Electrodynamics

Problem Sets

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The current version of this work can be found at:
`http://people.phys.ethz.ch/~nbeisert/lectures/`.

Contents

1.1. Vector calculus

In this problem, we recall a number of standard identities of vector calculus which we will frequently use in electrodynamics.

Definitions/conventions: We commonly write the well-known vectorial differentiation operators grad, div, rot using the vector $\vec{\partial}$ of partial derivatives $\partial_i := \partial/\partial x_i$ as

$$\operatorname{grad} F = \vec{\partial}F, \qquad \operatorname{div}\vec{A} = \vec{\partial}\cdot\vec{A}, \qquad \operatorname{rot}\vec{A} = \vec{\partial}\times\vec{A}. \tag{1.1}$$

The components of a three-dimensional vector product $\vec{a}\times\vec{b}$ are given by

$$(\vec{a}\times\vec{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} a_j b_k. \tag{1.2}$$

Here, ε_{ijk} is the totally anti-symmetric tensor in $\mathbb{R}^3$ with $\varepsilon_{123} = +1$.

a) Show that

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{and} \qquad \sum_{i,j=1}^{3} \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{kl}. \tag{1.3}$$

b) Show the following identities for arbitrary vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$:

$$\begin{aligned} \vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}), \\ \vec{a}\times(\vec{b}\times\vec{c}) &= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}, \\ (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}). \end{aligned} \tag{1.4}$$

c) Prove the following identities for arbitrary scalar fields F and vector fields $\vec{A}$:

$$\begin{aligned} \vec{\partial}\times(\vec{\partial}F) &= 0, \\ \vec{\partial}\cdot(\vec{\partial}\times\vec{A}) &= 0, \\ \vec{\partial}\times(\vec{\partial}\times\vec{A}) &= \vec{\partial}(\vec{\partial}\cdot\vec{A}) - \Delta\vec{A}, \\ \vec{\partial}\cdot(F\vec{A}) &= (\vec{\partial}F)\cdot\vec{A} + F\vec{\partial}\cdot\vec{A}. \end{aligned} \tag{1.5}$$

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1.2. Gauß's theorem

Consider the following vector fields $\vec{A}_i$ in two dimensions

$$\begin{aligned}\vec{A}_1 &= \big(3xy(y-x), x^2(3y-x)\big),\\ \vec{A}_2 &= \big(x^2(3y-x), 3xy(x-y)\big),\\ \vec{A}_3 &= \big(x/(x^2+y^2), y/(x^2+y^2)\big) = \vec{x}/\|x\|^2.\end{aligned} \tag{1.6}$$

a) Compute the flux of $\vec{A}_i$ through the boundary of the square Q with corners $\vec{x} = (\pm 1, \pm' 1)$

$$I_i = \oint_{\partial Q} \mathrm{d}x\, \vec{n}\cdot\vec{A}_i. \tag{1.7}$$

b) Compute the divergence of $\vec{A}_i$ and its integral over the area of this square Q

$$I'_i = \int_Q \mathrm{d}x^2\, \vec{\partial}\cdot\vec{A}_i. \tag{1.8}$$

1.3. Stokes' theorem

Consider the vector field

$$\vec{A} = (x^2 y, x^3 + 2xy^2, xyz). \tag{1.9}$$

a) Compute the integral along the circle S around the origin in the x,y-plane with radius R

$$I = \oint_S \mathrm{d}\vec{x}\cdot\vec{A}. \tag{1.10}$$

b) Compute the rotation $\vec{B}$ of the vector field $\vec{A}$

$$\vec{B} = \vec{\partial}\times\vec{A}. \tag{1.11}$$

c) Compute the flux of the rotation $\vec{B}$ through the disk S whose boundary is S, $\partial D = S$

$$I' = \int_D \mathrm{d}x^2\, \vec{n}\cdot\vec{B}. \tag{1.12}$$

1.4. Potential and electric field strength

Four point charges are placed at the corners $(a, 0)$, (a, a), $(0, a)$, $(0, 0)$ of a square. Find the potential and the electric field strength in the plane of this square. Sketch the field lines and equipotential lines of the following charge distributions:

a) $+q, +q, +q, +q$;

b) $-q, +q, -q, +q$;

c) $+q, +q, -q, -q$.

Hint: Using Mathematica, the commands `ContourPlot` and `StreamPlot` might come in handy.

2.1. Stable equilibrium

Two balls, each with charge $+q$, are placed on an insulating plate within the $z = 0$ plane where they can move freely without friction. Under the plate, a third ball with charge $-2q$ is fixed at $\vec{x} = (0, 0, -b)$. Treat the balls as point charges, and find stable positions for the two balls on the plate.

2.2. Energy stored in a parallel plate capacitor

Two plates with charges $+Q$ and $-Q$ and area A are placed parallel to each other at a (small) distance d. Find the energy stored in the electric field between them.

2.3. Electric field strength in a hollow sphere

A charged ball with homogeneous charge density ρ and radius R_A contains a spherical cavity with radius R_I that is shifted from the centre by the vector $\vec{a}$ with $\|\vec{a}\| < R_\mathrm{A} - R_\mathrm{I}$. Calculate the electric field strength within the cavity.

Hint: Use Gauß's theorem and the superposition principle to calculate the field strength.

2.4. delta-function

The delta-function is often used to describe the charge density of a point charge. It is defined through the following property when integrated over a smooth test function f with compact support:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\, \delta(x-a) = f(a). \tag{2.1}$$

a) Show that it can be written as the limit $\lim_{\epsilon\to 0} g_\epsilon(x) = \delta(x)$, where g_ϵ is defined as

$$g_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}}\, \mathrm{e}^{-x^2/(2\epsilon)}\,. \tag{2.2}$$

b) Show also that $\lim_{\epsilon\to 0} h_\epsilon(x) = \delta(x)$, where h_ϵ is defined as

$$h_\epsilon(x) = \frac{1}{2\pi \mathring{\imath}}\left(\frac{1}{x - \mathring{\imath}\epsilon} - \frac{1}{x + \mathring{\imath}\epsilon}\right). \tag{2.3}$$

c) Show, that the derivative of the delta-function satisfies

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\, \delta'(x-a) = -f'(a). \tag{2.4}$$

The properties of test functions ensure that there are no asymptotic contributions (boundary terms).

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2.5. Green functions in one dimension

Consider the region of space between two conducting, grounded, infinitely extended parallel plates that lie at $x = 0$ and $x = d$. Suppose that a third plate with uniform charge density σ is placed at $x = a$ where $0 < a < d$.

a) Show that finding the potential $\Phi(\vec{x})$ for $0 \leq x \leq d$ is equivalent to computing a Green function in one dimension, i.e. solving the equation

$$\frac{\partial^2}{\partial x^2} G(x, a) = -\delta(x - a) \tag{2.5}$$

with Dirichlet boundary conditions $G(0, a) = G(d, a) = 0$.

In order to simplify the solution of the following subproblems, change the reference frame in such a way that the charged plate is at $x = 0$ and the two conductors at $x = -a$ and $x = d - a$ respectively.

b) Divide the space in two regions, $-a < x < 0$ and $0 < x < d - a$, where there are no charges, and solve the two separate homogeneous Laplace equations for the potential. Integrating Poisson's equation from $x = -\epsilon$ to $x = +\epsilon$ and taking the limit $\epsilon \to 0$, determine the conditions that connect the two solutions in $x = 0$. Find the potential for the whole range $-a < x < d - a$, using the continuity of the potential at $x = 0$.

c) Directly integrate the differential equation for the potential and impose boundary conditions at $x = -a$ and $x = d - a$ to re-obtain the same result. In order to compare results, keep in mind the identities

$$\theta(x) = 1 - \theta(-x), \qquad 2x\theta(x) = x + |x|. \tag{2.6}$$

d) Transform the differential equation for the potential to Fourier space, solve it, and carry out the inverse transformation. Convince yourself that you can find a particular solution that is consistent with previous results.

3.1. Capacities

A simple capacitor consists of two isolated conductors with opposite charges of equal magnitude $Q_1 = +Q$ and $Q_2 = -Q$. In general both conductors will have different electrical potential and $U = \Phi_1 - \Phi_2$ denotes the potential difference. A characteristic quantity of the capacitor is the capacity C defined by

$$C = \frac{|Q|}{|U|}\,. \tag{3.1}$$

Calculate the capacities for the following cases:

a) two big parallel planar surfaces with area A and small distance $d \ll \sqrt{A}$;

b) two concentrical, conducting spheres with radii $a < b$;

c) two coaxial, conducting cylindrical surfaces of length L and radii $a < b \ll L$.

3.2. Imaginary dipoles

Consider two point charges q and q' at a distance d from each other, and a plane perpendicular to the line through q and q' in a distance αd from q.

a) Show that, in order for the plane to be at constant potential, one must have $q' = -q$ and $\alpha = 1/2$.
Hint: Look at the potential at large distance first.

Now consider a point charge at the (cartesian) coordinates $(a, b, 0)$ in the empty region of a space filled with a grounded conductor except for positive x and y.

b) Argue that introducing two mirror charges – with respect to the planes $x = 0$ and $y = 0$, respectively – is not sufficient to reproduce the boundary conditions of the conductor.

c) Exploiting the symmetry of the problem, introduce one more appropriate virtual charge and show explicitly that this makes the potential on the planes constant. Sketch the charge distribution.

Finally, consider a point charge in the empty region of a space filled with a grounded conductor except for the region of the angle $0 \le \varphi \le \pi/n$ (n integer).

d) Determine graphically the distribution of imaginary charges that reproduces the electric field of this charge in the empty region of space. What is the value of the electric field on the line where the two faces of the conductor meet?

e) *optional:* Sketch the electric field lines for the charge distribution of part d).

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3.3. Conducting sphere in an external electric field

A conducting sphere, bearing total charge Q, is introduced into a homogeneous electric field $\vec{E} = E\vec{e}_z$. How does the electric field change due to the presence of the sphere? How is the charge distributed on the surface of the sphere?

Hint: Motivate the following ansatz in spherical coordinates

$$\Phi(r, \vartheta, \varphi) = f_0(r) + f_1(r)\cos\vartheta, \tag{3.2}$$

and solve the Poisson equation $\Delta\Phi = 0$ using the Laplace operator

$$\Delta\Phi(r, \vartheta, \varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}\,. \tag{3.3}$$

A set of boundary conditions that completely fixes the solution is:

- at distances far from the sphere the homogeneous field has to dominate;
- the surface of the conducting sphere has to be an equipotential surface;
- the electric field has to satisfy Gauß's theorem.

3.4. Interior of a sphere with Neumann boundary conditions

The Green function for the interior of a sphere $B := \{x \in \mathbb{R}^3; \|x\| \leq R\}$ with radius R and Neumann boundary conditions is given by

$$G(x, y) = \frac{1}{4\pi\|x - y\|} + \frac{R}{4\pi\|y\|\|x - y'\|} - \frac{1}{4\pi R}\log\frac{\|y\|\|x - y'\| - \vec{y}\cdot(\vec{x} - \vec{y}')}{R^2}\,, \tag{3.4}$$

where $\vec{y}' := R^2\vec{y}/\|y\|^2$ is the inversion of the point $\vec{y}$ on the sphere.

a) Show that it satisfies the Laplace equation almost everywhere as well as the Neumann boundary condition for Green functions.

b) Show that it is symmetric in its two arguments. Interpret the value of $\vec{n}_y\cdot\vec{\partial}_y G$ on the boundary $y \in \partial B$.

c) Verify that $G(x, y)/\varepsilon_0$ describes the potential of the sum of a unit charge at y, a charge $q' = R/\|y\|$ at y' as well as a linear charge distribution with homogeneous charge density $\lambda = 1/R$ on the straight ray starting at y' and stretching out radially to infinity.

Hint: Sum and integrate the potential of the given charge distribution. Restrict the linear charge density to a maximal radius L at first. How can the divergent integral be understood or repaired?

4.1. Green functions in electrostatics

In this problem we analyse the electrostatic Green function in more detail. We consider Green functions G on a volume V with Dirichlet and Neumann boundary conditions on the surface ∂V.

a) Express the difference $G(y,z) - G(z,y)$ in terms of an integral over the surface ∂V. To do so, apply Green's second identity,

$$\int_V \mathrm{d}x^3 \left(\phi\Delta\psi - \psi\Delta\phi\right) = \oint_{\partial V} \mathrm{d}x^2\, \vec{n}\cdot\left(\phi\vec{\partial}\psi - \psi\vec{\partial}\phi\right) \tag{4.1}$$

with $\phi = G(y,x)$ and $\psi = G(z,x)$. Use that $\Delta_x G(y,x) = -\delta^3(x-y)$.

b) Show that a Green function $G_\mathrm{D}(x,y)$ with Dirichlet boundary conditions $G_\mathrm{D}(x,y) = 0$ for all $y \in \partial V$ must be symmetric in x and y.

c) Argue that $\vec{n}_y\cdot\vec{\partial}_y G_\mathrm{D}(x,y) \to -\delta^2(x-y)$ for $x \to \partial V$ and $y \in \partial V$. For the case $x \not\to y$ you can use the Dirichlet boundary condition for $G_\mathrm{D}(x,y)$. To understand the special case $x \to y$, use the above expression of Green's second identity.

d) Consider the (alternative formulation of the) Neumann boundary condition

$$\vec{\partial}_x\left[\vec{n}_y\cdot\vec{\partial}_y G_\mathrm{N}(x,y)\right] = 0 \quad \text{for all } y \in \partial V. \tag{4.2}$$

Show that $G_\mathrm{N}(x,y)$ is not symmetric in x and y in general. Construct a Green function $\tilde{G}_\mathrm{N}(x,y) = G_\mathrm{N}(x,y) + H(y) + K(x)$ that is symmetric in x and y. What properties must H and K have such that $\tilde{G}_\mathrm{N}$ is again a proper Green function?

4.2. Spherical cavity

Consider a spherical cavity with radius R and let the potential on its boundary be specified by an arbitrary function $U(\vartheta,\varphi)$.

a) Show that the potential within the cavity can be expressed as

$$\Phi(x) = \int \sin\vartheta'\,\mathrm{d}\vartheta'\,\mathrm{d}\varphi'\, \frac{R(R^2-r^2)\,U(\vartheta',\varphi')}{4\pi(r^2+R^2-2rR\cos\gamma)^{3/2}}\,, \tag{4.3}$$

where γ is the angle between x and x'. Determine $\cos\gamma$ in terms of the variables ϑ, φ, ϑ' and φ'.

Hint: Use the Green function obtained with the method of image charges and use spherical coordinates.

b) Write down the general solution of the Laplace equation in terms of spherical harmonics $Y_{\ell,m}$. Then use orthonormality relations to determine the coefficients for the given boundary condition.

c) Find the explicit potential $\Phi(x)$ inside the sphere for the boundary condition

$$U(\vartheta,\varphi) = U_0\cos\vartheta. \tag{4.4}$$

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4.3. Multipole moments of a cube

Assume that positive and negative point charges $\pm q$ are located on the corners of a cube with side length a. The point of origin is the centre of the cube, and the edges are aligned with the x,y,z-axes. Let the charge at $x, y, z > 0$ be positive. Charges on neighbouring corners have opposite signs.

a) Determine the positions of the charges in cartesian and spherical coordinates.

b) Determine the charge density in cartesian coordinates and, thereafter, in spherical coordinates, using

$$\delta^3(x - x_0) = \frac{1}{r^2 \sin\vartheta}\, \delta(r - r_0)\, \delta(\vartheta - \vartheta_0)\, \delta(\varphi - \varphi_0), \tag{4.5}$$

as well as $\sin\vartheta = \sin(\pi - \vartheta)$ and $\cos(\pi - \vartheta) = -\cos\vartheta$.

c) Calculate the spherical dipole, quadrupole and octupole moments of this charge configuration, using

$$Q_{\ell,m} = \int \mathrm{d}x^3\, \rho(x)\, r^\ell\, Y^*_{\ell,m}(\vartheta, \varphi), \qquad m = -\ell, -(\ell-1), \ldots, +(\ell-1), +\ell. \tag{4.6}$$

The required spherical harmonics are:

$$\begin{aligned}
Y_{00} &= 1, & Y_{11} &= -\sqrt{\frac{3}{2}}\, \sin\vartheta\, \mathrm{e}^{i\varphi}, \\
Y_{10} &= \sqrt{3}\, \cos\vartheta, & Y_{22} &= \sqrt{\frac{15}{8}}\, \sin^2\vartheta\, \mathrm{e}^{2i\varphi}, \\
Y_{21} &= -\sqrt{\frac{15}{2}}\, \cos\vartheta \sin\vartheta\, \mathrm{e}^{i\varphi}, & Y_{20} &= \sqrt{\frac{5}{4}}\, \big(3\cos^2\vartheta - 1\big), \\
Y_{33} &= -\sqrt{\frac{35}{16}}\, \sin^3\vartheta\, \mathrm{e}^{3i\varphi}, & Y_{32} &= \sqrt{\frac{105}{8}}\, \cos\vartheta \sin^2\vartheta\, \mathrm{e}^{2i\varphi}, \\
Y_{31} &= -\sqrt{\frac{21}{16}}\, \big(5\cos^2\vartheta - 1\big) \sin\vartheta\, \mathrm{e}^{i\varphi}, & Y_{30} &= \sqrt{\frac{7}{4}}\, \big(5\cos^3\vartheta - 3\cos\vartheta\big).
\end{aligned} \tag{4.7}$$

Furthermore: $Y_{\ell,-m} = (-1)^m\, Y^*_{\ell,m}$, and thus $Q_{\ell,-m} = (-1)^m\, Q^*_{\ell,m}$.

5.1. Expansion of the potential in Legendre polynomials

Suppose the potential on a spherical shell at radius R depends only on the polar angle ϑ and is given by $\Phi_0(\vartheta)$. Inside and outside the sphere there is empty space.

a) Find expressions for the potential $\Phi(r,\vartheta)$ inside and outside the sphere and for the charge density $\sigma(\vartheta)$ on the sphere.

Hint: The Legendre polynomials satisfy the following relations:

$$\int_0^\pi P_n(\cos\vartheta)P_m(\cos\vartheta)\sin\vartheta\,\mathrm{d}\vartheta = \delta_{n,m}\,\frac{2}{2n+1}\,. \tag{5.1}$$

b) Evaluate the expressions determined in part a) for $\Phi_0(\vartheta) = U\cos^2\vartheta$.

Hint: The first three Legendre polynomials are given as follows:

$$P_0(\cos\vartheta) = 1, \qquad P_1(\cos\vartheta) = \cos\vartheta, \qquad P_2(\cos\vartheta) = \tfrac{3}{2}\cos^2\vartheta - \tfrac{1}{2}. \tag{5.2}$$

Take now a spherical shell of radius R with uniform charge distribution $\sigma = Q/(4\pi R^2)$, except for a spherical cap at the north pole defined by the cone $\vartheta < \alpha$ in which $\sigma = 0$.

c) Show that the potential inside the sphere can be expressed as:

$$\Phi(r,\vartheta) = \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\big[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\big]\,\frac{r^\ell}{R^{\ell+1}}\,P_\ell(\cos\vartheta), \tag{5.3}$$

where for $\ell = 0$, $P_{\ell-1}(x) = -1$. What is the potential outside?

Hint: Use the following identities for the Legendre polynomials:

$$P_n(x) = \frac{1}{2n+1}\big(P'_{n+1}(x) - P'_{n-1}(x)\big). \tag{5.4}$$

d) Using the same configuration as in part c), find the magnitude and direction of the electric field at the origin.

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5.2. Rotation gymnastics

The rotation group (in N dimensions) is defined as the set of linear mappings of a vector space which leave the canonical scalar product invariant.

a) Prove that a linear transformation which leaves the norm of all vectors invariant also conserves the scalar product between two arbitrary vectors. Then show that any matrix preserving the norm of all vectors is orthogonal.

b) Show that the determinant of any orthogonal matrix is either $+1$ or -1.

Orthogonal matrices with negative determinant represent transformations that involve reflection. Since we are interested in rotations, let us restrict ourselves to the group of matrices with positive determinant, i.e. the special orthogonal group $\mathrm{SO}(N)$.

c) Write down the matrices that represent rotations of an infinitesimal angle $\delta\varphi$ around the i-th axis in three dimensions and subtract the identity from each of them. The resulting matrices are the generators of the rotations in 3D. Find a simple expression of the generators in terms of the totally antisymmetric tensor ε_{ijk}.

d) Show that infinitesimal rotations commute with each other up to higher order terms, whereas macroscopic rotations do not commute in general.

e) Write down the infinitesimal rotation of angle $\delta\varphi$ around a generic unit vector $\vec{n}$ (use the fact that $\vec{n}$ is left unchanged). By performing a large number of such rotations, extend the result to macroscopic angles φ around $\vec{n}$. Show that for every rotation with $\varphi \in (0, 2\pi)$ and arbitrary $\vec{n}$, there exists another representation with different φ', $\vec{n}'$.

5.3. Magnetic field of a finite coil

Consider a wire coiled up cylindrically along the z-axis. Let R be the radius of this cylindrical coil and L its length (it starts at $z = -L/2$ and ends at $z = +L/2$). Let $n = N/L$ be the winding number per unit length and I the (constant) current flowing through the wire. You may neglect boundary effects.

Calculate the z-component of the magnetic flux density B for points on the symmetry axis. Determine the magnetic field for $L \to \infty$ at constant n.

5.4. Current in a cylindrical wire

Consider a straight cylindrical wire of radius R oriented along the z-axis. The magnitude of the current density inside this wire depends on the distance from the centre of the wire as follows:

$$j(\rho) = j_0 \, \mathrm{e}^{-\rho^2/R^2} \, \theta(R - \rho), \tag{5.5}$$

where $\rho = \sqrt{x^2 + y^2}$ and $\theta(x)$ is the unit step function.

a) Find the current I flowing through the wire. Express j_0 through I.

b) Find the magnetic field inside and outside the wire as a function of the total current. Sketch the field lines, paying attention to the direction. Let the current flow into the positive z-direction.

6.1. Multipole expansion in spherical coordinates

We first recall the multipole expansion of the scalar potential. Given a localised distribution of charge described by the charge density $\rho(x')$, the potential $\Phi(x)$ outside the region where $\rho(x')$ is non-vanishing can be expanded as:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Q_{\ell,m}}{2\ell+1} \frac{1}{r^{\ell+1}} Y_{\ell,m}(\vartheta,\varphi), \tag{6.1}$$

where $Y_{\ell,m}(\vartheta,\varphi)$ denotes the spherical harmonics and $Q_{\ell,m}$ the multipole moments. The latter are defined as

$$Q_{\ell,m} = \int \mathrm{d}x'^3\, Y^*_{\ell,m}(\vartheta',\varphi')\, r'^{\ell} \rho(x'). \tag{6.2}$$

Consider now a circular loop of radius R lying in the x,y-plane centred at the origin with linear charge density $\lambda = \lambda_0 \cos\varphi$, where φ is the azimuthal angle measured in the x,y-plane.

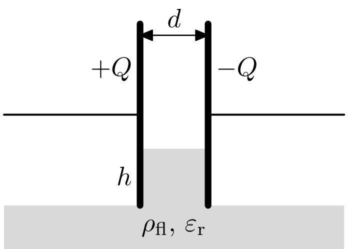
a) Express the charge density $\rho(x')$ given by the loop in spherical coordinates.

b) Compute the multipole moments $Q_{\ell,m}$ and use them to write down the multipole expansion of $\Phi(x)$ up to $\ell = 3$ for $\|x\| > R$. Discuss the nature of the resulting multipoles.

c) Calculate the electric field $\vec{E}(x)$ with $\|x\| > R$ of the dipole and quadrupole moments determined in part b).

6.2. Capacitor filled with dielectric

Consider a parallel plate capacitor with quadratic plates of edge length a and distance d between the plates. It is charged to the amount $\pm Q$ and subsequently separated from the voltage source. When the charged capacitor is placed on top of a dielectric fluid (with density ρ_{fl}, permittivity ε_{r}), the fluid rises between the plates up to a maximal height h_0.



a) Find the electrostatic energy $W_{\text{el}}(h)$ stored in the capacitor as a function of the height of the raised fluid h and the parameters defined above.

b) Find the potential energy $W_{\text{pot}}(h)$ of the fluid between the plates as a function of h.

c) Derive the defining equation for h_0 under the assumption that the total energy is minimised. Which amount of charge must be taken to the capacitor, such that the fluid rises up to half of the total height? Assume that $a = 20\,\text{cm}$, $d = 5\,\text{mm}$, $\varepsilon_{\text{r}} = 3$ and $\rho_{\text{fl}} = 0.8\,\text{g}/\text{cm}^3$.

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6.3. Magnetic field of a circular loop

Consider a conducting wire forming a circle of radius R in the centre of the x,y-plane. A constant current I flows counter-clockwise through this loop.

a) Calculate the magnetic field $\vec{B}$ at some point on the z-axis.

b) Now calculate the magnetic field $\vec{B}$ at an arbitrary point in the x,y-plane. You should obtain a result which is not analytically solvable, but can be expressed in terms of elliptic integrals. Check that it agrees with the result from part a) at the origin.

6.4. Magnetic moment of a rotating spherical shell

A spherical shell of radius R and charge Q (homogeneously distributed on the surface) is rotating around its z-axis with angular velocity $\vec{\omega} = \omega \vec{e}_z$.

a) Calculate the current density $\vec{\jmath}(x) = \vec{v}(x)\rho(x)$.

b) Calculate the magnetic moment $\vec{m} = \frac{1}{2}\int \mathrm{d}x^3\,(\vec{x}\times\vec{\jmath}(x))$ of the spherical shell.

c) Show that the leading behaviour of the magnetic field generated by this sphere for $\|x\| \gg R$ is that of a magnetic dipole, and write down the leading term of $\vec{B}$.

Hint: Use the Biot–Savart law and keep only the leading non-vanishing terms in $R/\|x\|$.

d) Now let $\vec{x}'$ be a vector such that $\vec{x}' \perp \vec{\omega}$ and $\|x'\| \gg R$. Calculate the lowest-order contribution of the force exerted by the magnetic field from the previous part on another identical sphere placed at a point $\vec{x}'$ and rotating with angular velocity $\vec{\omega}'$ parallel to $\vec{\omega}$. Due to the large distance between the spheres you can approximate them as two point-like objects carrying some magnetic moment.

6.5. Vector potential of a loop of wire

A circular loop of wire of radius a and negligible thickness carries a current I. The coordinates are chosen such that the wire lies in the x,y-plane. We are interested in the vector potential of the current density and want to obtain the corresponding magnetic field.

a) Find a vector potential for the loop using the formula

$$\vec{A} = \mu_0 \int \mathrm{d}x'^3\, \frac{\vec{\jmath}(x')}{4\pi\|x - x'\|}\,. \tag{6.3}$$

Hint: Expand $1/\|x - x'\|$ in spherical harmonics and use that $Y_{\ell,-m} = (-1)^m Y^*_{\ell,m}$.

b) Based on the above result, calculate the magnetic field at points x with $\|x\| < a$. In problem 6.3 we have seen that the magnetic field a) on the z-axis and b) in the x,y-plane of the loop always points in the z-direction. Can you recover this result?

Hint: Use that

$$P_\ell^m(x) = (-1)^m(1 - x^2)^{m/2}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,P_\ell(x). \tag{6.4}$$

c) Assume now that the current loop is put into an iron cavity with infinite relative permeability. What are the changes for the magnetic field? What would you have to do to solve the problem (no calculation)?

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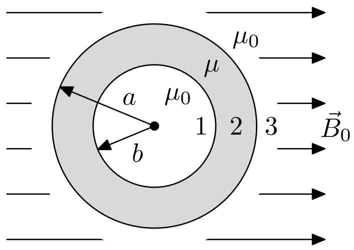
6.6. Iron pipe in a magnetic field

An infinitely long hollow cylinder (inner radius b, outer radius a) is placed with its axis orthogonally to an initially homogeneous magnetic field $\vec{B}_0$. The hollow cylinder is made of iron (permeability μ). The initial field $\vec{B}_0$ can be assumed to be sufficiently small not to saturate the iron, and the permeability μ is constant in the region of interest.



a) Derive the expression for $\vec{B}$ in the cavity ($r < b$).

Hint: Use the absence of free currents to describe the magnetic field H by means of a scalar potential Φ via $\vec{H} = -\vec{\partial}\Phi$. The Laplace equation holds in all three relevant regions of space. In cylindrical coordinates (r, φ, z) the Laplacian reads

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\,. \tag{6.5}$$

The boundary conditions of H and B at the surfaces as well as the behaviour for $r \to 0$ and $r \to \infty$ fix the constants in the solution of the differential equation.

b) Sketch the magnetic field lines in the full region of space, before and after the cylinder has been placed in the field. Consider also the cases of a paramagnetic ($\mu_\mathrm{r} > 1$), a diamagnetic ($\mu_\mathrm{r} < 1$), and a superconducting ($\mu_\mathrm{r} = 0$) cylinder.

6.7. Charged particle in an electromagnetic field

Consider a point particle carrying charge q in an electromagnetic field described by a vector potential A and a scalar potential Φ. The Lagrangian of the particle is given by

$$L(x, \dot{x}, t) = \tfrac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}}\cdot\vec{A}(x,t) - q\Phi(x,t), \tag{6.6}$$

where x is the position of the particle and m is its mass.

a) Determine the canonical momentum $\vec{p}$,

$$p_i = \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}_i}\,. \tag{6.7}$$

What is the relation between the canonical momentum p and the kinetic momentum $m\dot{x}$? Perform a Legendre transformation to determine the Hamiltonian,

$$H(x, p, t) = \vec{p}\cdot\dot{\vec{x}} - L(x, \dot{x}, t). \tag{6.8}$$

b) Use $\vec{B} = \vec{\partial}\times\vec{A}$ to explicitly verify

$$\sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j = \big(\dot{\vec{x}}\times\vec{B}\big)_i. \tag{6.9}$$

c) Start from the Hamiltonian equations

$$\dot{p}_i = -\frac{\partial H}{\partial x_i}\,, \qquad \dot{x}_i = \frac{\partial H}{\partial p_i}\,, \tag{6.10}$$

and derive the equation of motion for the charged particle in an electromagnetic field

$$m\ddot{\vec{x}} = q\big(\vec{E} + \dot{\vec{x}}\times\vec{B}\big). \tag{6.11}$$

7.1. Electric field of a moving wire

Consider an infinitely long wire carrying a constant current I in the z-direction, which is moving in the y-direction at constant speed v. We aim to find the electric field of this configuration.

a) Compute the magnetic field of a static wire in cylindrical coordinates.

b) Transform the y-coordinate such that the wire moves with velocity v, and compute the coordinate-transformed magnetic field B.

c) Compute the curl of the electric field generated by this time-varying magnetic field.

d) Use the translational symmetry in the z-direction and the Maxwell equations to solve for the electric field.

7.2. Mutual inductance

A small loop of wire with radius a is held at a distance z above the centre of a large loop with radius b. The planes of the two loops are parallel and perpendicular to the common axis.

a) Show that the magnetic field at distance z above the centre of a circular loop of radius R carrying a constant current I is given by:

$$\vec{B} = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}\,\vec{e}_z. \tag{7.1}$$

b) Suppose a current I flows in the big loop. Find the magnetic flux through the little loop. The little loop is so small that you may consider the field of the big loop to be essentially constant.

c) Suppose a current I flows in the little loop. Find the magnetic flux through the big loop. The little loop is so small that you may treat it as a magnetic dipole.

Hint: Recall that the magnetic field of a magnetic dipole $m\vec{e}_z$ is given in spherical coordinates by:

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\,(2\cos\vartheta\,\vec{e}_r + \sin\vartheta\,\vec{e}_\vartheta). \tag{7.2}$$

d) Find the mutual inductances, and confirm that $L_{12} = L_{21}$.

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7.3. Induction in a magnetic field

A homogeneous magnetic field B is aligned along the z-direction. Within this magnetic field, a conducting wire forming a circle of radius R rotates with circular velocity $\vec{\omega}$. Its rotational axis lies in the plane of the conductor and passes through its centre. Let ϑ be the angle between the rotational axis and the field direction. There is no voltage induced in the loop at time $t = 0$. Find the induced voltage in the conductor as a function of time.

7.4. Self-induction of a coaxial cable

A coaxial cable is represented by two coaxial conducting cylindrical shells with radii R_1 and R_2 with $R_1 < R_2$. A current I is flowing through each of the cylindrical shells along their axes in opposite directions. Compute the magnetic field of the cable and sketch the corresponding magnetic field lines. Then calculate the self-induction per unit length of this coaxial cable. Check that the formula for the magnetic energy of the cable holds:

$$W = \tfrac{1}{2}LI^2. \tag{7.3}$$

Hint: Determine the self-induction from the magnetic flux through a surface enclosed by the current via

$$\Psi = LI. \tag{7.4}$$

8.1. The Poynting vector

Maxwell's equations in vacuum are given by

$$\vec{\partial}\times\vec{E} = -\partial_t\vec{B}, \qquad \vec{\partial}\times\vec{B} = \mu_0\vec{\jmath} + \varepsilon_0\mu_0\partial_t\vec{E}, \tag{8.1}$$

and the speed of light is $c = 1/\sqrt{\varepsilon_0\mu_0}$.

a) Prove the following identity,

$$\frac{1}{2}\frac{\partial}{\partial t}\left(c^2\vec{B}^2 + \vec{E}^2\right) = -c^2\vec{\partial}\cdot\left(\vec{E}\times\vec{B}\right) - \frac{1}{\varepsilon_0}\,\vec{E}\cdot\vec{\jmath}. \tag{8.2}$$

b) Consider a particle with charge q moving in the electromagnetic field with velocity $\vec{v}$. Show that the time derivative of its kinetic energy is given by

$$\dot{W}_{\text{kin}} = q\vec{v}\cdot\vec{E}. \tag{8.3}$$

What is the equivalent for a continuous charge distribution?

The Poynting vector is defined as

$$\vec{S} := \varepsilon_0 c^2\vec{E}\times\vec{B}. \tag{8.4}$$

c) Prove Poynting's theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\varepsilon_0\int_V \mathrm{d}x^3\,\left(c^2\vec{B}^2 + \vec{E}^2\right) + W_{\text{kin}}\right) = -\oint_{\partial V}\mathrm{d}x^2\,\vec{n}\cdot\vec{S}, \tag{8.5}$$

where V is some time-independent volume and ∂V its surface. Interpret the physical meaning of each of the terms.

8.2. Net force by means of stress tensor

Consider the electrostatic field $\vec{E}$ induced by a uniformly charged solid ball of radius R and charge Q. Compute the net force $\vec{F}_H$ of the electric field on the northern half H of the ball with $z > 0$ using the Maxwell stress tensor and symmetry considerations.

Hint: The components of the electrostatics stress tensor T_{jk} with $j, k = x, y, z$ are given by

$$T_{jk} = \varepsilon_0\left(E_j E_k - \tfrac{1}{2}\delta_{jk}\vec{E}^2\right). \tag{8.6}$$

Compute the normal components of T on the surface ∂H and integrate.

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8.3. Invariant distance

The Lorentz boost with arbitrary direction and velocity is given by

$$t' = \gamma t - \frac{\gamma}{c^2}\,\vec{x}\cdot\vec{v}, \qquad \vec{x}' = \vec{x} - \gamma\vec{v}t + (\gamma - 1)\,\frac{\vec{x}\cdot\vec{v}}{\vec{v}^2}\,\vec{v}, \qquad \text{where} \qquad \gamma := \frac{1}{\sqrt{1 - \vec{v}^2/c^2}}\,. \tag{8.7}$$

a) Find a matrix $(\Lambda^{-1})^{\mu}{}_{\nu}$ (in $1+3$ block form) such that the above transformation takes the form $x'^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x^{\nu}$.

b) Verify that the matrix $\Lambda^{\mu}{}_{\nu}$ from part a) satisfies the condition

$$\Lambda^{\lambda}{}_{\mu}\eta_{\lambda\sigma}\Lambda^{\sigma}{}_{\nu} = \eta_{\mu\nu}. \tag{8.8}$$

Hint: Without loss of generality, you may rotate to a frame with $\vec{v} = v\vec{e}_z$.

c) The square distance between two spacetime points x_1 and x_2 is given by $s^2 = s^{\mu}s_{\mu}$ with the distance vector $s^{\mu} := x_1{}^{\mu} - x_2{}^{\mu}$. Show that it is a scalar quantity under Lorentz-transformations and, moreover, invariant under Poincaré-transformations.

9.1. Electromagnetic field tensor

The electromagnetic field tensor is given by

$$F_{\mu\nu} = -\partial_\mu A_\nu + \partial_\nu A_\mu = \begin{pmatrix} 0 & c^{-1}E_x & c^{-1}E_y & c^{-1}E_z \\ -c^{-1}E_x & 0 & -B_z & +B_y \\ -c^{-1}E_y & +B_z & 0 & -B_x \\ -c^{-1}E_z & -B_y & +B_x & 0 \end{pmatrix}. \tag{9.1}$$

a) Show that the electromagnetic field tensor is invariant under the following gauge transformation with an arbitrary scalar field Λ

$$A'_\mu = A_\mu + \partial_\mu \Lambda. \tag{9.2}$$

b) The dual electromagnetic field tensor is defined by

$$\tilde{F}_{\mu\nu} := \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{9.3}$$

Determine the matrix elements of $\tilde{F}_{\mu\nu}$ analogously to the above expression for $F_{\mu\nu}$.

c) Verify that the homogeneous Maxwell equations $\vec{\partial}\times\vec{E} = -\partial_t\vec{B}$ and $\vec{\partial}\cdot\vec{B} = 0$ can be expressed as

$$\partial^\mu \tilde{F}_{\mu\nu} = 0. \tag{9.4}$$

d) Compute the contractions $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$.

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9.2. Relativistic force

Two particles at a distance d move with identical velocity v in a direction perpendicular to their separation. Each particle carries the charge q.

a) Transform the electromagnetic fields due to one of the two particles from the rest frame of the particles to the frame of the observer. Then calculate the force between the particles in the frame of the observer.

b) The relativistic force four-vector is defined by $K^\mu := \mathrm{d}p^\mu/\mathrm{d}\tau$ with τ the proper time of the particle. Show that $\mathrm{d}/\mathrm{d}\tau = \gamma \mathrm{d}/\mathrm{d}t$, where t is the time in the frame of the observer. Use this to show that the components of four-force in the frame of the observer take the form

$$K^\mu = \gamma(\vec{F}\cdot\vec{v}/c, \vec{F}), \tag{9.5}$$

where $\vec{F}$ is the force on the particle in the frame of the observer.

Hint: Differentiate the invariant $p^\mu p_\mu$ w.r.t. t.

c) Write down the transformation law for K^μ between the rest frame of the particles and the frame of the observer. Verify that the force computed in part a) is consistent with the transformed force from the rest frame of the particles.

9.3. Fourier transform

The Fourier transformation in a three-dimensional space and its inverse are given by

$$\tilde{f}(\vec{k}) = \int \mathrm{d}x^3\, f(\vec{x})\, \mathrm{e}^{-\mathring{\imath}\vec{k}\cdot\vec{x}}, \qquad f(\vec{x}) = \int \frac{\mathrm{d}k^3}{(2\pi)^3}\, \tilde{f}(\vec{k})\, \mathrm{e}^{\mathring{\imath}\vec{k}\cdot\vec{x}}. \tag{9.6}$$

Prove the following identities for the Fourier transformation:

a) $h(x) = af(x) + bg(x) \implies \tilde{h}(k) = a\tilde{f}(k) + b\tilde{g}(k)$ $(a, b \in \mathbb{C})$.

b) $\vec{h}(x) = \vec{\partial} f(x) \implies \tilde{\vec{h}}(k) = \mathring{\imath}\vec{k}\tilde{f}(k)$.

c) $h(x) = f(x)g(x) \implies \tilde{h}(k) = (2\pi)^{-3}(\tilde{f} * \tilde{g})(k) := (2\pi)^{-3}\int \mathrm{d}k'^3\, \tilde{f}(k')\, \tilde{g}(k - k')$. This is also known as the convolution theorem.

d) $h(x) = f^*(x) \implies \tilde{h}(k) = \tilde{f}^*(-k)$.

e) $h(x) = \delta^3(x) \implies \tilde{h}(k) = 1$.

f) $\vec{h}(x) = \vec{\partial}\delta^3(x) \implies \tilde{\vec{h}}(k) = \mathring{\imath}\vec{k}$.

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9.4. Partially polarised light

An almost monochromatic electromagnetic wave is propagating in the z-direction and is described in complex notation by

$$\vec{B} = \vec{e}_z \times \vec{E}, \qquad \vec{E}(\vec{x}) = \vec{E}(t - z/c), \qquad \vec{E}(t) = \vec{E}_0(t)\, \mathrm{e}^{-\mathring{\imath}\omega_0 t}, \qquad \vec{E}_0(t) = \begin{pmatrix} E_1(t) \\ E_2(t) \\ 0 \end{pmatrix}. \quad (9.7)$$

We assume that the characteristic timescale of $\vec{E}_0$ (called coherence time τ), over which fluctuations of $\vec{E}_0$ are correlated, is much larger than the duration of a period (timescale $2\pi/\omega_0$), but is rapidly changing compared to the timescale of optical polarisation measurements. This allows us to average out the fluctuations in polarisation measurements and to treat $\vec{E}_0$ as a random variable. We indicate averages with the symbol $\langle\cdot\rangle$. We want to show that the full information about the polarisation of the wave is contained in the hermitian matrix

$$S = \begin{pmatrix} \langle E_1 E_1^* \rangle & \langle E_1 E_2^* \rangle \\ \langle E_2 E_1^* \rangle & \langle E_2 E_2^* \rangle \end{pmatrix} = S^\dagger. \quad (9.8)$$

a) Show that S can be parametrised as follows

$$S = s_0\sigma_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3, \quad (9.9)$$

with the Stokes parameters $s_i \in \mathbb{R}$ and the identity and Pauli matrices σ_i

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathring{\imath} \\ \mathring{\imath} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9.10)$$

Hint: The matrices of the form S span a vector space of real dimension 4.

b) By using the relation $\sigma_i\sigma_j = \sum_k \mathring{\imath}\varepsilon_{ijk}\sigma_k + \delta_{ij}\sigma_0$ $(i, j \neq 0)$ show that the matrices σ_i form an orthonormal basis for hermitian 2×2 matrices w.r.t. the scalar product $A \cdot B := \frac{1}{2}\operatorname{tr}(AB)$.

c) Express the Stokes parameters s_i in terms of the averages $\langle E_j E_k^* \rangle$.

d) Show that the matrix S is positive definite and compute its eigenvalues. Deduce the inequality $\sqrt{s_1^2 + s_2^2 + s_3^2} \leq s_0$.

e) Show that $s_0 = \sqrt{s_1^2 + s_2^2 + s_3^2}$ when the electromagnetic wave has a constant polarisation $\vec{E}_0 = \text{const.}$

f) Compute the Stokes parameters for waves with linear polarisation at angle α in the x,y-plane as well as for both circular polarisations.

Interpret all Stokes parameters s_i in terms of their physical meaning. What does the case $s_1 = s_2 = s_3 = 0$ mean?

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9.5. Group velocity

A one-dimensional wave packet $\phi(x,t)$ is moving in a dispersive medium, i.e. the angular velocity $\omega(k)$ depends non-linearly on the wave number k. At time $t = 0$ it takes a Gaußian shape

$$\phi(x) = \exp\left(-\frac{x^2}{2(\Delta x)^2}\right), \tag{9.11}$$

where we consider $\Delta x > 0$ as a measure for the spatial extent of the wave packet. The time dependency is given by

$$\phi(x,t) = \mathrm{Re} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \tilde{\phi}(k)\, \mathrm{e}^{ikx - i\omega(k)t}\,, \tag{9.12}$$

where $\tilde{\phi}(k)$ is the Fourier transform of $\phi(x)$ at time $t = 0$.

a) Show by completing the square that the Fourier transformed wave packet at $t = 0$ has a Gaußian profile. What is the relation between Δx and the analogous Δk? What does this mean?

Hint:

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \sqrt{2\pi}\,|\sigma|. \tag{9.13}$$

b) Show that after a time t, the maximum of the wave packet in space covers a distance of $v_\mathrm{g} t$, where the *group velocity* v_g is given by

$$v_\mathrm{g} = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}. \tag{9.14}$$

Here, k_0 denotes the wave number at the maximum of $\tilde{\phi}(k)$.

Hint: Expand $\omega(k)$ to first order in k around k_0, and evaluate the change in the maximum of the wave packet using (9.12). Use furthermore (9.13).

c) What is the speed of the individual phases? Under which circumstances do phase velocity and group velocity of the wave coincide?

d) Estimate how fast the wave packet is widening by finding an expression for the variation of the group velocity inside the pulse. Use the relation between Δk and Δx from part a), and interpret the result accordingly.

Hint: Estimate the variation as the difference between the group velocities at k_0 and $k_0 + \Delta k$ (analoguously to (9.14)), and determine Δv_g from an expansion of $\omega(k)$ around k_0 up to the first contributing order.

Electrodynamics — Problem Set 10

ETH Zurich, 2024 FS — Prof. N. Beisert

due: 13. 05. 2024

10.1. Wave packets and Fourier space

In this exercise we will solve the explicit time evolution of a plane electromagnetic wave packet by going to Fourier space. The wave packet at $t = 0$ is specified by:

$$\vec{E}_0 = \frac{2A}{\sqrt{2\pi\sigma}} \exp\left(-\frac{z^2}{2\sigma}\right)\vec{e}_x, \qquad \vec{B}_0 = 0, \tag{10.1}$$

where $\sigma > 0$ describes its width and $A \in \mathbb{R}$ its amplitude.

a) Transform the wave packet to Fourier space.

An electromagnetic wave in Fourier space takes the general form:

$$\begin{aligned}
\vec{E} &= \int \frac{\mathrm{d}k^3}{(2\pi)^3} \left[\vec{\alpha}(\vec{k})\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \vec{\alpha}(\vec{k})^*\, \mathrm{e}^{-i\vec{k}\cdot\vec{x}+i\omega t}\right], \\
\vec{B} &= \int \frac{\mathrm{d}k^3}{(2\pi)^3} \left[\vec{\beta}(\vec{k})\, \mathrm{e}^{i\vec{k}\cdot\vec{x}-i\omega t} + \vec{\beta}(\vec{k})^*\, \mathrm{e}^{-i\vec{k}\cdot\vec{x}+i\omega t}\right],
\end{aligned} \tag{10.2}$$

where $\omega = \omega(\vec{k}) := \|\vec{k}\|c$ is the angular velocity associated to a given wave number $\vec{k}$.

b) Show that the wave packet in Fourier space matches the general form of an electromagnetic wave at $t = 0$ provided that $\vec{\alpha}$ and $\vec{\beta}$ are chosen as:

$$\begin{aligned}
\vec{\alpha} &= (2\pi)^2 A \exp\left(-\tfrac{1}{2}\sigma k_z^2\right)\delta(k_x)\delta(k_y)\,\vec{e}_x, \\
\vec{\beta} &= (2\pi)^2 \frac{A}{c} \operatorname{sign}(k_z) \exp\left(-\tfrac{1}{2}\sigma k_z^2\right)\delta(k_x)\delta(k_y)\,\vec{e}_y.
\end{aligned} \tag{10.3}$$

c) Verify that the free Maxwell equations in Fourier space are satisfied:

$$\vec{k}\cdot\vec{\alpha} = \vec{k}\cdot\vec{\beta} = 0, \qquad \vec{\beta} = \frac{\vec{k}\times\vec{\alpha}}{\omega}, \qquad \vec{\alpha} = -\frac{c^2\vec{k}\times\vec{\beta}}{\omega}. \tag{10.4}$$

d) Compute the electromagnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ as explicit expressions in position space and show that they satisfy the Maxwell equations.

e) Compute the electromagnetic energy density per unit area in the x,y-direction

$$\frac{\mathrm{d}^2 W}{\mathrm{d}^2 A} = \frac{\varepsilon_0}{2} \int \mathrm{d}z\, \left(\vec{E}^2 + c^2\vec{B}^2\right). \tag{10.5}$$

Convince yourself that it is independent of time.

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10.2. Elliptically polarised waves

A wave $\vec{E}(\vec{x},t)$ with the wave vector $\vec{k} = k\vec{e}_z$ is given by

$$\vec{E} = \begin{pmatrix} A\cos(kz-\omega t) \\ B\cos(kz-\omega t+\varphi) \\ 0 \end{pmatrix}. \tag{10.6}$$

a) The path of the vector $\vec{E}(0,t)$ describes the polarisation of the wave. Show that it traces out an ellipse. For which values of A, B and φ is it a circle?

Hint: The equation of an ellipse is given by

$$aE_x^2 + 2bE_xE_y + cE_y^2 + f = 0, \tag{10.7}$$

where $b^2 - ac > 0$ and $f < 0$.

b) Show that for general A and B the wave can be written as a superposition

$$\vec{E}(\vec{x},t) = \vec{E}_+(z,t) + \vec{E}_-(z,t) \tag{10.8}$$

of two opposite circularly polarised waves

$$E_\pm(z,t) = \mathrm{Re}(A_\pm \vec{e}_\pm\, \mathrm{e}^{\imath kz-\imath\omega t}) \tag{10.9}$$

with $\vec{e}_\pm := \vec{e}_x \pm \imath\vec{e}_y$ and the constants $A_\pm$. Determine $A_\pm$ in terms of A, B, and φ.

Hint: Write the electric field as the real part of a complex vector with the phase $\mathrm{e}^{\imath kz-\imath\omega t}$ for a complex plane wave and express $\vec{e}_x$ and $\vec{e}_y$ in terms of $\vec{e}_\pm$.

10.3. Energy and momentum flux of a plane wave

Let us consider a real monochromatic plane wave travelling in the positive z-direction which is linearly polarised in the x-direction and has the amplitude E_0.

a) Compute the Poynting vector $\vec{S}$, and show that the intensity $I := \langle\|S\|\rangle$ of the wave reads

$$I = \tfrac{1}{2}c\varepsilon_0 E_0^2. \tag{10.10}$$

b) The Maxwell stress tensor T is defined as

$$T_{ij} := \varepsilon_0\big(E_iE_j - \tfrac{1}{2}\delta_{ij}\vec{E}^2\big) + \frac{1}{\mu_0}\big(B_iB_j - \tfrac{1}{2}\delta_{ij}\vec{B}^2\big). \tag{10.11}$$

Compute T for the given plane wave. What does it imply for the momentum flux?

c) How are momentum flux density and energy flux density related in this case?

11.1. Radiation from a linear antenna

Consider a thin linear antenna of length $2d$ aligned with the z-axis and centred at the origin. The antenna is excited with an oscillating sinusoidal current whose wavelength equals the length of the antenna. Thus $\lambda = 2d$ and $\omega = \pi c/d$, and the amplitude of the current is I_0.

Hint: Note that this antenna differs from the one discussed in the lecture. There, the sinusoidal current is symmetric with respect to the origin, whereas here it is antisymmetric.

a) Show that the vector potential $A(x, t)$ has the form:

$$\vec{A}(x,t) = \mu_0 I_0 \, \vec{e}_z \, \mathrm{e}^{-\mathring{\imath}\omega t} \int_{-d}^{d} \mathrm{d}z' \, \frac{\sin(kz')}{4\pi r'} \, \mathrm{e}^{\mathring{\imath}kr'} \,, \tag{11.1}$$

where $r' = \|x - x'\|$ with $\vec{x}' = z'\vec{e}_z$ and $k = \omega/c$ is the wave number.

For the remainder of this problem we will work in the radiation zone where $r \gg 2d = \lambda$ with $r = \|x\|$. With the unit vector $\vec{n}$ in the direction of $\vec{x}$, one approximates in this zone:

$$\|x - x'\| = r - \vec{n}\cdot\vec{x}' + \ldots, \tag{11.2}$$

b) Compute an exact expression for the power radiated per unit solid angle and plot the angular distribution as a function of the polar angle ϑ.

Hint: You can use the following identity for computing the vector potential:

$$\int \mathrm{d}x \sin(ax) \, \mathrm{e}^{-\mathring{\imath}bx} = \frac{\mathrm{e}^{-\mathring{\imath}bx}}{b^2 - a^2} \big(a\cos(ax) + \mathring{\imath}b\sin(ax)\big). \tag{11.3}$$

Furthermore, it will be helpful to use an expression for the power radiated per unit solid angle in terms of the Poynting vector.

c) Consider now an expansion in multipoles,

$$\mathrm{e}^{\mathring{\imath}k\vec{n}\cdot\vec{x}'} = \sum_{m=0}^{\infty} \frac{(\mathring{\imath}k)^m}{m!} \, (\vec{n}\cdot\vec{x}')^m, \tag{11.4}$$

and keep only the leading term in k. Within this approximation compute the power radiated per unit solid angle and plot the angular distribution as a function of ϑ.

d) Compare the results obtained in part b) and part c). Discuss whether the approximation of the multipole expansion is valid or not, and argue why.

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11.2. Radiation in relativistic circular motion

A particle of charge q moves with constant speed $v \approx c$ in a circular orbit of radius R in the x,z-plane. At time t' it passes through the origin $\vec{x}' = 0$ with velocity $\vec{v} = v\vec{e}_z$ and is accelerated with $\vec{a} = a\vec{e}_x = (v^2/R)\vec{e}_x$.

The electric field at position $\vec{x}$ and time $t = t' + \|x - x'\|/c$ of a relativistic point charge at $\vec{x}'$ with velocity $\vec{v}$ and acceleration $\vec{a}$ is given by

$$\vec{E}(x,t) = \frac{q}{4\pi\varepsilon_0} \frac{\|x - x'\|}{\big((\vec{x} - \vec{x}')\cdot\vec{w}\big)^3} \left[(1 - \beta^2)\vec{w} + \frac{1}{c^2}(\vec{x} - \vec{x}')\times(\vec{w}\times\vec{a})\right], \tag{11.5}$$

where $\vec{w} := (\vec{x} - \vec{x}')/\|x - x'\| - \vec{v}/c$ and $\beta := v/c$.

a) Using spherical coordinates and ignoring terms in (11.5) that fall off faster than $1/r$, show that the electric field measured by an observer at $\vec{x} = r\vec{e}_r$ at large distance r is

$$\vec{E}_{\text{rad}}(x,t) = \frac{qa}{4\pi\varepsilon_0 c^2 r} \frac{(\beta - \cos\vartheta)\cos\varphi\,\vec{e}_\vartheta + (1 - \beta\cos\vartheta)\sin\varphi\,\vec{e}_\varphi}{(1 - \beta\cos\vartheta)^3}. \tag{11.6}$$

For fields of radiation, $\vec{E}_{\text{rad}}$ is perpendicular to $\vec{x} - \vec{x}'$, so the Poynting vector for the radiation component of the fields due to a point charge is

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}\vec{E}^2_{\text{rad}}\frac{\vec{x} - \vec{x}'}{\|x - x'\|}. \tag{11.7}$$

b) Using (11.6) and (11.7) show that the Poynting vector at the position of the observer is

$$\vec{S}_{\text{rad}} = \frac{\mu_0 q^2 v^4}{16\pi^2 c r^2 R^2} \frac{(1 - \beta\cos\vartheta)^2 - (1 - \beta^2)\cos^2\varphi\sin^2\vartheta}{(1 - \beta\cos\vartheta)^6}\vec{e}_r. \tag{11.8}$$

Let us now consider the total power radiated out towards infinity by the relativistic point charge as measured on a sphere concentric to the orbit

$$P = r^2 \oint \mathrm{d}^2\Omega\,\vec{e}_r\cdot\vec{S}_{\text{rad}}\,\frac{\vec{x}\cdot\vec{w}}{r}. \tag{11.9}$$

c) As the particle moves relativistically, the power at the observation point and the retarded time t_{ret} differs from the power leaving the particle at time t. Show that the retardation effects defined by the relation $c(t - t_{\text{ret}}) = \|x - x'\|$ explain the factor $(\partial t_{\text{ret}}/\partial t)^{-1} = \vec{x}\cdot\vec{w}/r$ in the above expression for the emitted power P.

d) Show that the total radiated power evaluates to

$$P = \frac{\mu_0 q^2 v^4}{6\pi c R^2 (1 - \beta^2)^2}. \tag{11.10}$$

e) Calculate the energy radiated to infinity during one complete circle. For an electron ($m_e = 511\,\mathrm{keV}/c^2$) travelling with $\beta = 0.8$ and $R = 20\,\mathrm{m}$, how much energy is radiated in one orbit? Compare this value to the total relativistic energy of the electron.

12.1. Liénard–Wiechert potential

Consider a charge moving straight along the positive z-axis with a uniform velocity v starting at $z = 0$ at $t = 0$. We will show that its potential is given by

$$\Phi(\vec{x}, t) = \frac{q}{4\pi\varepsilon_0\sqrt{(z - vt)^2 + (1 - v^2/c^2)(x^2 + y^2)}}\,. \tag{12.1}$$

a) Calculate the Liénard–Wiechert potential directly in the observer frame.

b) Calculate the potential first in the rest frame of the charge. Then transform it back into the frame of the observer.

12.2. Reflection at an ideal conductor

A monochromatic wave with electric field $\vec{E}_1 = \mathrm{Re}[E\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}\,\vec{e}_x]$ with $\omega = kc$ propagates in empty space $z < 0$ and is incident to a perfect conductor occupying the space $z > 0$.

a) Calculate the magnetic field of the incident wave from Maxwell's equations.

b) Find the fields of the reflected wave, $\vec{E}_2$ and $\vec{B}_2$, such that the total fields fulfil the boundary conditions $E_\parallel = 0$ and $B_\perp = 0$ at the surface of the conductor. Compute the surface current in the plane $z = 0$.

c) Compute the time-averaged pressure on the conductor due to the induced Lorentz force density. Compare your result to the radiation pressure of the wave.

Hint: Assume that $\theta(z < 0)\delta(z) = \frac{1}{2}\delta(z)$.

12.3. Refraction of planar waves

A planar wave is incident perpendicularly onto a planar layer between two media. The indices of refraction of the three non-magnetic layers are n_1, n_2 and n_3. The thickness of the central layer is d, while the other two media each fill half spaces.

a) Calculate the reflection and transmission coefficients (i.e. the ratio of the reflected and transmitted wave with the incoming energy flux).

Hint: The time-averaged energy-flux density of a complex wave is given by

$$\big\langle \vec{S} \big\rangle = \frac{1}{2\mu_0}\,\mathrm{Re}\big(\vec{E}\times\vec{B}^*\big). \tag{12.2}$$

b) Let the medium with index n_1 be part of an optical system (e.g. a lens), and the medium with index n_3 be air ($n_3 = 1$). The surface of the first medium should have a layer of the medium with index n_2 of such a thickness that for a given frequency ω_0, there is no reflected wave. Determine the thickness d and the index of refraction n_2 of this layer.

⟶

12.4. Scattering of light

Classical light-scattering theory (known as Rayleigh theory) is used to describe light being scattered off small molecules (with an extension much smaller than the wavelength λ of the light). Here we consider electric and magnetic fields and the intensity of light scattered off small particles.

a) First consider a plane monochromatic light wave propagating in the x-direction, which is polarised in the z-direction. This wave is scattered off a small polarisable, but non-magnetic particle at the origin. The incident wave induces a dipole moment to the particle, that is proportional to the local field, $\vec{p}(t) = \alpha \vec{E}(0,t)$, where α is its polarisability. Determine first the magnetic field and then the electric field of the scattered wave at a far-away point $\vec{x}$, depending on the incident field E_0, the distance from the origin r, and the angle ϑ between $\vec{x}$ and the z-axis.

b) Calculate the intensity of this scattered light wave, at a point $\vec{x}$ far away from the scattering particle

Hint: Use the Poynting vector.

c) Use the wave-length dependence of the intensity of the scattered wave ($\propto 1/\lambda^4$) derived in the previous subproblem, to explain qualitatively the blue colour of the cloudless sky and the red colour of the sunrise and the sunset.

13.1. Rectangular waveguide

Consider a waveguide extended infinitely along the z-axis with a rectangular basis $0 < x < d_x$ and $0 < y < d_y$. Its surfaces are ideal conductors. Due to the geometry of the problem, you can make the following ansatz for propagating electromagnetic waves,

$$\vec{E}_4(x,y,z,t) = \mathrm{Re}\big(\vec{E}_3(x,y)\,\mathrm{e}^{ikz-i\omega t}\big),$$
$$\vec{B}_4(x,y,z,t) = \mathrm{Re}\big(\vec{B}_3(x,y)\,\mathrm{e}^{ikz-i\omega t}\big). \tag{13.1}$$

a) The 3-vectors $\vec{E}_3, \vec{B}_3$ split into 2-vectors $\vec{E}, \vec{B}$ (here: x- and y-components) and scalars e, b (z-component). From the Maxwell equations, derive equations for the x- and y-components $\vec{E}, \vec{B}$ in terms of their z-components e, b, and show that the following equations hold for the z-components,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] e = 0,$$
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] b = 0. \tag{13.2}$$

b) Express the boundary conditions $\vec{E}_\parallel = B_\perp = 0$ as conditions for the z-components of the fields.

c) Determine the solutions for so-called transverse magnetic waves (TM-waves) with $b = 0$.

d) Show that no transverse electromagnetic waves (TEM-waves) with $e = b = 0$ exist in a rectangular waveguide.

Hint: Use Gauß's theorem and Faraday's law as well as the boundary condition for $\vec{E}_\parallel$ to show that there are no TEM-waves in this waveguide.

13.2. TEM-mode of the coaxial waveguide

An electromagnetic wave propagates as the TEM-mode along the z-direction between two coaxial cylindrical conductors with radii $R_2 > R_1$ centred at the z-axis.

a) Find the field configuration for the TEM-mode explicitly.

b) Is the frequency bounded for this mode? What is the dispersion relation?

c) Calculate the average transport of power along the cylinder.

⟶

13.3. Optics via the principle of least action

Fermat's principle states that light travelling between two points in space $\vec{x}_1$ and $\vec{x}_2$ takes the path that minimises the optical length. The latter is given by

$$S = \int_{\vec{x}_1}^{\vec{x}_2} n(\vec{x})\,\mathrm{d}\ell\,, \tag{13.3}$$

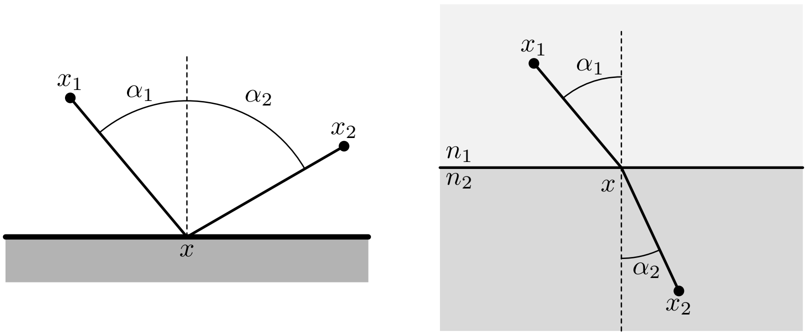
where $n(\vec{x})$ denotes the refractive index of the matter and $\mathrm{d}\ell = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ is the length of the infinitesimal element of the trajectory connecting $\vec{x}_1$ to $\vec{x}_2$. This is analogous to the principle of least action of the Lagrangian formalism.

Hint: It is convenient to parametrise the trajectory for this integral with a variable σ

$$S = \int_{\sigma_1}^{\sigma_2} n\big(\vec{x}(\sigma)\big)\,\frac{\mathrm{d}\ell}{\mathrm{d}\sigma}\,\mathrm{d}\sigma = \int_{\sigma_1}^{\sigma_2} n\big(\vec{x}(\sigma)\big)\sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}\sigma\,. \tag{13.4}$$

The choice of σ does not contribute to the integral and therefore you may identify it (locally) with one of the coordinates.

a) Find the trajectory of light between two points in a homogeneous medium.



(13.5)

b) Now consider light that is reflected from a plane mirror. The light travels in vacuum from point $\vec{x}_1$ to some point $\vec{x}$ on the surface of the mirror, and then, again in vacuum, to some point $\vec{x}_2$. Minimise the optical length over all positions of $\vec{x}$ on the mirror, and compare the incident and emergent angles for the chosen value of $\vec{x}$.

c) Finally, consider light propagating between two points in space which are located in different media with the refractive indices n_1 and n_2, respectively. The boundary surface between the two media is a plane. Consider a light path between point $\vec{x}_1$ in the medium with n_1 and $\vec{x}_2$ in the medium with n_2, which passes through the point $\vec{x}$ on the boundary surface. Choose $\vec{x}$ that minimises the total optical length. Find the relationship between incident and emergent angles (Snell's law).