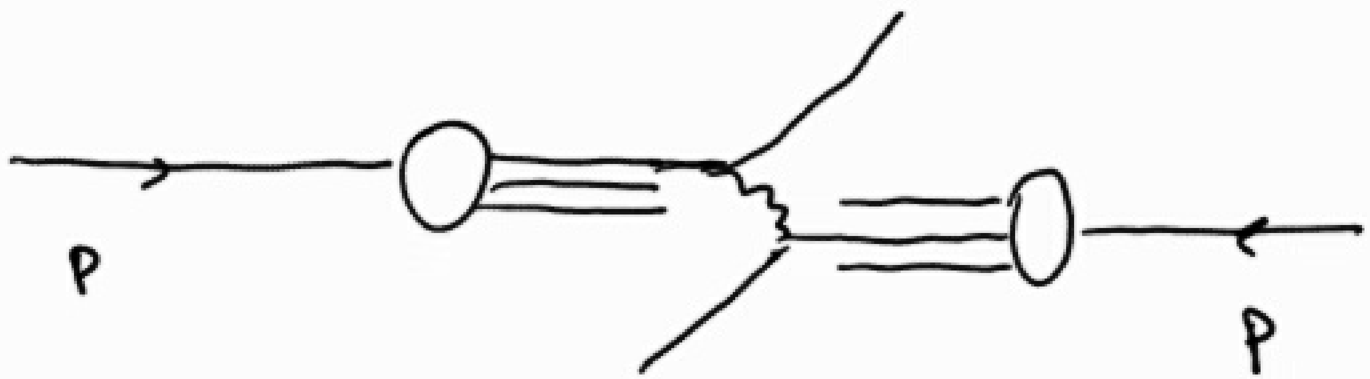
In this lecture series, we shall describe the Regge limit $s \gg |t|$ of QCD, as well as its multiparticle interactions (production in multi Regge kinematics), addressing the whole context as the high-energy limit of QCD.

We shall derive the Balitsky-Fadin-Kuraev-Lipatov eq. (BFKL) which resums the large logarithms, $\ln(s/Q^2)$, occurring in the high-energy limit, and we shall have a look at some state-of-the-art applications of the BFKL equation in QCD and in $N=4$ Super Yang-Mills.

To illustrate the high-energy limit, we shall consider the production of jets in hadron collisions.

We shall assume <u>factorisation</u>, i.e.



we distinguish two regions:

- a long-distance (soft) region, whose interactions are characterised by $d \sim 1$ fm, $t \sim 10^{-23}$ sec, $E \sim \Lambda_{QCD} \simeq 200$ MeV
- a short-distance (hard) region, whose interactions are characterised by $d \sim 10^{-2}$ fm, $t \sim 10^{-25}$ sec, $E \sim Q \simeq 20$ GeV

An intuitive picture of factorisation: (due to G. Sterman)

- in the proton rest frame, the interaction time between partons is $t_{int} \sim 10^{-23}$ sec; likewise, the flight time of one proton through the other is

$$t_{pp} \simeq \frac{1\,\text{fm}}{c} \sim 10^{-23}\,\text{sec}$$

: we see no difference in the interaction time between partons within the proton, or between partons which scatter among the colliding protons.

- in the centre-of-mass (c.m.) frame (say, the lab frame at LHC)

$$\gamma_p = \frac{E_p}{m_p} \simeq 10^4$$

- each proton looks like a pancake, Lorentz contracted in the direction of the collision

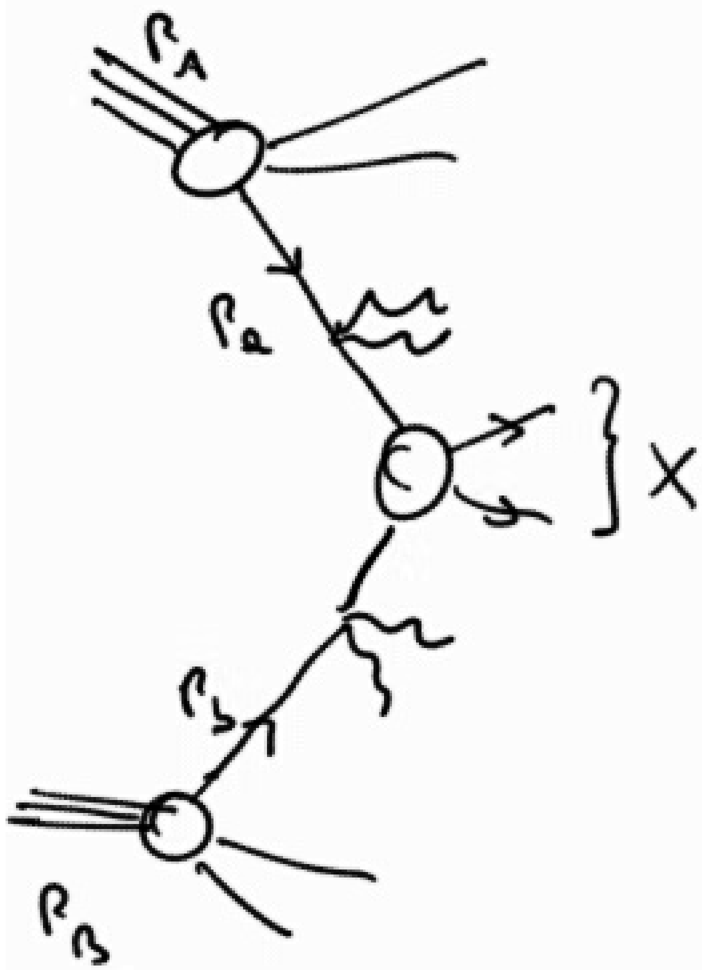
Accordingly, the time it takes to parton a within proton A to go through proton B will be Lorentz contracted : $t_{pp} \sim \frac{1\,fm}{\gamma_p} \frac{1}{c} \sim 10^{-22}\,sec$

- the interaction time of partons within the proton is time dilated : $t_{int} \sim 10^{-13}\,sec$

Thus parton a sees proton B as a frozen distribution of partons $\Rightarrow$ factorization

Factorization:

$$\sigma_X = \sum_{a,b} \int dx_1 dx_2\, f_{a/A}(x_1,\mu_F^2)\, f_{b/B}(x_2,\mu_F^2)\, \hat{\sigma}_{ab\to X}\left(x_1, x_2, \{p_i\}; \alpha_s(\mu_{R,F}^2), \frac{Q^2}{\mu_{R,F}^2}\right)$$



$f_{a/A}(x,\mu_F^2)$: PDF

μ_F : factorisation scale

μ_R : renormalisation scale

$x_{1,2}$: momentum fraction of parton within the proton

$\hat{\sigma}$: partonic cross section, to be computed as an expansion in α_s

$$\hat{\sigma} = \alpha_s^n \left(c_0 + c_1\alpha_s + c_2\alpha_s^2 + \cdots\right)$$

X : final state $\quad X = W, Z, H, t\bar{t}$, jets, ..

In factorisation, we assume there is only one relevant hard scale $Q \gg \Lambda_{QCD}$, which may be related to the mass of the heavy object produced, or to the transverse energy of jets. However, if the hadron c.m. energy $\sqrt{s} \gg Q$, or the parton c.m. energy $\sqrt{\hat{s}} = \sqrt{x_1 x_2 s} \gg Q$ then we have a third (semihard) region, where

$$\ln \frac{s}{Q^2} = \ln \frac{1}{x_1} + \ln \frac{\hat{s}}{Q^2} + \ln \frac{1}{x_2} \gg 1$$

we can have that either $\ln \frac{1}{x}$ is large, or $\ln \frac{\hat{s}}{Q^2}$ is large, or both are large.

$\ln 1/x \gg 1$ controls the small-x evolution of the PDFs, on top of the DGLAP evolution

$\ln \frac{\hat{s}}{Q^2} \gg 1$ are the large logarithms occurring in the partonic scattering, which may be resummed through the BFKL equation.

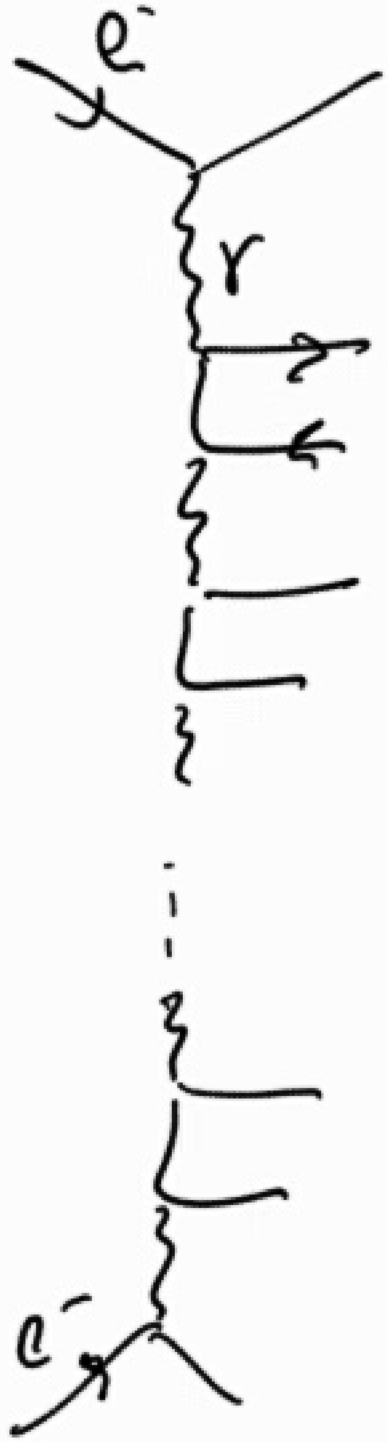
In these lectures, we shall deal mostly with the large $\ln \frac{\hat{s}}{Q^2}$.

We shall also refer to $\ln \frac{\hat{s}}{Q^2} \gg 1$ as to the <u>high-energy limit</u>

Large $\ln \frac{\hat{s}}{Q^2}$ were first considered in the high-energy limit of QED.

Firstly, using power counting arguments, one sees that in the limit $s \gg Q^2$, the largest contribution comes from the exchange of the highest-spin particle in the t channel; for QED, that is the exchange of a photon in the t channel. That accounts for the total cross section approaching a constant value at $s \gg Q^2$ (Pomeranchuk '58).

Now, suppose that a fermion pair is emitted along the photon exchanged in the t channel.



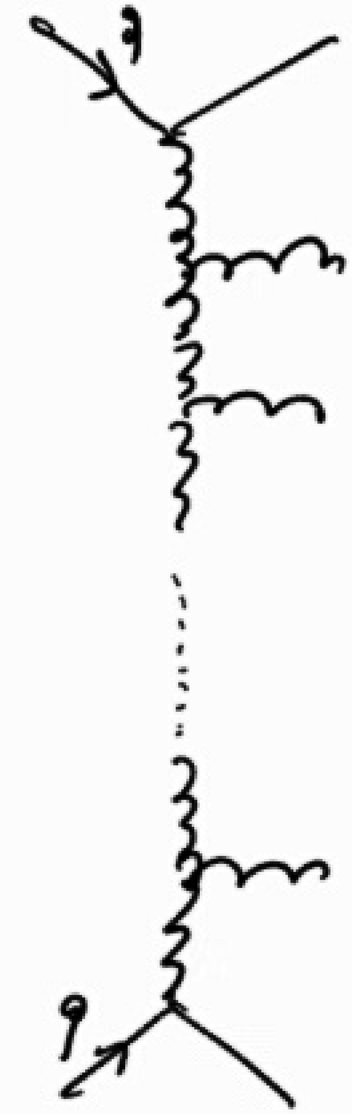
Then the $O(\alpha^2)$ corrections may contain a large logarithm $\ln \frac{s}{Q^2}$

One may consider resumming the powers of $\left(\alpha^2 \ln \frac{s}{Q^2}\right)$

Frolov, Gribov, Lipatov '71

The same will be done in QCD. But due to the gluon self-coupling, one resums the powers of $\left(\alpha_s \ln \frac{s}{Q^2}\right)$

BFKL 76-78



We shall begin the description of the high-energy limit with the simplest process where gluon exchange in t channel occurs: parton-parton scattering. We shall introduce the kinematic variables, the parton kinematics and dynamics.

Consider a boost along the beam axis. The rapidity y of a particle is given by $\tanh y = \frac{p_{\parallel}}{E}$ (see Ex. 1)

where $p_{\parallel}$ is the momentum along the beam axis and E is the energy.

That relation can be inverted: $y = \frac{1}{2} \ln \frac{E + p_{\parallel}}{E - p_{\parallel}}$

Energy and longitudinal momentum scale like $\cosh y$ and $\sinh y$, with constant of proportionality given by the transverse mass

$$E = m_\perp \cosh y \qquad m_\perp = \sqrt{m^2 + p_\perp^2}$$
$$p_\parallel = m_\perp \sinh y$$

Thus a 4-momentum is parametrised as

$$p = (m_\perp \cosh y, p_x, p_y, m_\perp \sinh y)$$

We shall often use light-cone coordinates

$$p^\pm = E \pm p_\parallel$$

then the 4-momentum is given by

$$p = (m_\perp e^{y}, m_\perp e^{-y}; p_x, p_y)$$

and the scalar product of 2 momenta is

$$2p\cdot q = p^+q^- + p^-q^+ - 2p_\perp\cdot q_\perp$$

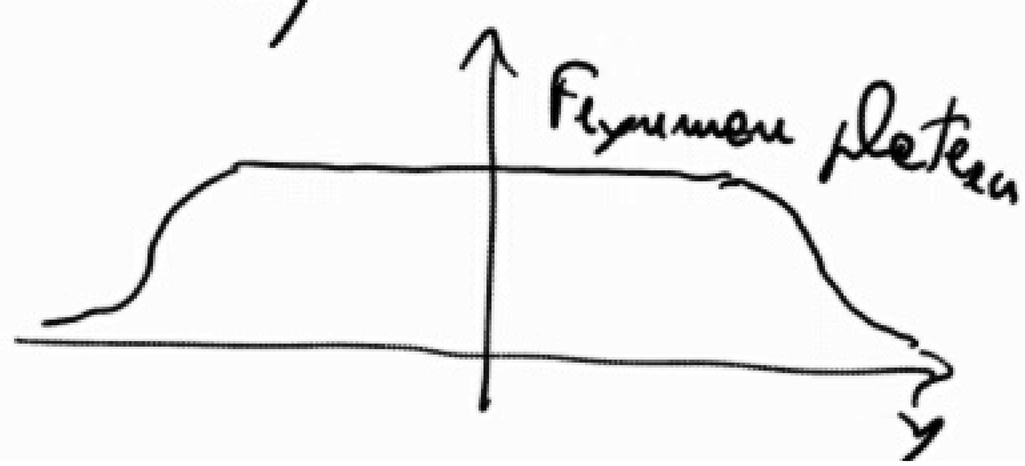
In terms of rapidity, the phase space measure is

$$\frac{d^3p}{2E(2\pi)^3} = \frac{dy\, d^2p_\perp}{4\pi(2\pi)^2}$$

since $dp_\parallel = m_\perp \cosh y \, dy$
$= E\, dy$

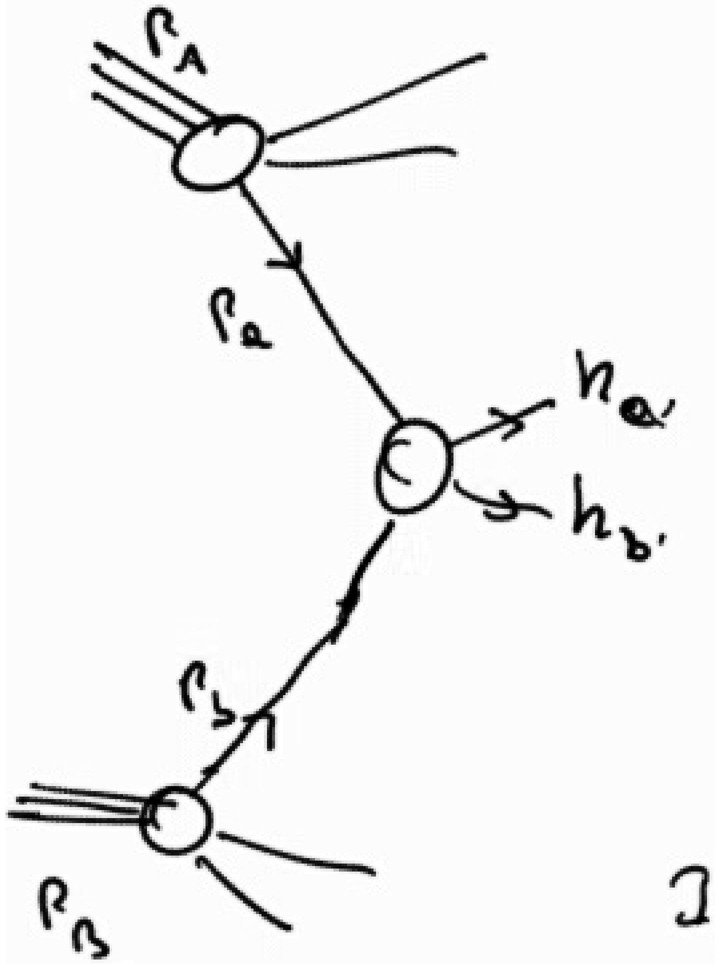
which transforms additively under boosts in the beam direction.
Thus the shape of the multiplicity distrib. $\frac{dN}{dy}$ is boost invariant

In a typical proton-proton collision, low $p_\perp$ particles fill up the phase space in y uniformly (Feynman plateau)



Consider the scattering of parton a in proton A with parton b in proton B.

In light-cone coord's,
$$p_a = (x_a\sqrt{s}, 0; \vec{0}_\perp)$$
$$p_b = (0, x_b\sqrt{s}; \vec{0}_\perp)$$
with $\vec{k}_\perp = (k_x, k_y)$



The momenta of the outgoing partons are

$$k_i = (k_{i\perp} e^{y_i}, k_{i\perp} e^{-y_i}; \vec{k}_{i\perp}) \qquad i = a', b'$$

Transverse momentum conservation requires that $|\vec{k}_{a'\perp}| = |\vec{k}_{b'\perp}| = k_\perp$

Introduce the rapidity of the c.m. frame $\bar{y} = \frac{y_{a'} + y_{b'}}{2}$ (boost)

and the rapidity difference $\Delta y = 2y^* = y_{a'} - y_{b'}$

in terms of the rapidity y^* of a parton in the c.m. frame

Light-cone momentum conservation yields:

$$x_a = \frac{k_\perp}{\sqrt{s}} e^{y_{a'}} + \frac{k_\perp}{\sqrt{s}} e^{y_{b'}} = 2\frac{k_\perp}{\sqrt{s}} e^{\bar{y}} \cosh y^*$$

(show it)

$$x_b = \frac{k_\perp}{\sqrt{s}} e^{-y_{a'}} + \frac{k_\perp}{\sqrt{s}} e^{-y_{b'}} = 2\frac{k_\perp}{\sqrt{s}} e^{-\bar{y}} \cosh y^*$$

Taking the ratio, we obtain: $\bar{y} = \frac{1}{2} \ln \frac{x_a}{x_b}$

In agreement with the definition of a boost, $\bar{y} = 0$ when the c.m. frame coincides with the lab frame.

The Mandelstam invariants are

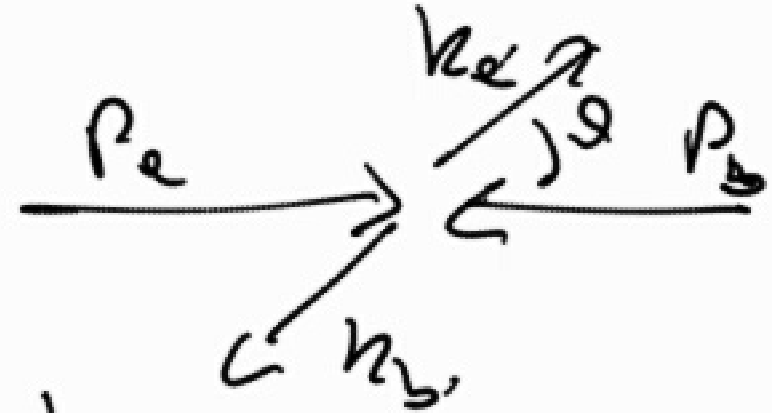
$$\hat{s} = (P_a + P_b)^2 = (k_{a'} + k_{b'})^2 = 4 k_\perp^2 \cosh^2 y^*$$

*) $$\hat{t} = (P_a - k_{a'})^2 = -2 k_\perp^2 e^{-y^*} \cosh y^*$$ (show it)

$$\hat{u} = (P_a - k_{b'})^2 = -2 k_\perp^2 e^{y^*} \cosh y^*$$

Note that the Mandelstam invariants depend on $k_\perp$ and y^*;
they do not depend on $\bar{y}$, which is determined by x_a and x_b.

The usual relation between the invariants and the scattering angle ϑ is



$$\hat{t} = -\frac{\hat{s}}{2}(1-\cos\vartheta) \qquad \hat{u} = -\frac{\hat{s}}{2}(1+\cos\vartheta)$$

using it in $*$), we obtain that $y^* = -\ln\tan\vartheta/2$

In fact, for massless particles $\tanh y = \frac{p_\parallel}{|\vec{p}|} = \cos\vartheta$

then $y = \frac{1}{2}\ln\frac{|\vec{p}|+p_\parallel}{|\vec{p}|-p_\parallel} = \frac{1}{2}\ln\frac{1+\cos\vartheta}{1-\cos\vartheta} = -\ln\tan\vartheta/2$

which is usually called <u>pseudorapidity</u> and denoted by η

For massless particles $\eta = y$

η is usually preferred by experimenters,

because easier to measure (just need to measure ϑ)

however in general $\eta \neq y$, with y transforming

additively under boosts.