is preserved, see Brioullin zones)

spinons are spin-1/2: doublets → scattering matrix

tensor op rank 2

$$S(u,v) = \frac{\Gamma\left(1-\frac{i}{2}(u-v)\right)\Gamma\left(\frac{1}{2}+\frac{i}{2}(u-v)\right)}{\Gamma\left(1+\frac{i}{2}(u-v)\right)\Gamma\left(\frac{1}{2}-\frac{i}{2}(u-v)\right)}\left(\frac{u-v}{u-v+i}\,\mathrm{id} + \frac{i}{u-v+i}\,\mathrm{ex}\right)$$

difference form (difference of rap. u,v): $S(u,v) = S(u-v)$

rapidities $u = \frac{2}{\pi}\,\mathrm{artanh}\,\tan\left(\frac{1}{2}p + \frac{1}{4}\pi\right)$

up to prefactor same S-matrix as for magnons in SU(3) chain.

7.5 Spectrum Overview

- Ferromagnetic vacuum: $E=0$ $P=0$ $J=L/2$

- Magnon excitations (finitely many of finite mode number)

$$E=\sum_n M_n \frac{4\pi^2 n^2}{L^2} \qquad P=\sum_n M_n \frac{2\pi n}{L} \qquad J^? = L/2 - \sum_n M_n$$

- large number of magnons $M\sim L$ at finite mode number → non-linear terms

$$E\sim\frac{1}{L} \qquad -\pi < P < +\pi \qquad J\sim L$$

← described by continuous Heisenberg model (field theory)

Bethe Eq.

- Spinon excitations of anti-ferro-magnetic vacuum (come in pairs)

dispersion relation $e\sim -\sin(-p)$ $\quad p, e \sim O(L^0)$

- anti-ferromagn. vacuum $E=2L\log 2$ $\quad P=\frac{1}{2}\pi L \pmod 4$

$J=0$

$$E_0 - E = \sum_n \frac{2\pi^2 (4n)}{L} \qquad P=\pi \mathbb{Z} + \sum \frac{2\pi n}{L} \qquad J \le \sum_n \frac{1}{2}$$

lattice system

Chapter 8

Quantum Integrability

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8. Quantum Integrability

8.1 R-Matrix Formalism

Recall scattering matrices encountered so far

$$S_{ab}^{cd}(u,v) = \frac{(u-v)\,\delta_a^c\delta_b^d + i\,\delta_a^d\delta_b^c}{u-v-i}$$

for many magnon flavours in SU(N) $N \geq 3$ chains, also spinors $a,b,c,d = \uparrow,\downarrow$

Here introduce an operator R (R-matrix)

$$R_{ab}^{cd}(u,v) = \frac{(u-v)\,\delta_a^c\delta_b^d + i\,\delta_a^d\delta_b^c}{u-v+i}$$

R as tensor op

$R : \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}$

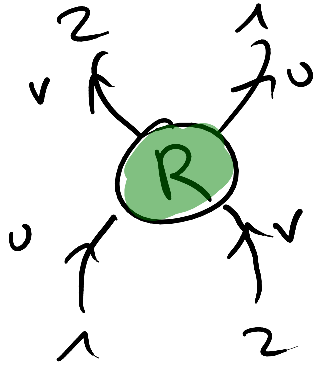
$$R = \frac{(u-v)\,\mathrm{id} + i\,\mathrm{ex}}{u-v+i}$$

for many sites $\mathbb{V}_k$ could use short cut notation R_{kl}

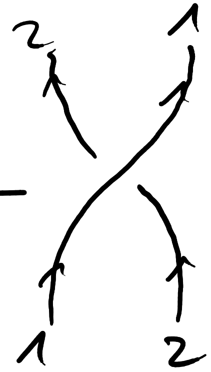
Graphical Representation

$R_{12} \sim R(u,v)$ $\quad u \to 1, \; v \to 2$

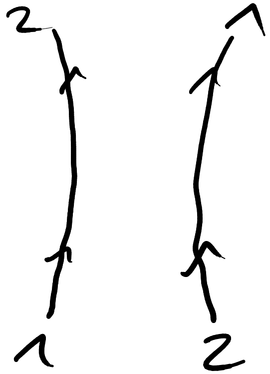
want to represent op. R and composition of it in diagrams



$= \dfrac{u-v}{u-v+i}$

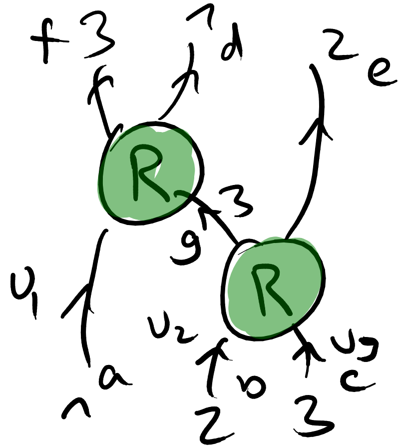


$+ \dfrac{i}{u-v+i}$



composition

$R_{13} R_{23} =$



in components:

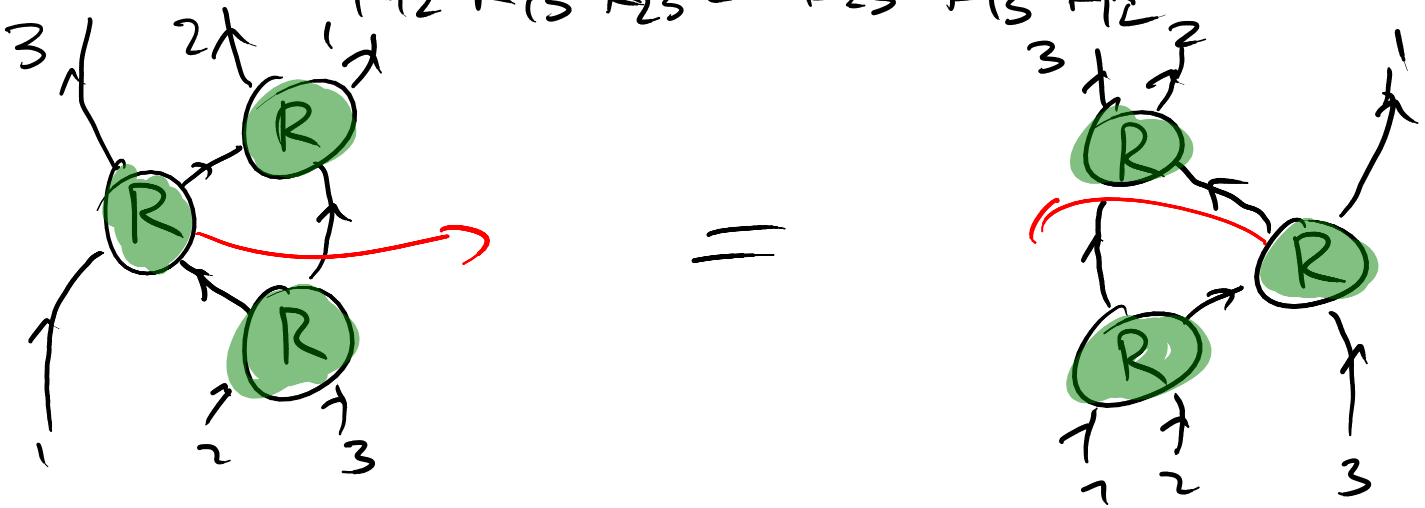
$R^{df}_{ag}(u_1, u_3)\, R^{eg}_{bc}(u_2, u_3)$

Properties of R-Matrices

For fact. scattering: Yang-Baxter-Eq.

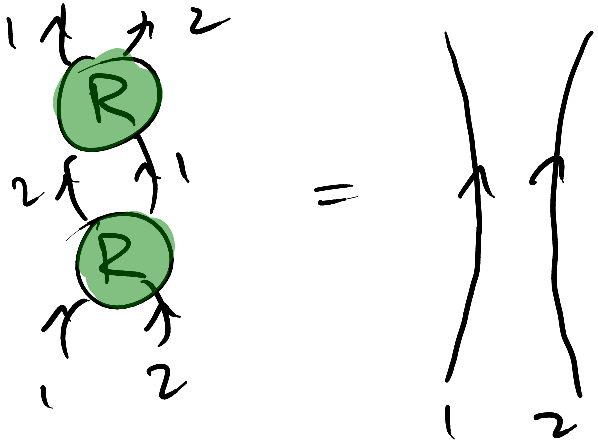
$$R_{12}(u_1 u_2)\, R_{13}(u_1 u_3)\, R_{23}(u_2 u_3) = R_{23}(u_2 u_3)\, R_{13}(u_1 u_3)\, R_{12}(u_1 u_2)$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$



YBE allows to deform /shift intersect. across lines

Similar property: $\quad R_{21} = (R_{12})^{-1} \quad$ or $\quad R_{21}\, R_{12} = \mathrm{id}_{12}$



$\simeq \mathrm{id}$

note

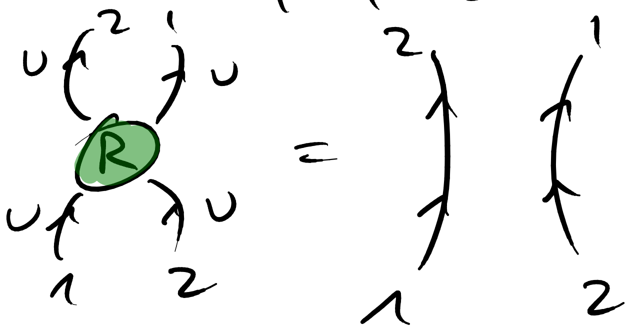
$$R_{21} := R_{21}(u_2, u_1)$$
$$= ex_{12}\, R_{12}(u_2, u_1)\, ex_{12}$$
$$= \dots = (R_{12})^{-1}$$

altogether:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad \text{and} \quad R_{12} R_{21} = \mathrm{id}.$$

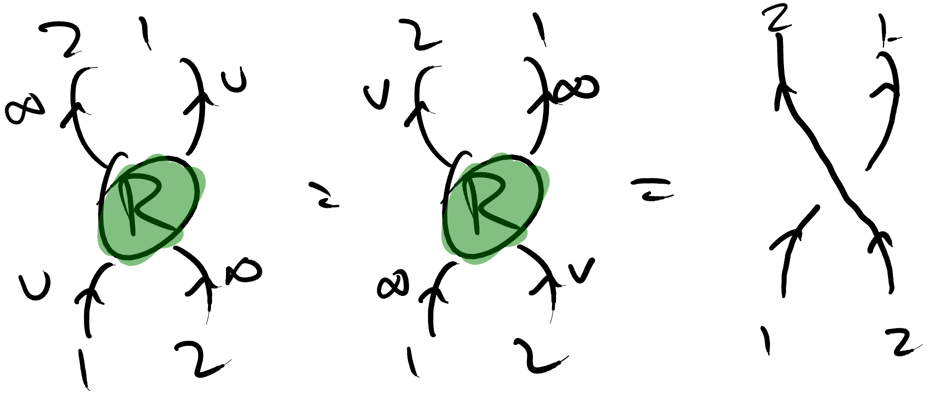
equivalent to permutation group S_N ← #sites two aux properties related to physics.

$R(u,u) = \mathrm{ex}$



for scattering: identical particles

for argument $u,v=\infty$ R trivializes $R(u,\infty) = R(\infty,v) = \mathrm{id}$



related to SU(N) symmetry of R.

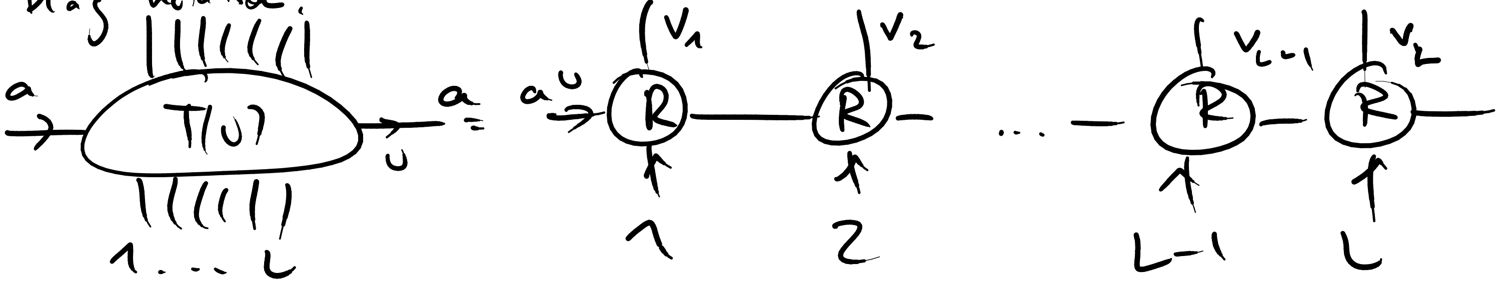
8.2 Charges

Monodromy and Traces

Closed boundary monodromy matrix $T(u)$ defined as

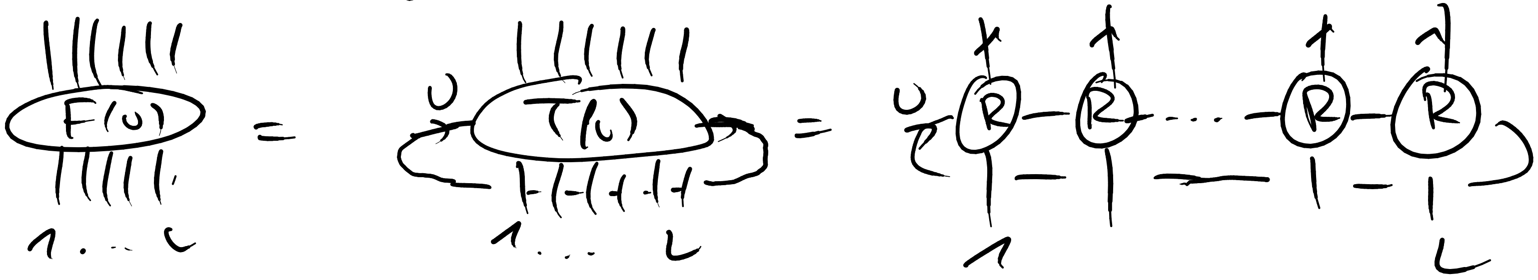
$$T_a(u) = R_{a,L} \cdot R_{a,L-1} \cdot \ldots \cdot R_{a,2} \cdot R_{a,1}$$

Diag notation:



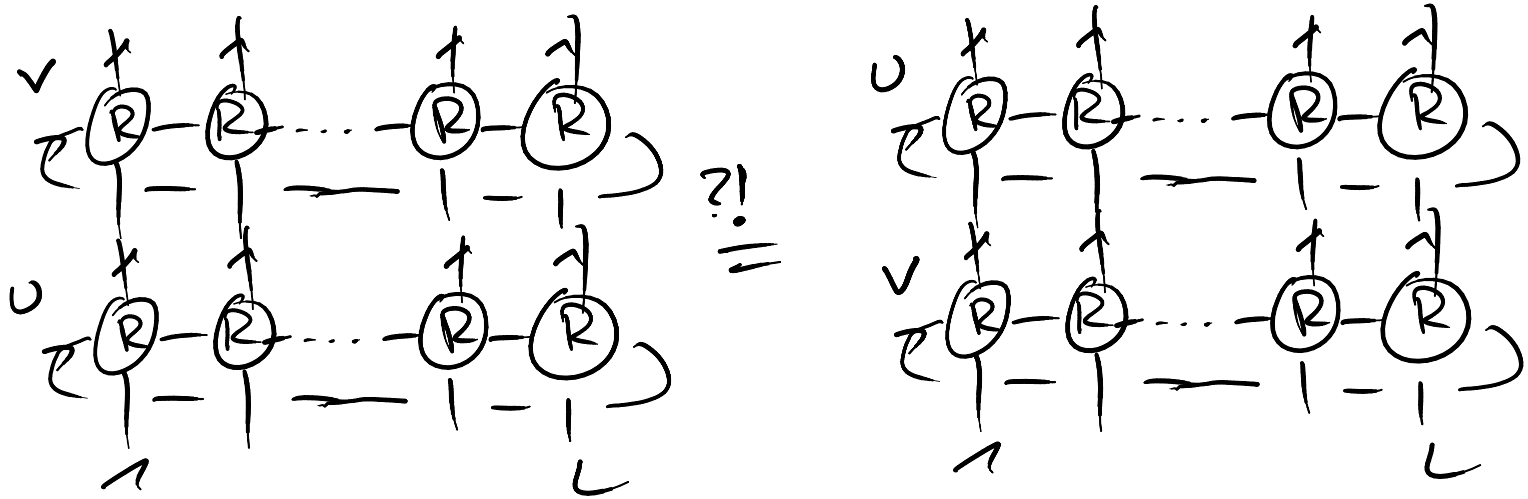
for a hom. spin chain all v_k equal $v_k = 0$.

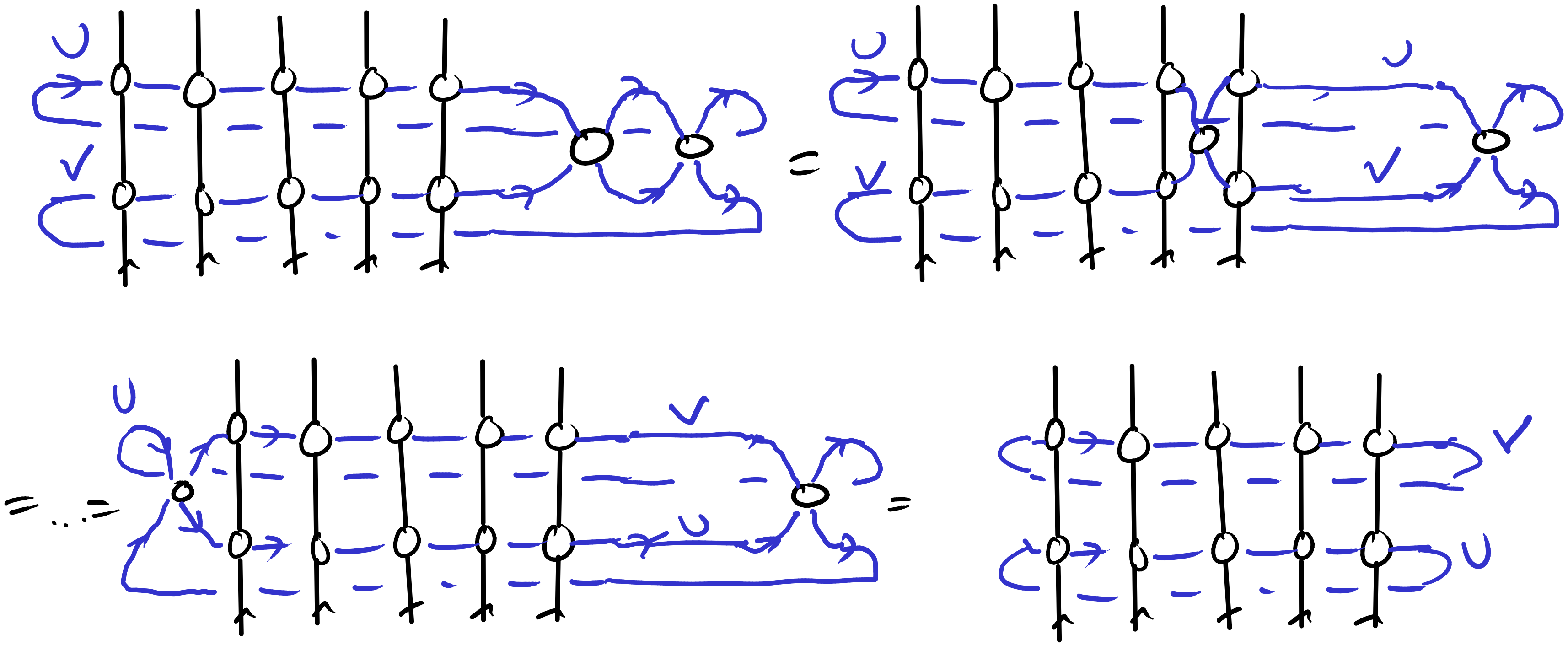
Trace $\quad F(u) = \mathrm{tr}_a\, T_a(u)$



in class mech: $\quad \{F(u), F(v)\} = 0$

in QM: $\quad [F(u), F(v)] \overset{?}{=} 0$





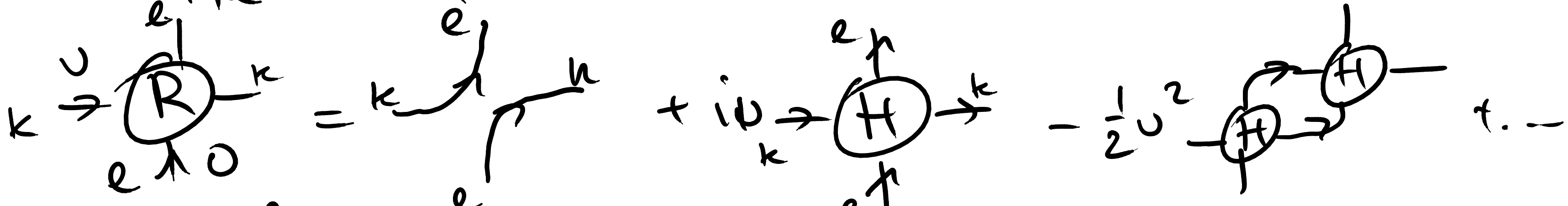
Local Charges

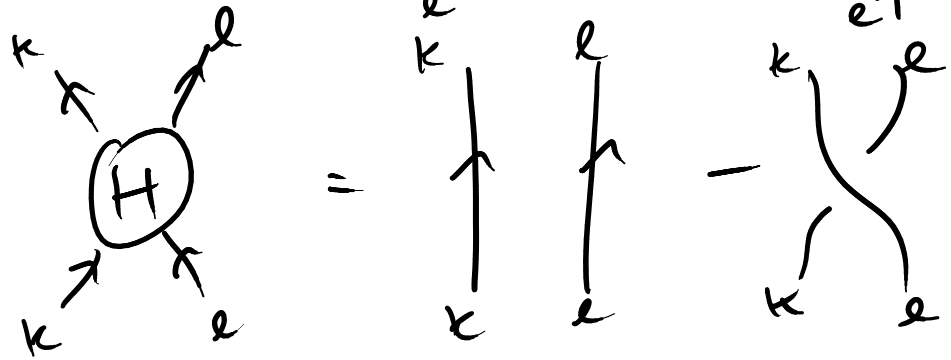
$F(u)$ is non-local operator, contains some local information

here: at point $u = 0 = v_k$. expand

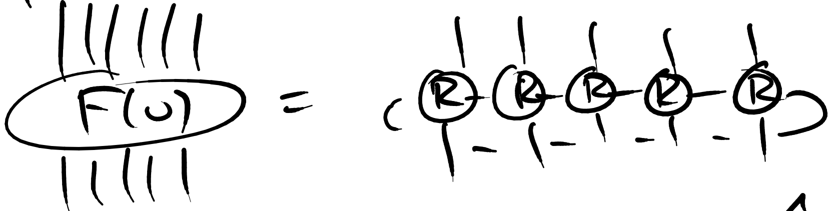
use $R_{a,j}(u,0) = ex_{a,j} + iu\, ex_{a,j} H_{a,j} - \frac{1}{2}u^2 ex_{a,j} H_{a,j}^2 + \dots$

$H_{k,l} = id_{k,l} - ex_{kl}$. kernel of Ham. op.

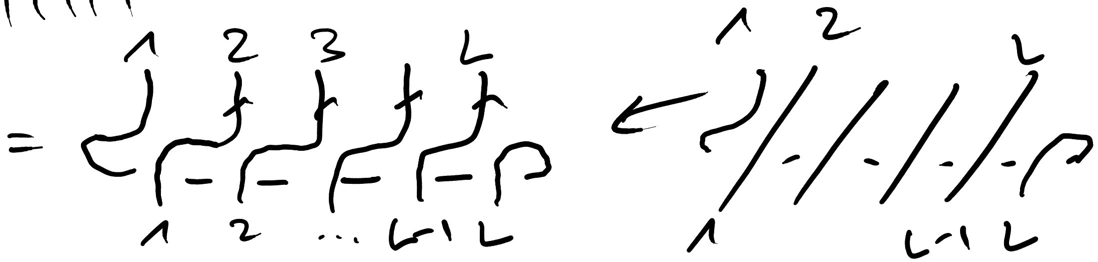




Expand $F(u)$

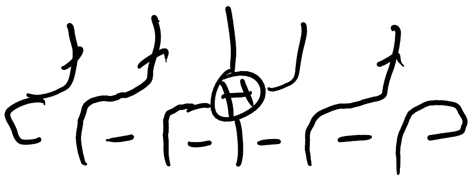


cyclic shift op.



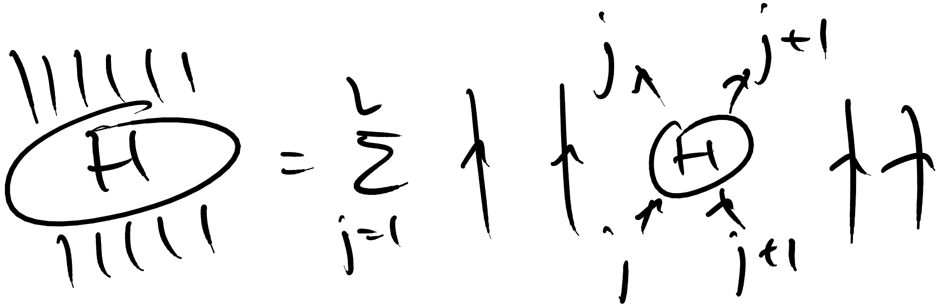
$\exp(iP)$

$+ \, iu \sum_{j=1}^{L}$



$\leftarrow \exp(iP) \, iu \, H$

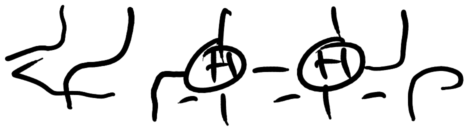
$+ \ldots$



$= \exp(iP) + iu \exp(iP) H + \ldots = \exp(iP + iu H + \ldots)$

at order u^2

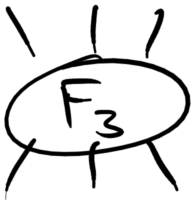
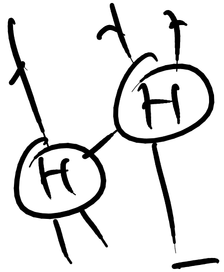
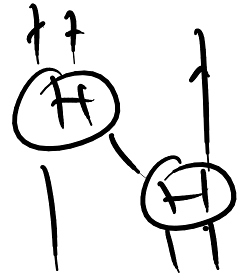
$F(u) = \dots \quad -u^2 \sum_{\substack{j<k=1 \\ |j-k|>1}}^{L}$ 

$-u^2 \sum_{j=1}^{L}$ 

$-\frac{1}{2}u^2 \sum_{j=1}^{L}$ 

[almost u^2 term of $\exp(iP + iuH)$]

$= \exp(iP + iuH + iu^2 F_3 + \dots)$ $\qquad = \frac{i}{2}[H_{j+1}, H_j]$

 $= \sum_{j=1}^{L}$  $\qquad$  $= \frac{i}{2}$  $- \frac{i}{2}$ 

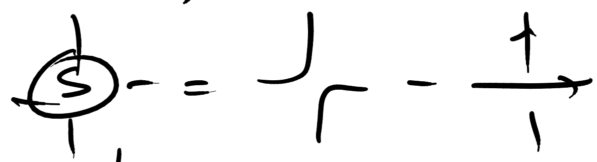
altogether: $F_2 = H$, $F_1 \sim P$ $\qquad [F_r, F_s] = 0$

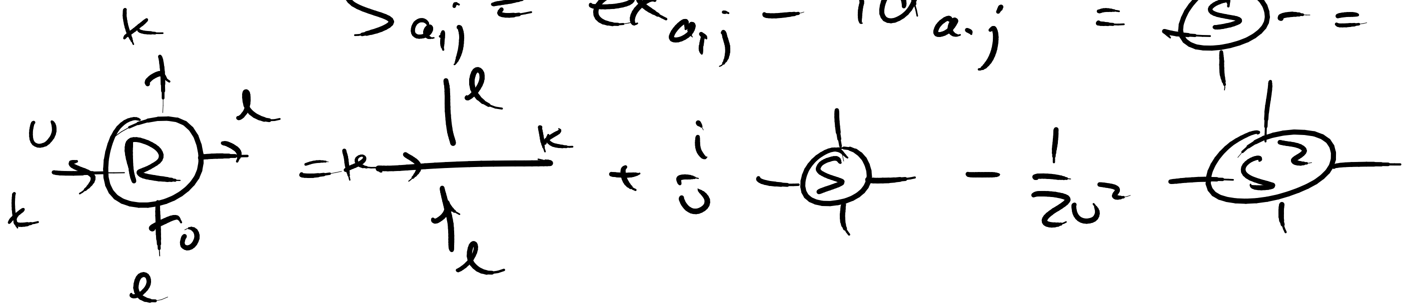
Multi-local Charges

consider the (symmetry) point $u=\infty$

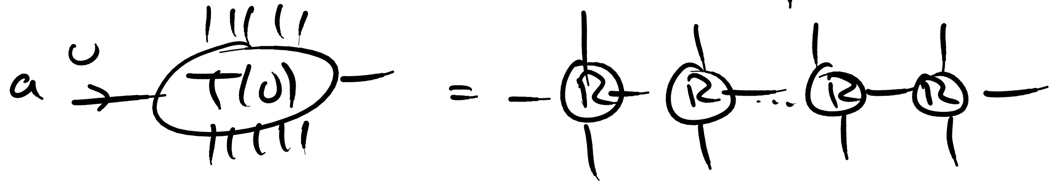
$$R_{a,j}(u,0) = \mathrm{id}_{a,j} + i u^{-1} S_{a,j} - \frac{1}{2} u^{-2} S_{a,j}^{2} + \dots$$

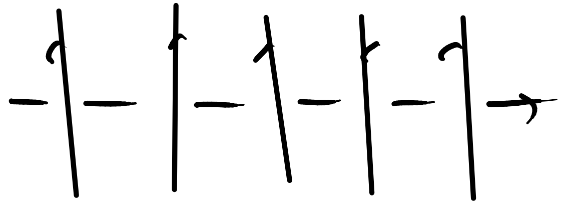
$$S_{a,j} = \mathrm{ex}_{a,j} - \mathrm{id}_{a,j} =$$

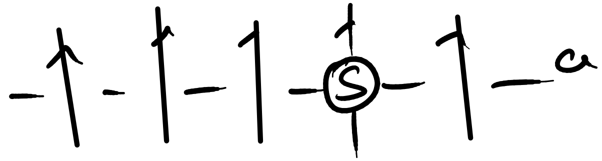


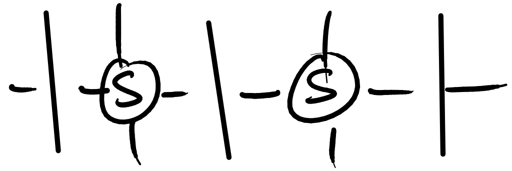


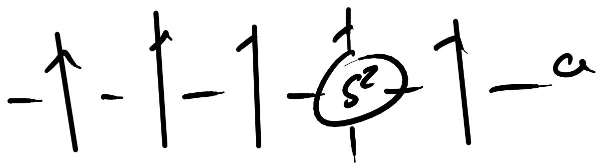
Expand $T(u)$ monodromy



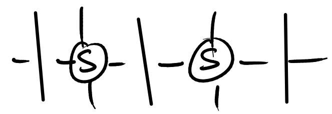
$=$  $\leftarrow$ id

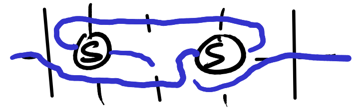
$+\frac{i}{U}\sum_{j=1}^{L}$  $\leftarrow \frac{i}{U} J_a$ $\quad J_a$ total ang. mom.

$-\frac{1}{U^2}\sum_{j\neq k=1}^{L}$ 

$-\frac{1}{2U^2}\sum_{j=1}^{L}$ 

$\Bigg\}$ almost $-\frac{1}{2U^2}(J_a)^2$

$Y_a = \frac{i}{2}\sum_{j\neq k=1}^{L}$ 

$-\frac{i}{2}\sum_{j\neq k=1}^{L}$ 

$= \exp\left(\frac{i}{U} J_a + \frac{i}{U^2} Y_a + \ldots\right)$

bi-local operator

commutator of $[S_{a,j}, S_{a,k}]$

8.3 Alternative Types of Bethe Ansätze

Algebraic Bethe Ansatz

use monodromy $T_a(u)$ in aux space a acts as 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

A, B, C, D are operators acting on chain / Hilbert space

obey an algebra: RTT-relations (Yang-Baxter eq)

$$R_{ab}(u,v)\, T_a(u)\, T_b(v) = T_b(v)\, T_a(u)\, R_{ab}(u,v)$$

in a basis $|\uparrow\rangle$, $|\downarrow\rangle$ and using that R is su(2) invariant

A, D preserve # up/down spins, B flips one $\downarrow$ to $\uparrow$, C flips $\uparrow$ to $\downarrow$

use as a framework of creation (B), annihilation (C) and charge (A, D) op as in QM II / QFT vacuum $|\downarrow\downarrow\ldots\downarrow\rangle = |0\rangle$ magnon states $|u_1 \ldots u_M\rangle = B(u_1)\ldots B(u_M)|0\rangle$.

states $B(u_1)\dots B(u_M)|0\rangle$ are eigenstates of $F(u) = A(u) + D(u)$

eigenvalue is

$$F(u) = \underbrace{\prod_{k=1}^{M} \frac{u - u_k - i/2}{u - u_k + i/2}}_{A(u)} + \underbrace{\left(\frac{u}{u+i}\right)^{L} \prod_{k=1}^{M} \frac{u - u_k + 3i/2}{u - u_k + i/2}}_{D(u)}$$

+ off-diagonal terms which cancel provided that

$$\left(\frac{u + i/2}{u - i/2}\right)^{L} = \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \quad \text{for all } k = 1 \dots M$$

Bethe eq!

expand $F(u) = \exp\left(iP + iu E + iu^2 F_3 + \dots\right)$

$$\exp(iP) = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} \qquad E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right) \qquad F_3 = \sum_{k=1}^{M} \left(\frac{i}{2(u_k + i/2)^2} - \frac{i}{2(u_k - i/2)^2}\right)$$

Algebraic Bethe Ansatz for Higher-Rank Chains / Symmetries

Here $su(2)$ $\quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$SU(N)$ $\quad T = \begin{pmatrix} A^1 & B^1 & * & * & * \\ C^1 & A^2 & B^2 & * & * \\ * & C^2 & \ddots & \vdots & B^{N-1} \\ * & * & & C^{N-1} & A^N \\ * & * & & & \end{pmatrix}$ $\quad$ $A^r \sim$ Cartan subalg. el.

B^r, C^r simple roots $(\pm)$

all other generators as products of A^r, B^r, C^r.

create magnon quasi particles from a vacuum $|N\,N \ldots N\rangle = |0\rangle$

by using $B^r\,|r+1\rangle \to |r\rangle$ populate all Hilbert space $(\mathbb{C}^N)^L$

eigenstates $|u_k^{(r)}\, u_l^{(s)} \ldots\rangle = B^r(u_k^{(r)})\, B^s(u_l^{(s)}) \ldots |0\rangle$

Analytic Bethe Ansatz

start with expression

$$F(u) = \prod_{k=1}^{M} \frac{u - u_k - i/2}{u - u_k + i/2} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^{M} \frac{u - u_k + 3i/2}{u - u_k + i/2}$$

what does follow? recall $F(u) \sim (R)^L$ $\quad R \sim \frac{*u + *1}{u+i}$

$$F(u) \sim \frac{P_L(u)}{(u+i)^L} \quad \text{(with q. op as coefficients)}$$

compare this to above $F(u)$: mismatch, add. poles at $u = u_k - i/2$

Residues $F(u_k - i/2 + \epsilon) \sim -\frac{i}{\epsilon} \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l - i}{u_k - u_l} + \frac{i}{\epsilon} \left(\frac{u_k - i/2}{u_k + i/2}\right)^L \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l}$

$$\sim -\frac{i}{\epsilon} \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l - i}{u_k - u_l} \left(1 - \left(\frac{u_k - i/2}{u_k + i/2}\right)^L \prod_{\substack{l=1 \\ l \neq k}}^{M} \frac{u_k - u_l + i}{u_k - u_l - i} \right) \leftarrow = 0 \text{ iff}$$

Bethe Eq. hold

Baxter Equation

$\tilde{F}(u) := (u+i/2)^L F(u-i/2)$ Polynomial in u

$$\tilde{F}(u) = (u+i/2)^L \prod_{k=1}^{M} \frac{u-u_k-i}{u-u_k} + (u-i/2)^L \prod_{k=1}^{M} \frac{u-u_k+i}{u-u_k}$$

introduce polynomial $Q(u) = \prod_{k=1}^{M} (u-u_k)$ ← poly. of Bethe roots

above eq. for $\tilde{F}(u)$ as

$$\tilde{F}(u)\,Q(u) = (u+i/2)^L\, Q(u-i) + (u-i/2)^L\, Q(u+i)$$

Difference eq.: Baxter eq. for $Q(u)$:

given some $\tilde{F}(u)$: defines 2-dim. space of solutions $Q(u)$

$Q(u)$ are polynomials in u (deg M) only for specific $\tilde{F}(u)$.

↑ iff Bethe Eq. hold.

Chapter 9

Quantum Algebra

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9. Quantum Algebra

9.1 Lie Algebra

Lie algebra g is vector space with Lie brackets $[\,,\,]$ as prod

$$[\cdot,\cdot] : g \times g \to g$$

• bilinear, • anti-symmetric • satisfy Jacobi-id

Repr of g: $\rho : g \to \mathrm{End}(W)$ (linear map on W)

such that Lie algebra is respected as a commutator of maps

$$[\rho(a), \rho(b)] = \rho([a,b]) \qquad a,b \in g.$$

Introduce a (imaginary) basis $J^a \in ig$:

$$[J^a, J^b] = i f^{ab}{}_c J^c \qquad f^{ab}{}_c \text{ structure const. for } g.$$

Invariant quad. form

$$M = c_{ab} J^a \otimes J^b$$

inverse of Cartan-Killing form

$$k(a,b) = \mathrm{tr}\ \rho_{ad}(a)\, \rho_{ad}(b) \qquad C^{ab} \sim k(J^a, J^b)$$

For $g = su(2) = so(3)$

$$c_{ab} = C^{ab} = \delta_{ab} \qquad f^{ab}{}_c = \varepsilon^{abc}$$

Loop Algebras

Can encode dep. on spectral par u into alg:

loop algebra $g[z, z^{-1}]$ is spanned by elements

$$J^a_n := z^n J^a \quad \text{where } n \in \mathbb{Z} \quad J^a \in g \text{ span } g.$$

n: loop level of a generator J^a_n.

loop alg is a Lie alg with

$$[J^a_m, J^b_n] = i f^{ab}{}_c J^c_{m+n}$$

$$[z^m J^a, z^n J^b] = z^{n+m} i f^{ab}{}_c J^c$$

Invariant quad form(s)

$$M_m = \sum_{k=-\infty}^{+\infty} c_{ab} J^a_k \otimes J^b_{m-k}$$

half loop algebras: only pos / non-neg levels

$$z\, g[z] \;/\; g[z]$$

evaluation repr.: given a rep ρ of g on $\mathbb{V}$ def

$$\rho_z : g[z, z^{-1}] \to \mathrm{End}(\mathbb{V})$$

$$\rho_z(J^a_n) := z^n \rho(J^a) \qquad z \in \mathbb{C} \text{ eval. par.}$$

eval. rep are useful for integrability due to enhanced irreducibility

two eval. rep $\rho'_{z'}$, $\rho''_{z''}$

$$\rho_{z', z''} = \rho'_{z'} \otimes 1 + 1 \otimes \rho''_{z''}$$ is irreducible if $\rho'_{z'}, \rho''_{z''}$ are and $z' \neq z''$

9.2 Classical Integrability

classical r-matrix r fits well into framework of Lie bialgebras.

Lie Bialgebra

Lie algebra g whose dual g^* is also a Lie algebra such that the two Lie algebra structures are compatible

dual of Lie brackets: $\mu^*: g^* \to g^* \otimes g^*$
such that for all $a, b \in g$, $c^* \in g^*$

$$c^*([a,b]) = \mu^*(c^*)(a \otimes b)$$

dual of dual Lie bracket is called Lie cobracket δ

$$\delta: g \to g \otimes g$$

- must be antisymmetric $\delta(a) \in g \wedge g$

dual Jacobi id $(1 + P_{12}P_{23} + P_{23}P_{12})(\delta \otimes 1)\delta(c) = 0$

Compatibility between $\mathfrak{g}, \mathfrak{g}^*$ / $[\,,\,], \delta$

$$\delta([a,b]) = [a, \delta(b)] + [\delta(a), b]$$

$$[a, b\otimes c] := [a,b]\otimes c + b\otimes [a,c].$$ as used for inv quad form M

Classical r-Matrix

A class r-matrix $r \in \mathfrak{g}\otimes\mathfrak{g}$ such that

$$\delta(a) = [r, a].$$

- anti-sym of δ implies that $r + P(r)$ is inv. elem. for $\mathfrak{g}$
- dual Jacobi id. requires that

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] \in \mathfrak{g}^{\otimes 3}$$ is inv. cubic elem to for $\mathfrak{g}$

$\Rightarrow$ $\mathfrak{g}$ is called coboundary ($[[r,r]] = 0 \Rightarrow \mathfrak{g}$ quasi-triangular)

Example

$$r(u,v) = \frac{-2 c_{ab} J^a \otimes J^b}{u-v} = \frac{-2M}{u-v}$$

compare to loop algebra basis with $J^a_n = u^n J^a$

Expand for $|u| \gg |v|$

want to express $r \in g[u,u^{-1}] \otimes g[v,v^{-1}]$

$$r = -2 \sum_{n=0}^{\infty} \frac{v^n}{u^{n+1}} M = -2 \sum_{n=0}^{\infty} c_{ab} J^a_{-n-1} \otimes J^b_n$$

r-matrix satisfies classical Yang-Baxter eq $[[r,r]] = 0$

symmetric part of expanded r:

$$r + P(r) = -2 \sum_{n=-\infty}^{+\infty} c_{ab} J^a_{-n-1} \otimes J^b_n = -2M_{-1}$$

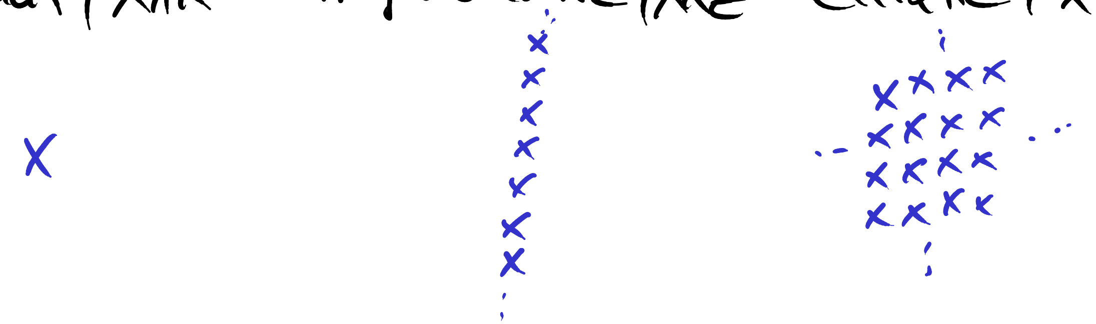
quad inv. form of loop alg. at level -1

Classification and Construction

Parametric solutions to class. YBE (difference form) have been classified by Belavin + Drinfeld:

Three distinct types (related to the pattern of poles in $\mathbb{C}$)

rational / XXX trigonometric / XXZ elliptic / XYZ



Towards construction useful to note sample $r \in u^{-1}\mathfrak{g}[u^{-1}] \otimes \mathfrak{g}[v]$

half loop algebras $u^{-1}\mathfrak{g}[u^{-1}]$ and $\mathfrak{g}[u]$ are related by conjugation w.r.t quad form $\langle , \rangle_{-1}$:

Full $\mathfrak{g}[u,u^{-1}]$ is classical double $d\mathfrak{g}[u] \to$ bialgebra structure on $\mathfrak{g}[u,u^{-1}]$ + classical r-matrix

9.3 Quantum Algebras

Enveloping Algebra

Put together Lie algebra g, corresponding Lie group G as well as products of all of their elements. $\rightarrow U(g)$

Define first tensor algebra $T(g)$: arbitrary polynomial's in elements of g with respecting order of letters in words.

$$J^a J^b J^c \neq J^a J^c J^b \quad \text{two indep. monomial's}$$

Env. algebra $U(g)$ is obtained by identifying Lie brackets with commutators:

$$J^a J^b - J^b J^a = [J^a, J^b] = i f^{ab}{}_c J^c$$

(should hold with any polynomial $X \ldots Y = X \ldots Y$)

$$U(g) = T(g) / \mathrm{span}(J^a J^b - J^b J^a - i f^{ab}{}_c J^c).$$

Why $U(g)$ in quantum physics?

- Lie algebra $g \subset U(g)$ as $g = \text{span}(J^a)$
- Can express products of Lie generators $J^a J^b$ or $J^b J^a$ etc. while respecting Lie alg structure
- Lie group $G \subset U(g)$ as $\{\exp(a) ; a \in g\}$
- Tensor products are naturally defined
- We can do non-trivial deformations of $U(g)$ for integr. syst.

Hopf Algebra

$U(g)$ has a natural Hopf algebra struct.

Hopf alg: biunital, biassociative, bialgebra with antipode consider some Hopf alg A over field $\mathbb{k}$

- Product μ, coproduct Δ are $\mathbb{k}$-linear (co)associative maps

$$\mu : A \otimes A \to A \qquad \Delta : A \to A \otimes A$$

bialgebra: compatibility between $\mu \Leftrightarrow \Delta$

$$\Delta(\mu(X \otimes Y)) = (\mu_{13} \otimes \mu_{24})(\Delta(X) \otimes \Delta(Y)).$$

Note: needed for consistency of tensor prod. representations.

$$\rho_{12}(X) := (\rho_1 \otimes \rho_2)(\Delta(X)).$$

- unit ϵ and counit η

$$\epsilon : \mathbb{k} \to A \qquad \eta : A \to \mathbb{k}$$

compatibility: $\mu(\epsilon(a) \otimes X) = aX$, $\eta_1(\Delta(X)) = X$

• antipode $\Sigma : A \to A$ satisfying

$$\mu(\Sigma, 1(\Delta(X))) = \epsilon(\eta(X))$$

if Σ exists it is unique.

Σ is anti-homomorphism of alg / coalg.

$$\mu(\Sigma(X) \otimes \Sigma(Y)) = \Sigma(\mu(Y \otimes X))$$

$$\Delta(\Sigma(X)) = (\Sigma \otimes \Sigma)(\tilde{\Delta}(X))$$

opposite coproduct

$\tilde{\Delta}(X) = P \circ \Delta(X)$

Σ incorporates negative / inverse of elements of A

<u>Example</u> for $A = U(g)$

$$\mu(X \otimes Y) = XY \quad \text{(modulo Lie brackets identification)}$$

coproduct:

$$\Delta(1) = 1 \otimes 1 \qquad \Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a \quad J^a \in g$$

$$\Delta(\exp(a)) = \exp(a) \otimes \exp(a) \qquad \exp(a) \in G$$

$(L-1)$-fold coproduct act on $A^{\otimes L}$

$$\Delta^{L-1}(1) = 1 \qquad \Delta^{L-1}(J^a) = \sum_{k=1}^{L} J^a_k$$

unit, counit

$$\epsilon(1) = 1, \quad \eta(1) = 1, \quad \eta(J^a) = 0 \qquad a \in g$$

antipode: $\quad \Sigma(1) = 1 \qquad \Sigma(J^a) = -J^a, \; \Sigma(\exp(a)) = \exp(-a)$

Universal R-Matrix

introduce univ R-Matrix $\quad R \in A \otimes A$

note R matrices of integr sys. are typically
repr. $(\rho_1 \otimes \rho_2) R$ rank-2 tensor operators on $V_1 \otimes V_2$

R relates $\Delta(X)$ with $\tilde{\Delta}(X)$ for any X

$$R \Delta(X) = \tilde{\Delta}(X) R \qquad \tilde{\Delta}(X) = R \Delta(X) R^{-1}$$

coproduct and opp. coproduct are not (necessarily)
the same (no cocommutativity) but they are
related by similarity transformation R

$\Rightarrow$ quasi-cocommutativity

ordering of factors in a tensor product matters only in terms of basis

Quasi-triangularity

$$\Delta_1(R) = R_{13} R_{23}$$
$$\Delta_2(R) = R_{13} R_{12}$$

imply the Yang-Baxter equation

$$R_{12}(R_{13}\ R_{23}) = R_{12}\,\Delta_1(R) = \tilde{\Delta}_1(R) R_{12}$$
$$= (R_{23} R_{13}) R_{12}$$

ultimately QT incorporates fusion
can interchangably ~~treat~~ treat 2 particles as 1 composite obj.

9.4 Yangian Algebra

Quantum algebra framework for XXX Heisenberg spin chain

Algebra

Yangian $Y(g)$ is def. of $U(g[u])$.

Generated by (products/polynomials in) level-zero gen $J^a \simeq J^a_0$
and level-one generator $Y^a \simeq J^a_1$, $a = 1 \dots \dim(g)$

$$[J^a, J^b] = i f^{ab}{}_c J^c \qquad \leftarrow \text{level-zero} \equiv g$$

$$[J^a, Y^b] = i f^{ab}{}_c Y^c \qquad \leftarrow Y^a \text{ transforms in adj of } g.$$

Plus Serre relation

$$[[J^a, Y^b], Y^c] + 2 \text{ cyclic} = \hbar^2 \cdot \text{"} J^3 \text{"}$$

Note higher levels J^a_n, $n > 1$ are expressed as commutators of Y's

Hopf algebra

coproduct $\Delta(1) = 1 \otimes 1$, $\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a$,

$$\Delta(Y^a) = Y^a \otimes 1 + 1 \otimes Y^a + i\hbar f^a{}_{bc} J^b \otimes J^c .$$

antipode $\Sigma(1) = 1$ $\Sigma(J^a) = -J^a$

$$\Sigma(Y^a) = -Y^a + \tfrac{i}{2}\hbar f^a{}_{bc} f^{bc}{}_d J^d$$

$$= -Y^a + i\hbar J^a \quad \text{for } g = su(2)$$

$\Sigma^2(J^a) = J^a$ $\Sigma^2(Y^a) = Y^a - 2i\hbar J^a$ (not involution)

Evaluation Representation as for $g[u]$

$\rho_u(1) = 1$ $\rho_u(J^a) = \rho(J^a)$ $\rho_u(Y^a) = u \cdot \rho(J^a)$

(r.h.s. of Serre relation $\stackrel{!}{=} 0$ for consistency)

Spin Chains

Homogeneous chain of L sites, use $u_j = 0$

$$\rho_0(1) = 1, \qquad \rho_0(J^a) = \rho(J^a), \quad \rho_0(Y^a) = 0.$$

Repr. ρ_{ch} on a chain of L sites

$$\rho_{ch} = (\rho_0 \otimes \dots \otimes \rho_0) \circ \Delta^{L-1}$$

note:

$$\Delta^{L-1}(J^a) = \sum_{j=1}^{L} J_j^a \quad , \quad \Delta^{L-1}(Y^a) = \sum_{j=1}^{L} Y_j^a + \hbar f^a{}_{bc} \sum_{j<k=1}^{L} J_j^b \, J_k^c$$

repr.

$$\rho_{ch}(J^a) = \sum_{j=1}^{L} \rho_j(J^a) \qquad \rho_{ch}(Y^a) = \hbar f^a{}_{bc} \sum_{j<k=1}^{L} \rho_j(J^b)\, \rho_k(J^c)$$

matches with monodromy $T(u)$ at $u = \infty$: $T(u) = \exp\left(\frac{i}{u} J + \frac{i}{u^2} Y + \dots\right)$

Symmetry

Let us consider Ham $H = \sum_k H_k \quad H_k = id_{k,k+1} - ex_{k,k+1}$

$[e_{ch}(J^a), H] = 0 \quad$ g is a symmetry of chain

$[e_{ch}(Y^a), H] \neq 0$ (by terms at boundary $j = 1, L$)

$Y(g)$ is not a symmetry of chain

is broken by periodic boundary conditions

is symmetry of bulk

is provides useful quantum operators (creation/annihilation)

if $Y(g)$ were symmetry $\Rightarrow$ large/full degeneracy of spectrum.

Magnon States

How does $Y(g)$ act on magnon states $(L=\infty)$ $|p_1, \ldots, p_M\rangle$

need to regularise J^z

$$\rho(J^z)_{reg} = \frac{1}{2}\sum(\sigma_j^z + id_j)$$

eigenvalue of $\rho(J^z)_{reg}$.

$$\rho(J^z)_{reg}\,|p_1 \ldots p_M\rangle = M \cdot |p_1 \ldots p_M\rangle$$

$$\rho(Y^z)_{ch} = \frac{i}{2}\hbar \sum_{j<k}\left(\sigma_j^- \sigma_k^+ - \sigma_j^+ \sigma_k^-\right)$$

$$\rho(Y^z)_{ch}\,|p\rangle = \frac{i}{2}\hbar \sum_{j<k}\left(e^{ipj}|k\rangle - e^{ipk}|j\rangle\right)$$

$$= \frac{i}{2}\hbar \sum_{k=1}^{\infty}\left(e^{-ipk} - e^{ipk}\right)\underbrace{\sum_j e^{ipj}|j\rangle}_{|p\rangle} = \frac{1}{2}\hbar\cot(p/2)\,|p\rangle = u\,|p\rangle$$

rap.

$u = \frac{1}{2}\hbar\cot(p/2)$ ← mag. mom.

$\hbar = 1$

R-Matrix

S matrix of (multiple flavours of) magnons is R

Symmetry of S extends to $Y(g)$: quasi-cocommutativity

$$R\,\Delta(X) = \tilde{\Delta}(X)\,R \qquad R \sim \frac{1}{u-v+i}\left((u-v)\,\mathrm{id} + i\,\mathrm{ex}\right)$$

can do for fund. eval. rep. with $X = J^a, Y^a$

$$\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a \qquad g = SU(N)$$

$$\overset{(\sim)}{\Delta}(Y^a) = u\,(J^a \otimes 1) + v\,(1 \otimes J^a) \pm \hbar f^a{}_{bc}\, J^b \otimes J^c$$

Q. cocomm. implies for $X = J^a$ $\quad R = R_1^{(u,v)}\,\mathrm{id} + R_2^{(u,v)}\,\mathrm{ex}$

for $X = Y^a$ implies $\quad i\hbar R_1 = (u-v) R_2$

$$R \sim (u-v)\,\mathrm{id} + i\hbar\,\mathrm{ex}$$

Tensor Products

R/S matrix acts on tensor product of two sites/particles

suppose $g = SU(N)$, site/part. repr. is fund.

from Lie repr $\quad \square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}$

fund $\otimes$ fund = sym $\oplus$ anti-sym.

$(\tfrac{1}{2}) \otimes (\tfrac{1}{2}) = (1) \oplus (0)$ $\quad$ for $SU(2)$

What changes in $Y(g)$?

consider 3 states:

can act with raising/lowering level-zero/level-one gen.

$|0\rangle = |\downarrow\downarrow\rangle \in \square\square$

$|s\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \in \square\square$

$|a\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \in \begin{smallmatrix}\square\\ \square\end{smallmatrix}$

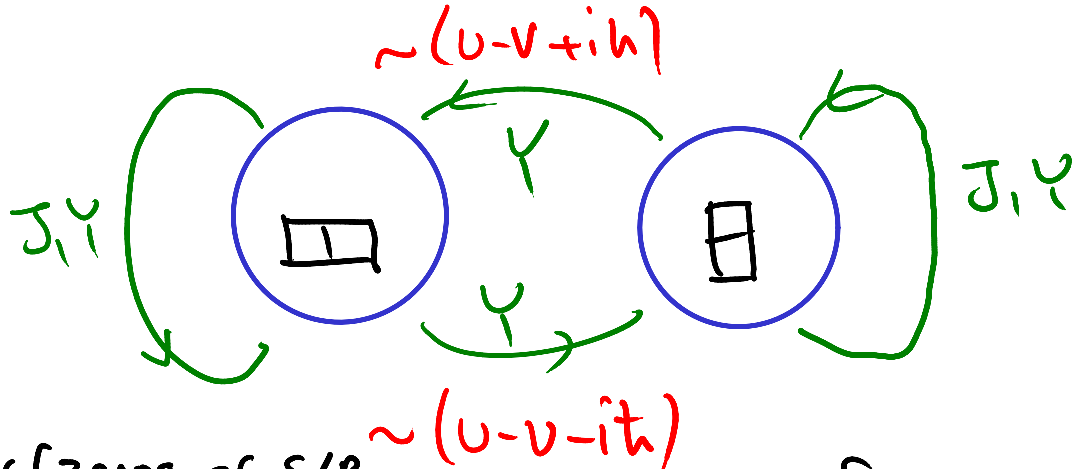
$$\Delta(J^+)|0\rangle = \tfrac{1}{2}|\uparrow\downarrow\rangle + \tfrac{1}{2}|\downarrow\uparrow\rangle = \tfrac{1}{2}|s\rangle$$

$$\Delta(Y^+)|0\rangle = \frac{1}{2}u\,|\uparrow\downarrow\rangle + \frac{1}{2}v\,|\downarrow\uparrow\rangle - \frac{i}{4}\hbar\,|\uparrow\downarrow\rangle + \frac{i}{4}\hbar\,|\downarrow\uparrow\rangle$$

$$= \frac{1}{4}(u+v)\,|s\rangle + \frac{1}{4}(u-v-i\hbar)\,|a\rangle$$

$$\Delta(J^-)\,|a\rangle = 0$$

$$\Delta(Y^-)\,|a\rangle = \frac{1}{2}(u-v+i\hbar)\,|0\rangle$$



$\square\square$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are unrelated by g

$\square\square$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ are generically related by $Y(g)$

poles/zeros of S/R

bound states of single-particle

→ tensor prod. is indecomposable but reducible.

for $u-v=\pm i\hbar$ one direction is forbidden.