Ex. 2

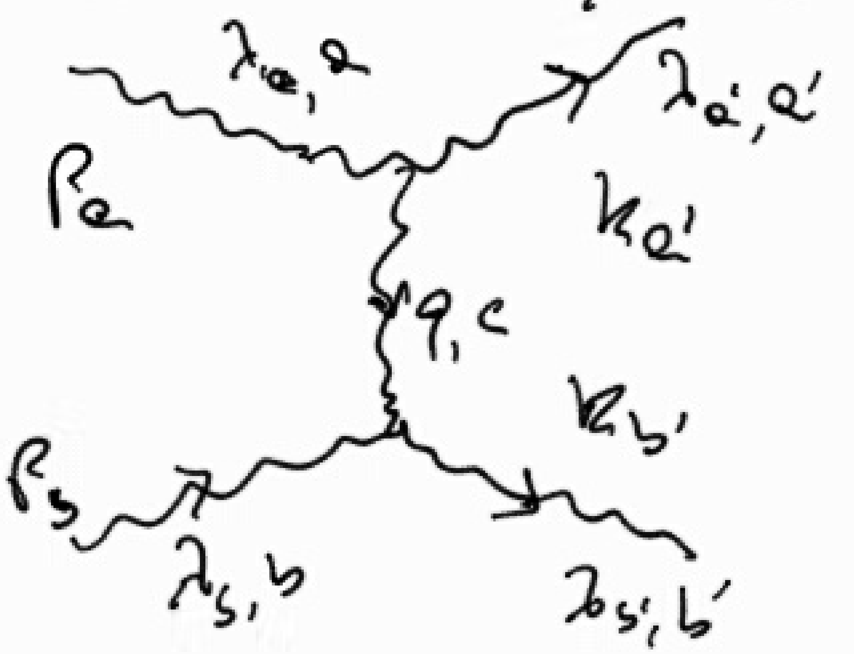
Consider gluon-gluon scattering at tree level.

The Feynman diagrams which contribute are

In the high-energy limit $s \gg |t|$ or $\Delta y \gg 1$,

we expect that only the one with gluon exchange in the crossed ($\hat{t}$) channel, yields a leading contribution, the others being power suppressed in $\frac{\hat{t}}{\hat{s}}$.

We shall compute then the one with gluon exchange in $\hat{t}$ channel, which per se is not gauge invariant. We may get, though, the right answer if we work consistently in a given gauge.



In light-cone coords, the non-vanishing components of the metric tensor are

$$g_{+-} = g_{-+} = \frac{1}{2} \qquad g_{xx} = g_{yy} = -1$$

Let me consider a basis of (polarization-like) unit vectors v_λ^μ, where λ is the helicity. I can write the metric tensor as

$$g^{\mu\nu} = v_\lambda^\mu \, g^{\lambda\lambda'} \, v_{\lambda'}^\nu = 2\left(v_+^\mu v_-^\nu + v_+^\nu v_-^\mu\right) - v_1^\mu \cdot v_2^\nu$$

Since $p_a = (x_a\sqrt{s}, 0; 0_2)$

$$p_b = (0, x_b\sqrt{s}; 0_2)$$

I can choose the light-cone vectors as:

$$v_+ = (1, 0; 0_2) = \frac{1}{x_a\sqrt{s}}\, p_a$$

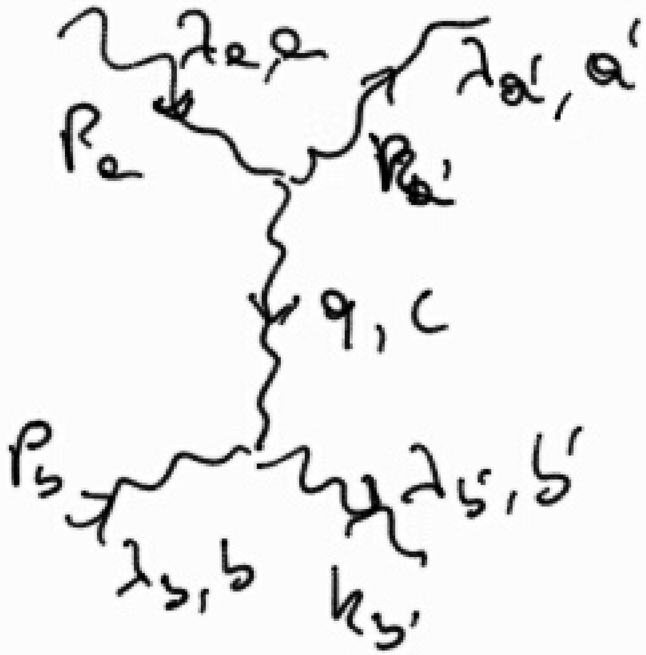
$$v_- = (0, 1; 0_2) = \frac{1}{x_b\sqrt{s}}\, p_b$$

and then write the metric tensor as

$$g^{\mu\nu} = 2\,\frac{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu}{\hat{s}} - \delta_\perp^{\mu\nu}$$

with $\delta_2^{\mu\nu}$ a Kronecker delta over the transverse components

So we can write the diagram as



$$iM^{aa'bb'}_{\lambda_a\lambda_{a'}\lambda_b\lambda_{b'}}$$

$$= g_s f^{aa'c}\left[g_{\mu_a\mu_{a'}}(p_a+k_{a'})_\nu + g_{\nu\mu_{a'}}(-k_{a'}+q)_{\mu_a} - g_{\mu_a\nu}(q+p_a)_{\mu_{a'}}\right]$$

$$\cdot(-i)\left[2\,\frac{p_a^\nu p_b^\rho + p_a^\rho p_b^\nu}{\hat{s}} - \delta_2^{\nu\rho}\right]\frac{1}{\hat{t}}$$

$$\cdot g_s f^{bb'c}\left[g_{\mu_b\mu_{b'}}(p_b+k_{b'})_\rho - g_{\rho\mu_{b'}}(k_{b'}+q)_{\mu_b} + g_{\mu_b\rho}(q-p_b)_{\mu_{b'}}\right]$$

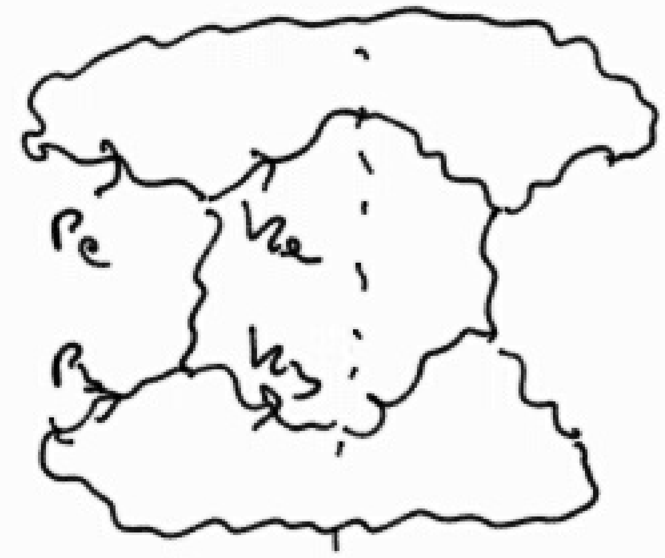
$$\cdot\,\epsilon^{\mu_a}_{\lambda_a}(p_a)\,\epsilon^{\mu_b}_{\lambda_b}(p_b)\,\epsilon^{\mu_{a'}}_{\lambda_{a'}}(k_{a'})\,\epsilon^{\mu_{b'}}_{\lambda_{b'}}(k_{b'})$$

$$\simeq -2ig_s^2 f^{aa'c} g_{\mu_a\mu_{a'}}\frac{\hat{s}}{\hat{t}} f^{bb'c} g_{\mu_b\mu_{b'}}\,\epsilon^{\mu_a}_{\lambda_a}(p_a)\,\epsilon^{\mu_b}_{\lambda_b}(p_b)\,\epsilon^{\mu_{a'}}_{\lambda_{a'}}(k_{a'})\,\epsilon^{\mu_{b'}}_{\lambda_{b'}}(k_{b'})$$

$$\cdot\left(1+O\left(t/s\right)\right)$$

where $q^2=\hat{t}$ and we used $p\cdot\epsilon_\lambda(p)=0$.

The leading contribution comes from combining the helicity-conserving terms in the 3-gluon vertices with a light-cone polarization mode in the gluon propagator.



Next, we square the amplitude and sum over helicities and colours.

Sum over helicities:

in QED, we can always use $\sum_{\lambda} \epsilon^{\mu}_{\lambda}(p)\,\epsilon^{\nu *}_{\lambda}(p) = -g^{\mu\nu}$ $*)$

because the longitudinal modes decouple from the physical transverse modes.

In QCD, that is not true. So, if we want to use $*)$ we must introduce ghosts. If we do not want to introduce ghosts, we must work in a physical gauge and write the sum over helicities as:

$$\sum_\lambda \epsilon_\lambda^\mu(p)\, \epsilon_\lambda^{\nu *}(p) = -\left[g_{\mu\nu} - \frac{n^\mu p^\nu + n^\nu p^\mu}{n \cdot p} + \frac{n^2 p^\mu p^\nu}{(n \cdot p)^2}\right]$$

where n is an arbitrary 4-vector, which is not collinear to p, $n \cdot p \neq 0$

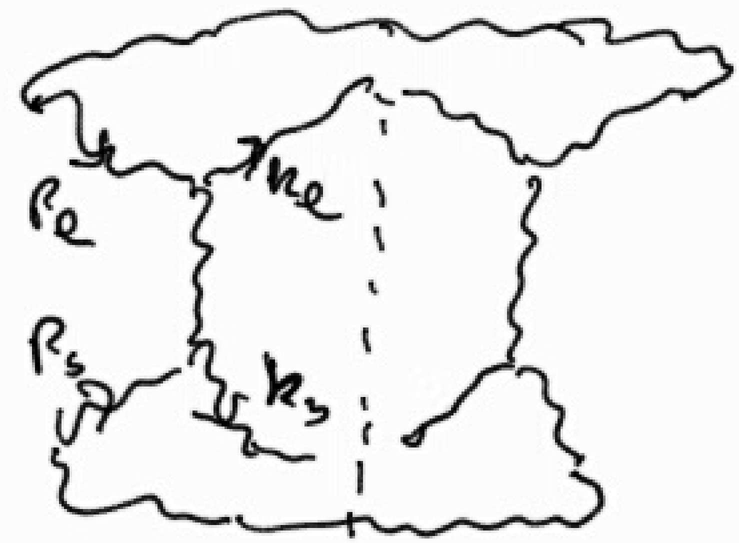
Let us consider the sum over the helicities of gluon p_a and fix $n = p_b$. Then

$$\sum_\lambda \epsilon_\lambda^\mu(p_a)\, \epsilon_\lambda^{\nu *}(p_a) = -\left(g^{\mu\nu} - \frac{p_a^\nu p_b^\mu + p_a^\mu p_b^\nu}{p_a \cdot p_b}\right) = \delta_\perp^{\mu\nu}$$

Next, we square the amplitude

Summing over the helicities of gluons

p_e and k_e, and contracting with the upper vertex:



$$g^{\mu_e \mu_{e'}} g^{\nu_e \nu_{e'}} \left[\sum_{\lambda_e} \epsilon^{\mu_e}_{\lambda_e}(p_e) \, \epsilon^{\nu_e *}_{\lambda}(p_e) \right] \left[\sum_{\lambda_{e'}} \epsilon^{\mu_{e'}}_{\lambda_{e'}}(k_{e'}) \epsilon^{\nu_{e'} *}_{\lambda_{e'}}(k_{e'}) \right] = 2 \left(1 + O\left(\frac{t}{s}\right) \right)$$

i.e. up to subleading terms, helicity is conserved in the jet-production vertices.

The computation just done is not gauge invariant: we cannot replace the polarization $\epsilon^{\mu}_{\lambda}(p)$ with the scalar one p^{μ} and obtain that the amplitude vanishes. However, we obtain the correct result if we stay in the gauge we just used, In fact, summing over colour

$$f^{aa'c} f^{a'ac'} f^{bb'c} f^{b'bc'} = C_A^2 (N_c^2 - 1)$$

Thus, summing over final helicities and colours, and averaging over initial ones, we obtain:

$$\overline{\sum} |M|^2 = \frac{1}{4(N_c^2-1)^2} 4 C_A^2 (N_c^2-1) \frac{4\hat{s}^2}{\hat{t}^2} g_s^4$$

$$= \frac{C_A^2}{N_c^2-1} \frac{4\hat{s}^2}{\hat{t}^2} g_s^4$$

$$= \frac{9}{2} \frac{\hat{s}^2}{\hat{t}^2} g_s^4$$

i.e. the correct result in the high-energy limit.

We can include the subleading diagrams, and make the amplitude gauge invariant by substituting $g^{\mu_a\mu_{a'}}$ with

$$\Gamma^{\mu_a\mu_{a'}} = g^{\mu_a\mu_{a'}} - \frac{p_a^{\mu_{a'}} p_b^{\mu_a} + p_b^{\mu_{a'}} k_{a'}^{\mu_a}}{p_a \cdot p_b} - \hat{t}\, \frac{p_b^{\mu_{a'}} p_b^{\mu_a}}{2\left(p_a \cdot p_b\right)^2}$$

and likewise in the lower vertex. (Fadin, Kuraev, Lipatov '77)

The effective amplitude

$$i\mathcal{M}^{aa'bb'}_{\lambda_a\lambda_{a'}\lambda_b\lambda_{b'}} = -2ig_s^2\, f^{aa'c}\, \Gamma_{\mu_a\mu_{a'}}\, \frac{\hat{s}}{\hat{t}}\, f^{bb'c}\, \Gamma_{\mu_b\mu_{b'}}\, \epsilon^{\mu_a}_{\lambda_a}(p_a)\, \epsilon^{\mu_{a'}}_{\lambda_{a'}}(k_{a'})\, \epsilon^{\mu_b}_{\lambda_b}(p_b)\, \epsilon^{\mu_{b'}}_{\lambda_{b'}}$$

is gauge invariant, up to subleading terms. In fact,

$$\Gamma^{\mu_a\mu_{a'}}\, p_{a\,\mu_a} = 0$$

$$\Gamma^{\mu_a\mu_{a'}}\, k_{a'\,\mu_{a'}} = O\left(\frac{t}{s}\right)$$

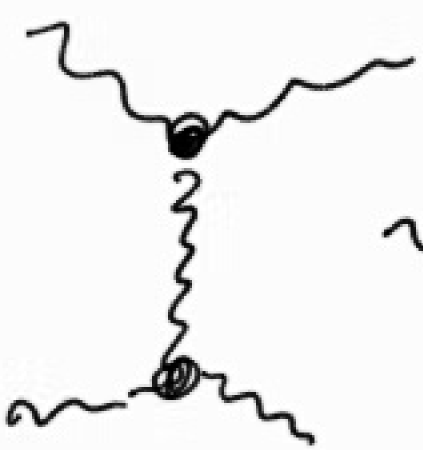
In addition, we can replace the sum over the helicities by

$$\sum_\lambda \epsilon^{\mu}_{\lambda}(p)\,\epsilon^{\nu *}_{\lambda}(p) = -g^{\mu\nu}$$

and we get the right helicity counting

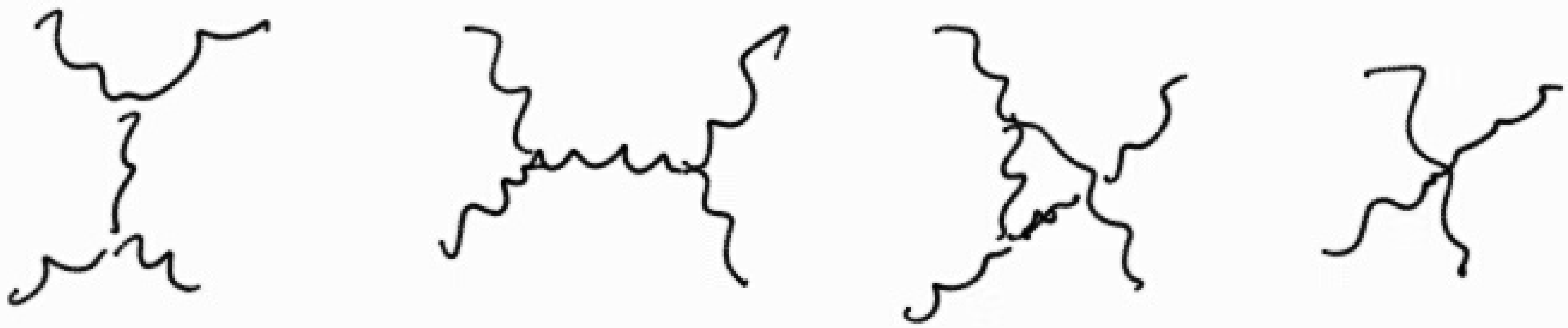
$$\Gamma^{\mu_a\mu_{a'}}\,\Gamma^{\mu_b\mu_{b'}}\left[\sum_{\lambda_a}\epsilon^{\mu_a}_{\lambda_a}(p_a)\,\epsilon^{\nu_a *}_{\lambda_a}(p_a)\right]\left[\sum_{\lambda_{a'}}\epsilon^{\mu_{a'}}_{\lambda_{a'}}(p_{a'})\,\epsilon^{\nu_{a'} *}_{\lambda_{a'}}(p_{a'})\right] = 2$$

It is important to remember that when we refer to the effective amplitude



$$\sim g_s^2 f^{aa'c}\,\Gamma_{\mu_a\mu_{a'}}\,\frac{2\hat{s}}{t}\,f^{bb'c}\,\Gamma_{\mu_b\mu_{b'}} \times \text{pol. vectors}$$

we really mean the full amplitude



in the high-energy limit $\hat{s} \gg |\hat{t}|$