5.4 Self-consistency equations for the superconducting gap

The anomalous correlation functions d_k and the superconducting gap Δ are determined from the self-consistency conditions (5.2.5), where the averages are calculated in the quadratic system (5.2.6) at a finite temperature T.

One of the possible ways to compute the anomalous average $\langle a_{-k\downarrow} a_{k\uparrow} \rangle$ is to re-express the a operators in terms of the quasiparticles γ and γ^+ and then use the equilibrium Fermi occupation numbers for the quasiparticles:

$$\begin{cases} \gamma_{k\uparrow}^+ = u_k a_{k\uparrow}^+ + v_k a_{-k\downarrow} \\ \gamma_{-k\downarrow} = v_k^* a_{k\uparrow}^+ - u_k^* a_{-k\downarrow} \end{cases} \quad \Rightarrow \quad \begin{cases} a_{k\uparrow}^+ = u_k^* \gamma_{k\uparrow}^+ + v_k \gamma_{-k\downarrow} \\ a_{-k\downarrow} = v_k^* \gamma_{k\uparrow}^+ - u_k \gamma_{-k\downarrow} \end{cases} \tag{5.4.1}$$

In terms of the quasiparticles $\gamma_{k\uparrow}^+$ and $\gamma_{-k\downarrow}$, the BCS Hamiltonian is diagonal, so we find

$$\langle a_{-k\downarrow} a_{k\uparrow} \rangle_T = v_k^* u_k \langle \gamma_{k\uparrow}^+ \gamma_{k\uparrow} - \gamma_{-k\downarrow} \gamma_{-k\downarrow}^+ \rangle_T = v_k^* u_k \left[2 n_F(\tilde{\varepsilon}_k) - 1 \right] = -v_k^* u_k \tanh \frac{\tilde{\varepsilon}_k}{2T} , \tag{5.4.2}$$

where $\tilde{\varepsilon}_k$ is the quasiparticle energy given by Eq. (5.3.4).

Substituting this into Eq. (5.2.7), we find the self-consistency equation for the gap

$$\Delta = \frac{g_0}{\mathcal{V}} \sum_k v_k^* u_k \tanh \frac{\tilde{\varepsilon}_k}{2T} . \tag{5.4.3}$$

Using Eq. (5.3.11) for u_k and v_k, we find

$$v_k^* u_k = \frac{\Delta}{2\tilde{\varepsilon}_k} . \tag{5.4.4}$$

Note that this quantity is significant only in the vicinity of the Fermi surface (since far away form the Fermi surface either u_k or v_k tends to zero).

We remark that $\Delta = 0$ is always a formal solution to the equations (5.4.3)–(5.4.4). But one can show that at low temperatures this solution does not correspond to a minimum of a free energy, but to its maximum. In other words, at low temperatures the $\Delta = 0$ solution is unstable, and the physically relevant solution is a nontrivial one. To find this nontrivial solution, we divide the equation by Δ and replace the sum over k by integration over energies:

$$\frac{1}{\mathcal{V}} \sum_k \quad \to \quad \nu_0 \int d\varepsilon , \tag{5.4.5}$$

where ν_0 is the density of electronic states (for free electrons) per unit volume and per spin projection and ε is the free-electron energy. Substituting equation (5.3.4) for $\tilde{\varepsilon}_k$ and shifting the integration variable to $\varepsilon = \varepsilon_k - \mu$, we finally find the self-consistency equation in the closed form

$$1 = g_0 \nu_0 \int d\varepsilon \frac{\tanh \frac{\sqrt{\varepsilon^2 + |\Delta|^2}}{2T}}{2\sqrt{\varepsilon^2 + |\Delta|^2}} . \tag{5.4.6}$$

This equation, in principle allows to determine Δ as a function of temperature (see Fig. 20).

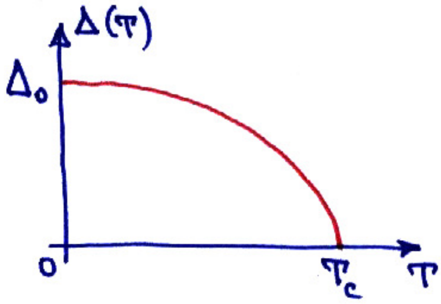

Figure 20: A sketch of the gap dependence on the temperature.

5.5 Superconducting gap at zero temperature

A subtle point in this calculation is that the integral (5.4.6) actually diverges logarithmically at large ε. Physically, this divergence is removed by introducing a cut-off at energies of the order of Debye energy ω_D (since the attraction mediated by phonons only extends to those energies).

At zero temperature, $\tanh(\ldots) \to 1$, and the equation (5.4.6) reduces to

$$1 = g_0\nu_0 \int_0^{\sim\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta_0^2}} = g_0\nu_0 \left[\ln \frac{\omega_D}{\Delta_0} + \text{const}\right] , \tag{5.5.1}$$

where const is a constant of order one. This gives the superconducting gap at zero temperature Δ_0 in the form

$$\Delta_0 = \text{const}\ \omega_D \exp\left(-\frac{1}{g_0\nu_0}\right) . \tag{5.5.2}$$

Note that the gap is exponentially small in g_0.

5.6 Superconducting transition temperature

In a similar way we can find the superconducting transition temperature T_c, with the only difference that now we neglect Δ in the self-consistency equation (5.4.6):

$$1 = g_0\nu_0 \int_0^{\sim\omega_D} d\varepsilon \frac{\tanh\frac{\varepsilon}{2T_c}}{\varepsilon} = g_0\nu_0 \left[\ln \frac{\omega_D}{T_c} + \text{const}\right] , \tag{5.6.1}$$

with some const of order one (but different from that in the calculation of Δ_0 above!). In other words, T_c is of the same order of magnitude as Δ_0.

Remarkably, one can determine the ratio T_c/Δ_0 without any ambiguity related to the cutoff. Namely, the difference of the integrals (5.5.1) and (5.6.1) is convergent and does not depend on the cut-off:

$$0 = \int_0^\infty d\varepsilon \left[\frac{\tanh\frac{\varepsilon}{2T_c}}{\varepsilon} - \frac{1}{\sqrt{\varepsilon^2 + \Delta_0^2}}\right] = \int_0^\infty dx \left[\frac{\tanh(x/2)}{x} - \frac{1}{\sqrt{x^2 + (\Delta_0/T_c)^2}}\right] . \tag{5.6.2}$$

From this equation, one finds the *universal* value for the ratio T_c/Δ_0:

$$T_c \approx 0.57\Delta_0\,. \tag{5.6.3}$$

[This value is easy to obtain by numerical methods. A more sofisticated analytic calculation gives

$$T_c = \left(\frac{e^C}{\pi}\right)\Delta_0\,, \tag{5.6.4}$$

where $C = 0.577\ldots$ is the Euler constant].

The relation (5.6.3) is in a remarkably good agreement with experimental values on many conventional superconductors, despite the simplifications made in the BCS theory.