# Heisenberg spin chain

Consider spin chain of $N$ sites, each w/ a spin $\frac{1}{2}$ particle (electr.)
The states are $|\uparrow\rangle$, $|\downarrow\rangle$ and any electron is in $a|\uparrow\rangle + b|\downarrow\rangle$


The spin operators $S_i^{x,y,z}$ act on each site $i$, and they satisfy:

$$[S_i^a, S_j^b] = \delta_{ij}\, \epsilon^{abc} S_i^c \qquad \text{w/ } S_i^a = \frac{1}{2}\sigma^a$$

The interaction is supposed to be between nearest neighbours.

The Hamiltonian is:

$$H = \frac{JN}{4} - J\sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} \qquad \text{w/ } \vec{S}_{N+1} = \vec{S}_1$$

Introduce raising/lowering op's $S^{\pm} = S_x \pm i S_y$, s.t.

$$S^+|\uparrow\rangle = 0 \qquad S^-|\uparrow\rangle = |\downarrow\rangle \qquad S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$$

$$S^+|\downarrow\rangle = |\uparrow\rangle \qquad S^-|\downarrow\rangle = 0 \qquad S^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$$

Can rewrite the Hamiltonian as:

$$H = \frac{JN}{4} - J\left[\sum_{i=1}^{N} \frac{1}{2}\left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right) + S_i^z S_{i+1}^z\right]$$

$S_i^{\pm} S_{i+1}^{\mp}$ are called hopping terms: they move a spin up or down to the nearest site.

This in in fact a XXX Heisenberg spin chain. A more general Hamiltonian, w/ $H \propto -\sum_{a=1}^{3}\sum_{i=1}^{N} J^a S_i^a S_{i+1}^a$ would correspond to a XYZ spin $\frac{1}{2}$ model.

The total spin is $\vec{S} = \sum_{i=1}^{N} \vec{S}_i$

It commutes w/ the Hamiltonian: $[H, \vec{S}] = 0$

which reflects the SU(2) symmetry of the model. The spin op's form an SU(2) algebra (as we'll see later, there is more symmetry)

The Hilbert space of the spin chain is $\mathcal{H} = \bigotimes_{n=1}^{N} V_n$ w/ $V_n \doteq \mathbb{C}^2$

Define a Lax operator: $L: V_n \otimes V_a \to V_n \otimes V_a$ in terms of an auxiliary space $V_a = \mathbb{C}^2$, s.t. $L_{n,a}(u) = u\, \mathbb{1}_n \otimes \mathbb{1}_a + i \sum_{j=1}^{3} S_n^j \otimes \sigma^j$

where $\mathbb{1}_n, S_n^j$ act on $V_n$, $\mathbb{1}_a, \sigma^j$ act on $V_a$

$\sigma^a$ are the Pauli matrices: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We may view the Lax operator as a matrix in the auxiliary space $V_a$

$$L_{n,a} = \begin{pmatrix} u + i S_n^z & i S_n^- \\ i S_n^+ & u - i S_n^z \end{pmatrix}$$

w/ entries being operators in $V_n$.

To define a commutation between entries of the Lax operator, equivalent to the Poisson bracket among entries of the Lax matrix, introduce the permutation operator $P = \frac{1}{2}\left(\mathbb{1}_n \otimes \mathbb{1}_a + \sum_i \sigma^i \otimes \sigma^i\right)$

s.t. $P(x \otimes y) = y \otimes x$

We may view $P$ as a matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$: $P = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$ (please check it!)

In fact,

$$\mathbb{1}_u \otimes \mathbb{1}_a = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \sigma^1\otimes\sigma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \quad \sigma^2\otimes\sigma^2 = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$$

$$\sigma^3\otimes\sigma^3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$ and one gets $$\frac{1}{2}\left(\mathbb{1}_u + \mathbb{1}_a + \sum_i \sigma^i\otimes\sigma^i\right) = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

In terms of $P$, we may write $L_{u,a} = \left(u - \frac{i}{2}\right)\mathbb{1}_{u,a} + iP_{u,a}$

Because we have 4 entries of the Lax operator, we'd need to write down in principle 16 commutators among the entries. We can do it in just one line, introducing 2 auxiliary spaces $V_a$ and $V_b$.

Let us define $L_{u,a}(u)$ acting on $V_u\otimes V_a$ and $L_{u,b}(v)$ acting on $V_u\otimes V_b$

Then $L_{u,a}(u)\,L_{u,b}(v)$ or $L_{u,b}(v)\,L_{u,a}(u)$ act on $V_u\otimes V_a\otimes V_b$

Firstly, let's establish the commutation relations of permutations

$$P_{u,a}P_{u,b} = P_{a,b}P_{u,a} = P_{u,b}P_{b,a} \qquad \text{w/} \quad P_{b,a} = P_{a,b}$$

Proof: let's test it on some tensor product $S_u\otimes a\otimes b$

$$P_{u,a}P_{u,b}\, S_u\otimes a\otimes b = P_{u,a}\, b\otimes a\otimes S_u = b\otimes S_u\otimes a$$

$$P_{a,b}P_{u,a}\, S_u\otimes a\otimes b = P_{a,b}\, a\otimes S_u\otimes b = b\otimes S_u\otimes a$$

$$P_{u,b}P_{b,a}\, S_u\otimes a\otimes b = P_{u,b}\, S_u\otimes b\otimes a = b\otimes S_u\otimes a$$

Introducing the R operator: $R_{ab}(u-v) = (u-v)\mathbb{1}_{a,b} + i P_{a,b}$ acting on $V_a \otimes V_b$, we can write the commutation relations of the Lax matrix compactly as

$$R_{ab}(u-v)\, L_{n,a}(u)\, L_{n,b}(v) = L_{n,b}(v)\, L_{n,a}(u)\, R_{ab}(u-v).$$

They are called Fundamental Commutation Relations (FCR)

Proof: ① $= R_{ab}(u-v)\, L_{n,a}(u)\, L_{n,b}(v)$

$$= \left[(u-v)\mathbb{1}_{a,b} + i P_{a,b}\right]\left[\left(u-\tfrac{i}{2}\right)\mathbb{1}_{n,a} + i P_{n,a}\right]\left[\left(v-\tfrac{i}{2}\right)\mathbb{1}_{n,b} + i P_{n,b}\right]$$

$$= (u-v)\Big[\left(u-\tfrac{i}{2}\right)\left(v-\tfrac{i}{2}\right)\mathbb{1}_{n,a,b} + i\left(u-\tfrac{i}{2}\right)\mathbb{1}_a \otimes P_{n,b} + i\left(v-\tfrac{i}{2}\right)P_{n,a}\otimes \mathbb{1}_b - P_{n,a}\otimes P_{n,b}\Big]$$

$$+\, i\Big[\left(u-\tfrac{i}{2}\right)\left(v-\tfrac{i}{2}\right)\mathbb{1}_n \otimes P_{a,b} + i\left(u-\tfrac{i}{2}\right)P_{a,b}\otimes P_{n,b} + i\left(v-\tfrac{i}{2}\right)P_{a,b}\otimes P_{n,a} - P_{a,b}\otimes P_{n,a}\otimes P_{n,b}\Big]$$

② $= L_{n,b}(v)\, L_{n,a}(u)\, R_{ab}(u-v)$

$$= \left[\left(v-\tfrac{i}{2}\right)\mathbb{1}_{n,b} + i P_{n,b}\right]\left[\left(u-\tfrac{i}{2}\right)\mathbb{1}_{n,a} + i P_{n,a}\right]\left[(u-v)\mathbb{1}_{a,b} + i P_{a,b}\right]$$

$$= \Big[\left(v-\tfrac{i}{2}\right)\left(u-\tfrac{i}{2}\right)\mathbb{1}_{n,a,b} + i\left(v-\tfrac{i}{2}\right)\mathbb{1}_b \otimes P_{n,a} + i\left(u-\tfrac{i}{2}\right)\mathbb{1}_a \otimes P_{n,b} - P_{n,b}\otimes P_{n,a}\Big](u-v)$$

$$+\, i\Big[\left(v-\tfrac{i}{2}\right)\left(u-\tfrac{i}{2}\right)\mathbb{1}_n \otimes P_{a,b} + i\left(v-\tfrac{i}{2}\right)P_{n,a}\otimes P_{a,b} + i\left(u-\tfrac{i}{2}\right)P_{n,b}\otimes P_{a,b} - P_{n,b}\otimes P_{n,a}\otimes P_{a,b}\Big]$$

where we use that $P_{ab} P_{na} P_{nb} = P_{ba} P_{nb} P_{ba} = P_{nb} P_{na} P_{ab}$.

The only pieces left over are

$$① = -(u-\cancel{v})\, P_{na} P_{nb} - \left(u - \tfrac{i}{2}\right) \underset{P_{nb} P_{na}}{\underset{\shortparallel}{P_{ab} P_{nb}}} - \left(\cancel{v} - \tfrac{i}{2}\right) \underset{P_{na} P_{nb}}{\underset{\shortparallel}{P_{ab} P_{na}}}$$

$$= -\left(u - \tfrac{i}{2}\right)\left(P_{na} P_{nb} + P_{nb} P_{na}\right)$$

$$② = -(u-\cancel{v})\, P_{nb} P_{na} - \left(\cancel{v} - \tfrac{i}{2}\right) \underset{P_{nb} P_{na}}{\underset{\shortparallel}{P_{na} P_{ab}}} - \left(u - \tfrac{i}{2}\right) \underset{P_{na} P_{nb}}{\underset{\shortparallel}{P_{nb} P_{ab}}}$$

$$= -\left(u - \tfrac{i}{2}\right)\left(P_{nb} P_{na} + P_{na} P_{nb}\right)$$

✓

which proves that ① = ②

Define a monodromy matrix $T_a(u) = L_{N,a}(u) \cdots L_{1,a}(u)$

It's a $2 \times 2$ matrix on the auxiliary space, whose entries are operators on the Hilbert space of the spin chain,

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Using repeatedly the FCR, one gets the FCR for $T$:

$$R_{ab}(u-v)\, T_a(u)\, T_b(v) = T_b(v)\, T_a(u)\, R_{ab}(u-v)$$

To prove it, it's enough to consider $T_a(u) = L_{n+1,a}(u)\, L_{n,a}(u)$

then $R_{ab} L_{n+1,a} L_{n,a} L_{n+1,b} L_{n,b} = R_{ab} L_{n+1,a} L_{n+1,b} L_{n,a} L_{n,b}$ (since $L_{n,a} L_{n+1,b}$ commu

$= L_{n+1,b} L_{n+1,a} L_{n,b} L_{n,a} R_{ab}$ (use FCR on LL

✓ $= L_{n+1,b} L_{n,b} L_{n+1,a} L_{n,a} R_{ab}$ (since $L_{n,b} L_{n+1,a}$ commu

The monodromy is a polynomial in u

$$T(u) = u^N + i\,u^{N-1} \underbrace{\sum_n S_n^a}_{S^a} \otimes \sigma^a + \cdots$$

where to next-leading we see the total spin $S^a$ appearing.

The transfer matrix is defined as the trace on the auxiliary space

$$\tau(u) = \mathrm{tr}_a\, T_a = A(u) + \Delta(u)$$

Rewrite the FCR as

$$T_a(u)\,T_b(v) = R_{ab}^{-1}(u-v)\,T_b(v)\,T_a(u)\,R_{ab}(u-v)$$

then the cyclicity of the trace implies that : $[\tau(u), \tau(v)] = 0$

So $\tau(u) = 2u^N + \sum_{i=0}^{N-2} Q_i u^i$ (the order N-1 is missing since $\mathrm{tr}\,\sigma^a = 0$)

where $Q_i$ are N-1 commuting operators. One of them must be H.

To see this, consider the special point $u = i/2$. Then $L_{n,a}(\tfrac{i}{2}) = i P_{n,a}$

and $T_a(\tfrac{i}{2}) = i^N P_{N,a} P_{N-1,a} \cdots P_{1,a}$

$= i^N P_{12} P_{23} \cdots P_{N-1,N} P_{N,a}$ using the commutations of the P's

Then $\tau(\tfrac{i}{2}) = i^N P_{12} P_{23} \cdots P_{N-1,N}$ since $\mathrm{Tr}_a P_{N,a} = \mathbb{1}_a$

In addition for any u, $\frac{d}{du} L_{n,a}(u) = \mathbb{1}_{n,a}$

So $\left.\frac{dT_a}{du}\right|_{u=i/2} = i^{N-1} \sum_n P_{N,a} \cdots \hat{P}_{n,a} \cdots P_{1,a}$ where $\hat{P}$ means that it is missing

Using the commutation of the $P$'s, we can rewrite it as

$$\frac{dT_a}{du}\bigg|_{u=i/2} = i^{N-1}\sum_n P_{12}P_{23}\cdots P_{n-1,n+1}\cdots P_{N-1,N}\,P_{N,a}$$

then
$$\frac{dz}{du}\bigg|_{u=i/2} = i^{N-1}\sum_n P_{12}P_{23}\cdots P_{n-1,n+1}\cdots P_{N-1,N}$$

So
$$\frac{1}{z(i/2)}\frac{dz}{du}\bigg|_{u=i/2} = \frac{d\ln z(u)}{du}\bigg|_{u=i/2} = \frac{1}{i}\sum_n \frac{P_{n-1,n+1}}{P_{n-1,n}P_{n,n+1}}$$

(use $P^2=1$)

$$= \frac{1}{i}\sum_n \frac{P_{n-1,n+1}P_{n,n+1}}{P_{n-1,n}}$$

$$= \frac{1}{i}\sum_n \frac{P_{n+1,n}\cancel{P_{n,n-1}}}{\cancel{P_{n-1,n}}}$$

Since $P = \frac{1}{2}\left(1_{n,a} + \sum_i \sigma^i\otimes\sigma^i\right)$, one can rewrite $H$ as

$$H = -J\frac{1}{2}\sum_n P_{n,n+1} + \frac{JN}{4}$$

thus $H = -J\frac{i}{2}\frac{d}{du}\ln z(u)\Big|_{u=i/2} + \frac{JN}{4}$ is one of the $N-1$ commuting operators generated by the transfer. The $N^{th}$ can be taken to be a spin $S^z$ component as we'll show now.

Another point where monodromy and transfer matrices become simpler is $u\to\infty$. In fact $\lim_{u\to\infty} L_{n,a} = u\,1_{n,a}$

so $\lim_{u\to\infty} T(u) = u^N + i\,u^{N-1}\sum_i S^i\otimes\sigma^i + \cdots$

Consider the FCR w/ $v \to \infty$. One gets

$$\left[(u-v)+i P_{ab}\right] T_a(u)\left(v^N + i v^{N-1} S^a \otimes \sigma^a\right)_b$$

$$= \left(v^N + i v^{N-1} S^a \otimes \sigma^a\right)_b T_a(u)\left[(u-v)+i P_{ab}\right]$$

The leading term in $v$ cancels out.

The subleading term, after using $P = \frac{1}{2}\left(\mathbb{1} + \sigma^a \otimes \sigma^a\right)$, implies that ~~because~~:

$$\left[T(u), S^a + \frac{1}{2}\sigma^a\right] \otimes \sigma^a = 0$$

i.e. $\left[S^a, T(u)\right] = \frac{1}{2}\left[T, \sigma^a\right]$

in particular $\left[S^z, T(u)\right] = \frac{1}{2}\left[T, \sigma^z\right] = \frac{1}{2}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix}$

and so $\left[S^z, A\right] = \left[S^z, D\right] = 0$

because $\tau = \operatorname{tr} T = A + D \quad \Rightarrow \left[S^z, \tau\right] = 0$

Analogously, one can check that

$$\left[S^+, A\right] = -\left[S^+, D\right] = -C \qquad \Rightarrow \left[S^+, \tau\right] = 0$$

$$\left[S^-, A\right] = -\left[S^-, D\right] = B \qquad \Rightarrow \left[S^-, \tau\right] = 0$$

So $S^a$ is the generators of symm. and commutes w/ $H$ and $\tau$.

This $SU(2)$ symm. is local, since the total spin op. acts on one site

To go beyond the local symm., consider the next term in the expansion of the monodromy matrix as $u\to\infty$

$$\lim_{u\to\infty} T(u) = u^N + i u^{N-1}\sum_{n=1}^{N} S_n^j \otimes \sigma^j - u^{N-2}\sum_{m<n} S_m^j S_n^k \otimes \sigma^j\sigma^k + \dots$$

the $O(u^{N-2})$ term is non local. Using $\sigma^j\sigma^k = \delta^{jk} + i\epsilon^{jkl}\sigma^l$

and $S^\pm = S^x \pm i S^y$ $\quad \sigma^\pm = \dfrac{\sigma^x \pm i\sigma^y}{2}$, we see that the $O(u^{N-2})$

term generates 3 structures

$$\sum_{m<n} S_m^j S_n^k \otimes \sigma^j\sigma^k \propto \sum_{m<n}\left(\hat{S}^z \otimes \sigma^z + \hat{S}^+ \otimes \sigma^+ + \hat{S}^- \otimes \sigma^-\right)$$

w/
$$\begin{cases} \hat{S}^z = S_m^+ S_n^- - S_m^- S_n^+ \\ \hat{S}^- = S_m^z S_n^- - S_m^- S_n^z \\ \hat{S}^+ = S_m^z S_n^+ - S_m^+ S_n^z \end{cases}$$

We commute them with $\mathcal{H}$. For example, for $\hat{S}^z$ we find

$$[\mathcal{H}, \hat{S}^z] \propto \sum_n \sum_{i<j}\left[S_n^a S_{n+1}^a, S_i^+ S_j^- - S_i^- S_j^+\right]$$

If $|i-j| \geq 2$, we must evaluate the commutator $[S_n^a S_{n+1}^a, S_i^\pm] = 0$

because $S_i^\pm$ commutes w/ $\mathcal{H}$. Then the only term which survives is

$$[\mathcal{H}, \hat{S}^z] \propto \sum_n \left[S_n^a S_{n+1}^a, S_n^+ S_{n+1}^- - S_n^- S_{n+1}^+\right]$$
$$= \sum_n S_n^z - S_{n+1}^z$$

which vanishes on periodic b.c's or on infinitely long chains.

Finally, using $[S^z, S^+] = S^+$ $[S^z, S^-] = -S^-$ $[S^+, S^-] = 2S^z$

one can show that
$$[\hat{S}^z, S^+] = [S^z, \hat{S}^+] = \hat{S}^+$$
$$[\hat{S}^z, S^-] = [S^z, \hat{S}^-] = -\hat{S}^- \quad \text{(please check!)}$$
$$[\hat{S}^+, \hat{S}^-] = [S^+, \hat{S}^-] = 2\hat{S}^z$$

i.e. a loop algebra.

However, when take e.g.
$$[\hat{S}^+, \hat{S}^z] = \hat{\hat{S}}^+ + \text{cubic term in } S \quad \text{w/ } \hat{\hat{S}}^+ \text{ an operator}$$

of yet a higher level. The algebra with this structure is called Yangian algebra.