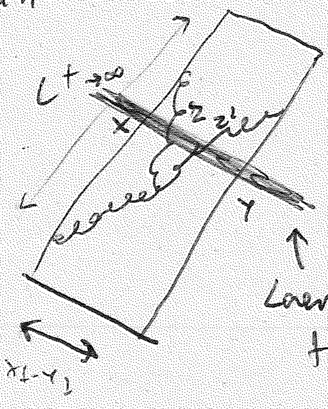
Lecture 3

Loop corrections

Let's start from planar limit. The non-planar corrections are "similar" but add a lot of notational and technical complications.

Basic idea: include more partons in projectile's WF

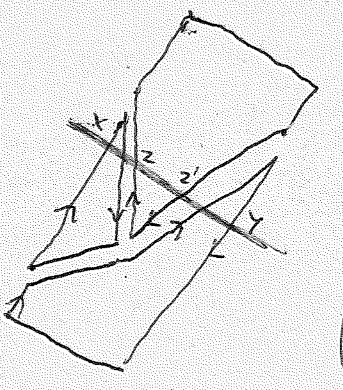


JJ or U large limit $\sim \int U(x, z, z'; y) \cdot \begin{pmatrix} \frac{\#}{2} \log^2 k_{max}/k_{min} \\ + \# \log k_{max}/k_{min} \\ + \text{finite} \end{pmatrix}$

At one-loop, the $\log k_{max}$ term gave us the LL BK equation.
At two-loops, the amplitude contains a $\log^2$ term that's just iterating it.
The two-loop BK comes from the single log term, ~~so the~~

As usual for RG eqs.

At large-N_c, the phase can be factorized into dipoles:

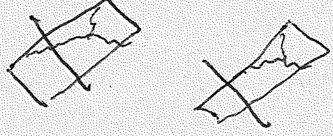


$\sim U(x,z)\, U(z,z')\, U(z',y)$,

(Since the eikonal phase depends only on transverse position at crossing, the "angles" in the picture don't matter)

②

From virtual corrections, one gets graphs $\sim UU$ and $\sim U$



Again, $\langle 0|U|0\rangle = 1$ must be a stable fixed-point, so these are related:

$$\partial_\eta U_{xy} \sim \int_{z,z'} K^{(2)}(x,z,z';y)\left[U_{xz}U_{zz'}U_{z'y} - \tfrac{1}{2}U_{xz}U_{zy} - \tfrac{1}{2}U_{xz'}U_{z'y}\right]$$
$$+ \int_{z} K^{(2)}(x,z,y)\left[U_{xz}U_{zy} - U_{xy}\right]$$

$= \underline{\text{NLO BK}}$

The NLO kernels have been known for ~ 10 years [Balitsky + Chirilli].

~~One~~ At higher loops, always get <u>strings of dipoles</u> [SCH '13]

3-loops: $$\partial_\eta U_{xy} \sim \int_{zz'z''} K(x,z,z',z'';y)\left[U_{xz}U_{zz'}U_{z'z''}U_{z''y} - \cdots\right]$$
$$+ \int \qquad [UUU - UU]$$
$$+ \int \qquad [UU - U]$$

3-loop kernels known, at least in $N=4$, currently. [SCH + Herranen '16]

<u>Linearization</u>. For small dipoles, (or "strict large N_c", see below), $U = 1 - \mathcal{U}$ ← small.

Expanding in $\mathcal{U}$ one gets a linear eq with at most 2 integrals:

$$\partial_\eta \mathcal{U}_{xy} \sim \int_{z,z'} \mathcal{U}_{zz'}\, K(x,y,z,z';a_s N_c) \quad \text{for some kernel } K.$$

Plugging in the power-laws $\mathcal{U}_{xy} = |x-y|^{1+i\nu} e^{im\arg(x-y)}$ as at 1-loop,

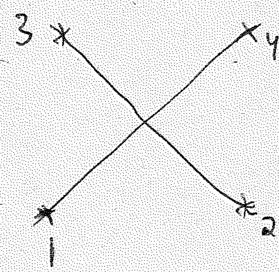
this reproduces the <u>BFKL Pomeron trajectory</u>

(to 2-loops in QCD, and 3-loops in $N=4$)

Perturbative amplitudes

When both projectile / targets are perturbative, amplitude is computable.

Ex: $\gamma^*\gamma^* \to \gamma^*\gamma^*$



Rapidity OPE for projectile / target:

$$J_4 J_1 \underset{\text{large boost}}{\sim} \int_{x_\perp, y_\perp} C(x_\perp, y_\perp)\, U(x_\perp, y_\perp) + g_s \int_{x,y,z} C(x,y,z)\, U(x,y,z) + \dots$$

$$J_3 J_2 \sim \int_{x,y} C(x,y)\, \bar{U}(x,y) + \dots$$

null Wilson lines along x^- direction.

Important: $\langle 0 | T\, U(x,y)\, \bar{U}(x',y') | 0 \rangle = 1 - \mathcal{O}(1/N_c^2)$

(exercise 3)

"large-N_c factorization".

$\Rightarrow$ to compute the (connected) planar amplitude, the linearized field is small: $\boxed{\mathcal{U} \sim 1/N_c^2}$ Keep only one dipole

$\Rightarrow$ To leading order in $1/N_c$, but all orders in 't Hooft coupling $\lambda = g^2 N_c$,

$$J_4 J_1 \underset{\text{large boost}}{\sim} \langle 0 | J_4 J_1 | 0 \rangle + \int_{x,y} C(x,y,\lambda)\, \mathcal{U}(x,y)$$

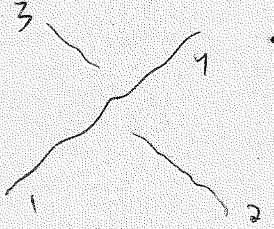
$\Rightarrow$ large-N_c amplitude controlled by Pomeron exchange.

Works even at strong coupling (AdS/CFT)!

Ex 2: <u>Parton scattering</u>

Consider $gg \to gg$.

OPE: $a_\lambda(p_4)\, a^\dagger_{\lambda'}(p_1) \sim \delta_{\lambda\lambda'}\, C(\underbrace{p_1+p_4}_{q_\perp})\, U(q_\perp) + \int UU + \cdots$



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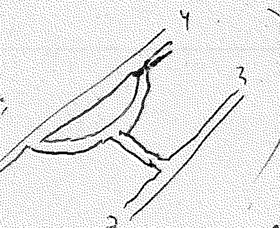
Again, in planar limit, $U \to 1 - \mathcal{U}$ $\sim \frac{1}{N^2}$

$\sim \delta_{\lambda\lambda'}\, C(q_\perp, \lambda)\, U(q_\perp)\, \left(1 + O(1/N_c^2)\right)$

$\Rightarrow$ Both projectile and target gluons get replaced by single <u>Wilson line</u>

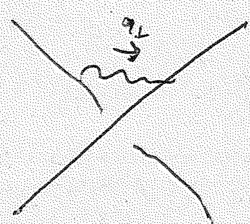
ex:



$\Rightarrow \mathcal{M}(1234) \longrightarrow \langle 0 | T\, U(q_\perp)\, \bar{U}(q_\perp) | 0 \rangle \cdot C_L(q_\perp) C_R$

At leading order, $C_L \approx q^+$, $C_L C_R \approx s$,

and $\langle 0 | U \bar{U} | 0 \rangle$ comes from single-gluon exchange.



~~Take~~ $U \sim 1 + i g T^a \int_{-\infty}^{\infty} dx^+ A_+^a + \cdots$

$\Rightarrow q^+ = q^- = 0$ $\quad \Rightarrow \langle U \bar{U} \rangle \approx g^2 \cdot \frac{i}{q_\perp^2} \quad \Rightarrow \mathcal{M} = g^2 \frac{s}{t}$.

Usual gluon exchange (at $s \gg t$).

RG: What's RG eq for single $\mathcal{U}$? Start from linearized BK and take "spectator" to ∞.

$$\partial_\eta \mathcal{U}_{xy} = \frac{\alpha_s N_c}{2\pi} \int \frac{d^2 z\, (x-y)^2}{(x-z)^2 (z-y)^2} \left(\mathcal{U}_{xz} + \underline{\mathcal{U}_{zy} - \mathcal{U}_{xy}} \right)$$

Take $y \to \infty$: $\partial_\eta \mathcal{U}_x \to \frac{\alpha_s N_c}{2\pi} \int \frac{d^2 z}{(x-z)^2} \cdot \left(\mathcal{U}_z - \mathcal{U}_x \right)$

Putting in $\mathcal{U}_x \sim e^{i q_\perp \cdot x}$ gives IR divergent integral.

<u>Fortunately</u>, BK can be derived in any dimension!

$$\hookrightarrow e^{i q \cdot x} \frac{\alpha_s N_c\, \Gamma(1-\epsilon)}{2\pi} \int \frac{d^{2-2\epsilon} z}{\pi (z^2)^{1-\epsilon}} \left(e^{i q \cdot z} - 1 \right) \approx \frac{\alpha_s N_c}{2\pi \epsilon} \left(\frac{q^2}{\mu^2} \right)^{-\epsilon} \equiv \alpha_g(q) \quad \text{gluon Regge trajectory.}$$

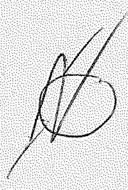
Result:

⇒ 2→2 scattering in planar limit takes form:

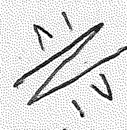
$$M(1234) \xrightarrow[(s\gg t)]{\text{Regge}} s^{a_g(\frac{t}{\mu^2},\lambda)}\, C(t/\mu^2,\lambda)\,, \quad \underline{\text{to all orders in } \lambda},$$

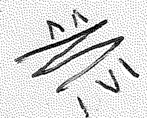
(a some "trajectory" $a(t,\lambda)$ = eigenvalue of evolution on $U(q_\perp)$.

For higher-point amplitudes ($2\to4$, $3\to3$), can get non-trivial "dipoles",



like 

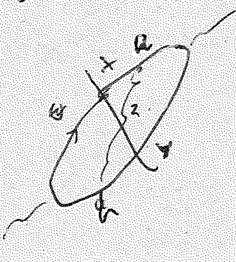
and $2\to6$: 

$2\to8$:  etc.

Eigenvalues more complicated, but controlled (at large N_c) in $N=4$ SYM or 1-loop QCD, by $\underline{\text{integrable spin chain}}$.

Beyond planar limit.

We still have an expansion in terms of Wilson lines:



$$\sim \mathrm{Tr}(U_y^\dagger T^a U_x T^b)\cdot U^{ab}_{adj}(z)$$

Perturbatively, trick is to expand in "$U-1$".

Optimal way: write $\boxed{U(z) = e^{igW^a(z)T^a}}$ ← Reggeized gluon field. (as in pion effective Lagrangian, $U=e^{i\pi}$)

Note:

1. $W^a = \int_{-\infty}^{\infty} dx^+ A_+^a + \frac{g}{2} f^{abc} \int_{x_1^+<x_2^+} A_+^b(x_1) A_+^c(x_2) + \ldots$

   = operator which creates Reggeized gluon

2. Same W^a works in any rep.

3. Gauge-inv. for gauge-transforms which vanish at ∞
   (= same sense as "S-matrix is gauge-inv.")

$\Rightarrow$ Perturbative projectiles can be expanded in powers of (gW).

Ex: Dipole: $U(x,y) = \frac{1}{N_c} \mathrm{Tr}\left(U(x)\, U^\dagger(y) \right)$

$$= 1 - \frac{g^2}{4N_c} (W_x - W_y)^a (W_x - W_y)^a + O(W^4).$$

The LO correlator of W's, we've just computed in mom-space:

$$\langle 0 | T\, W^a(p)\, \bar{W}^b(p') | 0 \rangle = \delta^{ab} (2\pi)^{2-2\epsilon} \delta^{2-2\epsilon}(p+p') \frac{i}{p^2}.$$

In coord space: $\langle W^a(x)\, \bar{W}^b(y) \rangle = \frac{\delta^{ab}}{4\pi}\, i \log\left(\frac{(x-y)^2}{\mu^2} \right) + O(\epsilon)$

(Note $\langle 0 | T\, W W | 0 \rangle = 0$: parallel WL's don't scatter, need lines which actually scatter: ).

$\Rightarrow$ ~~Dipole-dipole scattering $\langle 0 | U \bar{U} | 0 \rangle =$~~

~~$= \dots \log^2(x$~~

For more Reggeons, Wick's thm: $\langle W_1 W_2 \bar{W}_3 \bar{W}_4 \rangle = \langle W_1 \bar{W}_3 \rangle \langle W_2 \bar{W}_4 \rangle + \langle W_1 \bar{W}_4 \rangle \langle W_2 \bar{W}_3 \rangle + O(g^2).$

$\Rightarrow$ Solution to exercise 3:

$$\langle 0 | T\, U(x,y)\, \bar{U}(x',y') | 0 \rangle = 1 + 2 \left(\frac{g^2}{4N_c} \right)^2 \langle (W_x - W_y)^a (\bar{W}_{x'} - \bar{W}_{y'})^b \rangle^2$$

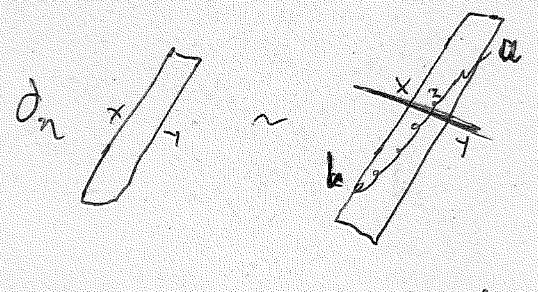
$$\boxed{= 1 - 2 \left(\frac{\alpha_s}{4N_c} \right)^2 (N_c^2 - 1) \log^2 \left(\frac{(x-x')^2 (y-y')^2}{(x-y')^2 (y-x')^2} \right)}$$

conformally-inv. cross-ratio.

$\left(\simeq 1 - \# \frac{\lambda^2}{N_c^2} \text{ at large } N_c \text{, as claimed} \right).$

Once we have expanded projectile / target in powers of W,
the energy dependence is fixed by Evolution eq.

To understand it at finite N_c, look at dipole:



$$\partial_\eta U(x,y) \sim \int_z \mathrm{Tr}\left(U_F^\dagger(y) T^a U_F(x) T^b \right) U_{adj}^{ab}(z)$$

This contains 3 types of WL's: U_F, $U_F^\dagger$, U_{adj}.

Notationally, the trick is to introduce <u>color rotation operators</u>: $U_R \to T_R^a U_R$.

More precisely, introduce rotation of color at the point x:

$$T_L^a(x) = \left(T^a U(x)\right)\frac{\delta}{\delta U(x)}, \qquad T_R^a(x) = \left(U(x) T^a\right)\frac{\delta}{\delta U(x)}$$

Left / right (= future / past) rotations commute: $[T_L, T_R] = 0$

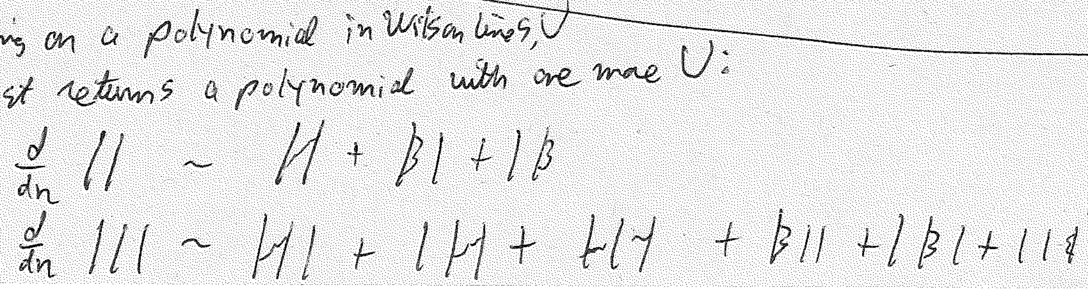
$$[T_R^a, T_R^b] \sim i f^{abc} T_R^c\, \delta^2(x-x')$$

Then, the BK eq. can be uplifted to a form that works on any # of WL's:

$$\boxed{\begin{aligned}\frac{d}{d\eta} = \frac{\alpha_s}{2\pi^2} \int d^{2-2\epsilon}x\, d^{2-2\epsilon}y\, d^{2-2\epsilon}z\, K(x,y;z) \\ \cdot\Big[U_{adj}^{ab}\left(T_L^a(x) T_R^b(y) + T_L^a(y) T_R^b(x)\right) - T_L^a(x) T_L^b(y) \\ - T_R^a(x) T_R^b(y) \Big]\end{aligned}}$$

(Balitsky – JIMWLK eq.)

Acting on a polynomial in Wilson lines, U
it just returns a polynomial with one more U:



= Sum over pairwise interaction

The kernel can be written in any dimension as:

$$K(x,y;z) = \frac{\Gamma(1-\epsilon)^2}{\pi^{-2\epsilon}}\,\mu^{2\epsilon}\;\frac{(x-z)\cdot(y-z)}{\left[(x-z)^2(z-y)^2\right]^{1-\epsilon}}\,.$$

In practice, this can:

- Be simulated numerically (Monte-Carlo)
- Be solved by expanding in Reggeized gluons W^a.

The expansion is straightforward (but tedious) algebra.

[SCH 1309.6521]

Just use: $U(x) = e^{igW^a(x)T^a}$

$$igT_L^a(x) = \frac{\delta}{\delta W^a(x)} + \frac{g}{2} f^{abc} W^c(x)\frac{\delta}{\delta W^b(x)} + \frac{g^2}{12} WW\frac{\delta}{\delta W} + \frac{g^4}{720} WWWW\frac{\delta}{\delta W}$$

$+ \cdots$ (Baker-Campbell-Hausdorff formula)

Plugging in gives:

$$\frac{d}{d\eta} = \frac{\alpha_s}{2\pi^2}\int d^{2-2\epsilon}x\, d^{2-2\epsilon}y\, d^{2-2\epsilon}z\; K(x,y;z)\, f^{aa'c} f^{bb'c}\,(W(x)-W(z))^{a'}(W(y)-W(z))^{b'}\frac{\delta^2}{\delta W^a(x)\,\delta W^b(y)}$$

$$+\frac{\alpha_s C_A}{2\pi^2}\int d^{2-2\epsilon}x\, d^{2-2\epsilon}z\,(W(z)-W(x))^a\frac{\delta}{\delta W^a(x)}$$

Second line gives the <u>gluon Regge trajectory</u> for each gluon, first line gives sum over <u>pairwise interactions</u> (BKP).

At large N_c, in $D=4$, ~~only~~ nearest-neighbor interactions survive

$\Rightarrow$ <u>Integrable spin chain</u> (Vittorio's lectures)

$\frac{d}{d\eta}$ |||| ~ H|| + |H| + ||H (open chain)

2→2 scattering, general structure (finite N_c)

To calculate 2→2 partonic amplitudes, use (rapidity) OPE:

$$a(p_4)\,a^{+}(p_1) \underset{\text{large boost}}{\sim} g W\,(1+O(g^2)) + g^2 WW + g^3 WWW + \cdots$$

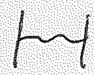
Since the coefficients are suppressed by more Reggeons,

at leading-log need only first term:

$$\mathcal{M} \sim \langle W \bar{W} \rangle .$$

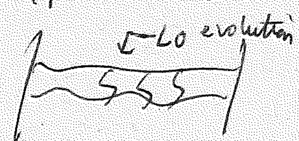
The rapidity evolution is diagonal, $\frac{d}{d\log s} W(p) = a_g(p)\, W(p)$

⇒ get Regge form $\mathcal{M} \sim g^2 \frac{s^{a_g(t)}}{t}$ (leading log).



At NLL: need two-gluon exchange. (in addition to NLO trajectory).

Because of the explicit g's, two-gluons only needed at LO:



At NNLL, similarly, only need LO ingredients in 3-reggeon exchange

Also, 1→3 and 3→1 pieces in evolution

(tested recently ...)

Solution to exercises from lecture 2.

1. Start from BFKL (1-loop) eigenvalue, w/ $m=0$:

$$j-1 = \tilde{a}\left[2\psi(1) - \psi\left(\tfrac{1+i\nu}{2}\right) - \psi\left(\tfrac{1-i\nu}{2}\right)\right].$$

Set $\Delta = 2+i\nu$ (scaling dimension) and expand:

$$(*) \qquad j-1 = \tilde{a}\left[\frac{-2}{\Delta-3} + 0 + 0 + \frac{1}{2}\zeta_3(\Delta-3)^2 + \frac{1}{8}\zeta_5(\Delta-3)^4 + \cdots\right].$$

Solving for $\Delta-3$ is conceptually tricky since the BFKL series is very singular $\left(\sim \frac{a^L}{(\Delta-3)^{2L-1}}\right)$.

It's easier to go the other way, DGLAP $\to$ BFKL.

$$\text{DGLAP} \quad \boxed{\underbrace{(\Delta-3)-(j-1)}_{\text{twist}} = \frac{-2\tilde{a}}{j-1} + (\cdots)} \qquad (*a).$$

The useful regime is: $\tilde{a} \ll j-1 \ll \sqrt{\tilde{a}}$ $\Rightarrow$ $\Delta-3 \simeq \frac{-2\tilde{a}}{j-1}$.

($\tilde{a} \ll j-1$: so that "$\cdots$" is small; $j-1 \ll \sqrt{\tilde{a}}$: so that $\frac{\tilde{a}}{j-1} \gg j-1$.)

~~Then we have:~~ Then, ~~…~~ term

multiply $(*a)$ by $\frac{(j-1)}{\Delta-3}$ $\Rightarrow$ $j-1 = \frac{-2\tilde{a}}{\Delta-3} + \left[\frac{(j-1)^2}{\Delta-3} + \frac{j-1}{\Delta-3}(\cdots)\right]$.

The bracket is "small", so this can be solved iteratively.

$\Rightarrow$ a term $\left(\frac{\tilde{a}}{j-1}\right)^L$ in ~~DGLAP~~ $(\cdots)$ gives $j-1 \approx -\frac{\tilde{a}}{2}\left(\frac{\Delta-3}{-2}\right)^{L-2}$ in BFKL.

Subleading poles $\frac{\tilde{a}^L}{(j-1)^{L-m}}$ gives a^{m+1} in BFKL $\Rightarrow$ ignore.

$\Rightarrow$ $(*)$ can only come from: $\boxed{\Delta - j - 2 = \frac{-2\tilde{a}}{j-1} + 0 + 0 - 4\zeta_3\left(\frac{\tilde{a}}{j-1}\right)^4 - 4\zeta_5\left(\frac{\tilde{a}}{j-1}\right)^6 + \cdots}$

~~2. Soluti~~

Solution to problem 2:

(Show conformal symmetry of BK under inversion: $x_\perp \to \frac{x_\perp}{x_\perp^2}$.)

The basic fact is: $(x_\perp - y_\perp)^2 \to \left(\frac{x_\perp}{x_\perp^2} - \frac{y_\perp}{y_\perp^2}\right)^2 = \frac{(x_\perp - y_\perp)^2}{x_\perp^2\, y_\perp^2}$.

If we do the inversion in $z_\perp$ too, get an extra Jacobian:

$$d^2z \to \frac{d^2z}{z^4}$$

$$\Rightarrow \quad \frac{d^2z\,(x-y)^2}{(x-z)^2(z-y)^2} \to \frac{d^2z}{\cancel{z^4}}\,\frac{\frac{(x-y)^2}{\cancel{x^2y^2}}}{\frac{(x-z)^2}{\cancel{x^2z^2}}\,\frac{(y-z)^2}{\cancel{y^2z^2}}} = \frac{d^2z\,(x-y)^2}{(x-z)^2(z-y)^2} \checkmark$$

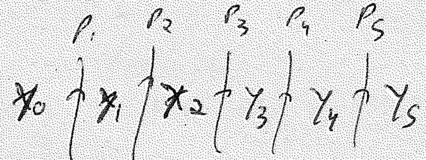
Invariant!

In fact, the form of the one-loop BK kernel is the only possibility consistent with this symmetry.

The 1-loop BKP has a further symmetry in fact, in the planar limit.

If one considers ~~the~~ a chain of Reggeons in mom. space:

$$W(p_1)\,W(p_2)\,W(p_3)\cdots W(p_n)$$



then, in terms of dual coords:

$$y_i = \sum_{j=1}^{i} p_j\,,$$

the evolution eq. in y's is the same in mom. and coord space!

This is related to the Wilson loop / amplitude duality in $N=4$ SYM and is the reason why the chain is integrable.

(conformal sym + "dual" conformal sym = Yangian)

(Problem 3 ~~solution~~ was given in lecture).