√Ex. 4. Maximally helicity-violating (MHV) amplitudes in MRK

Amplitudes in MRK can also be evaluated as helicity amplitudes. The method is less general, because for an arbitrary number of partons it is limited to (N)MHV configurations, but it is much more compact so it is worth examining it.

We shall consider the Parke-Taylor multi-gluon amplitudes in the MHV configuration $(--+\ldots++)$ whose convention is to take the momenta as all outgoing.

Then, for $n+2$ produced partons, momentum conservation is

$$P_a + P_b + \sum_{i=0}^{n+1} P_i = 0$$

In light-cone coordinates, that is:

$$\begin{cases} P_a^+ = -\sum_{i=0}^{n+1} P_i^+ \\ P_b^- = -\sum_{i=0}^{n+1} P_i^- \\ \sum_{i=0}^{n+1} P_{i_\perp} = 0 \end{cases}$$

We consider massless Dirac spinors $\not{p}\,\psi(p) = 0$ and in particular their chiral projection $\psi_\pm(p) = \frac{1 \pm \gamma_5}{2}\,\psi(p)$

We use the usual conventional shorthand

$$\psi_\pm(p) = |p\pm\rangle \qquad \overline{\psi_\pm(p)} = \langle p\pm|$$

Using the chiral representation of the γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

and the normalization condition $\langle p\pm|\gamma^\mu|p\pm\rangle = 2p^\mu$

the spinors are

(up to an overall phase)

$$\psi_+(p_i) = \begin{pmatrix} \sqrt{p_i^+} \\ \frac{p_{\perp i}}{|p_{\perp i}|}\sqrt{p_i^-} \\ 0 \\ 0 \end{pmatrix}; \quad \psi_-(p_i) = \begin{pmatrix} 0 \\ 0 \\ \frac{p_{\perp i}^*}{|p_{\perp i}|}\sqrt{p_i^-} \\ -\sqrt{p_i^+} \end{pmatrix}$$

(please check it)

$$\Psi_+(p_e) = i \begin{pmatrix} \sqrt{-p_e^+} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Psi_-(p_e) = i \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{-p_e^+} \end{pmatrix}$$

$$\Psi_+(p_b) = -i \begin{pmatrix} 0 \\ \sqrt{-p_b^-} \\ 0 \\ 0 \end{pmatrix} \qquad \Psi_-(p_b) = -i \begin{pmatrix} 0 \\ 0 \\ \sqrt{-p_b^-} \\ 0 \end{pmatrix}$$

The spinors of incoming partons must be continued to negative energy after complex conjugation, so e.g. $\overline{\Psi_+(p_e)} = i\,(0 \;\; 0 \;\; \sqrt{-p_e^+} \;\; 0)$

$$\overline{\Psi_-(p_e)} = i\,(0 \;\; -\sqrt{-p_e^+} \;\; 0 \;\; 0)$$

The spinor products are defined as

$$\langle p k \rangle = \langle p- | k+ \rangle = \overline{\Psi_-(p)}\,\Psi_+(k)$$

$$[p k] = \langle p+ | k- \rangle = \overline{\Psi_+(p)}\,\Psi_-(k)$$

Then, with the explicit spinor representation above, we get:

$$\langle p_i p_j \rangle = p_{i\perp} \sqrt{\frac{p_j^+}{p_i^+}} - p_{j\perp} \sqrt{\frac{p_i^+}{p_j^+}}$$

$$\langle p_a p_i \rangle = -i \sqrt{\frac{-p_a^+}{p_i^+}}\, p_{i\perp}$$

$$\langle p_i p_b \rangle = i \sqrt{-p_b^- p_i^+}$$

$$\langle p_a p_b \rangle = -\sqrt{\hat{s}}$$

where we have used the mass-shell condition

$$|p_{i\perp}|^2 = p_i^+ p_i^-$$

It is easy to check the identities

$$\langle ij \rangle = -\langle ji \rangle$$

$$[ij] = -[ji]$$

$$\langle ij \rangle^* = \mathrm{sign}(p_i^0 p_j^0)[ji]$$

$$\langle ij \rangle [ji] = 2 p_i \cdot p_j = \hat{s}_{ij}$$

$$\langle i+ | \not{k} | j+ \rangle = [ik]\langle kj \rangle$$

$$\langle i- | \not{k} | j- \rangle = \langle ik \rangle [kj]$$

Furthermore, for the gluon polarization we use the representation

$$\epsilon_\mu^\pm(p,k) = \pm \frac{\langle p\pm | \gamma_\mu | k\pm \rangle}{\sqrt{2}\langle k\mp | p\pm \rangle} \qquad k = \text{arbitrary light like momentum}$$

with properties

$$\epsilon_\mu^{\pm *}(p,k) = \epsilon_\mu^\mp(p,k)$$

$$\epsilon_\mu^\pm(p,k)\cdot p = \epsilon_\mu^\pm(p,k)\cdot k = 0$$

and

$$\sum_{\lambda=\pm} \epsilon_\mu^\lambda(p,k)\,\epsilon_\rho^{\lambda *}(p,k) = -g_{\mu\rho} + \frac{p_\mu k_\rho + p_\rho k_\mu}{p\cdot k}$$

which is equivalent to use a physical gauge.

With the explicit spinor representation we introduced above, the gluon polarizations we choose are

$$\epsilon_\mu^+(p_i,p_a) = -\frac{p_{i_\perp}^*}{p_{i_\perp}}\left(\frac{\sqrt{2}\,p_{i_\perp}}{p_i^-}, 0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)$$

$$\epsilon_\mu^+(p_i,p_b) = \left(0, \frac{\sqrt{2}\,p_{i_\perp}^*}{p_i^+}, \frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right)$$

(please check it!)

$$\epsilon_\mu^+(p_a,p_b) = \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right)$$

$$\epsilon_\mu^+(p_b,p_a) = -\left(0, 0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)$$

An n-gluon helicity amplitude at tree level can be colour decomposed as

$$M_n = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{tr}\left(T^{\sigma_1} \cdots T^{\sigma_n}\right) A\left(g_{\sigma_1}, \ldots, g_{\sigma_n}\right)$$

where S_n/Z_n are non-cyclic permutations of n elements.

The T's are the colour matrices in the fundamental repr.
We shall use the normalization $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$.
The colour coefficients $A(g_{\sigma_1}, \ldots, g_{\sigma_n})$ are gauge invariant and depend on momenta and helicities of the n gluons

An alternative color decomposition of the n-gluon tree-level amplitude is

(Frizzo, Maltoni, VDD 99)

$$M_n = \frac{(ig)^{n-2}}{2} \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma_2} c_1} f^{c_1 a_{\sigma_3} c_2} \cdots f^{c_{n-3} a_{\sigma_{n-1}} a_n} A(g_1, g_{\sigma_2}, \ldots, g_{\sigma_{n-1}}, g_n)$$

The advantage is that we deal with $(n-2)!$ permutations only.

For the MHV configuration, $(--+\cdots+)$, we have

$$A(g_1, \ldots, g_n) = 2^{n/2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle (n-1) n \rangle \langle n1 \rangle}$$

with i, j the gluons of negative helicity

Now, we shall evaluate the MHV amplitude in MRK

In MRK, the spinor products become

$$\langle p_i p_j \rangle \simeq -p_{j\perp}\sqrt{\frac{p_i^+}{p_j^+}} \qquad \text{for } p_i^+ \gg p_j^+$$

$$\langle p_a p_b \rangle = -\sqrt{p_0^+ p_{n+1}^-}$$

$$\langle p_a p_i \rangle = -i\sqrt{\frac{p_0^+}{p_i^+}}\, p_{i\perp} \qquad \text{(please check it)}$$

$$\langle p_i p_b \rangle = i\sqrt{p_i^+ p_{n+1}^-}$$

We choose A and B as the gluons of negative helicity, and suppose to have $n+4$ gluons $(A^-, 0^+, \ldots, (n+1)^+, B^-)$

We evaluate firstly the colour configuration $[A, 0, \ldots, n+1, B]$

$$\frac{\langle p_a p_b \rangle^4}{\langle p_a p_0 \rangle \langle p_0 p_1 \rangle \cdots \langle p_{n+1} p_b \rangle \langle p_b p_a \rangle}$$

in MRK, it becomes

$$\frac{\hat{s}^2}{-i\, p_{0_\perp} (-p_{1_\perp}) \sqrt{\frac{p_0^+}{p_1^+}} (-p_{2_\perp}) \sqrt{\frac{p_1^+}{p_2^+}} \cdots (-p_{n+1_\perp}) \sqrt{\frac{p_n^+}{p_{n+1}^+}}\; i\sqrt{p_{n+1}^+ p_{n+1}^-}\, \sqrt{\hat{s}}}$$

$$= (-1)^{n+1} \frac{\hat{s}}{p_{0_\perp} p_{1_\perp} \cdots p_{n+1_\perp}}$$

It is easy to see that any permutation of the $n+2$ gluons is power suppressed with respect to the one we just computed.

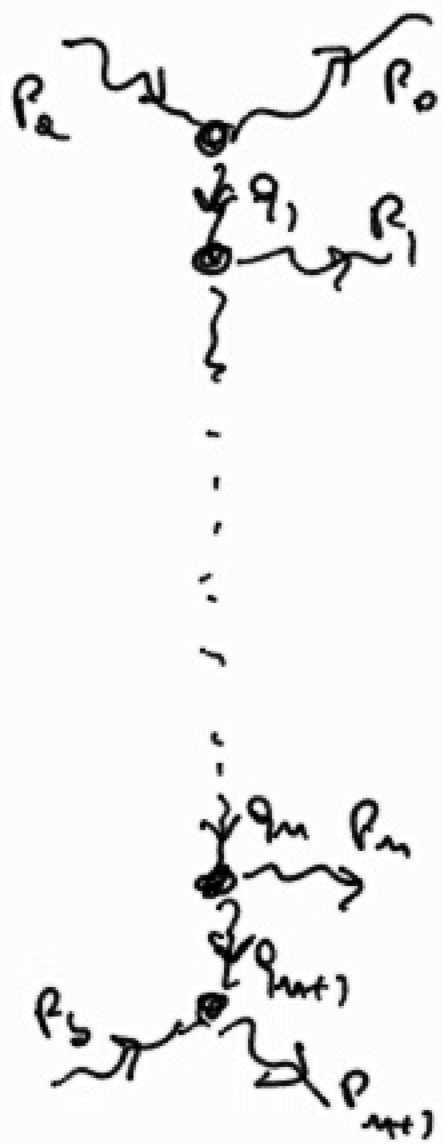
Using the DFM colour decomposition, the MHV amplitude in MRK immediately becomes

$$M(g_A^-, g_1^+, \ldots, g_{n+1}^+, g_B^-)$$

$$= (-1)^{n+1} 2^{\frac{n+4}{2}} \frac{\hat{s}}{\prod_{i=0}^{n+1} p_{i_\perp}} \frac{(ig)^{n+2}}{2} f^{a d_0 c_1} f^{c_1 d_1 c_2} \cdots f^{c_{n+1} d_{n+1} b}$$

$$= 2(-1)^{n} 2^{n/2} (ig)^{n+2} \frac{\hat{s}}{\prod_{i=0}^{n+1} p_{i_\perp}} f^{a d_0 c_1} f^{c_1 d_1 c_2} \cdots f^{b d_{n+1} c_{n+1}}$$

The amplitude for $gg \to ggg$ scattering we computed in Ex. 3 generalizes to the production of $n+2$ partons (Fadin, Kuraev, Lipatov 1977)



$$
\begin{aligned}
M^{a b d_0 \dots d_{n+1}}_{\lambda_a \lambda_0 \dots \lambda_{n+1} \lambda_b} = \; & 2\hat{s} \left(i g f^{a d_0 c_1} \Gamma^{\mu_a \mu_0} \right) \epsilon^{\lambda_a *}_{\mu_a}(p_a)\, \epsilon^{\lambda_0}_{\mu_0}(p_0) \\
& \frac{1}{\hat{t}_1} \left(i g f^{c_1 d_1 c_2} C^{\mu_1}(q_1, q_2) \right) \epsilon^{\lambda_1}_{\mu_1}(p_1) \\
& \vdots \\
& \frac{1}{\hat{t}_n} \left(i g f^{c_n d_n c_{n+1}} C^{\mu_n}(q_n, q_{n+1}) \right) \epsilon^{\lambda_n}_{\mu_n}(p_n) \\
& \frac{1}{\hat{t}_{n+1}} \left(i g f^{b d_{n+1} c_{n+1}} \Gamma^{\mu_b \mu_{n+1}} \right) \epsilon^{\lambda_{n+1}}_{\mu_{n+1}}(p_{n+1})\, \epsilon^{\lambda_b *}_{\mu_b}(p_b)
\end{aligned}
$$

with

$$
\Gamma^{\mu_a \mu_0} = g^{\mu_a \mu_0} - \frac{p_a^{\mu_0} p_b^{\mu_a} + p_b^{\mu_0} p_a^{\mu_a}}{p_a \cdot p_b} - \hat{t}_1 \frac{p_b^{\mu_0} p_b^{\mu_a}}{2 (p_a \cdot p_b)^2}
$$

$$
\Gamma^{\mu_b \mu_{n+1}} = g^{\mu_b \mu_{n+1}} - \frac{p_a^{\mu_b} p_b^{\mu_{n+1}} + p_a^{\mu_{n+1}} p_{n+1}^{\mu_b}}{p_a \cdot p_b} - \hat{t}_{n+1} \frac{p_a^{\mu_{n+1}} p_a^{\mu_b}}{2 (p_a \cdot p_b)^2}
$$

$$C^\mu(q_i, q_{i+1}) = (q_i + q_{i+1})_2^\mu + \left(\frac{\hat{S}_{bi}}{\hat{S}} + 2\,\frac{\hat{t}_i}{\hat{S}_{ai}}\right) P_a^\mu - \left(\frac{\hat{S}_{ai}}{\hat{S}} + \frac{2\hat{t}_{i+1}}{\hat{S}_{bi}}\right) P_b^\mu$$

and with

$$q_1 = P_a - P_0$$

$$q_2 = q_1 - P_1$$

$$\vdots$$

$$q_{n+1} = q_n - P_n = P_{n+1} - P_b$$

the Γs are the helicity-conserving vertices on the sides, and the Cs are the vertices for the emission of the gluons along the ladder, and $\hat{t}_i = q_i^2 = -|q_{i_\perp}|^2 = -q_{i_\perp} q_{i_\perp}^*$

When we contract with the gluon polarization vectors of the MHV configuration, we obtain

$$\Gamma^{\mu_a \mu_0} \epsilon^{+*}_{\mu_a}(p_a, p_b) \, \epsilon^{+}_{\mu_0}(p_0, p_b) = 1$$

$$\Gamma^{\mu_b \mu_{n+1}} \epsilon^{+*}_{\mu_b}(p_b, p_a) \, \epsilon^{+}_{\mu_{n+1}}(p_{n+1}, p_a) = \frac{p^*_{n+1_\perp}}{p_{n+1_\perp}}$$

(note that here A and B are incoming, so their helicities are reversed)

$$C(q_i, q_{i+1}) \cdot \epsilon^*(p_i, p_a) = \sqrt{2} \, \frac{q^*_{i_\perp} q_{i+1_\perp}}{p_{i_\perp}}$$

$p_i = q_i - q_{i+1}$

we note that the simplicity of the C vertex we knew from its square is manifest at fixed helicities

So the FKL amplitude in the MHV configuration becomes

$$M(--+\cdots+) = 2\hat{s}\,(ig)^{n+2} f^{a d_0 c_1} f^{c_1 d_1 c_2} \cdots\cdots f^{b d_{n+1} c_{n+1}}$$

$$\cdot \frac{1}{(-q_{1_\perp} \cancel{q^*_{1_\perp}})} \sqrt{2}\, \frac{\cancel{q^*_{1_\perp}}\, \cancel{q_{2_\perp}}}{p_{1_\perp}}\, \frac{1}{(-\cancel{q_{2_\perp}}\, \cancel{q^*_{2_\perp}})} \cdots\cdots \sqrt{2}\, \frac{\cancel{q^*_{n_\perp}}\, \cancel{q_{n+1_\perp}}}{p_{n_\perp}}\, \frac{1}{(-\cancel{q_{n+1_\perp}}\, \cancel{q^*_{n+1_\perp}})}\, \frac{\cancel{p^*_{n+1_\perp}}}{p_{n+1_\perp}}$$

use $q_{1_\perp} = -p_{0_\perp}$ and $q_{n+1_\perp} = p_{n+1_\perp}$

so $$M(--+\cdots+) = 2\hat{s}\,(-1)^n\, 2^{n/2}\, (ig)^{n+2} \frac{\hat{s}}{\prod_{i=0}^{n+1} p_{i_\perp}} f^{a d_0 c_1} f^{c_1 d_1 c_2} \cdots f^{b d_{n+1} c_{n+1}}$$

which coincides with the MHV amplitude in MRK

we computed before

Using the MHV amplitude, please check that:

- besides the helicity configuration $(A^-, 0^+, 1^+, --, n^+, (n+1)^+, B^-)$ we just evaluated, only the helicity configurations:

$$(A^+, 0^-, 1^+, --, n^+, (n+1)^+, B^-)$$

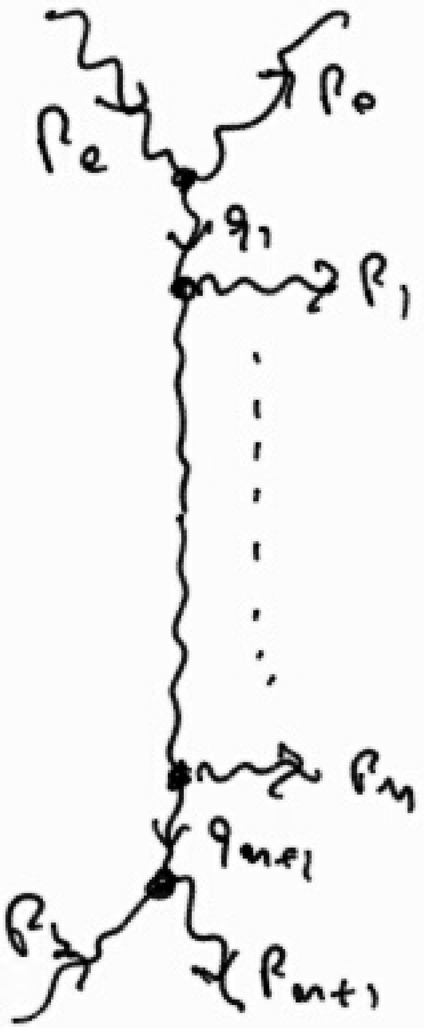
$$(A^-, 0^+, 1^+, ----, n^+, (n+1)^-, B^+)$$

$$(A^+, 0^-, 1^+, --, n^+, (n+1)^-, B^+)$$

are leading in MRK. They correspond to flipping the helicities on the upper and/or on the lower vertices Γ.

If instead we flip the helicities of one or two gluons along the ladder, we get a subleading contribution. Thus, helicity must be conserved in the upper and lower vertices.

We can summarize it, by writing the multi-gluon amplitude at fixed helicities in MRK as:



$$
\begin{aligned}
M^{a b d_0 \cdots d_{n+1}}_{\lambda_a \lambda_b \lambda_0 \cdots \lambda_{n+1}} = 2\hat{s}\, & \left(i g f^{a d_0 c_1} C_{gg}^{\lambda_a \lambda_0}(p_a, p_0) \right) \\
& \frac{1}{\hat{t}_1} \left(i g f^{c_1 d_1 c_2} C_g^{\lambda_1}(q_1, q_2) \right) \\
& \quad \vdots \\
& \frac{1}{\hat{t}_n} \left(i g f^{c_n d_n c_{n+1}} C_g^{\lambda_n}(q_n, q_{n+1}) \right) \\
& \frac{1}{\hat{t}_{n+1}} \left(i g f^{b d_{n+1} c_{n+1}} C_{gg}^{\lambda_b \lambda_{n+1}}(p_b, p_{n+1}) \right)
\end{aligned}
$$

with

$$
\begin{cases}
C_{gg}^{-+}(p_a, p_0) = 1 \\
C_{gg}^{-+}(p_b, p_{n+1}) = \dfrac{p^*_{n+1\perp}}{p_{n+1\perp}}
\end{cases}
\qquad \text{and} \qquad \left(C_{gg}^{-+} \right)^* = C_{gg}^{+-}
$$

This is confirmed by the FKL amplitude, for which we obtain:

$$p^{\mu_a \mu_0} \epsilon^{-*}_{\mu_a}(p_a, p_b) \epsilon^{-}_{\mu_0}(p_0, p_b) = 1 \qquad \text{(please check it)}$$

$$p^{\mu_b \mu_{n+1}} \epsilon^{-*}_{\mu_b}(p_b, p_a) \epsilon^{-}_{\mu_{n+1}}(p_{n+1}, p_a) = \frac{p_{n+1_\perp}}{p^*_{n+1_\perp}}$$

In order to conserve helicity on the upper and lower vertices and to flip the helicity of one of the gluons along the ladder, we need 3 gluons with negative helicity. That is an NMHV configuration, which for general kinematics has a more complicated analytic structure.

For the MHV configuration, and for a gluon with positive helicity along the ladder we have

$$C_g^+(q_i, q_{i+1}) = \sqrt{2}\, \frac{q^*_{i_\perp} q_{i+1_\perp}}{p_{i_\perp}}$$

Using a NMHV amplitude in MRK, one obtains that

$$C_g^-(q_i, q_{i+1}) = \sqrt{2}\, \frac{q_{i_\perp} q^*_{i_\perp}}{p^*_{i_\perp}}$$

i.e. $(C^+)^* = C^-$

This is confirmed by the FKL amplitude, from which we obtain that $C(q_i, q_{i+1}) \cdot \epsilon^-(p_i, p_e) = \sqrt{2}\, \frac{q_{i_\perp} q^*_{i+1_\perp}}{p^*_{i_\perp}}$

so flipping the helicity of a gluon along the ladder, just like for the upper and lower vertices, amounts to just a change of phase

Also note that we immediately see that

$$\sum_{\lambda_i} C_g^{\lambda_i}(q_i, q_{i+1})\, C_g^{\lambda_i *}(q_i, q_{i+1}) = 4\, \frac{q_{i_\perp}^2\, q_{i+1_\perp}^2}{k_{i_\perp}^2}$$

which shows that the simplicity of the squared central-emission vertex is immediately given by the vertex itself at fixed helicity rather than by the original, Lorentz-covariant, expression of the vertex.

It will be even more so when we consider the NLO corrections to the vertex.

Suppose that one of the incoming partons is a quark. The colour decomposition of the tree-level amplitude for $q\bar{q} + (n-2)$ gluons is

$$M_n(q_1, g_2, \ldots, g_{n-1}, \bar{q}_n)$$
$$= g^{n-2} \sum_{\sigma \in S_{n-2}} \left(T^{a_{\sigma_2}} \cdots T^{a_{\sigma_{n-1}}}\right)_{i_1 \bar{i}_n} A_n(q_1, g_{\sigma_2}, \ldots, g_{\sigma_{n-1}}, \bar{q}_n)$$

In particular, we consider the MHV amplitude

$$A_n\left(\bar{q}_a^-, q_0^+ ; g_1^+, \ldots, g_k^-, \ldots, g_{n+1}^+, g_b^+\right) = 2^{\frac{2+n}{2}} \frac{\langle p_a p_k \rangle^3 \langle p_0 p_k \rangle}{\langle p_a p_0 \rangle \cdots \langle p_{n+1} p_b \rangle \langle p_b p_a \rangle}$$

where the k^{th} gluon has negative helicity

By evaluating the spinor products in MRK, it is easy to see that the leading helicity configurations are the one for which $k = b, n+1$, so again the 2 partons of negative helicity are at the upper and lower vertices, where helicity is conserved.

By crossing the outgoing antiquark $\bar{q}_a^-$ into the incoming quark q_a^+, we obtain the amplitude for $q_a g_b \to q_0 g_1 \cdots g_{n+1}$

$$M^{a b d_0 \cdots d_{n+1}}_{\lambda_a \lambda_b \lambda_0 \cdots \lambda_{n+1}}(q_a g_b \to q_0 g_1 \cdots g_{n+1}) = 2\hat{s}\left(g T^{c}_{0\bar{a}}\, C^{\lambda_a \lambda_0}_{\bar{q}q}(p_a, p_0)\right)$$

$$\frac{1}{\hat{t}_1}\left(i g f^{c_1 d_1 c_2}\, C^{\lambda_1}_{g}(q_1, q_2)\right)$$

$$\vdots$$

$$\frac{1}{\hat{t}_n}\left(i g f^{c_n d_n c_{n+1}}\, C^{\lambda_n}_{g}(q_n, q_{n+1})\right)$$

$$\frac{1}{\hat{t}_{n+1}}\left(i g f^{b d_{n+1} c_{n+1}}\, C^{\lambda_b \lambda_{n+1}}_{gg}(p_n, p_{n+1})\right)$$

which is identical to the multi-gluon amplitude in MRK, except for the upper vertex $C^{-+}_{\bar{q}q}(p_a, p_0) = -i$

with $\left[C^{\lambda_a \lambda_0}_{\bar{q}q}(p_a, p_0)\right]^* = s\, C^{-\lambda_a -\lambda_0}_{\bar{q}q}(p_a, p_0)$ $\qquad s = -\,\mathrm{sign}(\bar{q}^0 q^0)$

Likewise, for a quark line on the lower vertex we obtain

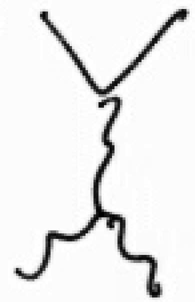
$$C_{\bar{q}q}^{\lambda_b \lambda_{n+1}} = i\sqrt{\frac{p^*_{n+1_\perp}}{p_{n+1_\perp}}}$$

In a similar way, one can evaluate $\bar{q}g \to \bar{q}g \cdots g$

Consider in particular the scattering $qg \to qg$

Squaring the amplitude we wrote above, and summing (averaging) over final (initial) colours and helicities we obtain

$$\overline{\sum} |M_{qg\to qg}|^2 = \frac{C_A C_F}{N_c^2 - 1} \frac{4\hat{s}^2}{\hat{t}^2} g_s^4 \qquad \text{(please check it)}$$

Thus $\left|$  $\right|^2 \Big/ \left|$  $\right|^2 = \frac{C_A}{C_F} = 9/4$

as stated in the lectures