In this lecture, we introduce a color decomposition of the scattering amplitude in the $\hat{t}$ channel, and derive some dispersion relations.

These are propedeutic tools to introduce the BFKL equation for the octet and singlet exchange in the $\hat{t}$ channel.

In order to analise the colour structure, we decompose the amplitude in terms of the $SU(3)$ representations occurring in the product $\underline{8} \otimes \underline{8}$ of the two gluons exchanged in the $\hat{t}$ channel

$$M^{aa'bb'}_{\mu_a\mu_{a'};\mu_b\mu_{b'}} = g^{\mu_a\mu_{a'}} g^{\mu_b\mu_{b'}} \sum_T P^{aa'}_{bb'}(T)\, A^T(\hat{s},\hat{t})$$

with $A^T(\hat{s},\hat{t})$ colourless amplitudes

and $P^{aa'}_{bb'}(T)$ colour projectors

$$P^{aa'}_{bb'}(T)\, P^{bb'}_{cc'}(T') = P^{aa'}_{cc'}\, \delta_{TT'}$$

Now $\underline{8} \otimes \underline{8} = (\underline{8} \otimes \underline{8})_S + (\underline{8} \otimes \underline{8})_A$

with $(\underline{8} \otimes \underline{8})_S = \underline{1} + \underline{8}_S + \underline{27}$

$(\underline{8} \otimes \underline{8})_A = \underline{8}_A + \underline{10} + \overline{\underline{10}}$

We shell used the projectors parity under $\hat{s} \leftrightarrow \hat{u}$ crossing

$$P^{aa'}_{b'b}(T) = (-)^T \, P^{aa'}_{bb'}(T)$$

with

$$(-)^T = \begin{cases} -1 & \text{for } (\underline{8} \otimes \underline{8})_A \\ +1 & \text{for } (\underline{8} \otimes \underline{8})_S \end{cases}$$

and we shell used the explicit projectors

$$P^{aa'}_{bb'}(\mathbb{1}) = \frac{1}{N_c^2 - 1} \delta^{aa'} \delta_{bb'}$$

$$P^{aa'}_{bb'}(\underline{8}_A) = \frac{1}{N_c} f^{aca'} f^{bcb'}$$

The colourless amplitude $A^T(\hat{s},\hat{t})$ can be decomposed in t channel partial wave amplitudes,

$$A^T(\hat{s},\hat{t}) = \sum_\ell (2\ell+1)\, A^T_\ell(\hat{s},\hat{t})\, P_\ell(z_t)$$

where ℓ is the angular momentum, $P_\ell(z_t)$ are Legendre polynomials, $z_t = -\cos\vartheta_t$ is the scattering angle in the t channel physical region, which can be obtained by crossing the channel $\hat{s}$ and $\hat{t}$

s-channel

$$t = -\frac{s}{2}(1-\cos\vartheta) \quad \xrightarrow{\text{cross}} \quad s = -\frac{t}{2}(1-\cos\vartheta_t)$$

$$u = -\frac{s}{2}(1+\cos\vartheta) \qquad\qquad u = -\frac{t}{2}(1+\cos\vartheta_t)$$

Note that $z_t = -\cos\vartheta_t = -\left(\frac{2s}{t}+1\right)$

Because the amplitude is invariant under $\hat{s} \leftrightarrow \hat{u}$ crossing,

$$M^{aa'bb'}(s,t,u) = M^{aa'b'b}(u,t,s)$$

and the projectors parity is $P^{aa'}_{b'b}(T) = (-1)^T P^{aa'}_{bb'}(T)$

we obtain the parity of the colourless amplitude under $\hat{s} \leftrightarrow \hat{u}$ crossing $A^T(-z_t, t) = (-1)^T A^T(z_t, t)$

We write the amplitude through a dispersion relation,

$$A(\hat{s},\hat{t}) = \int_{-\infty}^{\infty} \frac{ds'}{2\pi i} \frac{\mathrm{Disc}\, A(s',t)}{s'-s}$$

with $\mathrm{Disc}\, A(s',t) = A(s+i\epsilon,t) - A(s-i\epsilon,t)$

i.e. with the integral over the branch cuts of the complex $\hat{s}$

We choose $\hat{t}$ to be unphysical, i.e. real and negative $t<0$, so we consider the physical s & u channels.

Then the cuts are at $-\hat{t} < \hat{s} < \infty$ for the s channel, and $-\hat{t} < \hat{u} < \infty$ for the u channel. Because $u = -s-t$,

$$\Downarrow$$

$$-\infty < \hat{s} < 0$$

So
$$A(s,t) = \int_{-\infty}^{0} \frac{ds'}{2\pi i} \frac{\text{Disc}\, A(s',t)}{s'-s} + \int_{-t}^{\infty} \frac{ds'}{2\pi i} \frac{\text{Disc}\, A(s',t)}{s'-s}$$

Because $z_t = -\left(\frac{2s}{t}+1\right)$ we can also write the dispersion relation in the z_t complex plane

$$A(s,t) = \int_{-\infty}^{-1} \frac{dz'_t}{2\pi i} \frac{\text{Disc}\, A(z'_t,t)}{z'_t - z_t} + \int_{1}^{\infty} \frac{dz'_t}{2\pi i} \frac{\text{Disc}\, A(z'_t,t)}{z'_t - z_t}$$

Note that in the t channel physical region, $z_t = -\cos\theta_t$ so $-1 < z_t < 1$, while here the branch cuts are at $z_t < -1$ and $z_t > 1$, where the s and u channel are physical and the t channel is unphysical.

Using the orthogonality condition

$$\int_{-1}^{1} dz \, P_m(z) \, P_n(z) = \frac{2}{2n+1} \delta_{mn}$$

we may invert the partial wave expansion

$$A^T(s,t) = \sum_{\ell} (2\ell+1) \, A^T_\ell(s,t) \, P_\ell(z_t)$$

at $-1 < z_t < 1$, to obtain the amplitude for the ℓ^{th} wave

$$A^T_\ell(s,t) = \frac{1}{2} \int_{-1}^{1} dz_t \, P_\ell(z_t) \, A^T(s,t)$$

Next, introduce the Legendre function,

$$Q_\ell(z') = \frac{1}{2} \int \frac{dz}{z'-z} P_\ell(z)$$

Note that $Q_\ell(-z') = \frac{1}{2}\int \frac{dz}{-z'-z} P_\ell(z) = (-)^\ell \frac{1}{2}\int \frac{dw}{-z'+w} P_\ell(w)$

$$= (-)^{\ell+1} Q_\ell(z')$$

Also,

$$\text{Disc } A^T(-z_t, t) = A^T(-z+i\epsilon, t) - A^T(-z-i\epsilon, t)$$

$$= (-)^T \left[A^T(z-i\epsilon, t) - A^T(z+i\epsilon, t)\right]$$

$$= (-)^{T+1} \text{Disc } A^T(z_t, t)$$

so the amplitude for the ℓ^{th} wave is

$$A_\ell^T(s,t) = \frac{1}{2}\int_{-1}^{1} dz_t\, P_\ell(z_t) \left(\int_{-\infty}^{-1} \frac{dz'}{2\pi i} \frac{\text{Disc } A^T(z'_t, t)}{z'-z} + \int_1^\infty \frac{dz'}{2\pi i} \frac{\text{Disc } A^T(z', t)}{z'-z} \right)$$

$$= \int_{-\infty}^{-1} \frac{dz'}{2\pi i} Q_\ell(z')\, \text{Disc } A^T(z'_t, t) + \int_1^\infty \frac{dz'}{2\pi i} Q_\ell(z')\, \text{Disc } A^T(z'_t, t)$$

$$= \left[1 + (-1)^{\ell+T}\right] \int_1^\infty \frac{dz'}{2\pi i} Q_\ell(z')\, \text{Disc } A^T(z'_t, t)$$

Now, suppose that $f(z)$ is a function which is analytic at $z=0, \pm 1, \pm 2, \dots$ and vanishes faster than $\frac{1}{|z|}$ as $z \to \infty$

Consider the function $F(z) = \dfrac{\pi f(z)}{\sin(\pi z)}$ which has poles at $z=0, \pm 1, \pm 2$

$$\operatorname{Res} F(z)\Big|_{z=n} = \lim_{z \to n} (z-n) \frac{\pi f(z)}{\sin(\pi z)} = \lim_{z \to n} \frac{\pi f(z)}{\frac{d}{dz}\sin(\pi z)} = \lim_{z \to n} \frac{f(z)}{\cos(\pi z)} = (-)^n f(n)$$

The poles of $f(z)$ at $z=z_i$ are also poles of $F(z)$, different from the ones at $z=n$

Because $f(z)$ vanishes faster than $\frac{1}{|z|}$ as $z \to \infty$,

$\oint F(z)\, dz = 0$ over a circle centered at the origin and of radius $R \to \infty$

By the residue theorem,

$$\oint F(z)\,dz = 2\pi i \left(\sum_n \mathrm{Res}\, F(z)\Big|_{z=n} + \sum_{z_i} \mathrm{Res}\, F(z)\Big|_{z=z_i} \right) = 0$$

So

$$\sum_n (-)^n f(n) = -\sum_{z_i} \mathrm{Res}\, F(z)\Big|_{z=z_i} = -\frac{1}{2\cancel{\pi} i} \oint dz\, \frac{\cancel{\pi} f(z)}{\sin(\pi z)}$$

encircling all the poles of $f(z)$ at $z = z_i$

In particular, we can take a path parallel to the imaginary axis, and to the right of all the poles. Then

$$\sum_n (-)^n f(n) = -\frac{1}{2i} \oint \frac{f(z)}{\sin(\pi z)}$$

which is called Sommerfeld-Watson representation

If we apply it to the partial wave expansion,

$$A^T(s,t) = \sum_\ell (2\ell+1) A_\ell^T P_\ell(z_t) = \sum_\ell (-)^\ell (2\ell+1) A_\ell^T P_\ell(-z_t)$$

$$= -\frac{1}{2i} \int_{\delta - i\infty}^{\delta + i\infty} d\ell \, (2\ell+1) A_\ell^T(s,t) \frac{P_\ell(-z_t)}{\sin(\pi \ell)}$$

So far, we have described the general analytic structure of scattering amplitudes. Now, we consider the high-energy limit. Then $z_t = -\left(\frac{2s}{t}+1\right) \to -\frac{2s}{t} \gg 1$

We take the asymptotic values of Legendre polynomials and functions

$$P_\ell(z) \to \frac{1}{\sqrt{\pi}} \frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)} (2z)^\ell$$

$$Q_\ell(z) \to \sqrt{\pi} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} (2z)^{-(\ell+1)}$$

Replacing the ℓ^{th} wave amplitude into the SW representation of the amplitude

$$A^T(s,t) = \int \frac{d\ell}{2i}(2\ell+1)\frac{P_\ell(-z_t)}{\sin(\pi\ell)}\left[1+(-)^{\ell+T}\right]\int_1^\infty \frac{dz'}{2\pi i} Q_\ell(z')\, \mathrm{Disc}\, A^T$$

and using the asymptotics of P_ℓ and Q_ℓ, we obtain

$$A^T(s,t) = \frac{1}{4\pi}\int_{\delta-i\infty}^{\delta+i\infty} d\ell \left[(-)^\ell + (-)^T\right]\frac{z^\ell}{\sin(\pi\ell)} F_\ell^T(t)$$

where $F_\ell^T = \int_1^\infty dz \quad z^{-(\ell+1)}\, \mathrm{Disc}\, A^T(z_t', t)$

is the Mellin Transform of Disc A^T

Because $z = -\left(\frac{2s}{t}+1\right) = 2e^{y}-1$

one can change variable to the rapidity y

and write

$$A^T(s,t) = \frac{1}{4\pi}\int_{\delta-i\infty}^{\delta+i\infty} dl \left[(-)^l + (-)^T\right] \frac{e^{ly}}{\sin(\pi l)} \mathbb{F}_l^T(t)$$

where $\mathbb{F}_l^T(t) = \int_0^\infty dy\, e^{-ly}\, \mathrm{Disc}\, A^T(z_t, t)$

is the Laplace transform of $\mathrm{Disc}\, A^T$