3.4 Fermi liquid theory

Refs: [AM] Chapter 17, [PC], Chapter 7.

Landau Fermi liquid theory was introduced to describe low-energy degrees of freedom of a Fermi gas with interactions in a non-perturbative way (to complement the perturbative diagrammatic approach). It was originally introduced for ^{3}He, but can also be applied to electrons in metals.

The main idea of the Fermi liquid theory is that, in a Fermi gas, even in the presence of interactions, the low-lying structure of excitations is the same as in the non-interacting one. The only two differences are:

- The elementary excitations (named *quasiparticles*) interact with each other;
- Quasiparticles are no longer given by the creation and annihilation operators for physical particles.

We first formulate the theory and discuss some of its consequences and then comment in more detail on its justification and on its relation to the diagrammatic approach.

3.4.1 Interaction of quasiparticles

The main physical assumption of the Fermi liquid theory is that the low-lying states of the interacting system are adiabatically connected to the noninteracting one (i.e., if we continuously switch on the interaction, the spectrum is also modified continuously and smoothly). Then the resulting excitation spectrum may be parametrized by the same occupation numbers n_k as the original one, provided the state is close to the ground state. The theory is restricted to low excitation energies and to low temperatures. In the ground state, the occupation numbers $n_k^{(\mathrm{GS})}$ are one and zero below and above the Fermi surface, respectively. We can express the energy of the eigenstates of the interacting system in terms of the deviations of the occupation numbers from the ground-state values $\delta n_k = n_k - n_k^{(\mathrm{GS})}$. Furthermore, at low energies we can expand the total energy of the system with respect to δn_k. To the second order, the expansion reads

$$E/V = \int \frac{d^3k}{(2\pi)^3}\, \varepsilon_k^{(0)}\, \delta n_k + \frac{1}{2} \iint \frac{d^3k}{(2\pi)^3}\, \frac{d^3k'}{(2\pi)^3} f_{kk'}\, \delta n_k \delta n_{k'}\,. \tag{3.4.1}$$

The linear coefficients $\varepsilon_k^{(0)}$ are the quasiparticle energies (with the Fermi energy subtracted), and the quadratic coefficients $f_{kk'}$ describe pairwise interactions between quasiparticles.

The quasiparticle energies $\varepsilon_k^{(0)}$ are renormalized with repsect to the bare spectrum of electrons. We can introduce the Fermi velocity of quasiparticles

$$v_F = \left.\frac{\partial \varepsilon_k^{(0)}}{\partial k}\right|_{k_F}\,. \tag{3.4.2}$$

Furthermore, it is convenient to introduce the effective mass m^* as

$$m^* = k_F/v_F\,. \tag{3.4.3}$$

Note that this definition of the effective mass differs from the effective mass in semiconductors: in the latter case the effective mass describes the spectrum curvature at the bottom (or top) of the band, while in our case, it is related to the renormalization of the Fermi velocity at the Fermi surface.

If we consider a spinful case (e.g., electrons), then the occupation numbers n_k are, in fact, 2×2 matrices with spin indices, $n_{k;\alpha\beta} = \langle a^+_{k;\alpha} a_{k;\beta}\rangle$. The f-function, in turn, also acquires spin structure,

$$E_{\text{int}}/V = \frac{1}{2}\iint \frac{d^3k}{(2\pi)^3}\,\frac{d^3k'}{(2\pi)^3} f_{kk';\alpha\gamma,\beta\delta}\,\delta n_{k;\alpha\beta}\delta n_{k';\gamma\delta}\,. \tag{3.4.4}$$

Note that we had already encounter a similar structure in the Hatree–Fock approximation, where the exchange term also had a spin structure. If we further assume that the interaction is inariant with respect to the overall spin rotation (which is the case for exchange-type interaction), then this restricts the f function to

$$f_{kk';\alpha\gamma,\beta\delta} = \frac{\pi^2}{k_F m^*}\left[F_{kk'}\,\delta_{\alpha\beta}\delta\gamma\delta + G_{kk'}\,\sigma_{\alpha\beta}\sigma_{\gamma\delta}\right]\,, \tag{3.4.5}$$

where

$$\nu = \frac{m^* k_F}{\pi^2} \tag{3.4.6}$$

is the density of states of quasiparticles (including spin) at the Fermi surface. With this normalization, the coefficients G and F are dimensionless.

If the Fermi liquid is isotropic (which we will assume further for simplicity), then the Fermi surface is spherical and both $F_{kk'}$ and $G_{kk'}$ depend only on the angle ϑ between k and k'. In this model, the effective mass m^* is constant along the Fermi surface.

3.4.2 Relation between effective mass and interaction in the Galilean-invariant case

In the Galilean-invariant case (invariance with respect to the choice of an inertialreference frame), there is a relation between the mass renormalization m^* and the interaction parameter $F(\vartheta)$. This Galilean invariance is satisfied in liquid ^{3}He, but not for electrons in metals.

The relation between m^* and $F(\vartheta)$ follows from the condition that the total momentum coincides with the physical mass current. We will omit here the derivation [see, e.g., the book of Lifshitz and Pitaevskii, *Statistical Physics Part 2* (volume 9 of the Course of Theoretical Physics)] and only mention the result:

$$\frac{m^*}{m} = 1 + \int \frac{d\Omega}{4\pi}\,F(\vartheta)\cos\vartheta\,, \tag{3.4.7}$$

where $d\Omega$ denotes integration over the solid angle.

3.4.3 Occupation numbers of quasiparticles

Adding or removing a quasiparticle at a wave vector k changes the total energy of the system by the *energy of the quasiparticle*,

$$\tilde{\varepsilon}_k = \varepsilon^{(0)}_k + \int \frac{d^3k'}{(2\pi)^3} f_{kk'}\,\delta n_{k'}\,. \tag{3.4.8}$$

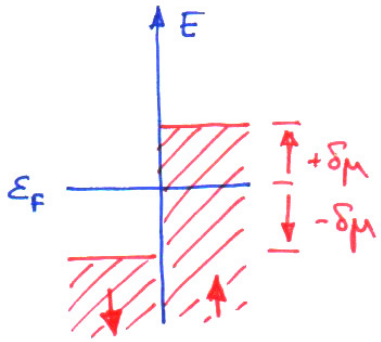

Figure 15: The shifts of the chemical potential for spin-up and spin-down quasiparticles in an externsal magnetic field.

Therefore, at a finite temperature T, the occupation numbers of *quasiparticles* obeys the usual Fermi distribution:

$$n_k = \frac{1}{e^{\tilde{\varepsilon}_k/T} + 1}\,. \tag{3.4.9}$$

Note that this relation is, in fact, a complicated self-consistent condition on n_k, since $\tilde{\varepsilon}_k$ in the right-hand side is itself a functional of n_k.

3.4.4 Renormalization of the specific heat

For the specific heat $C_V = (\partial E/\partial T)$, the first term in (3.4.1) gives the leading contribution at low temperatures. As a result, the specific heat is given by the same expression as for a free Fermi gas, but with the renormalized mass:

$$C_V = V\,\frac{m^* k_F T}{3} = V\,\frac{\pi^2}{3}\nu T\,. \tag{3.4.10}$$

The interaction term can be shown to give a higher-order (in T) contribution to C_V.

3.4.5 Renormalization of the spin succeptibility

The magnetic spin succeptibility is also renormalized. Indeed, without interaction, in a magnetic field, the spin-up and spin-down electrons acquire opposite shifts in energy, and thus the total spin of the gas is proportional to the external field. In a Fermi liquid, one also needs to take interaction into account.

In an external magnetic field $\mathbf{H}$, the energy shift of an electron (and of a quasiparticle) is

$$\delta\varepsilon = -\beta\sigma\mathbf{H}\,, \tag{3.4.11}$$

where β is the magnetic moment of an electron (the same as of a quasiparticle). Without loss of generality, we assume that the field is applied along the z direction. Then the chemical potential of spin-up qasipraticles is shifted by some value $\delta\mu$ and that of spin-down quasiparticles is shifted by $-\delta\mu$ (see Fig. 15). This shift of the chemical potential is determined from minimizing the total energy

$$E_{\rm tot}/V = \nu\,\frac{(\delta\mu)^2}{2} + \langle G_{kk'}\rangle\,\nu\,\frac{(\delta\mu)^2}{2} - \nu\,\delta\mu\,\beta H\,. \tag{3.4.12}$$

Here the first two terms come from the two terms in (3.4.1) and $\langle G_{kk'} \rangle$ is the G function in (3.4.5) averaged over the Fermi surface:

$$\langle G_{kk'} \rangle = \frac{4}{\nu^2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} G_{kk'} \, \delta(\varepsilon_k) \, \delta(\varepsilon_{k'}) \, . \tag{3.4.13}$$

By minimizing (3.4.12) with respect to $\delta\mu$ gives

$$\delta\mu = \frac{\beta H}{1 + \langle G_{kk'} \rangle} \, , \tag{3.4.14}$$

so the magnetic succeptibility is

$$\chi = \beta\nu \, \frac{\delta\mu}{H} = \frac{\beta^2 \nu}{1 + \langle G_{kk'} \rangle} \, . \tag{3.4.15}$$

The interaction affects the renormalization of ν (the same renormalization as in the specific heat (3.4.10)) and also produces the term $\langle G_{kk'} \rangle$ in the denominator of (3.4.15).

3.4.6 Renormalization of the electron spectral weight

The quasiparticles carry the same charge and the same spin as electrons, but are not identical to electrons: they contain not only one-electron component, but also three- and more electron components. Some of the properties of the electron gas (e.g. STM tunneling) are expressed in terms of single-electron operators, and for them the overlap between electrons and quasiparticles is important.

We can denote this overlap by

$$Z_k = \left| \langle k | a_k^+ | \mathrm{GS} \rangle \right|^2 \tag{3.4.16}$$

for the quasiparticles $|k\rangle$ above the Fermi surface and

$$Z_k = \left| \langle k | a_k | \mathrm{GS} \rangle \right|^2 \tag{3.4.17}$$

for the quasiparticles below the Fermi surface. The spectral weights Z_k are real numbers between 0 and 1.

We can further relate these spectral weights to the residues of the Green's function. By inserting the full basis of quasiparticle states between the electron creation/annihilation operators in (3.2.28), we find

$$G^c(\omega, k) = \frac{Z_k}{\omega - \varepsilon_k^{(0)} + i\delta \, \mathrm{sign}\, \omega} + \text{regular part} \tag{3.4.18}$$

(note that $\varepsilon_k^{(0)}$ already has the chemical potential subtracted, in our notation). In other words, Z_k expresses the renormalization of the residue of the Green's function in its pole.

By comparing this expression with the self-energy correction (3.2.43), we can identify:

$$\varepsilon_k^{(0)} = \varepsilon_k - \mu + \mathrm{Re}\, \Sigma_{\varepsilon_k^{(0)}}(k) \tag{3.4.19}$$

(renormalization of the spectrum) and

$$Z_k = \left(1 - \frac{\partial \Sigma_\omega(k)}{\partial \omega}\Big|_{\omega=\varepsilon_k^{(0)}}\right)^{-1} \tag{3.4.20}$$

(renormalization of the spectral weight).

An important assumption in this identification is the absence of the imaginary part of the self energy,

$$\mathrm{Im}\, \Sigma_{\varepsilon_k^{(0)}}(k) = 0\,. \tag{3.4.21}$$

In fact, $\mathrm{Im}\, \Sigma_{\varepsilon_k^{(0)}}(k)$ defines the decay rate of the quasiparticle excitation. Neglecting the decay of quasiparticles is one of the key assumptions of the Fermi liquid theory. More precisely, one assumes that the decay rate of the quasiparticle is much smaller than their energy,

$$|\,\mathrm{Im}\, \Sigma_{\varepsilon_k^{(0)}}(k)| \ll \varepsilon_k^{(0)}\,. \tag{3.4.22}$$

One can see (either from the diagrammatic analysis of $\Sigma_{\varepsilon_k^{(0)}}(k)$ or from the Fermi golden rule) that the decay of quasiparticles is given by available decay processes (which conserve both momentum and energy). Up to the energy-momentum dependence of the corresponding matrix elements, this rate is proportional to the phase space volume of available quasipaticles to decay into. One can show that, for a generic Fermi surface (without nesting), such a phase-space volume scales as $[\varepsilon_k^{(0)}]^2/\varepsilon_F$. This justifies the Fermi liquid theory at energies and temperatures much lower than ε_F.

Problem Set 9

Problem 9.1

(a) Consider a Fermi liquid in an empty space with the Green's function given by (3.4.18). Consider the momentum distribution of *physical electrons*

$$n_k^e = \langle a_k^+ a_k \rangle \,. \tag{3.4.23}$$

Show that the jump of n_k^e at the Fermi surface equals Z_k (see Fig. 16):

$$\lim_{\delta k \to +0} (n^e_{k_F - \delta k} - n^e_{k_F + \delta k}) = Z_{k_F} \,. \tag{3.4.24}$$

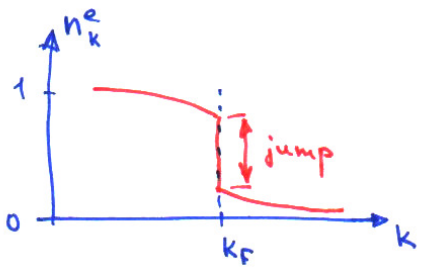

Figure 16: The jump in the momentum distribution of electrons in a Fermi liquid.

(b) In the experimental paper S. Huotari et al, Phys. Rev. Lett. **105**, 086403 (2010), the jump of n_k^e is measured in sodium (Na). Note that, in the presence of a crystal potential, the relation between the jump of n_k^e and Z_k is more complicated:

$$\lim_{\delta k \to +0} (n^e_{k_F - \delta k} - n^e_{k_F + \delta k}) = |\Psi_k^{(0)}|^2 Z_k \,, \tag{3.4.25}$$

where $\Psi_k^{(0)}$ is the amplitude of the plane wave with the wave vector k in the Bloch state Ψ_k:

$$\Psi_k^{(0)} = \langle e^{-ikx} \Psi_k \rangle \tag{3.4.26}$$

(with the average taken over a unit cell of the lattice). Explain this formula.

Problem 9.2*

Show that there is no contribution to the specific heat from the interaction of quasi-particles, to the leading order in T.