Hamilton mechanics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad H = H(\{q_i\},\{p\};t)$$

can write it also as

$$\dot{\vec{\omega}} = J \frac{\partial H}{\partial \vec{\omega}} \qquad \begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \left(\begin{array}{c|c} 0 & \mathbb{1}_n \\ \hline -\mathbb{1}_n & 0 \end{array}\right) \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}$$

$$J = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$ is $2n \times 2n$ symplectic matrix : $J^T = -J$

A transform. ~~is canonical~~ $Q_i = Q_i(\{q\},\{p\};t)$ is canonical
$P_i = P_i(\{q\},\{p\};t)$

if there is a fn. $K = K(\{Q\},\{P\};t)$ s.t.

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Take a restricted (i.e. does not depend on t) canonical Transf.

$$\vec{\Omega} = \vec{\Omega}(\vec{\omega}) \quad \rightarrow \quad \dot{\vec{\Omega}} = M \dot{\vec{\omega}} \quad \text{or} \quad \dot{\Omega}_i = \underbrace{\frac{\partial \Omega_i}{\partial \omega_j}}_{M_{ij}} \dot{\omega}_j$$

then Ham. eq's : $\dot{\vec{\Omega}} = M J \frac{\partial H}{\partial \vec{\omega}}$

$$\frac{\partial H}{\partial \omega_i} = \frac{\partial H}{\partial \Omega_j} \frac{\partial \Omega_j}{\partial \omega_i} \quad \Rightarrow \quad \frac{\partial H}{\partial \vec{\omega}} = M^T \frac{\partial H}{\partial \vec{\Omega}}$$

$$\dot{\vec{\Omega}} = M J M^T \frac{\partial H}{\partial \vec{\Omega}}$$

but if canonical $\dot{\vec{\Omega}} = J \frac{\partial H}{\partial \vec{\Omega}} \quad \Rightarrow \quad \boxed{M J M^T = J}$ which ~~defines~~ is a property of sympl. matr.

sympl. group $J \in Sp(2n, \mathbb{R})$

The goal of a can. transf. is to change variables so the system can be more easily integrated.

Ex. if a coord. is cyclic: $\frac{\partial H}{\partial q_i} = 0 \Rightarrow p_i = \text{const.}$

E.g. the harm. osc.: $H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$ w/ $k = m\omega^2$

$$\overset{\uparrow}{=} \frac{p^2}{2m} + \frac{kq^2}{2}$$

take the (inverse) can. transf. $\begin{cases} p = \sqrt{2m\omega P}\cos Q \\ q = \sqrt{\frac{2P}{m\omega}}\sin Q \end{cases}$

then $K = H = \omega P = E$ $\left|\begin{array}{l} \text{now } \frac{\partial H}{\partial Q} = 0 \Rightarrow P = \text{const.} \\ \dot{Q} = \frac{\partial H}{\partial P} = \omega \Rightarrow Q(t) = \omega t + \alpha \end{array}\right.$

so $\begin{cases} p = \sqrt{2mE}\cos(\omega t + \alpha) \\ q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha) \end{cases}$ ✓

note that $\frac{q^2}{\frac{2E}{m\omega^2}} + \frac{p^2}{2mE} = 1$ is an ellipse

In general, suppose that $F(\{q\},\{p\},t)$ is an integral of the eq. of motion: $\frac{dF}{dt}=0$

now: $$\frac{dF}{dt}=\frac{\partial F}{\partial t}+\sum_i\left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial p_i}\dot{p}_i\right)$$

$$= \;\;'' \;+\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$=\frac{\partial F}{\partial t}+\{F,H\}$$ w/ $\{F,G\}=\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right)$

Poisson br.

so for an IEM: $\cancel{\frac{\partial F}{\partial t}+\{F,H\}=0}$ $\Rightarrow \frac{\partial F}{\partial t}=\{H,F\}$

if the IEM F does not depend on t: $\{H,F\}=0$

$$\{q_i,q_k\}=\{p_i,p_k\}=0 \qquad \{q_i,p_j\}=\delta_{ij}$$

$$\{\vec{\omega},\vec{\omega}\}=J$$

go to new variables: $\vec{\Omega}=\vec{\Omega}(\vec{\omega})$

~~then~~ $\cancel{\{\vec{\Omega},\vec{\Omega}\}=\left(\frac{\partial\vec{\Omega}}{\partial\vec{\omega}}\right)^T J\left(\frac{\partial\vec{\Omega}}{\partial\vec{\omega}}\right)}$

$$\{\Omega_i,\Omega_k\}=\frac{\partial\Omega_i}{\partial\omega_j}\{\omega_j,\omega_e\}\frac{\partial\Omega_k}{\partial\omega_e}$$

$$\{\vec{\Omega},\vec{\Omega}\}=\frac{\partial\vec{\Omega}}{\partial\vec{\omega}}J\left(\frac{\partial\vec{\Omega}}{\partial\vec{\omega}}\right)^T=MJM^T\overset{\text{if canonical}}{=}J=\{\vec{\omega},\vec{\omega}\}$$

so the P.B. is a canonical invariant.

Properties of P.B: antisymm: $\{F, G\} = -\{G, F\}$

linear: $\{aF + bD, G\} = a\{F, G\} + b\{D, G\}$

prod: $\{FD, G\} = \{F, G\}D + F\{D, G\}$

Jacobi id: $\{\{F, D\}, G\} + \{\{D, G\}, F\} + \{\{G, F\}, D\} = 0$

Poisson theo: If I_1, I_2 are IEM, then

$$I_3 = \{I_1, I_2\} \text{ is an IEM}$$

use Jacobi id. to prove it.

I_3 depends on I_1, I_2

Two IEM are in involution if $\{I_1, I_2\} = 0$

Ex: $\{J_i, J_j\} = \epsilon_{ijk} J_k$ for free particle

$\{P_i, J_j\} = \epsilon_{ijk} p_k$ Kepler problem

If J_x, J_y are IEM $\Rightarrow$ J_z is IEM

however J_x, J_y are not in involution

$\{J^2, J_z\} = 0 \Rightarrow J^2, J_z$ are in involution

A system is called (Liouville) integrable if it has n IEM I_j $j=1,..n$ which are in involution $\{I_i, I_j\} = 0$ $\forall i,j$

Consider 2 matrices M, L. If the eq. of motion is

$\dot{L} = [M, L]$ then M, L are a <u>Lax pair</u>

L generates IEM : take $F_k = \mathrm{tr}\, L^k$

then $\dot{F}_k = \mathrm{tr}\, \dot{L}^k = -k\, \mathrm{tr}\left(L^{k-1} [L, M] \right)$

$$= -k\, \mathrm{tr}\left(L^k M - L^{k-1} M L \right) = 0$$

$\Rightarrow F_k = \mathrm{tr}\, L^k = \text{const.} \Rightarrow$ eigenvalues of L are constants (IEM)

Note that $L(t) = g(t) L(0) g^{-1}(t)$ $\quad M = \dot{g}(t) g^{-1}(t)$

is a solution to Lax eq : $\dot{L} = [M, L]$

in fact : $\dot{L}(t) = \dot{g}(t) L(0) g^{-1}(t) - g(t) L(0) \dot{g}(t) g^{-2}(t)$

$$= ML - LM$$

Ex: the harmonic oscillator

Define the Lax pair : $L = \begin{pmatrix} p & m\omega q \\ m\omega q & -p \end{pmatrix}$ $\qquad M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix}$

then $\dot{L} = [M, L] \Rightarrow \begin{pmatrix} \dot{p} & m\omega\dot{q} \\ m\omega\dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -m\omega^2 q & \omega p \\ \omega p & m\omega^2 q \end{pmatrix}$ (check it)

which yields the usual eq.'s : $\begin{cases} \dot{p} = -m\omega^2 q \\ \dot{q} = \frac{p}{m} \end{cases}$

furthermore, $F_1 = \mathrm{tr}\ L = 0$

$$L^2 = \begin{pmatrix} p^2 + m^2\omega^2 q^2 & 0 \\ 0 & p^2 + m^2\omega^2 q^2 \end{pmatrix} = \begin{pmatrix} 2mE & 0 \\ 0 & 2mE \end{pmatrix}$$

so $F_2 = \mathrm{tr}\ L^2 = 4mH \Rightarrow \boxed{H = \frac{1}{4m} \mathrm{tr}\ L^2}$

One can consider Lax pairs dependent on a spectral param. λ

Then $F_k(\lambda) = \mathrm{tr}\ L^k(\lambda) = \sum_i F_{k,i} \lambda^i$

then each $F_{k,i}$ is a conserved quantity or IEM

R-matrix

L, H $\quad L$ diagonalizable : $L = U \Lambda U^{-1}$

Λ diagonal $\Rightarrow \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$; $\lambda_i \quad i=1,..,k$ are conserved quantities

Let E^i be N basis vectors $\Rightarrow E^i_j$: matrix unities

s.t. $E^i_j E^k = \delta_j^k E^i$

we can write $\quad L = L^i_j E^j_i \quad$ i.e. L^i_j are entries of Lax matr.

We want to compute P.B's between L^i_j, i.e. compute $\{L^i_j, L^k_e\}$

Embed Lax matrix in double tensor prod:

$$L_1 = L \otimes 1\!\!1 = L^i_j E^j_i \otimes 1\!\!1$$

$$L_2 = 1\!\!1 \otimes L = L^i_j \, 1\!\!1 \otimes E^j_i$$

so the index refers to the space in which the matrix is embedded.
Consider matr. which act on tensor prod:

$$T_{12} = T^{ij}_{ke} E_i^k \otimes E_j^l$$

$$T_{21} = T^{ij}_{ke} E_j^l \otimes E_i^k \qquad \text{is } T_{12} \text{ permuted}$$

Eigenvalues of Lax matrix are in involution iff there is fu. r_{12}

s.t. $\quad \{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] \quad$ *) (no proof given)

where $\quad r_{12} = q_{12} + \frac{1}{2}[k_{12}, L_2]$

w/ $\quad k_{12} = \{U_1, U_2\} U_1^{-1} U_2^{-1} \qquad q_{12} = U_2 \{U_1, \Lambda_2\} U_1^{-1} U_2^{-1}$

Poisson brackets satisfy Jacobi id's. The one related to $*]$ requires that

$$\left\{ L_1 , [r_{12}, r_{13}] + [r_{12}, r_{23}] + \{L_2, r_{13}\} - \{L_3, r_{12}\} \right\} + \text{cycl. perm.} = 0$$

If r is a constant, (i.e. does not depend on PS) then Jacobi is satisfied if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{23}, r_{13}] = 0 \qquad \square)$$

If r is antisymm, then $r_{21} = -r_{12}$

and $\square)$ is called <u>Yang-Baxter</u> eq.

For the harmonic oscillator,

$$\cancel{r = \frac{1}{g}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{g}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

{r is dynamical; it is not a constant}

$$r_{12} = -\frac{\omega}{4mH}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes L$$

Lax and Monodromy

After having looked at 1-dim. systems (which depend on t) in classical mechanics, let us look at field theories in 1+1 dimensions. One calls a field theory integrable if the field eq's satisfy $\partial_t L - \partial_x M = [M, L]$ for a pair of matrices L, M.

This is called a Lax connection. The nomenclature can be understood if choose $M = A^{(t)}$ $\quad L = A^{(x)}$

then
$$\partial_t A^{(x)} - \partial_x A^{(t)} + [A^{(x)}, A^{(t)}] = 0$$

~~The Lax connection can be seen as a compact;~~

Consider a system w/ a Lax connection and a spectral param. u and periodic b.c.'s. Introduce the monodromy matrix as the path ordered exp:

$$T(u) \equiv P \exp \int_0^{2\pi} dx \, L(x, t, u)$$
$$= 1 + \int_0^{2\pi} dx_1 \, L(x_1) + \frac{1}{2} \int_0^{2\pi} dx_1 \int_{x_1}^{2\pi} dx_2 \, L(x_1) L(x_2) + \dots$$

It can be shown that ~~∂_t~~

$$\boxed{\partial_t T(u) = [M(0), T]}$$

Then the transfer matrix, defined as $\tau = \mathrm{tr}\, T(u)$,

has $\partial_t \tau = \mathrm{tr}\, \partial_t T(u) = \mathrm{tr}[M(0), T] = 0$

so $\tau(u) = \sum_i Q_i u^i$ is conserved for all values of u.

So, expanding τ ~~about~~ as a series in u, about some point, generates an infinite tower of IEM.

P.S. Monodromy comes from the auxiliary linear problem to the Lax connect.

Suppose there is a fn. ψ s.t. $\begin{cases} (\partial_x - L)\psi = 0 \\ (\partial_t - M)\psi = 0 \end{cases}$

then $\begin{cases} \partial_t \partial_x \psi = \partial_t (L\psi) = (\partial_t L)\psi + LM\psi \\ \partial_x \partial_t \psi = \partial_x (M\psi) = (\partial_x M)\psi + ML\psi \end{cases}$

so $\partial_t \partial_x \psi = \partial_x \partial_t \psi \iff \partial_t L - \partial_x M = [M, L]$ i.e. the Lax connect.

Now, suppose ~~that~~ there is a matrix A s.t. $\psi(2\pi, t) = A\,\psi(0, t)$

then $0 = \partial_t [\psi(2\pi, t) - A\psi(0,t)]$

$0 = M(2\pi)\psi(2\pi, t) - (\partial_t A)\psi(0,t) - A M(0)\psi(0,t)$ since $M(2\pi) = M(0)$

$0 = M(0) A \psi(0,t)$

$\Rightarrow (\partial_t A - [M(0), A])\psi(0,t) = 0$

so $\partial_t A = [\mathcal{H}(0), A]$ i.e. A satisfies the same eq. as T

therefore we can take $\psi(2\pi, t) = T(u)\,\psi(0,t)$

r matrix

The condition $\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$

for the eigenvalues of a Lax matrix to be in involution in a mech. system

translates in a field theory in the condition:

$$\{L_1(x,t,u), L_2(y,t,v)\} = [r_{12}(u-v), L_1(x,t,u) + L_2(y,t,v)]\,\delta(x-y)$$

w/ $r_{21}(-u) = r_{12}(u)$

Imposing the Jacobi identity, this implies:

$$[r_{12}(u_1-u_2), r_{13}(u_1-u_3)] + [r_{12}(u_1-u_2), r_{23}(u_2-u_3)] + [r_{13}(u_1-u_3), r_{23}(u_2-u_3)] = 0$$

which is again the classical Yang-Baxter eq.

On the elements of the monodromy, this implies that:

$$\{T_1(u), T_2(v)\} = [r_{12}(u-v), T_1(u)\,T_2(v)]$$

which is called Sklyanin bracket.

Taking the trace in both spaces, this implies for the transfer matrix

that: $\{\tau(u), \tau(v)\} = 0$

so the conserved quantities of the transfer matrix are in involution