We have described the parton kinematics.
We shall state now some simple facts about two-jet production, and then move on to the parton dynamics.

The factorization formula for 2-jet production is

$$\frac{d\sigma}{dy_a\, dy_b\, d^2k_{a_\perp} d^2k_{b_\perp}} = \sum_{ij} \int dx_a\, dx_b\, f_{i/A}(x_a, \mu_F^2)\, f_{j/B}(x_b, \mu_F^2) \frac{d\hat{\sigma}_{ij}}{dy_a\, dy_b\, d^2k_a\, d^2k_b}$$

with

$$d\hat{\sigma}_{ij} = \frac{(2\pi)^4 \delta^4(p_a + p_b - k_a - k_b)}{2\hat{s}} \frac{dy_a\, d^2k_{a_\perp}}{4\pi\,(2\pi)^2} \frac{dy_b\, d^2k_{b_\perp}}{4\pi\,(2\pi)^2} |M_{ij}|^2$$

making the momentum-conserving δ function explicit, that becomes

$$\frac{d\hat{\sigma}_{ij}}{dy_{e}\,dy_{j}\,d^2k_{e_\perp}\,d^2k_{j_\perp}} = \frac{|M_{ij}|^2}{16\pi^2\hat{s}}\,\delta^2(k_{e_\perp}+k_{j_\perp})\,\delta\left(x_a\sqrt{s} - k_{e_\perp}e^{y_e} - k_{j_\perp}e^{y_j}\right)$$
$$\cdot\,\delta\left(x_b\sqrt{s} - k_{e_\perp}e^{-y_e} - k_{j_\perp}e^{-y_j}\right)$$

If we plug it into the factorisation formula, we get

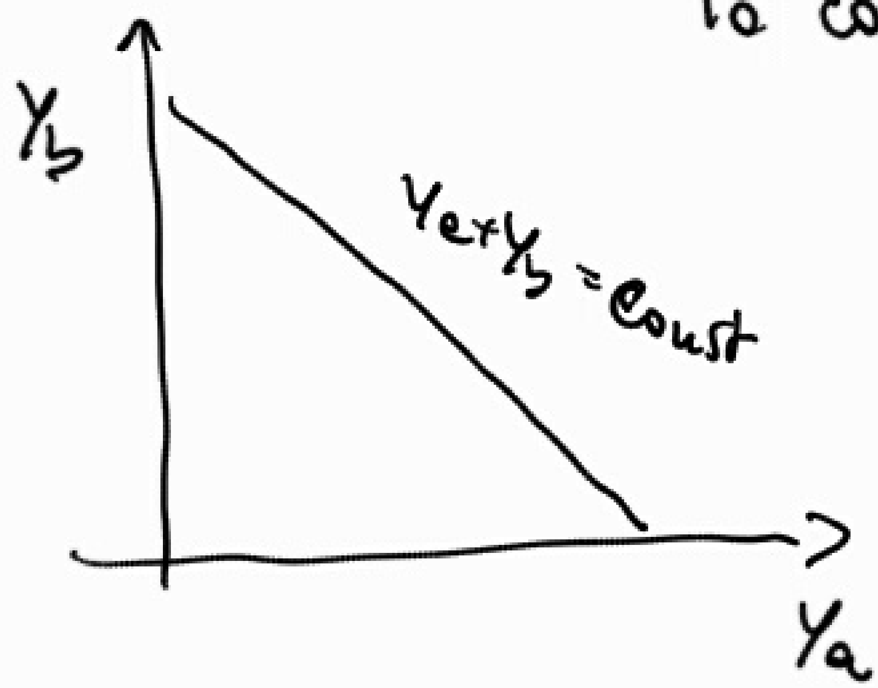
$$\frac{d\sigma}{dy_e\,dy_j\,dk_\perp^2} = \sum_{ij} x_a f_{i/A}(x_a,\mu_F^2)\,x_b f_{j/B}(x_b,\mu_F^2)\,\frac{d\hat{\sigma}}{d\hat{t}}$$

where we labelled $\dfrac{d\hat{\sigma}}{d\hat{t}} = \dfrac{|M_{ij}|^2}{16\pi\hat{s}^2}$

The dynamics is contained in $|M_{ij}|^2$, which depends on $k_\perp$, y^* but not on $\bar{y}$: $|M_{ij}|^2 = |M_{ij}|^2(k_\perp, y^*)$

So if we are interested to study the dynamics, it's convenient to fix $\bar{y}$ or to integrate it out, since its variation only modifies the PDFs.

That is, on the plane of the jet rapidities, it is convenient to consider the lines of fixed $\bar{y}$.



Let us consider now what the squared matrix elements are. We write down explicitly some of them:

use $|M_{ij}|^2 = g_s^4 \, |\mathcal{M}_{\bar{ij}}|^2$

$$|\mathcal{M}_{qq' \to qq'}|^2 = \frac{4}{9} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

$$|\mathcal{M}_{q\bar{q} \to q'\bar{q}'}|^2 = \frac{4}{9} \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}$$

$$|\mathcal{M}_{gg \to q\bar{q}}|^2 = \frac{1}{6} \frac{\hat{t}^2 + \hat{u}^2}{\hat{t}\hat{u}} - \frac{3}{8} \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}$$

$$|\mathcal{M}_{qg \to qg}|^2 = \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} - \frac{4}{9} \frac{\hat{s}^2 + \hat{u}^2}{\hat{s}\hat{u}}$$

$$|\mathcal{M}_{gg \to gg}|^2 = \frac{9}{2} \left(3 - \frac{\hat{t}\hat{u}}{\hat{s}^2} - \frac{\hat{s}\hat{u}}{\hat{t}^2} - \frac{\hat{t}\hat{s}}{\hat{u}^2} \right)$$

Let us now look at the limit $\Delta y = 2y^* \gg 1$

In this limit, $\begin{cases} x_a \simeq \dfrac{k_2}{\sqrt{s}} e^{y_a} \\ x_b \simeq \dfrac{k_2}{\sqrt{s}} e^{-y_b} \end{cases}$

so each PDF gets contribution from one parton only.

The Mandelstam invariants become:

$$\hat{s} = 4 k_2^2 \cosh^2 y^* \Rightarrow k_2^2 e^{2y^*}$$

$$\hat{t} = -2k_2^2 \cosh y^* e^{-y^*} \Rightarrow -k_2^2$$

$$\hat{u} = -2k_2^2 \cosh y^* e^{y^*} \Rightarrow -k_2^2 e^{2y^*}$$

thus $\hat{u} = -\hat{s} - \hat{t} \simeq -\hat{s}$ and $-\hat{t} \ll \hat{s}$

a large logarithm has arisen $\ln \frac{\hat{s}}{-\hat{t}} \simeq 2y^* = \Delta y$ which equals the rapidity interval between the 2 outgoing partons.

Note that in this limit the SME become:

$$|\mathcal{M}_{gg\to gg}|^2 \simeq \frac{9}{2}\frac{\hat{s}^2}{\hat{t}^2}$$

$$|\mathcal{M}_{qg\to qg}|^2 \simeq 2\,\frac{\hat{s}^2}{\hat{t}^2}$$

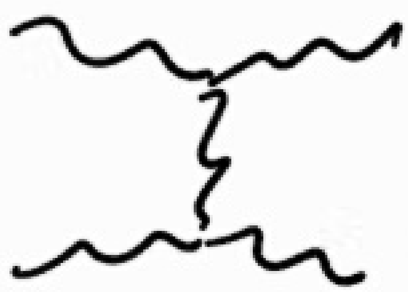
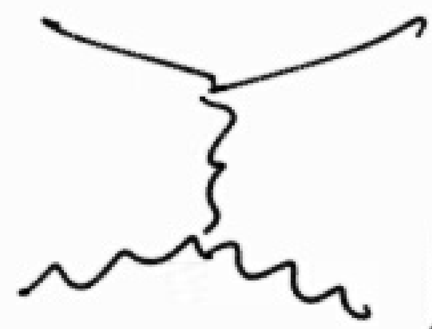
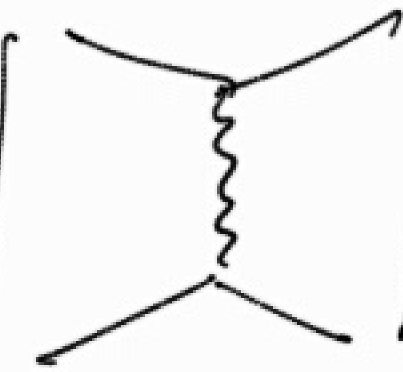
$$|\mathcal{M}_{qq'\to qq'}|^2 \simeq \frac{2}{9}\frac{\hat{s}^2}{\hat{t}^2}$$

(these will be computed in the tutorials)

while

$$|\mathcal{M}_{gg\to q\bar{q}}|^2 \simeq \frac{1}{6}\frac{\hat{u}}{\hat{t}} \qquad |\mathcal{M}_{q\bar{q}\to q'\bar{q}'}|^2 \simeq \frac{4}{9}$$ are subleading

thus, pictorially:

$\Big|$  $\Big|^2 = \dfrac{C_A}{C_F}\Big|$  $\Big|^2 = \left(\dfrac{C_A}{C_F}\right)^2\Big|$  $\Big|^2$

where $\dfrac{C_A}{C_F} = \dfrac{9}{4}$

so the 3 processes above have the same power behaviour in $\hat{s}/\hat{t}$, and differ only by the colour strength in the helicity-conserving (jet-production) vertices

The other SME, which do not feature gluon exchange in the t channel, are suppressed by powers of $\hat{s}/\hat{t}$.

Note also the $\dfrac{d\hat{\sigma}_{ij}}{d\hat{t}} = \dfrac{|\mathcal{M}_{ij}|^2}{16\pi\hat{s}^2} = O\left(\dfrac{1}{\hat{t}^2}\right)$

so the cross section does not fall off with $\hat{s}$

($\to$ pomeron)

and because $\theta \to 0$ as $\Delta y \gg 1$, it shows the power divergence typical of Rutherford scattering

For future reference, we note that the 2-parton phase space we used in the factorization formula is

$$d\Phi_2 = \frac{dy_{a'}\, d^2k_{a'\perp}}{4\pi\,(2\pi)^2}\;\frac{dy_{b'}\, d^2k_{b'\perp}}{4\pi\,(2\pi)^2}\,(2\pi)^4\,\delta^4\left(p_a + p_b - k_{a'} - k_{b'}\right)$$

with

$$\delta^4\left(p_a + p_b - k_{a'} - k_{b'}\right) = \delta^2\left(k_{\perp a'} + k_{\perp b'}\right)$$
$$\cdot\, 2\,\delta\left(x_a\sqrt{s} - k_{\perp a'}e^{y_{a'}} - k_{\perp b'}e^{y_{b'}}\right)\delta\left(x_b\sqrt{s} - k_{\perp a'}e^{-y_{a'}} - k_{\perp b'}e^{-y_{b'}}\right)$$

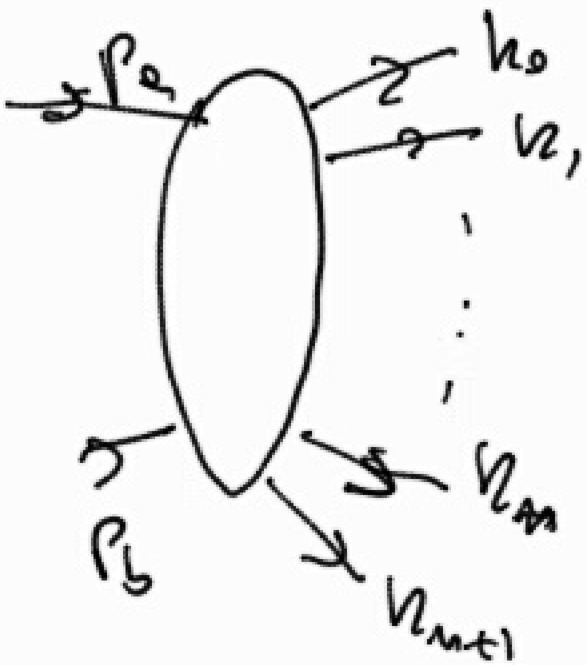
we can fix the rapidities using light-cone momentum cons.

$$\int dy_{a'}\, dy_{b'}\, \delta\left(x_a\sqrt{s}\ \cdots\ \right)\delta\left(x_b\sqrt{s}\ \cdots\ \right)$$
$$= \frac{1}{\hat{s}}\,\frac{\cosh\left(y_{a'} - y_{b'}\right) + 1}{\sinh\left(y_{a'} - y_{b'}\right)} \xrightarrow{\Delta y \gg 1} \frac{1}{\hat{s}}$$

So in the limit, $\Delta y \gg 1$, the 2-parton phase space becomes

$$d P_2 = \frac{1}{2\hat{s}} \frac{d^2 k_{a'_2}}{(2\pi)^2} \frac{d^2 k_{b'_2}}{(2\pi)^2} (2\pi)^2 \delta^2 (k_{a'_2} + k_{b'_2})$$

Let us assume that in the collision of two partons $n+2$ partons are produced.



The light-cone momenta are

$$p_a = (p_a^+, 0; 0_\perp) \qquad \text{with} \quad p_a^+ = x_a\sqrt{s}$$

$$p_b = (0, p_b^-; 0_\perp) \qquad p_b^- = x_b\sqrt{s}$$

$$k_i = (k_{i\perp} e^{y_i}, k_{i\perp} e^{-y_i}; k_{i\perp}) \qquad i = 0, \ldots, n+1$$

Momentum conservation yields

$$\begin{cases} p_a^+ = \sum_i k_{i\perp} e^{y_i} \\ p_b^- = \sum_i k_{i\perp} e^{-y_i} \\ \sum_{i=0}^{n+1} \vec{k}_{i\perp} = 0 \end{cases}$$

The Mandelstam invariants are:

$$\hat{s} = (p_a + p_b)^2 = p_a^+ p_b^- = \sum_{i,j} k_{i_\perp} k_{j_\perp} e^{y_i - y_j}$$

$$\hat{s}_{ai} = (p_a - k_i)^2 = -p_a^+ k_i^- = -\sum_j k_{i_\perp} k_{j_\perp} e^{-(y_i - y_j)} \qquad t\text{-type}$$

$$\hat{s}_{bi} = (p_b - k_i)^2 = -p_b^- k_i^+ = -\sum_j k_{i_\perp} k_{j_\perp} e^{y_i - y_j} \qquad u\text{-type}$$

$$\hat{s}_{ij} = (k_i + k_j)^2 = k_i^+ k_j^- + k_i^- k_j^+ - 2\vec{k}_{i_\perp} \cdot \vec{k}_{j_\perp}$$

$$= 2 k_{i_\perp} k_{j_\perp} \cosh(y_i - y_j) - 2\vec{k}_{i_\perp} \cdot \vec{k}_{j_\perp}$$

Note that the boost invariance of the c.m. frame implies that the invariants depend only on rapidity differences

Now, we introduce the <u>multi-Regge kinematics</u> (MRK), i.e. we assume that the partons are strongly ordered in rapidity and have comparable transverse momentum:

$$y_0 \gg y_1 \gg \cdots \gg y_{n+1}$$

$$k_{0_2} \simeq k_{1_2} \simeq \cdots k_{n+1_2}$$

This is equivalent to require a strong ordering in light-cone coordinates:

$$p_0^+ \gg p_1^+ \gg \cdots \gg p_{n+1}^+$$

$$p_0^- \ll p_1^+ \ll \cdots \ll p_{n+1}^-$$

Momentum conservation is approximated by

$$\begin{cases} p_a^+ = x_a\sqrt{s} \simeq k_{0_\perp} e^{y_0} \\ p_b^- = x_b\sqrt{s} \simeq k_{n+1_\perp} e^{-y_{n+1}} \\ \sum_{i=0}^{n+1} \vec{k}_{i_\perp} = 0 \end{cases}$$

So, to leading power, the momenta of the incoming partons are entirely fixed by the jet-production vertices. The contribution of partons $1, \ldots, n$ is power suppressed.

The Mandelstam invariants are approximated by:

$$\hat{s} \simeq k_0^+ k_{n+1}^- = k_{0_\perp} k_{n+1_\perp} e^{y_0 - y_{n+1}}$$

$$\hat{s}_{ai} \simeq -k_0^+ k_i^- = -k_{0_\perp} k_{i_\perp} e^{y_0 - y_i}$$

$$\hat{s}_{bi} \simeq -k_i^+ k_{n+1}^- = -k_{i_\perp} k_{n+1_\perp} e^{y_i - y_{n+1}}$$

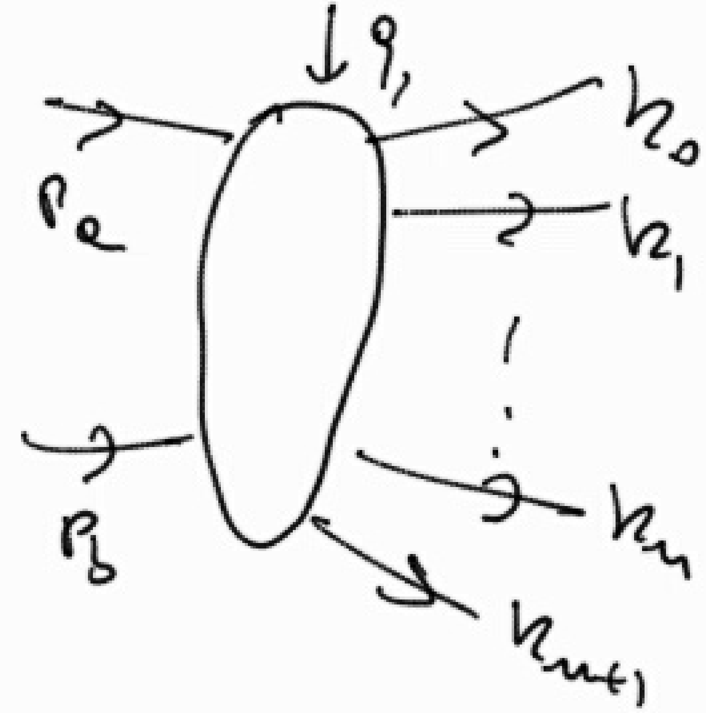
$$\hat{s}_{ij} \simeq k_{i_\perp} k_{j_\perp} e^{|y_i - y_j|}$$

the strong ordering in rapidity implies that

$$\hat{s} \gg \hat{s}_{ij} \gg k_{i_\perp}^2$$

with the condition that $\prod_{i=0}^{n} \hat{s}_{i,i+1} = \hat{s} \prod_{i=1}^{n} k_{i_\perp}^2$

Let us consider the momenta exchanged in the t channel. The first is



$$q_1 = p_a - k_0$$

In light-cone coordinates

$$q_1 = \left(\sum_{i=1}^{n+1} k_i^+ , -k_0^- ; -\vec{k}_{0_\perp} \right)$$

in MRK, $q_1 \simeq \left(k_{1_\perp} e^{y_1} , -k_{0_\perp} e^{-y_0} ; -\vec{k}_{0_\perp} \right)$

Squaring: $q_1^2 = -k_{0_\perp} k_{1_\perp} e^{-(y_0 - y_1)} - k_{0_\perp}^2 \simeq -k_{0_\perp}^2$

thus only the transverse degrees of freedom are relevant for the momentum transfer q_1.

The second momentum exchanged in T channel is

$$q_2 = q_1 - k_1$$

In light-cone coordinates:

$$q_2 = \left(\sum_{i=2}^{n+1} k_i^+ , -k_0^- - k_1^- ; -\vec{k}_{0_\perp} - \vec{k}_{1_\perp} \right)$$

in MRK, it is approximated by

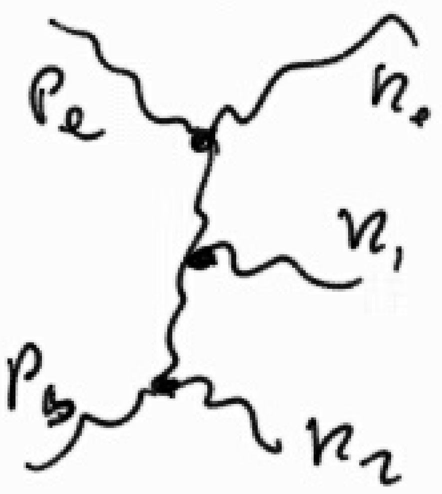
$$q_2 \simeq \left(k_{2_\perp} e^{y_2} , -k_{1_\perp} e^{-y_1} ; -\vec{k}_{0_\perp} - \vec{k}_{1_\perp} \right)$$

Squaring: $q_2^2 = -k_{1_\perp} k_{2_\perp} e^{-(y_1 - y_2)} - |\vec{k}_{0_\perp} + \vec{k}_{1_\perp}|^2 \simeq -|k_{0_\perp} + k_{1_\perp}|^2$

The picture iterates, and one can show that in all the momentum transfers $q_{i+1} = q_i - k_i$ with $i = 1, \dots, n$ only the transverse degrees of freedom are relevant.

This kinematical fact has far-reaching consequences.
Firstly, it will allow us to write an iterative equation in transverse momentum space, the BFKL equation.
Secondly (something that has been appreciated only recently) the functions which will describe any BFKL ladder will be single-valued functions.
(For the BFKL equation at leading log, they are single-valued harmonic polilogarithms)

In the tutorials, we show that the amplitude for 3-gluon production has a ladder structure



$$M \simeq 2\hat{s}\left(i g f^{a c_1 d_0}\Gamma^{\mu_a \mu_0}\right)\frac{1}{\hat{t}_1}\left(i g f^{c_1 d_1 c_2} C^{\mu_1}(q_1, q_2)\right)\frac{1}{\hat{t}_2}\left(i g f^{b c_2 d_2}\Gamma^{\mu_b \mu_2}\right)$$

and its square, summed (averaged) over final (initial) helicities and colors

$$\sum_{hel,col} \left|M_{gg\to ggg}\right|^2 = \frac{16 g^6 C_A^3}{N_c^2-1}\,\frac{\hat{s}^2}{k_{0_\perp}^2 k_{1_\perp}^2 k_{2_\perp}^2}$$

The 3-parton phase space is

$$dP_3 = \prod_{i=0}^{2} \frac{dy_i \, d^2k_{\perp i}}{4\pi\,(2\pi)^2} (2\pi)^4 \delta^4\left(p_a + p_b - \sum_{i=0}^{2} k_i\right)$$

However, we have seen that in MRK, light-cone momentum conservation is

$$p_a^+ = x_a\sqrt{s} \simeq k_{0\perp} e^{y_0}$$

$$p_b^- = x_b\sqrt{s} \simeq k_{n+1\perp} e^{-y_{n+1}}$$

irrespective of how many partons are produced.

Thus we can still fix two rapidities, as we have done for two partons, using light-cone momenta

$$\lim_{y_0 - y_2 \gg 1} \int dy_0 \, dy_2 \, \delta(x_a\sqrt{s} - \cdots)\,\delta(x_b\sqrt{s} - \cdots) = \frac{1}{\hat{s}}$$

and the 3-parton phase space becomes

$$dP_3 = \frac{1}{2\hat{s}} \frac{d^2k_{a_\perp}}{(2\pi)^2} \left(\frac{dy_1\, d^2k_{1_\perp}}{4\pi\,(2\pi)^2} \right) \frac{d^2k_{2_\perp}}{(2\pi)^2} (2\pi)^2 \delta^2(k_{a_\perp} + k_{1_\perp} + k_{2_\perp})$$

The cross section for 3-gluon production is

$$d\hat{\sigma}_{gg\to ggg} = \frac{1}{2\hat{s}} dP_3 \cdot |M_{gg\to ggg}|^2$$

We can integrate out gluon k_1 over the rapidity range $\Delta y = y_0 - y_2$ and use transverse momentum cons.

and obtain

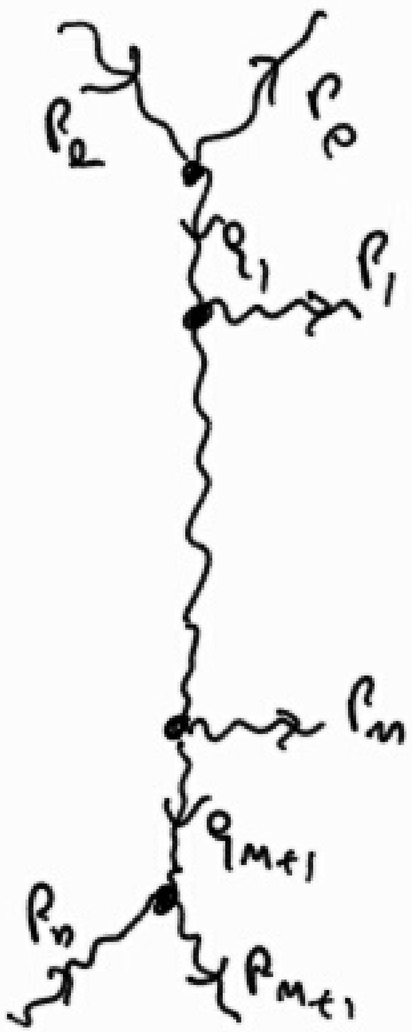
$$\frac{d\sigma_{gg\to ggg}}{dk_{\perp_2}^2\, dk_{\perp_2}^2\, d\cos\varphi} = \frac{C_A^3 \alpha_s^3}{4\pi} \frac{\Delta y}{k_{\perp_2}^2 k_{\perp_2}^2 \left(k_{\perp_2}^2 + k_{\perp_2}^2 + 2 k_{\perp_2} k_{\perp_2} \cos\varphi \right)}$$

with φ the azimuthal angle between $k_{\perp_2}$ and $k_{\perp_2}$

Because $\Delta y = \ln \frac{\hat{s}}{k_{\perp_2} k_{\perp_2}}$, we see the rise of a

large logarithm

In the tutorials, we show that the tree-level n-gluon amplitude in MRK has the form,



$$
\begin{aligned}
M^{a b d_0 \cdots d_{n+1}}_{\lambda_a \lambda_b \lambda_0 \cdots \lambda_{n+1}} = 2\hat{s} & \left(i g f^{a d_0 c_1} C^{\lambda_a \lambda_0}_{gg}(p_a, p_0) \right) \\
& \frac{1}{\hat{t}_1} \left(i g f^{c_1 d_1 c_2} C^{\lambda_1}_{g}(q_1, q_2) \right) \\
& \vdots \\
& \frac{1}{\hat{t}_n} \left(i g f^{c_n d_n c_{n+1}} C^{\lambda_n}_{g}(q_n, q_{n+1}) \right) \\
& \frac{1}{\hat{t}_{n+1}} \left(i g f^{b d_{n+1} c_{n+1}} C^{\lambda_b \lambda_{n+1}}_{gg}(p_b, p_{n+1}) \right)
\end{aligned}
$$

In the tutorials, we also show that to leading-log accuracy the real part of the one-loop 4-gluon amplitude has the form $\mathrm{Re}\, M_4^{(1)} = M_4^{tree}\, \alpha(\hat{t}) \ln \frac{\hat{s}}{-\hat{t}}$

where $\alpha(\hat{t}) = \alpha_s N_c \hat{t} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 (q-k)_\perp^2}$

$$= 8\pi \alpha_s N_c c_\Gamma \left(\frac{\mu^2}{-t}\right)^{\epsilon} \frac{1}{\epsilon} \qquad \text{in dim. reg.}$$

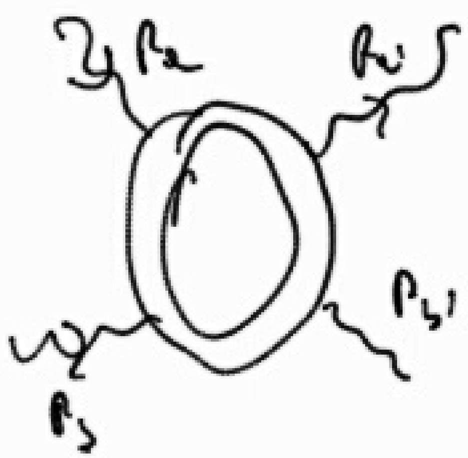
with $\quad c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$

$\alpha(\hat{t})$ is the gluon Regge trajectory

In fact, in 1976 Lipatov made the ansatz that to leading log accuracy the loop corrections were obtained to all orders by dressing the propagator of the gluon exchanged in the $\hat{t}$ channel $\frac{1}{\hat{t}} \to \frac{1}{\hat{t}}\left(\frac{\hat{s}}{-\hat{t}}\right)^{\alpha(\hat{t})}$

(for ladder diagrams, this was first realised by B. Lee, Sawyer '62)

so the one-loop 4-gluon amplitude would be written as



$$M_4^{1\text{-loop}} = 2\hat{s}\left(i g f^{a a' c}\, C_{gg}^{\lambda_a \lambda_{a'}}(p_a, p_{a'})\right)$$
$$\frac{1}{\hat{t}}\left(\frac{\hat{s}}{-\hat{t}}\right)^{\alpha(\hat{t})}$$
$$\left(i g f^{b b' c}\, C_{gg}^{\lambda_b \lambda_{b'}}(p_b, p_{b'})\right)$$

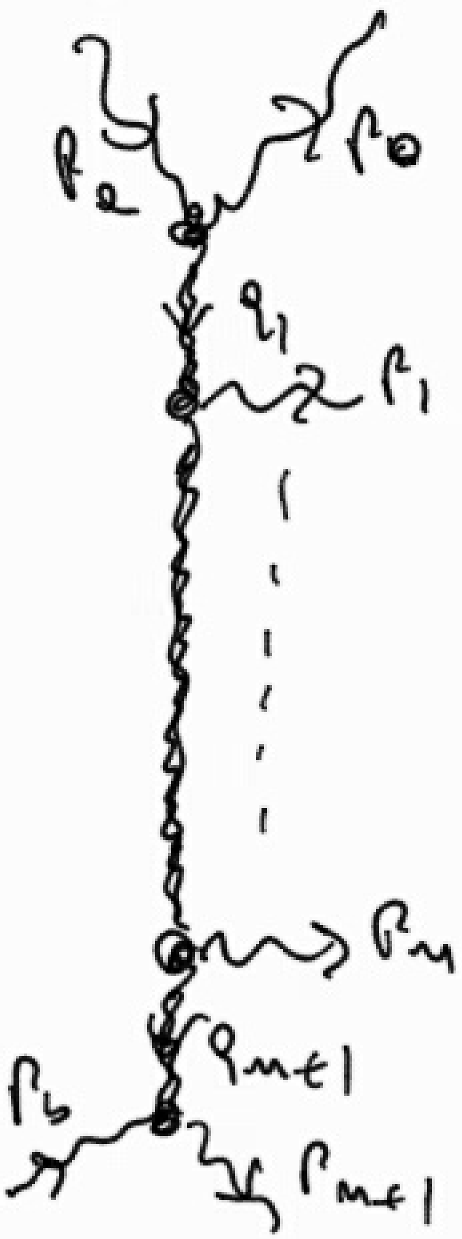
Note that regulating the gluon Regge Trajectory

$$\left(\frac{s}{-t}\right)^{\alpha(t)} = e^{\alpha(t) \ln \frac{s}{-t}}$$

with a cutoff μ $\quad \alpha(t) \simeq - \frac{\alpha_s N_c}{2\pi} \ln \frac{-t}{\mu^2}$

we have a product, with minus sign, of a collinear logarithm with a logarithm of type $\ln \hat{s}/\hat{t}$, like in a Sudakov form factor. So the loop amplitude is infrared sensitive, and vanishes as $\mu \to 0$, which is the typical IR behaviour of gauge theories first observed in QED (Bloch-Nordsieck).

To LL accuracy, Fadin-Kuraev-Lipatov (1977) showed that the behaviour of the loop 4-gluon amplitude generalises to n gluons,



$$
\begin{aligned}
M^{a b d_0 \cdots d_{n+1}}_{\lambda_a \lambda_b \lambda_0 \cdots \lambda_{n+1}} = 2\hat{s} & \left(i g f^{a d_0 c_1} C^{\lambda_a \lambda_0}_{gg}(p_a, p_0) \right) \\
& \frac{1}{\hat{t}_1} \left(\frac{\hat{s}_1}{-\hat{t}_1} \right)^{\alpha(\hat{t}_1)} \left(i g f^{c_1 d_1 c_2} C^{\lambda_1}_{g}(q_1, q_2) \right) \\
& \vdots \\
& \frac{1}{\hat{t}_n} \left(\frac{\hat{s}_n}{-\hat{t}_n} \right)^{\alpha(\hat{t}_n)} \left(i g f^{c_n d_n c_{n+1}} C^{\lambda_n}_{g}(q_n, q_{n+1}) \right) \\
& \frac{1}{\hat{t}_{n+1}} \left(\frac{\hat{s}_{n+1}}{-\hat{t}_{n+1}} \right)^{\alpha(\hat{t}_{n+1})} \left(i g f^{b d_{n+1} c_{n+1}} C^{\lambda_b \lambda_{n+1}}_{gg}(p_b, p_{n+1}) \right)
\end{aligned}
$$