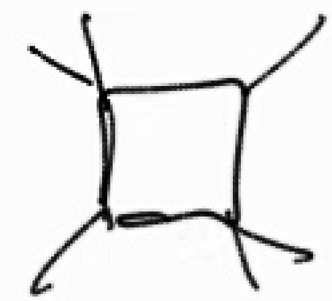
In this last lecture, we shall consider some applications of BFKL and MRK to amplitudes in $N=4$ Super Yang Mills, a theory which is maximally supersymmetric and conformally invariant, so its β function vanishes (also LL BFKL, having only energy logs, and not collinear logs, has a vanishing β function, but as we have seen, the NLL corrections do not).

In particular, we shall consider the 't Hooft limit of large N_c, with $\lambda = g^2 N_c$ fixed, such that only planar diagrams contribute.

At any order in the coupling, colour-ordered MHV amplitudes in planar $\mathcal{N}=4$ SYM can be written as tree-level amplitudes times a momentum dependent loop coefficient $M_n^{(L)} = M_n^{(0)} m_n^{(L)}$

(this would not be true in QCD)

In particular, at one loop $m_n^{(1)} = \sum_{pq} F^{2me}(p,q,P,Q)$



the n-pt amplitude can be written as a sum of "2-mass easy" boxes, and at all loops the MHV amplitude is written as (Bern Dixon Smirnov 2005)

$$m_n = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + \mathrm{Const}^{(l)} + E_n^{(l)}(\epsilon)\right)\right] + R$$

where $f^{(\ell)}(\epsilon)$ is a 2^{nd} order polynomial in ϵ $(d = 4-2\epsilon)$

$$f^{(\ell)}(\epsilon) = \frac{\hat{\gamma}_K^{(\ell)}}{4} + \epsilon \frac{\ell}{2} \hat{\mathcal{G}}^{(\ell)} + \epsilon^2 f_2^{(\ell)}$$

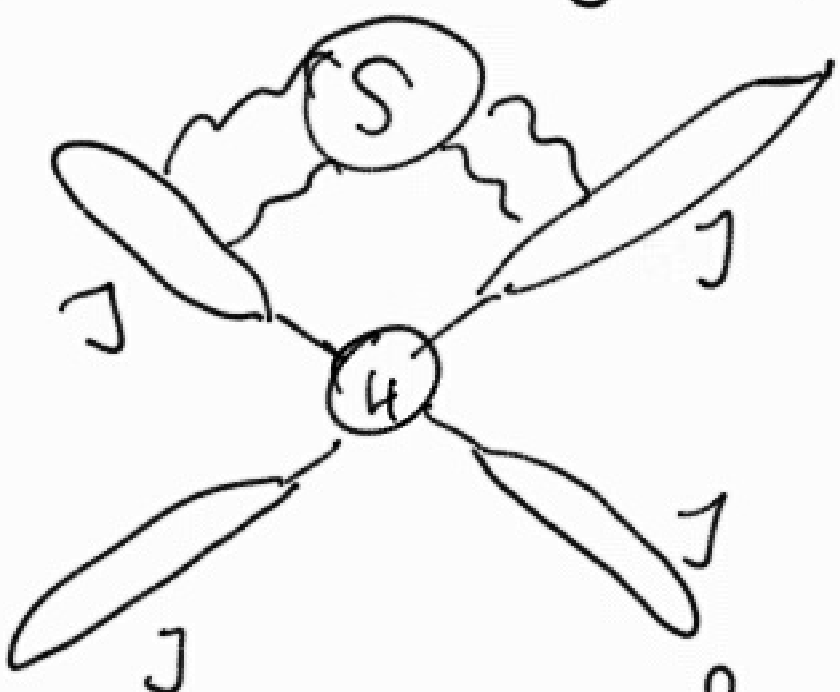
$\hat{\gamma}_K^{(\ell)}$ is the ℓ-loop cusp anomalous dimension

$\hat{\mathcal{G}}^{(\ell)}$ is the ℓ-loop collinear anomalous dim.

$$E_n^{(\ell)}(\epsilon) = O(\epsilon)$$

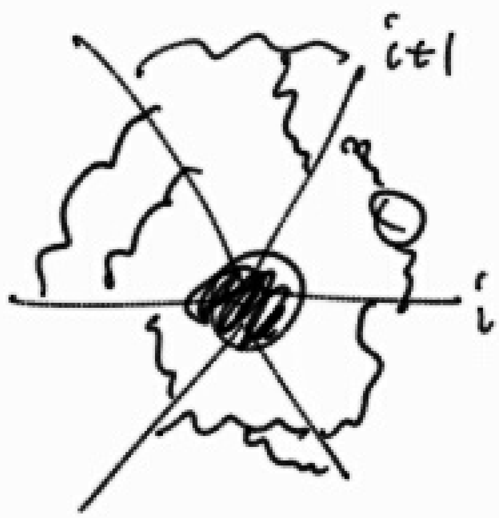
and R is a remainder function, which is finite in ϵ and is present only for 6 or more points and at 2 or more loops. R is a function of conformally invariant cross ratios.

The InfraRed (IR) structure is fixed by the factorization of a multi-leg amplitude, like in QCD, in terms of



jet, or collinear, functions, a soft function, and a hard (regular) function. However, colour-wise, the planar limit of $N=4$ SYM is trivial, and one can absorb the soft function into the jet functions, and be left with a factorised amplitude

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$



where each slice is the square root of a Sudakov form factor, which can be integrated because, since the β function vanishes, the coupling runs only through the dimension $\bar{\alpha}(\mu^2)\,\mu^{2\epsilon} = \bar{\alpha}(\lambda^2)\,\lambda^{2\epsilon}$

Thus the Sudakov form factor has a simple solution,

$$\ln\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha(\mu^2), \epsilon\right)\right] =$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^{n}\left(\frac{-Q^2}{\mu^2}\right)^{-n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right]$$

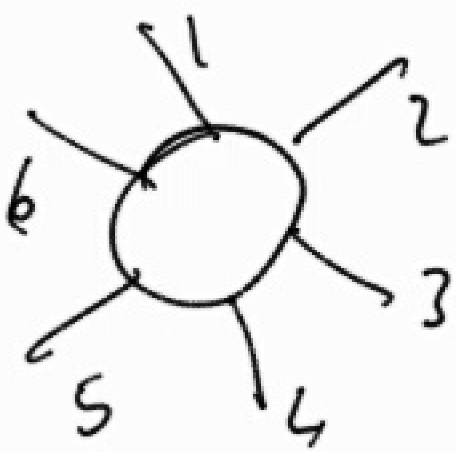
which yields the IR structure of the planar $N=4$ amplitudes and so of the BDS formula.

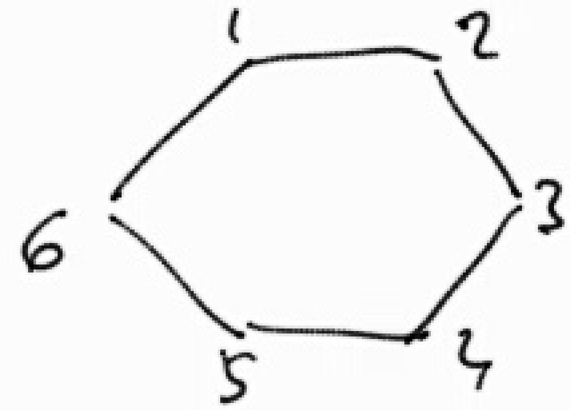
However, the BDS formula says much more: it says that the (conformal and dual conformal) symmetries of planar $N=4$ SYM completely fix also the finite part of 4- and 5-pt amplitudes to all loops. Beyond 5 points, the amplitudes can depend also on conformal cross ratios, through the remainder R. The symmetries fix which cross ratios the remainder R depends upon, but not the functional dependence of R on the cross ratios. That must be computed or derived otherwise. It is better to specify the cross ratios in dual space where to each leg p_i corresponds a segment $x_i - x_{i+1}$

The leg p_i is massless $p_i^2 = 0$ $\Rightarrow$ the segment $x_i - x_{i+1}$ is null-like $(x_i - x_{i+1})^2 = x_{i,i+1}^2 = 0$

Momentum conservation $\sum_{i=1}^{n} p_i = 0$ is fulfilled if $x_{n+1} = x_1$

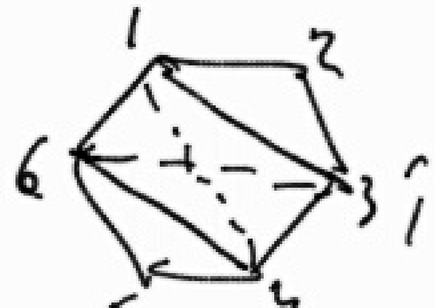
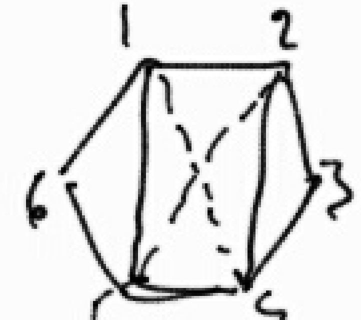
Then to an n-pt. amplitude corresponds an n-side polygon. In addition, the Mandelstam invariants are





$$x_{k,k+j}^2 = (p_k + \cdots + p_{k+j-1})^2$$

For $n = 6$, the cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$$  ; $$u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}$$  ; $$u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$ 

In MRK, the amplitude factorises as in QCD.

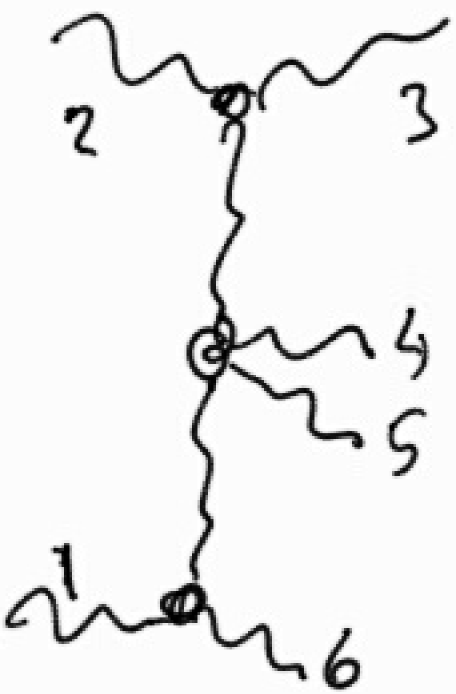
In particular, the building blocks of the multigluon amplitude, i.e. the Regge trajectory and the central-emission vertex, occur already in the 4-pt and 5-pt amplitudes, so they can be determined from there.

Because the remainder function R occurs first at 6 points, this implies that the remainder R vanishes in MRK.
This argument relies on a simple picture of multi-Regge factorisation where octet exchange in the t channel yields only Regge poles. We know that this picture holds true

for the real part of the amplitude in QCD at NLL level. This holds true also in $N=4$ SYM in Euclidean space (where all Mandelstam invariants are negative and the amplitudes are real).

However, starting from 6 points the amplitude develops a cut when analytically continued to some regions of Minkowski space. Accordingly, it acquires a discontinuity, which can be described by a BFKL-like dispersion relation (more on this later)

Let us consider now a 6-pt MHV amplitude in the QMRK :



$$p_3^+ \gg p_4^+ \simeq p_5^+ \gg p_6^+$$

$$p_3^- \ll p_4^- \simeq p_5^- \ll p_6^-$$

Please check that the cross ratios :

$$u_1 = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2} = \frac{s_{12}\, s_{45}}{s_{123}\, s_{345}}$$

$$u_2 = \frac{x_{24}^2\, x_{15}^2}{x_{25}^2\, x_{14}^2} = \frac{s_{23}\, s_{56}}{s_{234}\, s_{123}}$$

$$u_3 = \frac{x_{35}^2\, x_{26}^2}{x_{36}^2\, x_{25}^2} = \frac{s_{34}\, s_{61}}{s_{345}\, s_{234}}$$

do not take limiting values, like 0 or 1, in the QMRK

In the QMRK above, they become (please check)

$$u_1 \simeq \frac{s_{45}}{(p_4^+ + p_5^+)(p_4^- + p_5^-)}$$

$$u_2 \simeq \frac{|p_{3\perp}|^2\, p_5^+ p_6^-}{\left(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-\right)(p_4^+ + p_5^+)\, p_6^-}$$

$$u_3 \simeq \frac{|p_{6\perp}|^2\, p_3^+ p_4^-}{p_3^+ (p_4^- + p_5^-)\left(|p_{3\perp} + p_{4\perp}|^2 + p_5^+ p_4^-\right)}$$

It can be shown that the remainder function $R_6^{(2)}$ of the 2-loop 6-pt amplitude, $R_6^{(2)}(u_1, u_2, u_3)$, when computed in the QMRK above, is exact also in generic kinematics.

(Duhr, Smirnov, VDD 2009)

Because the one-loop iterated part of the amplitude in the BDS formula is made of $\log s_{ij}$ and $\mathrm{Li}_2(1-u_{ij})$, in the QMRK above the functional dependence of the amplitude on them is not modified.

So the whole amplitude $A_6^{(L)}$, when computed in QMRK with the two gluons emitted along the t-channel ladder forming a cluster with no rapidity ordering, turns out to be exact in general kinematics!

Likewise, one can show that the remainder $R_7^{(2)}$, when computed in the QMRK:

$$p_3^+ \gg p_4^+ \simeq p_5^+ \simeq p_6^+ \gg p_7^+$$

$$p_3^- \ll p_4^- \simeq p_5^- \simeq p_6^- \ll p_7^-$$

is exact also in general kinematics, by examining how the cross ratios $u_{ij} = \dfrac{x_{i,j+1}^2\, x_{i+1,j}^2}{x_{ij}^2\, x_{i+1,j+1}^2}$ (which for the 7-pt amplitude are 7 in d dimensions, reduced to 6 in 4 dimensions) scale in QMRK.

So the amplitude $A_7^{(L)}$, computed in the QMRK of a cluster of 3 gluons emitted along the ladder, is exact in general kinematics.

This feature can be generalised to the amplitude in the QMRK:

$$p_3^+ \gg p_4^+ \simeq p_5^+ \simeq \dots \simeq p_{n-1}^+ \gg p_n^+$$

$$p_3^- \ll p_4^- \simeq p_5^- \simeq \dots \simeq p_{n-1}^- \ll p_n^-$$

with a cluster of $(n-4)$ gluons emitted along the ladder. By power scaling arguments, it can be shown that the remainder $R_n^{(L)}$, computed in the QMRK above, is exact in general kinematics, and so is the MHV amplitude $M_n^{(L)}$.

As we said previously, the 6-pt amplitude in MRK

$$p_3 \gg p_4 \gg p_5 \gg p_6$$

develops a cut when, starting from the Euclidean space where all Mandelstam invariants $s_{i,j}$ are negative, we analytically continue to the Minkowsky region where $s_{12} > 0$, $s_{45} > 0$. The ensuing discontinuity can be described by a BFKL-like equation for the octet exchanged in the $\hat{t}$ channel.

(Bartels, Lipatov, Sabio-Vera 2008)

Firstly, we continue to Minkowski with the prescription $-s_{ij} = s_{ij}\, e^{-i\pi}$. Because $u_1 = \dfrac{s_{12}\, s_{45}}{s_{123}\, s_{345}}$ it acquires a phase $u_1 \to u_1\, e^{-2i\pi}$ (which leads to the discontinuity). The 3 cross ratios in the MRK become

$$u_1 \simeq 1 - \frac{|p_{4_\perp} + p_{5_\perp}|^2}{s_{45}}$$

$$q_1 = -(p_2 + p_3)$$

$$u_2 = \frac{|q_{1_\perp}|^2\, |p_{5_\perp}|^2}{|q_{2_\perp}|^2\, s_{45}}$$

$$q_2 = q_1 - p_4$$

$$q_3 = q_2 - p_5 = p_1 + p_6$$

$$u_3 = \frac{|q_{3_\perp}|^2\, |p_{4_\perp}|^2}{|q_{2_\perp}|^2\, s_{45}}$$

Note that $u_1 \to 1$ and $u_2, u_3 \to 0$.

Then we define the reduced cross ratios

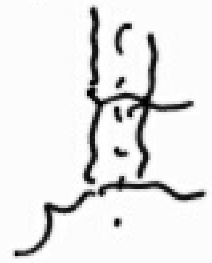
$$\tilde{u}_2 = \frac{u_2}{1-u_1} = \frac{|q_{12}|^2\,|p_{52}|^2}{|q_{22}|^2\,|p_{42}+p_{52}|^2}$$

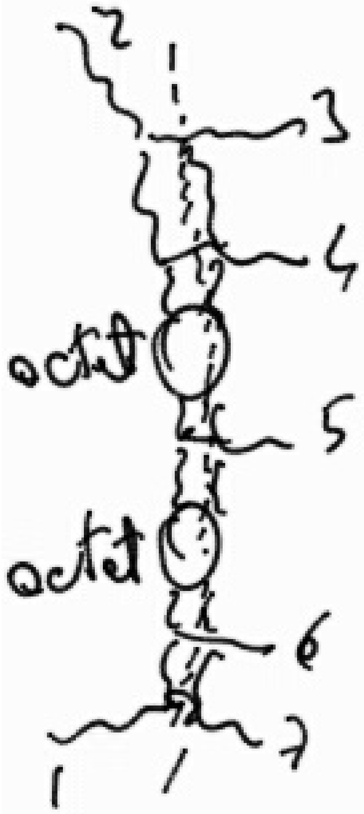
$$\tilde{u}_3 = \frac{u_3}{1-u_1} = \frac{|q_{32}|^2\,|p_{42}|^2}{|q_{22}|^2\,|p_{42}+p_{52}|^2}$$

and then take their ratio, and define a complex variable

$$\frac{\tilde{u}_3}{\tilde{u}_2} = \frac{|q_{32}|^2\,|p_{42}|^2}{|q_{12}|^2\,|p_{52}|^2} \;\rightarrow\; w = \frac{q_{32}\,p_{42}}{q_{12}\,p_{52}}$$

Then one writes a dispersion relation for the octet exchanged in the t channel in terms of the complex variable w. This accounts for the imaginary part of the remainder function R_6, and so of the MHV amplitude A_6, and resums the logarithms $\log(1-u_1)$.

The structure of the dispersion relation is factorized into some impact factors   and an octet ladder in between.



This structure iterates itself. So at 7 points, we just need to add a central-emission vertex between 2 octet ladders.

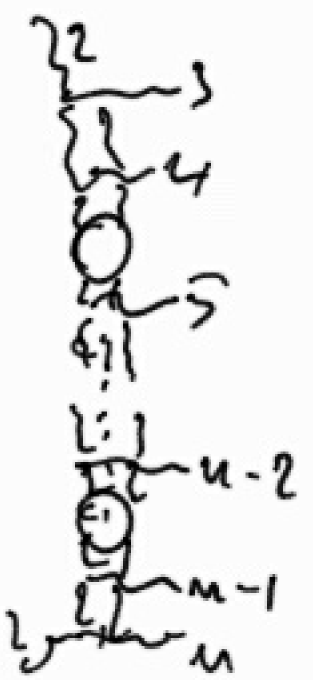
The cut is now obtained by continuing to Minkowski

$s_{12}, s_{456}, s_{45}, s_{56} > 0$

Then again one of cross ratios acquires a phase and the amplitude a discontinuity, which is described by a dispersion relation, which is a function of two complex variables

$$w_1 = \frac{q_{3_2}\, p_{4_2}}{q_{1_2}\, p_{5_2}} \qquad w_2 = \frac{q_{4_2}\, p_{5_2}}{q_{2_2}\, p_{6_2}}$$

This structure generalizes to n points, by inserting more octet ladders and central emission vertices.



One can then write a dispersion relation, which is a function of $(n-5)$ complex variables

A few things are worth noting, at LLA:

- the two-loop n-point remainder function $R_n^{(2)} = R_n^{(2)}(w_1, \dots, w_{n-5})$ factorizes into a sum of 6-pt remainders $R_n^{(2)} = \sum_{i=1}^{n-5} R_6^{(2)}(w_i)$

(Bartels, Kormilitzin, Lipatov, Prygarin 2011)

- because there are only transverse momenta which never vanish, in the dispersion relation, the functions which describe it must be single-valued.

It can be shown that those functions are SV iterated integrals on the Riemann spheres with punctures, $M_{0,n-2}$

In fact, for $n=6$ one has $M_{0,4}$, which coincides with the SVHPLs described before.

This allows us to completely solve the dispersion relation at LLA for any number of points and loops. And in fact, to compute also many non-MHV amplitude in MRK at LLA.