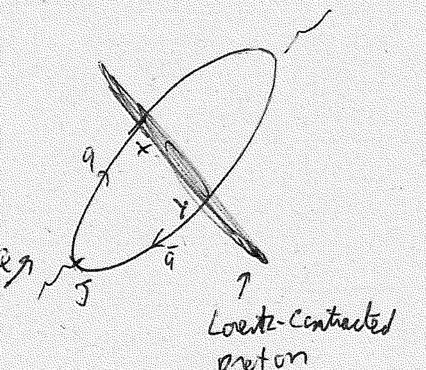
Lecture 2

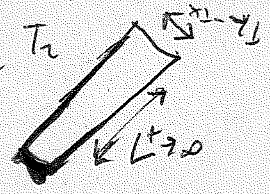
Review of last time, la DIS:

$$J^\mu(Q)\, J^\nu(-Q) \underset{\text{large boost}}{\sim} \int d^{2-2\epsilon}x_\perp\, d^{2-2\epsilon}y_\perp\; U(x_\perp, y_\perp)\; C(Q, x_\perp, y_\perp)$$



eikonal phase

$U = \frac{1}{N_c} \mathrm{Tr}$



Depends on two transverse coordinates and a rapidity cut-off

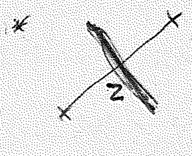
$$\Rightarrow \frac{d}{d\eta} U \sim \int_z (UU - U) \quad \text{BK eq.}$$

Last time's exercise was to compute $C(Q, x_\perp, y_\perp)$ "photon wavefunction" to leading order. Let's do it here for scalar loop and $J^\mu \to \phi^\dagger \phi \equiv \mathcal{O}$, $Q = (Q^+, 0, Q_\perp)$

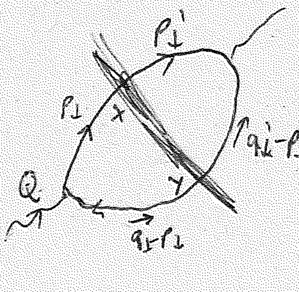
Solution:

From supplement last time, scalar propagator across shock is:

$$\langle \phi(x^+, p^+, p_\perp)\; \phi^\dagger(x'^+, p'^+, p'_\perp) \rangle_{\text{shock}} = \frac{\delta(p^+ + p'^+)}{2p^+} \int d^{2-2\epsilon}z\; U(z)\, e^{iz\cdot(p_\perp - p'_\perp)}\, e^{-i\frac{(p_\perp^2 + m^2)x^+ + (p_\perp'^2 + m^2)(-x'^+)}{2p^+}}$$



Consider the loop:



① Energy p^+ is conserved and must be positive $\Rightarrow p^+ = zQ^+$, $0 < z < 1$.

② Transverse mom. not conserved $\Rightarrow$ need $\int [dp_\perp][dp'_\perp]$ $\quad dp_\perp = \frac{d^{2-2\epsilon}p_\perp}{(2\pi)^{2-2\epsilon}}$.

③ Since $Q^- = 0$, get two time integrals: $\int_{-\infty}^{0} dx'^+ \int_0^\infty dx^+$

[stop at shock since all pos-E modes propagate forward!]

The time integrals (x^+) can be done directly since they only appear in phases.

$$\Rightarrow \mathcal{O}(Q)\mathcal{O}(-Q) \sim Q^+ \int d^2x_\perp d^2y_\perp U(x,y) \int_0^1 \frac{dz}{(2Q^+z)(2Q^+(1-z))}$$

$$\cdot \int [dp][dp'] \frac{1}{\frac{p_\perp^2+m^2}{2Q^+z} + \frac{(q_\perp+p_\perp)^2+m^2}{2Q^+(1-z)}} \frac{1}{\frac{p_\perp'^2+m^2}{2Q^+z} + \frac{(q_\perp-p_\perp')^2+m^2}{2Q^+(1-z)}} e^{i(p-p')\cdot(x-y)}$$

Cancelling factors $q2Q^+, z, 1-z$, we thus get

$$C(x_\perp, y_\perp, Q) = Q^+\int_0^1 dz\, z(1-z) \underbrace{\left[\int \frac{[dp]\, e^{ip\cdot(x-y)}}{(1-z)p_\perp^2 + z(q_\perp-p_\perp)^2+m^2}\right]}_{= K_0(\tilde{Q}|x-y|) \text{ times a phase}} \left[\, cc \,\right].$$

where $\tilde{Q} = \sqrt{z(1-z)q^2 + m^2}$

$$\boxed{\Rightarrow \frac{1}{Q^+} C(x_\perp, y_\perp, Q) = \int_0^1 dz\, z(1-z) K_0(\tilde{Q}|x-y|)^2}$$

Lessons: 1. Dipole size related to invariant mass (M, Q) NOT CM energy $s = 2Q^+p^- = Q^2/x$

$\Rightarrow$ Expect α_s to run with Q or "dipole size", NOT $\sqrt{s}$.

2. For actual γ and quarks,

$$\Phi_L = 2N_c \sum_f \frac{\alpha_{EM}}{\pi} \int_0^1 dz\, z(1-z)\, 4\tilde{Q}^2 K_0(\tilde{Q}|x-y_\perp|)^2,$$

$$\Phi_T = \dots$$

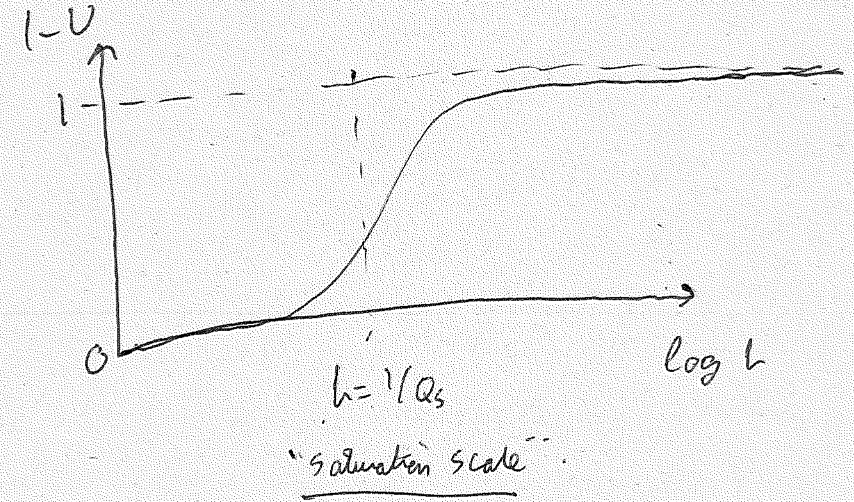
(See Kovchegov + McLerran ph/9903246; Donnachie et al "Pomeron phys & QCD" Ch. 8).

What do we expect for $U(x,y)$ in DIS?

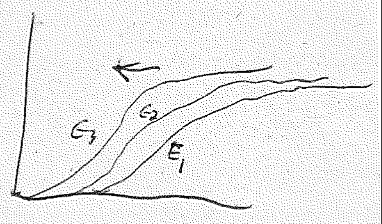
- For simplicity take $x-y \ll$ proton size, $\Rightarrow U = U(x-y) \equiv U(b)$.

- As $b \to 0$, the $q\bar{q}$ charges basically cancel : expect $U \to 1$
  "color transparency"

- As $b\Lambda_{QCD} \gg 1$ (or before), expect $U \to 0$ if the proton's color field is "strong" in any sense.



- With increasing E ($\propto \log 1/x$), more soft gluons $\Rightarrow$ curve moves up/left



- Useful analogy in literature:

BK equation for $\mathcal{N} = 1-U$:

$$\partial_\eta \mathcal{N} \sim S(\mathcal{N} - \mathcal{N}\mathcal{N})$$

$$\Leftrightarrow \quad \partial_t f = D\,\partial_x^2 f(x) + f(x)\big(1-f(x)\big)$$

$\partial_x^2 \Rightarrow \dfrac{\partial^2}{\partial \log b^2}$

"Reaction diffusion" eq.

$\mathcal{N} = 0$: unstable fixed point "transparency"
$\mathcal{N} = 1$: stable fixed point "opaque"

This eq. is used to describe e.g. population spreading; nonlinearity stops growth when the density is too high

"Gluons multiply until they start killing each other".

- RD equation exhibits <u>travelling waves</u>: front moves at constant velocity.
  This is also seen in solutions (numerical) to BK.

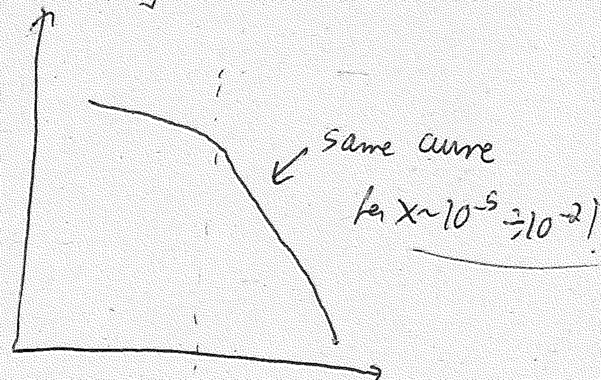
I will not pretend to be a phenomenologist,

but a nice idea is that, after some evolution, the shape approaches a universal one:

$U(b,x) \to U(b\,Q_s(x))$.

$\Rightarrow$ cross-section $\sigma[x, Q^2] \to \sigma[Q^2/Q_s^2(x)]$.

This is seen qualitatively in HERA data:



The right part of plot is essentially DGLAP (large Q^2), and is linear.

The break is (putatively) where the nonlinear effects, e.g. the fact that U saturates at 1, kicks in.

In practice, $Q_s \sim 1 \div 2$ GeV $\Rightarrow$ a bit low for perturbation thy.

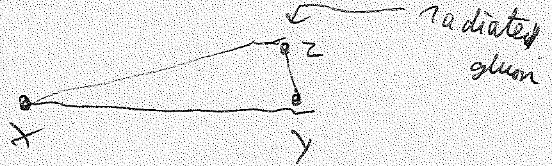
(Recall however that: $m_\tau \simeq 1.8$ GeV and hadronic τ decays, described perturbatively, give one of the best measurements of α_s).

So Q_s is not necessarily "too low", but factors of 2 (Q_s vs $Q_s/2$ or $2Q_s$) matter a lot! $\Rightarrow$ Need some control over corrections.

Running coupling

As mentioned, α_s should run with some transverse scale (not tot. energy).

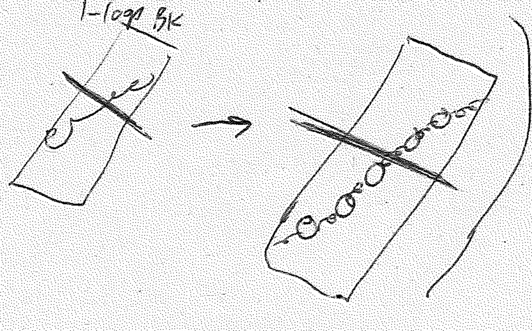
But which? $x-y$, $x-z$, $y-z$?



If they were all of the same order, this wouldn't matter so much.

But there are some large log region ($\log(Q/Q_s)$) where the sizes can be parametrically different.

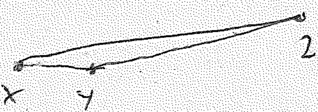
The consensus (from looking at fermion loop chains: and trying to minimize the loop effects)



is that one should use the smallest of the three scales.

e.g., if one end of the dipole moves by a small bit $|z-y| \ll |x-y|$ as above, then the relevant physics is at the scale $|z-y|$.

- If a very big dipole pair is generated



, use the size of the original $|x-y|$ dipole.

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*[Another natural constraint is that $\alpha_s(Q_{??})$ should be consistent with the BFKL eq., in resummations as described below.] I do not know if this has been tested.

Let's understand the growth in linear regime.

write: $U(x,y) = 1 - \mathcal{U}(x,y)$ ← small.

Here, ignore [illegible] dep.

BK: $\partial_\eta \mathcal{U}(b) = \frac{\alpha_s N_c}{2\pi} \int \frac{d^2 z}{\pi} \frac{b^2}{z^2 (b-z)^2} \Big(\mathcal{U}(z) + \mathcal{U}(b-z) - \mathcal{U}(b) - \mathcal{U}(z)\mathcal{U}(b-z) \Big)$

nonlinear

Scale-free, so power-law eigenfunctions:

$$\mathcal{U}_{\nu,m}(b) = |b|^{1+i\nu} e^{im \arg b}.$$

(m = azimuthal angular mom., remember we're in 2D!).

The evolution is diagonal, with eigenvalue:

$$\partial_\eta \mathcal{U}_{\nu,m} = \big(j(\nu,m) - 1\big) \mathcal{U}_{\nu,m}$$

rescale $z \to z/b$

with
$$\big(j(\nu,m)-1\big) = \frac{\alpha_s N_c}{2\pi} \int \frac{d^2 z}{\pi} \frac{1}{z^2 |1-z|^2} \Big(|z|^{1+i\nu} e^{im \arg z} + |1-z|^{1+i\nu} e^{im \arg(1-z)} - 1 \Big)$$

Note: 1. we call the eigenvalue $j-1$ because the energy growth in Regge theory is characterized by spin: $\sigma \sim (1/x)^{j-1}$.

2. The integral converges. Claude will describe methods to do such "Fourier-Mellin" integrals.

3. The exponent $1+i\nu$: fixed by self-adjointness (basically dimensional analysis, see exercise 3)

linearized BK eigenvalue:

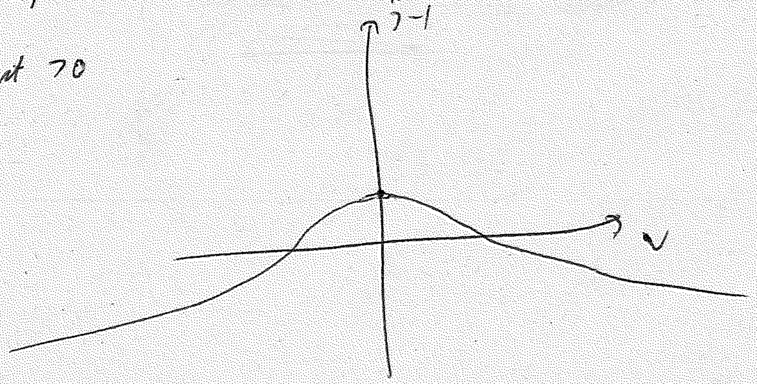
$$j(\nu, m) = 1 + \frac{\alpha_s N_c}{\pi}\left[2\psi(1) - \psi\left(\frac{1+|m|+i\nu}{2}\right) - \psi\left(\frac{1+|m|-i\nu}{2}\right)\right]$$

$$\left(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\right)$$

Exactly the same as BFKL eigenvalue!

(⇒ linearized BK, around $U=1$ "transparent" = BFKL).
More general, direct map described on Friday.

Key feature: intercept > 0



$$j(0,0) = 1 + \frac{\alpha_s N_c}{\pi} 4 \log 2 > 1$$

"BFKL Pomeron".

BFKL vs DGLAP.

Within linear regime, the small-dipole (large Q^2) limit of BFKL should match with small-x limit of DGLAP solutions!

Recall, in DIS, one define moments of PDFs: $\sigma_j(Q^2) \sim \int_0^1 dx\, x^{j-2}\, \sigma[x, Q^2]$.

Dependence on Q^2 is fixed by RG. Ignoring running coupling, for illustration,

$$\sigma_j(Q) = \sigma_j(Q_0) \cdot \left(\frac{Q_0}{Q}\right)^{\Delta(j)-j-2}$$

← twist of operators with twist ≈ 2, like $\mathrm{tr}\left(F^{\mu}{}_{+} (D_+)^{j-2} F_{\mu +}\right)$

Taking the inverse Mellin transform gives:

$$(*)\quad \text{DGLAP:}\quad \sigma[x, Q^2] \sim \int_{-i\infty}^{i\infty} dj\, x^{1-j} \left(\frac{Q_0}{Q}\right)^{\Delta(j)-j-2} \sigma_j(Q_0)$$

(+ subleading twists)

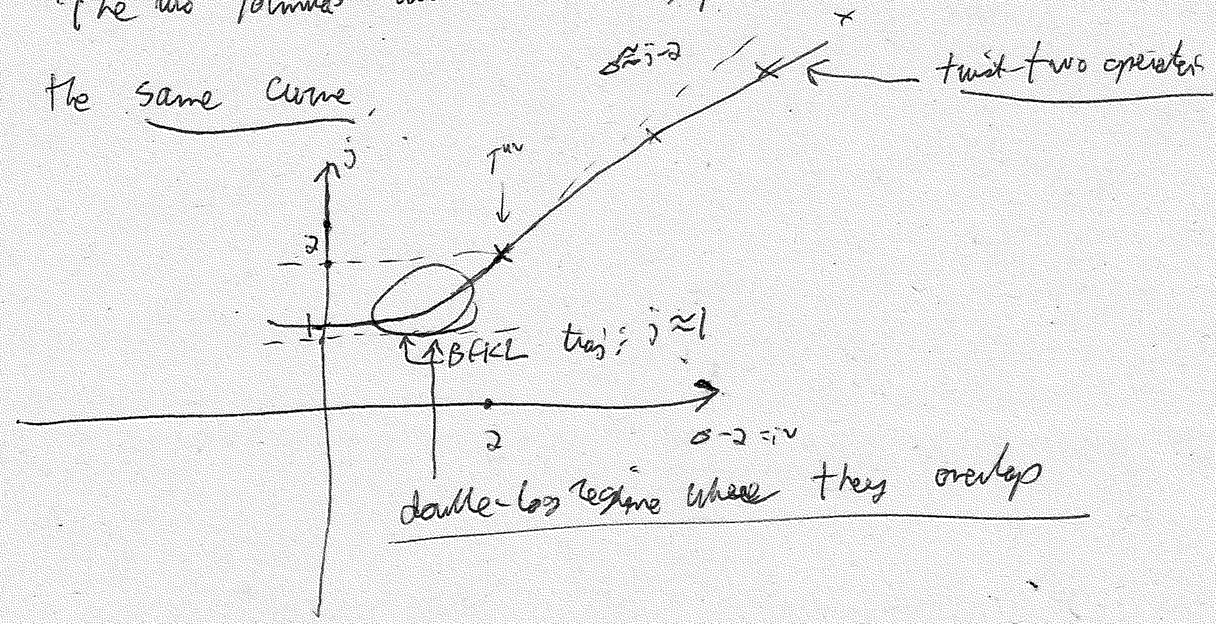
In BFKL / linearized BK, the evolution variable is x instead of Q^2, and the eigenfunctions labelled by $Q^{i\nu}$ are summed over. Setting $\Delta = 2 + i\nu$,

$$(**)\quad \text{BFKL:}\quad \sigma[x, Q^2] \sim \int_{-i\infty}^{i\infty} d\Delta \left(\frac{x Q_0}{Q}\right)^{1-j(\Delta)} \left(\frac{Q_0}{Q}\right)^{\Delta-3} c(x_0)$$

(+ subleading powers of x)

(the "integration constant" $x \to x\frac{Q_0}{Q}$ in BFKL case was taken so that the two exponents match).

$\Rightarrow$ The two formulas will be consistent, provided that $\Delta(j)$ and $j(\Delta)$ trace the same curve.



(See discussions in: Korchemsky et al ph/0306250

Polchinski et al th/0603115

SCH + Heinonen 1604.07417)

BFKL vs DGLAP: let's see how they match, near $(j,\Delta)=(1,3)$.

The 1-loop DGLAP (gluonic) anomalous dimension has <u>pole</u> at $j\to 1$:

$$\Delta(j)-j-2 \approx -\frac{2\tilde{a}}{j-1} \qquad \tilde{a} \equiv \frac{a_s N_c}{\pi}.$$

This pole reflects that the plot starts bending significantly there.

The 1-loop BFKL eigenvalue similarly diverges:

$$j(\Delta)-1 \approx \frac{-2\tilde{a}}{\Delta-3}. \qquad \text{The poles match!}$$

From 1-loop DGLAP, one can predict <u>leading poles</u> in BFKL.

The input is that the DGLAP series is in powers of $\Delta-j-2 \sim \frac{\tilde{a}^L}{(j-1)^L}$:

at most one pole per loop order.

$\Rightarrow$ in the regime $\tilde{a} \ll j-1 \ll 1$, we can truncate:

$$\Delta-3-(j-1) = \frac{-2\tilde{a}}{(j-1)} + \text{smaller}.$$

This is a quadratic eq. in $j-1$. Sol:

$$(j-1) = \frac{\Delta-3 \pm \sqrt{(\Delta-3)^2+8\tilde{a}}}{2}$$

~~The DGLAP ...~~

Take $\oplus$ sign, so that $j-1 \approx \Delta-3$ ("track 2") at $j>1$.

Then, for $\Delta<3$, the two terms $\approx$ cancel:

$$j-1 = \frac{-2\tilde{a}}{\Delta-3} + \frac{4\tilde{a}^2}{(\Delta-3)^3} + \# \frac{\tilde{a}^3}{(\Delta-3)^5} + \cdots$$

$\Rightarrow$ leading poles $\frac{\tilde{a}^L}{(\Delta-1)^{2L-1}}$ in BFKL <u>all predicted by DGLAP</u>.

It's actually possible to <u>resum</u> all those terms into a modified BK eq, allegedly improving the convergence of P.T. (see Iancu et al 1502.05642)

Exercises.

1. We've seen how LO DGLAP → leading poles $\frac{a^L}{(\Delta-3)^{2L-1}}$ in BFKL.

Conversely, LL BFKL → leading poles $\frac{a^L}{(j-1)^L}$ in DGLAP.

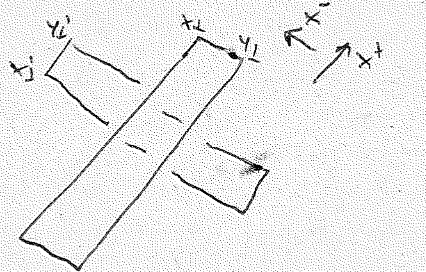
⇒ starting with BFKL eigen: (for $m=0$) $j(\Delta)-1 = \frac{a N_c}{\pi}\left[2\psi(1)-\psi\left(\frac{1+i\nu}{2}\right)-\psi\left(\frac{1-i\nu}{2}\right)\right]$, $\Delta=2+i\nu$,

invert the curve to find: $\Delta(j)-j-2 = \frac{2a_sN_c}{\pi}\frac{1}{j-1} + \#\left(\frac{a_sN_c}{\pi(j-1)}\right)^2 + \#(\,)^3 + \#(\,)^4 + \dots$

2. <u>Conformal symmetry</u>: Show that $\mathcal{U}(x_\perp,y_\perp)$ and $\mathcal{U}\left(\frac{x_\perp}{x_\perp^2},\frac{y_\perp}{y_\perp^2}\right)$ obey the *same* BK equation.

3. Compute the scattering amplitudes between two dipoles:

$$\langle 0|T\,U(x_\perp,y_\perp)\,\bar U(x'_\perp,y'_\perp)|0\rangle$$, at leading order (two-gluon-exchange)



3a) Argue that a target with wave-function $T(p,\nu) = \int d^2x\,d^2y\; e^{ip\cdot\left(\frac{x+y}{2}\right)}\,|x-y|^{-3+i\nu}\,\bar U(x,y)$,

will give rise, as $p\to 0$, to the translation-invariant exp. value <u>$\langle U(x,y)\rangle_T \propto |x-y|^{1+i\nu}$</u>.

Conclude that, for $p=0$, states with different ν are orthogonal:

$$\langle 0|T\,T(0,\nu)\,\bar T(0,\nu')|0\rangle \propto (2\pi)^2\delta^2(p+p')\,\delta(\nu+\nu')$$

This explains the "$1+i\nu$" eigen.