# Introduction to Integrability

Lecture Slides, Chapter 5

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The current version of this work can be found at: http://people.phys.ethz.ch/~nbeisert/lectures/.

# 5. Inverse Scattering Method

## 5.1 Solitons and Scattering Processes

### Infinite chains

* array of spins $\vec{S}_j$, $-\infty < j < +\infty$.
* NN Hamiltonian as before, $\qquad H = \sum_{j=-\infty}^{+\infty} H_j$
* Poisson brackets as before

Two asymptotic regions with prescribed asymptotics of spins

$$\lim_{j\to-\infty} \vec{S}_j = \vec{S}_L \qquad \lim_{j\to+\infty} \vec{S}_j = \vec{S}_R \qquad \left(\begin{array}{l}\text{some}\\ \text{prescribed}\\ \text{decay}\end{array}\right)$$

### Solitons

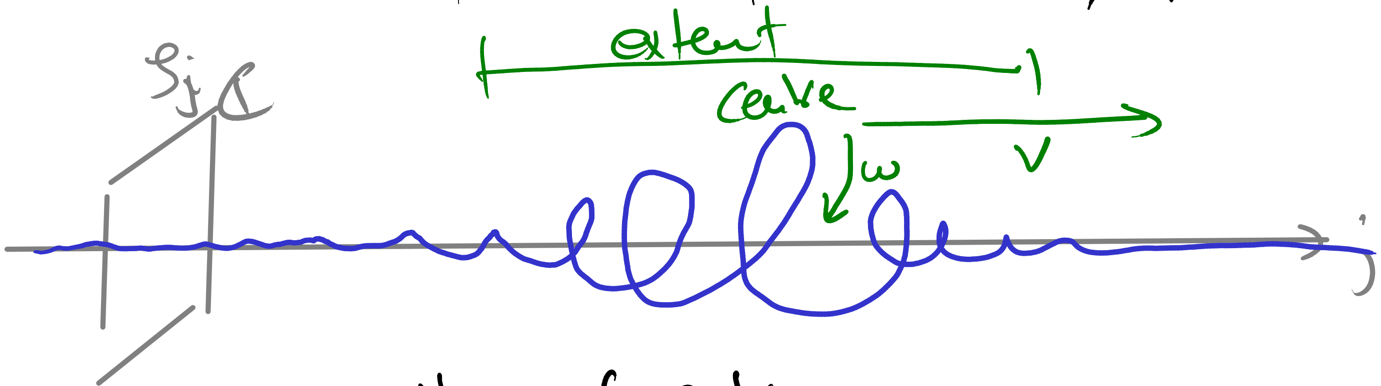
* exact solutions to non-linear (wave) equation of motion
* localised to a spatial region, exponential decay away from it.
* propagate with a stable shape, constant velocity. * scattering is elastic

Concrete soliton solution for spin chain (stereographic proj $\rho_j$)

$$\rho_j(t) = \frac{2\sin(k-k^*)}{\sin(k^*)\beta_j(t) + \sin(k)/\beta_j(t)^*}$$

$$\beta_j(t) := \mu \exp(-ik(2j-1) + 2i\sin^2(k)\,t)$$

2 parameters $k \in \mathbb{C}$, $\operatorname{Im} k > 0$, $k \equiv k + \pi\mathbb{Z}$; $\mu \in \mathbb{C}$




$\log|\mu| \sim$ position of centre

$\mu/|\mu| \sim$ orientation in x-y-plane

$$v = \frac{\operatorname{Im}\sin^2(k)}{\operatorname{Im} k} \qquad \omega = \frac{2\operatorname{Im}(k^*\sin^2(k))}{\operatorname{Im} k}$$

## Soliton Scattering

* solitons occupy a finite region
* place several solitons on chain at sufficient distance
* they would evolve independently
* will collide for different velocities
* will eventually move apart as two wave packets
* elastic: properties of wave packets are preserved
* two solitons are exchanged (spatial order)
* also for scattering of three or more solitons: same
* sequence of two-soliton scatterings
* factorised, elastic scattering.

# Inverse Scattering Method

* construct arbitrary solutions for spin chain with $\infty$ extent.
* based on an auxiliary scattering problem using Lax transport

Start with state $\vec{S}_j$ → Lax transport $\mathcal{L}_j(u)$

→ auxiliary linear problem, auxiliary vector array $\psi_j(u)$

linear problem: superposition of solutions, basis solutions

formulate (aux) scattering problem

compare solutions in far left / far right.

solutions have prescribed asymptotics → label basis for L/R

Scattering matrix $T(u)$ translates between bases L/R

Introduced scattering transformation $\vec{S}_j \to T(u)$

can be inverted, (even in practical

inverse scattering transformation $T(u) \to \vec{S}_j$

(+ bound states)

$$k \leftrightarrow u$$

benefits:

* time evolution can be imposed on $T(u)$ easily

$$T(k) \to T(k,t)$$

construct time evolution of any initial configuration

$$\vec{S}_j \to T(k) \to T(k,t) \to \vec{S}_j(t)$$

even better:

construct states/solutions in terms of abstract ~~aux~~ scattering data:

$$T(k) \to T(k,t) \to \vec{S}_j(t)$$

used for constructing soliton solutions and scattering processes

## 5.2 Auxiliary Scattering Problem

Based on some (any) configuration / state $\vec{S}_j$
with asymptotics $\lim_{j\to\mp\infty} \vec{S}_j = \vec{S}_{L/R}$

**Aux. linear Problem**

elem. Lax transport $\tilde{\mathcal{L}}_j$

$$\tilde{\mathcal{L}}_j(u) = \mathrm{id} + \frac{i}{u}\bar{S}_j \qquad \bar{S}_j := \vec{S}_j \cdot \vec{\sigma}$$

* unimodular ($\det \overset{!}{=} 1$)
$$\mathcal{L}_j(u) := \frac{-u}{\sqrt{u^2+1}}\tilde{\mathcal{L}}_j(u)$$

* introduce wave number $k$ to replace $u$:

$$u = u(k) := -\cot(k)$$

$$\Rightarrow \mathcal{L}_j(k) := \mathcal{L}_j(u(k)) = \cos(k)\,\mathrm{id} - i\sin(k)\,\bar{S}_j$$

aux. lin. prob. for 2-vector $\psi_j(k)$ : $\psi_j(k) = \mathcal{L}_j(k)\psi_{j-1}(k)$

spell out linear eq. $\psi_j(k) = \cos(k)\psi_{j-1}(k) - i\sin(k)\bar{S}_j\psi_{j-1}(k)$

## Scattering Problem

consider asymptotic behaviour of solutions $\psi_j(k)$

solve $\psi_j(k)$ by finite Lax transport

$j'$ fixed position

$$\psi_j(k) = \mathcal{L}_j(k)\,\mathcal{L}_{j-1}(k)\ldots\mathcal{L}_{j'+1}(k)\,\psi_{j'}(k)$$

since solves eq. uniquely, $\psi_{j'}(k) \in \mathbb{C}^2$ arbitrary, 2 indep compl. solution

consider left asymp. region $j \to -\infty$ asymptotic behaviour of $\bar{S}_j$

$$\psi_j(k) \sim (\cos(k)\,\mathrm{id} - i\sin(k)\,\bar{S}_L)\,\psi_{j-1}(k)$$

eigenvalues $\{e^{\pm ik}\}$

two distinguished solutions with asymptotics

$$\psi^1_{L,j}(k) \sim V^1_L\, e^{-ikj} \qquad \psi^2_{L,j}(k) \sim V^2_L\, e^{+ikj}$$

vectors are characterised by $\bar{S}_L V^1_L = +V^1_L \quad \bar{S}_L V^2_L = -V^2_L$

$$\bar{S}_L V_L = V_L \sigma^z$$

combine $\psi^1_{L,j}$ + $V^2_{L,j}$ into 2x2 matrix $\quad V_L := (V^1_L \ V^2_L)$

$$\psi_{Lj}(k) := \left(\psi^1_{Lj}(k) \ \ \psi^2_{Lj}(k)\right) \sim V_L \exp(-ikj\,\sigma^z)$$

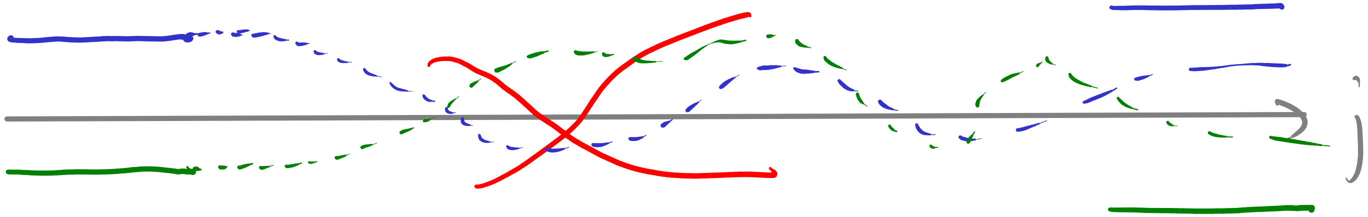
write solution to aux lin. prob.

$$\psi_{Lj}(k) = \lim_{j' \to -\infty} W_{jj'}(k)\, V_L \exp(-ikj'\sigma^z)$$
$$W_{jj'}(k) \exp(-ikj'\bar{S}_L)\, V_L$$

$\to \det \psi_{L,j}(k) = \det V_L \stackrel{!}{=} 1.$

Similar analysis for far right $\psi^{1,2}_{R,j}(k)$ , $j \to \infty$

$$\psi_{Rj}(k) = \lim_{j' \to +\infty} W_{j'j}(k)^{-1}\, V_R \exp(-ikj'\sigma^z) \qquad \det \psi_{Rj} \stackrel{!}{=} 1$$




Space of solutions in 2D.  Lax scattering matrix

$$\Psi_{Lj}(k) \overset{!}{=} \Psi_{Rj}(k)\, T(k)$$

$$\Rightarrow T(k) = \lim_{j,j'\to\mp\infty} V_R^{-1}\exp(ik_{j'}\bar{S}_R)\, W_{j'j}(k)\exp(-ik_j\bar{S}_L)\, V_L$$

## Time evolution

Lax equation

$$\frac{d}{dt}\mathcal{L}_j(k) = M_j(k)\,\mathcal{L}_j(k) - \mathcal{L}_j(k)\, M_{j-1}(k)$$

$$M_j(k) = 2i\sin(k)^2\,\frac{\bar{S}_j + \bar{S}_{j+1}}{2 + w(\bar{S}_j\cdot\bar{S}_{j+1})} - \sin(k)\cos(k)\,\frac{[\bar{S}_j, \bar{S}_{j+1}]}{2 + w(\ldots)}$$

asymptotic regions: $\lim_{j \to \mp\infty} M_j(k) = M_{L/R}(k) := i \sin^2(k)\, \bar{S}_{L/R}$

$\overset{\text{ev. eq.}}{\Rightarrow}$ $\frac{d}{dt} \psi_j(k) = M_j(k)\, \psi_j(k) - i \sin^2(k)\, \psi_j(k)\, \sigma^z$

time evolution of $T(k)$

$$\frac{d}{dt} T(k) = [M(k), T(k)] \qquad M(k) = i \sin^2(k)\, \sigma^z$$

$T^{11}(t) = T^{11}(t_0)$ const!

$T^{12}(t) = \exp\left(+2i \sin^2(k)\,(t-t_0)\right) T^{12}(t_0)$

$T^{21}(t) = \exp\left(- \;\ldots \right) T^{21}(t_0)$

$T^{22}(t) = T^{22}(t_0)$

# Analytical Structure

analyse dependence on $k$ of $T(k)$. so far $k \in \mathbb{R}$
but we apply complex analysis with $k \in \mathbb{C}$

dependence on $k$ is $2\pi$ periodic $\to$ chain is a lattice
Brillouin zone

actually a $\pi$-Periodicity (note $v = -\cot(k)$)

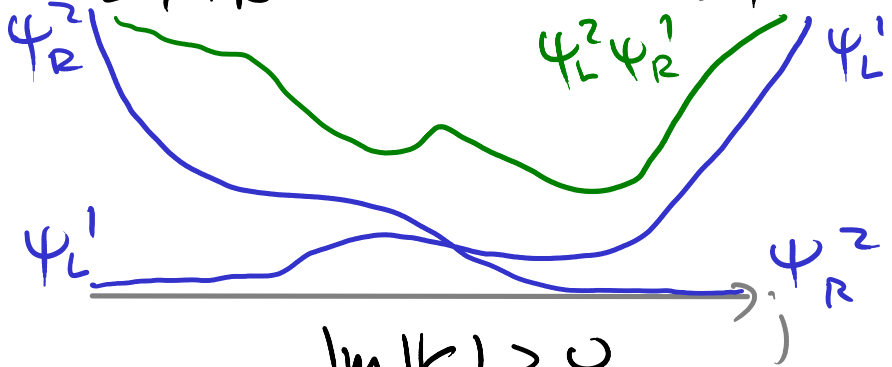
$$\psi_j(k+\pi) = (-1)^j \psi_j(k) \Rightarrow T(k+\pi) = T(k)$$

analyticity / holomorphic $\quad \mathcal{L}$ depends analytically on $v, k$
$\Rightarrow W_{jj'}(k)$ is analytic but potential issues when $j, j' \to \pm\infty$
investigate plane wave factor

$$\exp(\pm ikj) = \exp(\mp \operatorname{Im}(k) j) \exp(\pm i \operatorname{Re}(k) j)$$

solutions $\psi^{1,2}_{L/R}(k)$ decays or diverges exponentially depending on solution, sign $\mathrm{Im}(k)$, on L/R

$\mathrm{Im}\, k > 0$ : $\psi^1_L, \psi^2_R \to 0$ $\psi^2_L, \psi^1_R \to \infty$ in respective asympt. region




for $T(k)$ use $\det \psi(k) = 1$

$\Rightarrow \psi^1_L, \psi^2_R$ are analytical in $k$ for $\mathrm{Im}\, k > 0$

$$T^{11}(k) = \psi^1_{Lj}(k)^T \epsilon\, \psi^2_{Rj}(k)$$

$\Rightarrow T^{11}(k)$ is analytical for $\mathrm{Im}\, k > 0$

$T^{12,21,22}$ are not necessarily analytical.

$$\det T(k) = \frac{\det \Psi_{L,j}}{\det \Psi_{R,j}} = \frac{\det V_L}{\det V_R} \overset{!}{=} 1 .$$

Reality conditions $\vec{S}_j \in \mathbb{R}^3$

$$\epsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow T(k)^* = \epsilon T(k^*) \epsilon^{-1}.$$

$$V^*_{L/R} = \epsilon V_{L/R} \epsilon^{-1}$$

exact relationship for 2x2 matrices $T$

$$T^{-1} = (\det T)^{-1} \epsilon T^T \epsilon^{-1}$$

combine to unitarity statement $T(k)^+ = T(k^*)^{-1}$

$$T^{11}(k)^* = T^{22}(k^*) \qquad T^{12}(k)^* = -T^{21}(k^*)$$

# Bound States

Special points $k = k_n \in \mathbb{C}$ related to bound states

bound states: normalisable solutions of aux lin. prob.

for generic $k \in \mathbb{C}$ no normalisable solutions diverge left/right or both.

need solution that decays in both asymp. regions, assume $\mathrm{Im}\, k > 0$

$$\psi_j(k) \sim e^{\pm ikj} \quad \text{for } j \to \pm\infty$$

$$\psi_j(k) \sim \psi^1_{L,j}(k) \sim \psi^2_{Rj}(k)$$

need $k = k_n$ such that $\psi^1_{Lj}(k_n) \sim \psi^2_{Rj}(k_n)$

$\psi^1_R$ diverges R

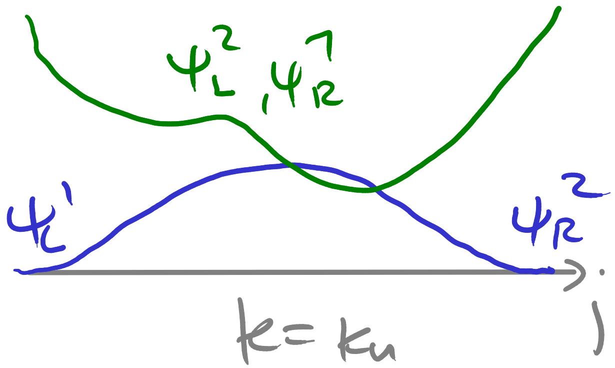
$\psi^2_R$ ~~decays~~ decays

$$\overset{\text{dec}}{\psi^1_L} = T^{11} \overset{\text{div}}{\psi^1_R} + T^{21} \overset{\text{dec}}{\psi^2_R} \Rightarrow T^{11}_{(k_n)} \overset{!}{=} 0$$

$$\psi^2_L = T^{12} \psi^1_R + T^{22} \psi^2_R$$

$$\psi^1_{Lj}(k_n) = T^{21}(k_n)\,\psi^2_{Rj}(k_n)$$

careful: poles in $T(k)$




want to understand coefficient of prop $\psi^1_L / \psi^2_R$. work harder

zero in $T^{11}(k) = \psi^1_{Lj}(k_n)^T \epsilon\, \psi^2_{Rj}(k_n) \Rightarrow \psi^1_L \sim \psi^2_R$

bound state

extract coefficient by derivative in $k$

$$T^{11\prime}(k_n) = \psi^{1\prime}_{Lj}(k_n)\,\epsilon\,\psi^2_{Rj}(k_n) + \overbrace{\psi^1_{Lj}(k_n)^T\epsilon\,\psi^{2\prime}_{Rj}(k_n)}^{\to 0 \text{ for } j\to+\infty}$$

* analyse for $j\to+\infty$: $\psi^1_L$ is bound state $\Rightarrow$ decays, $\psi^{2\prime}_R$ also decays

2nd part $\to 0$

* analyse for $j\to-\infty$: first part $\to 0$

iteration relation $\psi^{1\prime}_j(k)^T\epsilon\,\psi^2_j(k) - \psi^{1\prime}_{j-1}(k)^T\epsilon\,\psi^2_{j-1}(k) = i\psi^1_j(k)^T\epsilon\,\sigma_j\,\psi^2_j(k)$

$\sum_{j=-\infty}^{+\infty}$ telescopes $\qquad \sum$

$$\ldots \Rightarrow T^{11}(k_n) = \sum_{j=-\infty}^{+\infty} i \psi^1_{Lj}(k_n)^T \epsilon \vec{S}_j \psi^2_{Rj}(k_n)$$

can express factor of proportionality.

## Conserved Charges

extract useful data / conserved charges from $T(k)$

Expansion around $k=0$ ($u=\infty$) yields non-local charges

$$T(k) = V_R^{-1}\left(\mathrm{id} - ik\,\vec{\sigma}\cdot\Delta\vec{J} + O(k^2)\right)V_L$$

step fn.

$$\Delta\vec{J} := \sum_{j=-\infty}^{+\infty}\left[\vec{S}_j - \theta_{j\le 0}\vec{S}_L - \theta_{j>0}\vec{S}_R\right]$$

also $u=\pm i$ describes $H, P$ $\quad \mathrm{Im}\,k \to \pm\infty$

$$T^{11}(+i\infty) = \exp\left(-\tfrac{1}{2}H - \tfrac{i}{2}P\right) \quad T^{22}(-i\infty) = \exp\left(-\tfrac{1}{2}H + \tfrac{i}{2}P\right)$$

## 5.3 Reconstruction

Aux lin. prob. defines for every $\vec{s}_j$ : basis of sol. $\varphi_j(k)$, construct $T(k)$

detailed Fourier analysis of $T(k)$ $\varphi_j(k) \to \Phi_{jj'}$

scattering data

Fourier Transform $\varphi_j(k) \xrightarrow{k \to j'} \Phi_{j,j'}$

$$\varphi_{Lj}(k) = V_L \exp(-ikj\sigma^2)$$ ← asymptotic behavior

$$+ \sum_{n=0}^{\infty} \Phi_{j,j-2n}\, 2i \sin(k\sigma^2) \exp(-ik(j-2n-1)\sigma^2)$$

* analyse solution $\varphi_{Lj}(k) = \lim_{j' \to -\infty} \mathcal{L}_j(k) \ldots \mathcal{L}_{j'+1}(k) V_L \exp(-ikj'\sigma^2)$
  $(j-j')$ copies of $\mathcal{L}(k)$, eigenvalues $e^{\pm ik}$ $\Rightarrow$ bounds range for $n$ !
* even modes : $k \equiv k+\pi$
* prefactor $\sin(k\sigma^2)$ acts as difference operator $\varphi_{jj'} \to \varphi_{j,j'+1} - \varphi_{j,j'-1}$
  $k=0$ and $k=\pi$ asymptotics is exact.

## Gelfand Levithan Marchenko Equation

reconstruct $\Psi$ from scattering data through summation equation.

$$\psi_{Lj} = \psi_{Rj} T \qquad \det T = 1$$

$$0 = +\left(\frac{1}{T^{11}(k)} - V_L V_R^{-1}\right)\psi^2_{Rj}(k) - \underbrace{\left(\psi^2_{Lj}(k) - V_L V_R^{-1}\psi^2_{Rj}\right)}_{\text{Plain FT}} + \underbrace{\frac{T^{12}(k)\,\psi^1_{Lj}(k)}{T^{11}(k)}}_{\text{FT} \to \text{convolution}}$$

Fourier Traf.

$$\int_0^\pi \frac{dk\, e^{-ik(j-2m-1)}}{2\pi i \sin(k)} \ldots$$

contour integration

$T^{11}$ analytic, zeros at $k = k_n$ (bound states)

$\to$ Poles in $\frac{1}{T^{11}}$. contribute!

Residue theorem.

$\to \psi^2_{j,j-2m}$

$$V_L^1 N^{0\,12}_{2j-2m} - \sum_{n=0}^{\infty} \psi^1_{j,j-2n}\left(N^{0\,12}_{2j-2m-2n} - N^{0\,12}_{2j-2m-2n-2}\right)$$

$$N^{0\,12}_{2m} := \int_0^\pi \frac{dk\, e^{-ik(2m-1)}}{2\pi i \sin(k)} \frac{T^{12}(k)}{T^{11}(k)}$$

$$N^{X\,12}_{2n} := \mu_n \frac{\sin(k_n - k_n^*)}{\sin(k_n^*)} \exp(-ik_n(2m-1))$$

Assemble summation eq.

$$0 = \Psi^{2}_{j,j-2m} + V^{1}_{L} N^{12}_{2j-2m} - \sum_{n=0}^{\infty} \Psi^{1}_{j,j-2n} \left( N^{12}_{2j-2m-2n} - N^{12}_{2j-2m-2n-2} \right)$$

everything expressed in $k$-data.

Matrix eq.

$$N_{2m} = \begin{pmatrix} 0 & N^{12}_{2m} \\ N^{21}_{2m} & 0 \end{pmatrix}$$

$$0 = \Psi_{j,j-2m} + V^{1}_{L} N_{2j-2m} - \sum_{n=0}^{\infty} \Psi_{j,j-2n} \left( N_{2j-2m-2n} - N_{2j-2m-2n-2} \right)$$

$$N^{0\,12}_{2m} = \int_0^{\pi} \frac{dk\, e^{ik(2m-1)}}{2\pi i \sin(k)} \frac{\tau^{12}(k)}{\tau^{11}(k)} \qquad N^{\times 12}_{2m} = \sum_n \mu_n \frac{\sin(k_n - k_n^*)}{\sin(k_n^*)} e^{-ik_n(2m-1)}$$

time evol. for $N$

$$\frac{\tau^{12}(k,t)}{\tau^{11}(k,t)} = \exp\left(\pm 2i \sin^2(k)(t-t_0)\right) \frac{\tau^{12}(k,t_0)}{\tau^{11}(k,t_0)}$$

$$\mu_n(t) = \exp\left(\pm 2i \sin(k_n)(t-t_0)\right) \mu_n(t_0)$$

## Spin configuration

how to obtain $\vec{S}_j$ from $\Psi_{jj'}$

Fourier transformation of lin. aux. problem

$$\psi_j(k) = \psi_{j-1}(k)\cos(k\sigma^z) - i\,\bar{S}_j\,\psi_{j-1}(k)\,\sigma^z \sin(k\sigma^z)$$

use Fourier ansatz for $\psi_{l,j}(k) \to$ difference eq. $n>0$

$$2\Psi_{j,j-2n} = \Psi_{j-1,j-2n-1} + \bar{S}_j \psi_{j-1,j-2n-1}\sigma^z + \psi_{j-1,j-2n+1} - \bar{S}_j\,\psi_{j-1,j-2n+1}\sigma^z$$

$n=0$:
$$2\psi_{jj} = \psi_{j-1,j-1} + \bar{S}_j\,\psi_{j-1,j-1}\sigma^z + V_L - \bar{S}_j\,V_L\,\sigma^z$$

unit vector $\to$
$$\psi_{jj} - \bar{S}_j\,\psi_{jj}\,\sigma^z = V_L - \bar{S}_j V_L \sigma^z \qquad \text{solve for } \bar{S}_j$$

$$\vec{S}_j\cdot\vec{\sigma} = \bar{S}_j = \left(\psi_{jj} - V_L\right)\sigma^z\left(\psi_{jj}-V_L\right)^{-1}.$$

## 5.4 Soliton Solutions

Apply relations to construction of solitons with exact expressions

Start with scattering data in $N^{0}_{2m}$, $N^{\times}_{2m}$.

Assume reflectionless scattering $\Rightarrow$ $N^{0}_{2m} = 0$ $\quad T^{12}(k) = 0$

Single zero of $T^{11}(k)$ at $k = \kappa$ with $\operatorname{Im} \kappa > 0$

Ansatz
$$N^{\times 12}_{2m} = -N^{\times 21 *}_{2m} = \mu \frac{\sin(\kappa - \kappa^{*})}{\sin(\kappa^{*})} e^{-i\kappa(2m-1)}$$

$\mu \in \mathbb{C}$

$\vec{S}_L = \vec{e}_z$

$V_L = \mathrm{id}$

GLM eq. with ansatz for $\psi_{jj'}$

with $j'$ dependence a lin. comb. of $\exp(-i\kappa j')$ & $\exp(+i\kappa^{*} j')$

perform sums as geometric series

$\to$ rational function equations
$$\sum_{n=0}^{\infty} x^{n} = \frac{1}{1-x}$$

find solution

$$\Psi_{j,j-2n} = \frac{\sin(k-k^*)}{\sin(k)\sin(k^*)} \frac{1}{\mu\mu^* + e^{-2i(k-k^*)j}}$$

$$\begin{pmatrix} \mu\mu^* \sin(k^*)\, e^{-ik^*(2n+1)} & -\mu \sin(k)\, e^{ik(2n+1)-2ik^*j} \\ -\mu^* \sin(k^*)\, e^{-ik^*(2n+1)+2ikj} & -\mu\mu^* \sin(k)\, e^{ik(2n+1)} \end{pmatrix}$$

evaluate at $n=0 \Rightarrow \Psi_{jj} \Rightarrow \vec{S}_j$ $\qquad \mathrm{Im}\, k > 0$

$$S_j = \frac{2\mu^* \sin(k-k^*)}{\mu\mu^* e^{-ik(2j-1)} \sin(k^*) + e^{-ik^*(2j-1)} \sin(k)}$$

$A\vec{J} = 2\,\mathrm{Im}\cot(k)\vec{e}_z \Leftarrow k=0,\ k=\pm i\infty$

$H = 4\,\mathrm{Im}(k)$

$P = 4\,\mathrm{Im}\log\sin(k)$ $\quad T(k) = \mathrm{diag}\left(\frac{\cot(k)-\cot(k)}{\cot(k)-\cot(k^*)},\ \frac{\cot(k)-\cot(k^*)}{\cot(k)-\cot(k^*)}\right)$

## 5.4 Soliton Solutions (cont)

$$S_j = \frac{2\mu^* \sin(k-k^*)}{\mu\mu^* e^{-ik(2j-1)}\sin(k^*) + e^{-ik^*(2j-1)}\sin(k)} \qquad k \in \mathbb{C}, \quad \operatorname{Im} k > 0$$

**Features**

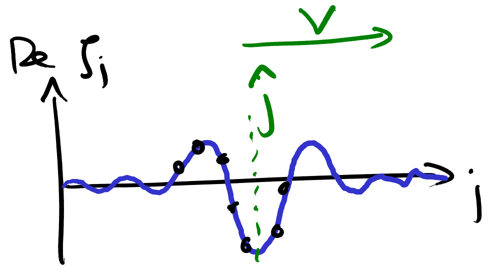
- $S_j \to S_L = 0$ at $j \to -\infty$
- grows exponentially away from $j = -\infty$ with $S_j \sim \exp(2ik^* j)$
- both terms in ~~den~~ compatible size for finite $j$ near $\hat{J}$

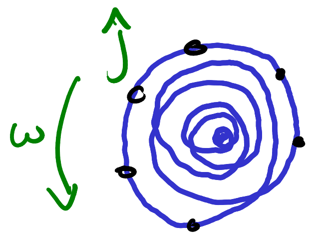
$$\hat{J} = -\frac{\log|\mu|}{2\operatorname{Im}(k)} + \frac{1}{2} \qquad S^z_{\hat{J}} = \frac{1-\cos(k)\cos(k^*)}{\sin(k)\sin(k^*)}$$

$$\hat{S} = S_{\hat{J}} = \frac{2\sin(k-k^*)}{\sin(k)+\sin(k^*)} \exp\left(\frac{k\log\mu + k^*\log\mu^*}{k-k^*}\right)$$

- after maximum, solution goes back exponentially to

$S_j \to S_R = 0$ at $j \to +\infty$ with $S_j \sim \exp(2ikj)$.

Sketch 


from north pole 


- strongly localised around $\hat{j}$ with exponential decay beyond characteristic width $\sim 1/\mathrm{Im}\,k$
- time evolution $\mu(t) = \mu(0)\exp(2i\sin^2(k)t)$
  - $\to$ velocity of centre $\hat{j}$ $\quad v = \dfrac{\mathrm{Im}\sin^2(k)}{\mathrm{Im}(k)}$
  - $\to$ angular velocity of orientation of centre $\hat{j}$ : $\omega = \dfrac{2\,\mathrm{Im}(k^*\sin^2(k))}{\mathrm{Im}(k)}$
- conserved charges

$$H = 4\,\mathrm{Im}(k) \qquad P = 4\,\mathrm{Im}\log\sin(k)$$

- angular mom deviation : $\Delta\vec{J} = 2\,\mathrm{Im}\cot(k)\,\vec{e}_z$

- scattering matrix in ISM approach

$$T(k) = \mathrm{diag}\left( \frac{\cot(k) - \cot(k)}{\cot(k) - \cot(k^*)}, \frac{\cot(k) - \cot(k^*)}{\cot(k) - \cot(k)} \right)$$

- Extract charges from expansion at $k=0$, $k = \pm i\infty$

$$T(k) \overset{k\approx 0}{=} id - 2i\,\mathrm{Im}\left(\cot(k)\right)\sigma^2 k + O(k^2) = id - i\Delta\vec{J}\cdot\vec{\sigma}\, k + O(k^2)$$

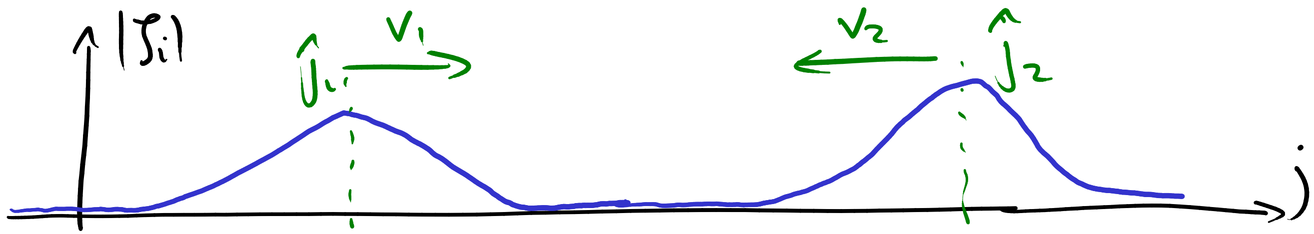
$$\mathrm{Im}\, k \to \pm\infty \sim u = -\cot(k) = \pm i$$

$$T^{11}(+i\infty) = e^{ik-ik^*}\frac{\sin(k^*)}{\sin(k)} = \exp\left(-\tfrac{1}{2}H - \tfrac{i}{2}P\right)$$

$$T^{22}(-i\infty) = e^{ik-ik^*}\frac{\sin(k^*)}{\sin(k)} = \exp\left(-\tfrac{1}{2}H + \tfrac{i}{2}P\right)$$

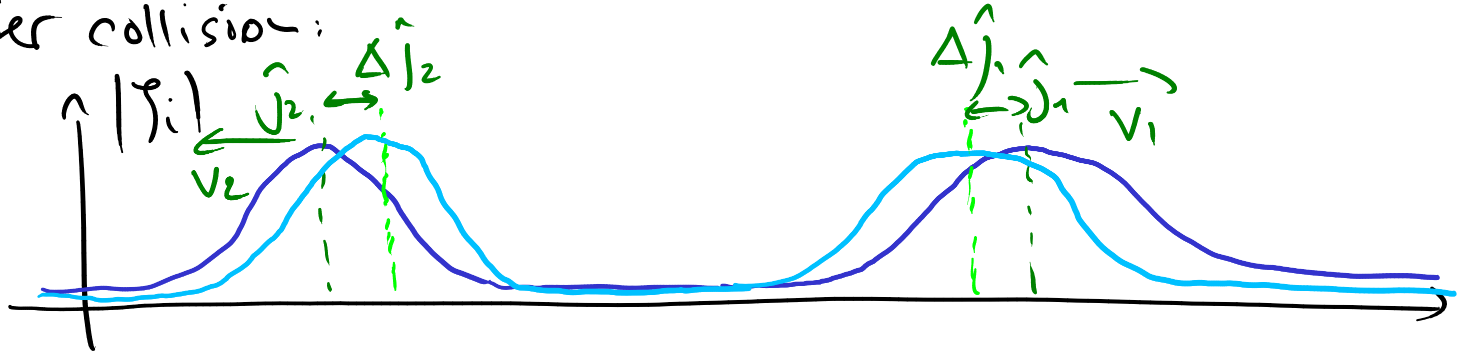
## Multiple Solitons

Solutions with two or more zeros of $\tau''(k)$

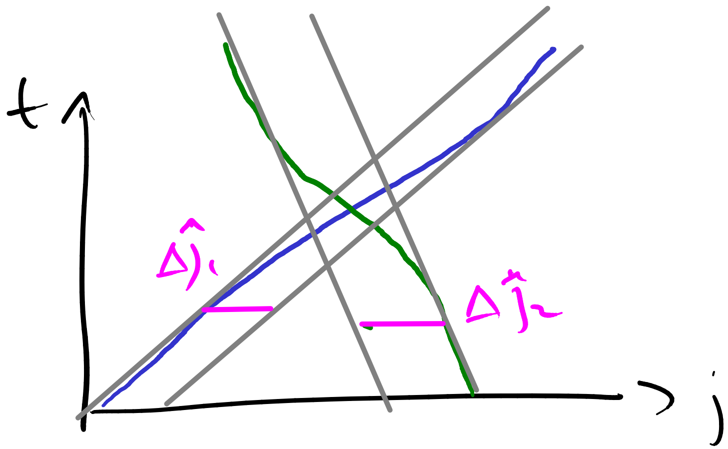



- in ISM provide scattering data with two $k_1$, $k_2$
- use ansatz for with 2 indep. exp. behaviours
- substitute → linear equations for unknowns.

after collision:



evolution of two centres in collision of solitons




compute relative shift by

either $\mu_2 \to 0$ or $\mu_1 \to \infty$

$\hat{j}_2 \to +\infty$ $\qquad$ $\hat{j}_1 \to -\infty$

compute centre $\hat{j}_1$ (trivial, does not depend on $k_2$)

compute centre $\hat{j}_2$ non-trivial, depends on $k_1$

$k_{\mathrm{eff}\,1} = k_1 \quad \mu_{\mathrm{eff},1} = \mu_1$

$k_{\mathrm{eff},2} = k_2 \quad \mu_{\mathrm{eff},2} = \left(\frac{\cot(k_2)-\cot(k_1)}{\cot(k_2)-\cot(\bar{k}_1)}\right)^2 \mu_2$

after scattering $1 \leftrightarrow 2$ : soliton scattering factor $S(k_2,k_1) = 1/\hat{J} = (T_1''(k_2))^{-2}$

$$\mu'_{\mathrm{eff},1} = S(k_1,k_2)^{-1}\mu_{\mathrm{eff},1} \qquad \mu'_{\mathrm{eff},2} = S(k_2,k_1)\,\mu_{\mathrm{eff},2}$$