ETH Zürich · Institute for Theoretical Physics

Classical Mechanics

Lecture script

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Abstract

The subject of the course is classical mechanics. The following topics are discussed:

- Galilean transformations and Newtonian mechanics
- Variational methods
- Principle of least action
- Lagrangian mechanics
- Symmetries and conservation laws
- Two-body systems
- Oscillations
- Rigid body dynamics
- Hamiltonian mechanics
- Hamilton–Jacobi equation
- Special relativity

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1

Principle of relativity and Galilean transformations

We will assume the existence of frames of reference (a system of coordinates) in which the motion of a free particle (on which no force is applied) is uniform, i.e. it moves with a constant velocity:

$$\text{free particle:} \quad \vec{v} = \text{constant}. \tag{1.1}$$

Such reference frames are called *inertial*.

The *principle of relativity* states that **the laws of physics take exactly the same form in all inertial frames.**

Let us assume that we know the trajectories (positions as functions of time) of all the particles which constitute a physical system as they are observed in one inertial reference frame S. Can we compute the trajectories in a different inertial frame S'? To achieve that, we need to transform the space and time coordinates $(\vec{r}, t)$ in the frame S to the corresponding coordinates $(\vec{r}\,', t')$ in the frame S'. A good guess for the (Cartesian) coordinate transformations are the so-called Galilean transformations:

$$\vec{r}\,' = \mathbf{R}\,\vec{r} - \vec{r}_0 - \vec{V}t, \quad t' = t_0 + t, \tag{1.2}$$

or, equivalently,

$$r_i' = \sum_{j=1}^{3} R_{ij} r_j - r_{0,i} - V_i t, \tag{1.3}$$

where $\vec{V}$ is the apparent velocity of a point at rest in the frame S' according to an observer in the frame S. The vector $\vec{r}_0$ is the position of the origin of S' as seen in S at the time $t = 0$, and t_0 is the time shown by the clock of S' at $t = 0$. The 3×3 matrix $\mathbf{R}$ describes the rotation of the axes of S' with respect to the axes of S and it satisfies

$$\mathbf{R}^T \mathbf{R} = \mathbf{1}_{3\times 3}\,, \tag{1.4}$$

where $\mathbf{1}_{3\times3}$ is the 3×3 identity matrix. **Remark:** the interpretation of $\vec{r}_0$ and $\vec{V}$ given above holds literally for $\mathbf{R} = \mathbf{1}$. For a general rotation, setting $\vec{r}\,' = 0$ in Eq. (1.2) shows that the origin of S' is at $\vec{r} = \mathbf{R}^T(\vec{r}_0 + \vec{V}t)$ in S, i.e. it starts at $\mathbf{R}^T\vec{r}_0$ and moves with velocity $\mathbf{R}^T\vec{V}$ (the components of $\vec{r}_0$, $\vec{V}$ refer to the axes of S'). In components, the orthogonality condition reads

$$R_{ij} R_{ik} = \delta_{jk}, \tag{1.5}$$

with a sum over i implied, where δ_{jk} is the Kronecker delta symbol. In addition, the determinant of the rotation matrix is

$$\det\mathbf{R} = 1. \tag{1.6}$$

It turns out that Galileo's transformations are only accurate for small velocities in comparison to the speed of light: $V \ll c$. We will discuss the correct generalisation of Galileo's transformations (the so-called Lorentz transformations) towards the end of this course. In the meantime, we will assume that Eq. (1.2) is valid; this is an excellent approximation for a plethora of phenomena.

1.1 Time intervals and distances in two inertial frames

Consider two events:

$$\text{Event A: } (t_A, \vec{r}_A), \qquad \text{Event B: } (t_B, \vec{r}_B),$$

as observed in the reference frame S.

In the reference frame S' these events are described by the time and space coordinates

$$\text{Event A: } (t'_A, \vec{r}'_A) = (t_A + t_0, \mathbf{R}\vec{r}_A - \vec{r}_0 - \vec{V}t_A),$$

$$\text{Event B: } (t'_B, \vec{r}'_B) = (t_B + t_0, \mathbf{R}\vec{r}_B - \vec{r}_0 - \vec{V}t_B).$$

The time difference of the two events is equal in the two frames S and S',

$$t'_A - t'_B = t_A - t_B. \tag{1.7}$$

Therefore, time intervals are measured to be the same in all inertial frames. Space distances are also the same in all inertial frames. For two events occurring at the same time t, their space distance is

$$\begin{aligned} \Delta r' &= \left|\vec{r}'_A - \vec{r}'_B\right| = |\mathbf{R}\,\vec{r}_A - \mathbf{R}\vec{r}_B| = \sqrt{(\vec{r}_A - \vec{r}_B)^T\mathbf{R}^T\mathbf{R}(\vec{r}_A - \vec{r}_B)} \\ &= \sqrt{(\vec{r}_A - \vec{r}_B)^T(\vec{r}_A - \vec{r}_B)} = |\vec{r}_A - \vec{r}_B| = \Delta r. \end{aligned} \tag{1.8}$$

1.2 Vectors, scalars and rotations

Under a rotation, the position $\vec{r}$ of a particle transforms as

$$\vec{r} \to \vec{r}' = \mathbf{R}\,\vec{r}, \quad \det\mathbf{R} = 1, \quad \mathbf{R}^T\mathbf{R} = 1. \tag{1.9}$$

The position is only one of many objects with the same transformation under rotations. We can formally define a vector $\vec{A}$ to be a set of three numbers which transforms in the same way as the position vector does under a rotation:

$$\vec{A} \to \vec{A}' = \mathbf{R}\,\vec{A}, \quad \det\mathbf{R} = 1, \quad \mathbf{R}^T\mathbf{R} = 1. \tag{1.10}$$

For example, time derivatives of the position vector (velocity, acceleration, ...) are also vectors:

$$\frac{d\vec{r}}{dt} \to \frac{d\vec{r}'}{dt'} = \frac{d(\mathbf{R}\,\vec{r})}{dt} = \mathbf{R}\,\frac{d\vec{r}}{dt}. \tag{1.11}$$

We define the dot product of two vectors as

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i. \tag{1.12}$$

The dot product is invariant under rotations,

$$\begin{aligned} \vec{A} \cdot \vec{B} \quad &\to \quad \vec{A}' \cdot \vec{B}' = \sum_i A'_i B'_i = \sum_{ijk} (R_{ij} A_j)(R_{ik} B_k) \\ &= \quad \sum_{ijk} R^T_{ji} R_{ik} A_j B_k = \sum_{jk} \delta_{jk} A_j B_k = \sum_k A_k B_k \\ &= \quad \vec{A} \cdot \vec{B}. \end{aligned} \tag{1.13}$$

All objects which are invariant under rotations are called *scalars.*

The cross product of two vectors (such as the angular momentum) is also a vector. Indeed, if $\vec{A}, \vec{B}$ are vectors, then their cross product transforms as

$$\vec{A} \times \vec{B} \to \vec{A}' \times \vec{B}' = \mathbf{R}(\vec{A} \times \vec{B}). \tag{1.14}$$

To prove this, consider the dot product

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \sum_i \left(\sum_{jk} \epsilon_{ijk} A_j B_k \right) C_i = \det \left(\vec{A}, \quad \vec{B}, \quad \vec{C} \right),$$

where the determinant is that of the matrix with columns $\vec{A}, \vec{B}, \vec{C}$. Under a rotation, this dot product is invariant,

$$\begin{aligned} \det \left(\vec{A}, \quad \vec{B}, \quad \vec{C} \right) \quad \to \quad & \det \left(\vec{A}', \quad \vec{B}', \quad \vec{C}' \right) = \det \left(\mathbf{R}\vec{A}, \quad \mathbf{R}\vec{B}, \quad \mathbf{R}\vec{C} \right) \\ & = \det \left\{ \mathbf{R} \left(\vec{A}, \quad \vec{B}, \quad \vec{C} \right) \right\} \\ & = \det(\mathbf{R}) \det \left(\vec{A}, \quad \vec{B}, \quad \vec{C} \right) \\ & = \det \left(\vec{A}, \quad \vec{B}, \quad \vec{C} \right), \end{aligned} \tag{1.15}$$

and it is therefore a scalar:

$$\begin{aligned} (\vec{A}' \times \vec{B}') \cdot \vec{C}' &= (\vec{A} \times \vec{B}) \cdot \vec{C} \\ \rightsquigarrow (\vec{A}' \times \vec{B}') \cdot (\mathbf{R}\vec{C}) &= (\vec{A} \times \vec{B}) \cdot \vec{C} \\ \rightsquigarrow \left[\mathbf{R}^T (\vec{A}' \times \vec{B}') \right] \cdot \vec{C} &= (\vec{A} \times \vec{B}) \cdot \vec{C} \\ \rightsquigarrow (\vec{A}' \times \vec{B}') \cdot \vec{C} &= \left[\mathbf{R} (\vec{A} \times \vec{B}) \right] \cdot \vec{C}. \end{aligned} \tag{1.16}$$

Since the above is valid for any vector $\vec{C}$, we conclude that

$$\vec{A}' \times \vec{B}' = \mathbf{R} (\vec{A} \times \vec{B}). \tag{1.17}$$

Finally, the gradient differential operator

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{1.18}$$

is also a vector. To see this, we first note that

$$x'_i = \sum_j R_{ij} x_j \rightsquigarrow \frac{\partial x'_i}{\partial x_j} = R_{ij} \tag{1.19}$$

and

$$x_i = \sum_j R^T_{ij} x'_j \rightsquigarrow \frac{\partial x_i}{\partial x'_j} = R^T_{ij} = R_{ji}. \tag{1.20}$$

Then

$$\vec{\nabla} \to \vec{\nabla}' : \quad \frac{\partial}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i} \frac{\partial}{\partial x_j} = \sum_j R_{ij} \frac{\partial}{\partial x_j}, \quad \text{i.e.} \quad \vec{\nabla}' = \mathbf{R}\vec{\nabla}. \tag{1.21}$$

The rotation matrix $\mathbf{R}$ is a particular case of a *Jacobian matrix*, namely the one for a change of variables. In general, given a vector-valued function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ which maps a vector $\vec{x} \in \mathbb{R}^n$ onto a vector $\vec{f}(\vec{x}) \in \mathbb{R}^m$, that is, given m functions, $f_1, \ldots, f_m$, each depending on n variables, $x_1, \ldots, x_n$, the *Jacobian matrix* is defined as

$$\mathbf{J}_{ij} = \frac{\partial f_i}{\partial x_j}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{1.22}$$

Note that the gradient of a scalar function, see Eq. (1.18), is the particular case of a Jacobian matrix with $m = 1$.

Jacobian matrices are also needed to invert functions. Recall that a function is continuously differentiable if its derivative is a continuous function; likewise, a vector-valued function is continuously differentiable if the entries of its Jacobian matrix are continuous functions. If $m = n$, we can define the determinant of the Jacobian matrix, usually called the *Jacobian*,

$$\det \mathbf{J} = \det\left(\frac{\partial f_i}{\partial x_j}\right), \qquad i, j = 1, \ldots, n.$$

The inverse function theorem states that if a continuously differentiable function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$ has a non-vanishing Jacobian at a point $\vec{x}$, then $\vec{f}$ is invertible near $\vec{x}$, and the inverse function is also a continuously differentiable function.

1.3 The form of laws in classical mechanics

The Galilean principle of relativity requires that all laws of physics are the same in all inertial reference frames. If we can cast our physical laws as vector equalities, it is then guaranteed that they will have the same form for all frames which are related by a rotation transformation.

$$\begin{aligned} \text{Physics law in frame } S\text{: } \vec{A} &= \vec{B} \\ \rightsquigarrow \mathbf{R}\vec{A} &= \mathbf{R}\vec{B} \\ \rightsquigarrow \text{Physics law in frame with rotated axes } S'\text{: } \vec{A}' &= \vec{B}'. \end{aligned} \tag{1.23}$$

The principle of relativity requires that the laws of physics take the same form not only under rotation transformations but also under "boosts", where two reference frames move with a constant velocity relative to each other:

$$\vec{r} \to \vec{r}' = \vec{r} + \vec{V}t. \tag{1.24}$$

Notice that the acceleration vector

$$\vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} = \vec{a}' \tag{1.25}$$

is the same for all inertial observers with parallel axes. Newton's law,

$$\vec{F} = m\vec{a}, \tag{1.26}$$

connects the acceleration, a boost-invariant quantity, to the force. As long as the latter is also boost invariant, the Galilean principle of relativity is satisfied.

We will see later that the acceleration vector is not boost invariant when using the correct principle of relativity as it is formulated in Einstein's special relativity. Therefore, we should not expect that the physical forces of nature are boost invariant either. Having said that, we will be able to ignore special relativity in most of our classical mechanics investigations, in the same way that Newton and many others were unaware of it when they laid the foundations of modern physics. In particular, our Newtonian understanding of gravity is consistent with the Galilean principle of relativity.

Consider an isolated system of N particles which interact with each other through the gravitational force only. The force acting on the i-th particle in a reference frame S is

$$m_i \frac{d^2\vec{r}_i}{dt^2} = -G \sum_{j\neq i} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{\left|\vec{r}_i - \vec{r}_j\right|^3}. \tag{1.27}$$

According to the principle of relativity, a Galilean transformation

$$(t, \vec{r}) \to (t', \vec{r}') = (t + t_0, \mathbf{R}\,\vec{r} + \vec{V}t + \vec{r}_0)$$

should leave Eq. (1.27) invariant. Indeed, this is the case. In the new frame,

$$\begin{aligned} m_i \frac{d^2\vec{r}_i{}'}{dt'^2} &= -G \sum_{j\neq i} m_i m_j \frac{\vec{r}_i{}' - \vec{r}_j{}'}{\left|\vec{r}_i{}' - \vec{r}_j{}'\right|^3} \\ \rightsquigarrow \mathbf{R}\, m_i \frac{d^2\vec{r}_i}{dt^2} &= -\mathbf{R}\, G \sum_{j\neq i} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{\left|\vec{r}_i - \vec{r}_j\right|^3} \\ \rightsquigarrow m_i \frac{d^2\vec{r}_i}{dt^2} &= -G \sum_{j\neq i} m_i m_j \frac{\vec{r}_i - \vec{r}_j}{\left|\vec{r}_i - \vec{r}_j\right|^3}. \end{aligned} \tag{1.28}$$

Question: Is the electromagnetic force boost invariant?

2

Variational methods

Classical mechanics is usually formulated in terms of Newton's laws in the form of second-order differential equations, as in Eq. (1.27). Alternatively, we can formulate the laws of classical mechanics by means of a so-called *variational principle*. It is based on a scalar quantity, called the action. Before we present this formulation, let us warm up with a few problems in physics and mathematics which use techniques of the calculus of variations.

2.1 The brachistochrone ($\beta\rho\alpha\chi\iota\sigma\tau\omega\varsigma\ \chi\rho\omega\nu\omega\varsigma$ = shortest time) problem

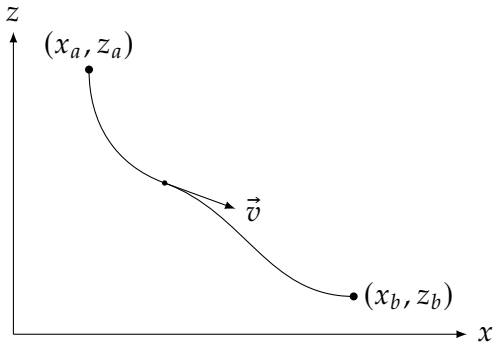


Figure 2.1. The brachistochrone problem: find which curve connecting (x_a, z_a) and (x_b, z_b) yields the fastest descent.

Consider a point-like object of mass m which is allowed to slide without friction in a vertical plane from a point (x_a, z_a) to a point (x_b, z_b), constrained to a curve $x = x(z)$, where z is the height and x is the horizontal position of the object. We can compute the velocity of the object at all points $(x(z), z)$ of its trajectory by using energy conservation. Assuming that the initial velocity is zero, we have

$$mgz_a = mgz + \frac{1}{2}mv^2 \rightsquigarrow v = \sqrt{2g(z_a - z)}. \tag{2.1}$$

The elementary distance segment on the curve $x = x(z)$ is given by Pythagoras' theorem as

$$ds = \sqrt{dx^2 + dz^2} = |dz|\sqrt{1 + \left(\frac{dx}{dz}\right)^2}, \tag{2.2}$$

and the magnitude of the velocity is

$$v = \frac{ds}{dt} \rightsquigarrow dt = \frac{ds}{v} = |dz| \sqrt{\frac{1 + \left(\frac{dx}{dz}\right)^2}{2g(z_a - z)}}. \tag{2.3}$$

The total time for the transition is

$$T = \int dt = \int_{z_b}^{z_a} dz \sqrt{\frac{1 + \left(\frac{dx}{dz}\right)^2}{2g(z_a - z)}}. \tag{2.4}$$

We are seeking the function $x(z)$ which minimises the time T.

2.2 Euler–Lagrange equations

The problem that we have just described belongs to a more general class of problems in the calculus of variations. We seek to find which function $y(t) = y_s(t)$ among all well-behaved functions $y(t)$ yields an extreme value (minimum or maximum) of the integral

$$I[y] = \int_{t_a}^{t_b} dt\, F\left(y(t), y'(t)\right), \tag{2.5}$$

given that the values of the function at the initial and final points of the integration domain, $y(t_a) = y_a$ and $y(t_b) = y_b$, are known.

Assume that we have found the curve $y_s(t)$ with

$$y_s(t_a) = y_a, \quad y_s(t_b) = y_b, \tag{2.6}$$

for which $I[y_s]$ is a minimum of $I[y]$.[1] Consider now a small deformation around $y_s(t)$,

$$y(t) = y_s(t) + \delta y_s(t), \quad \delta y_s(t) = \epsilon \eta(t), \quad \epsilon \to 0, \tag{2.7}$$

where $\eta(t)$ is a smooth function which vanishes at the boundaries,

$$\eta(t_a) = \eta(t_b) = 0, \tag{2.8}$$

so that $y(t_a) = y_a$ and $y(t_b) = y_b$ remain satisfied. Since $I[y_s]$ is an extremum, our small deformation (ignoring $O(\epsilon^2)$ terms) does not change its value,

$$\begin{aligned} I[y_s] &= I[y_s + \delta y_s] \\ &= \int_{t_a}^{t_b} dt\, F\left(y_s(t) + \delta y_s(t), y_s'(t) + \delta y_s'(t)\right). \end{aligned} \tag{2.9}$$

We observe that the operations of varying a function and taking its derivative with respect to its argument commute:

$$\delta y_s'(t) = \frac{d}{dt} \delta y_s(t) = \epsilon \eta'(t). \tag{2.10}$$

Thus we can Taylor-expand the arguments of F, and write

$$\begin{aligned} I[y_s] &= \int_{t_a}^{t_b} dt\, F\left(y_s(t), y_s'(t)\right) + \int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s} \epsilon \eta(t) + \frac{\partial F}{\partial y_s'} \epsilon \eta'(t)\right] + O(\epsilon^2) \\ &= I[y_s] + \epsilon \int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s} \eta(t) + \frac{\partial F}{\partial y_s'} \eta'(t)\right] + O(\epsilon^2). \end{aligned} \tag{2.11}$$

[1] Notice that $I[y]$ is a function of a function. Such objects are called *functionals*.

This yields

$$\begin{aligned} 0 &= \int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s}\eta(t) + \frac{\partial F}{\partial y_s'}\eta'(t)\right] \\ &= \int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s} - \frac{d}{dt}\frac{\partial F}{\partial y_s'}\right]\eta(t) + \int_{t_a}^{t_b} dt \, \frac{d}{dt}\left[\frac{\partial F}{\partial y_s'}\eta(t)\right] \\ &= \int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s} - \frac{d}{dt}\frac{\partial F}{\partial y_s'}\right]\eta(t) + \frac{\partial F}{\partial y_s'}\eta(t)\Big|_{t_a}^{t_b}. \end{aligned} \tag{2.12}$$

The last term is zero since $\eta(t)$ vanishes at the boundaries. Thus, the integral

$$\int_{t_a}^{t_b} dt \left[\frac{\partial F}{\partial y_s} - \frac{d}{dt}\frac{\partial F}{\partial y_s'}\right]\eta(t) = 0 \tag{2.13}$$

vanishes for every function $\eta(t)$. In order for this to happen, we must have

$$\frac{\partial F}{\partial y_s} - \frac{d}{dt}\frac{\partial F}{\partial y_s'} = 0. \tag{2.14}$$

This is the Euler–Lagrange equation.

A completely analogous derivation of Euler–Lagrange differential equations can be made for minimising multidimensional integrals which are functionals of many functions. Given a vector-valued function $\vec{y} : \mathbb{R}^n \to \mathbb{R}^m$ which maps a vector $\vec{x} \in \mathbb{R}^n$ onto a vector $\vec{y}(\vec{x}) \in \mathbb{R}^m$, that is, given m functions, $y_1, \ldots, y_m$, each depending on n variables, $x_1, \ldots, x_n$, we may consider the integral

$$I[y_1, \ldots, y_m] = \int dx_1 \ldots dx_n F\left[y_i(\vec{x}), \frac{\partial y_i(\vec{x})}{\partial x_j}\right]. \tag{2.15}$$

If the above integral takes a minimum or maximum value, then under small variations of the functions,

$$y_i(\vec{x}) \to y_i(\vec{x}) + \delta y_i(\vec{x}) = y_i(\vec{x}) + \epsilon\eta_i(\vec{x}), \quad i = 1, \ldots, m,$$

its value should be unchanged:

$$\begin{aligned} 0 &= \delta I[y_1, \ldots, y_m] \\ &= \delta \int dx_1 \ldots dx_n F\left[y_i(\vec{x}), \frac{\partial y_i(\vec{x})}{\partial x_j}\right] \\ &= \int dx_1 \ldots dx_n \left[\sum_{i=1}^{m} \frac{\partial F}{\partial y_i}\delta y_i(\vec{x}) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial F}{\partial(\partial_j y_i)}\delta\partial_j y_i(\vec{x})\right], \end{aligned} \tag{2.16}$$

where we have introduced the shorthand notation

$$\partial_j \equiv \frac{\partial}{\partial x_j}.$$

We use again that the operations of varying a function and taking its derivative with respect to its arguments commute,

$$\delta\partial_j y_i(\vec{x}) = \partial_j \delta y_i(\vec{x}), \tag{2.17}$$

and, after an integration by parts, we have

$$\begin{aligned} 0 &= \epsilon \int dx_1 \dots dx_n \sum_{i=1}^{m} \eta_i \left[\frac{\partial F}{\partial y_i} - \sum_{j=1}^{n} \partial_j \frac{\partial F}{\partial(\partial_j y_i)} \right] \\ &\quad + \epsilon \sum_{ij} \int dx_1 \dots dx_n \partial_j \left[\eta_i \frac{\partial F}{\partial(\partial_j y_i)} \right] . \end{aligned} \tag{2.18}$$

The second term is a total derivative in the x_j integration and it vanishes if we require that the variations $\delta y_i(\vec{x})$ vanish at the boundaries of the integration domain. Since the above identity must be valid for an arbitrary choice of functions $\eta_i(\vec{x})$, we arrive at the general Euler–Lagrange equations:

$$\frac{\partial F}{\partial y_i} - \sum_{j=1}^{n} \partial_j \frac{\partial F}{\partial(\partial_j y_i)} = 0 , \qquad i = 1, \dots, m . \tag{2.19}$$

A particularly useful case is that of a vector-valued function of one variable, $\vec{y} : \mathbb{R} \to \mathbb{R}^m$ which maps a variable $x \in \mathbb{R}$ onto a vector $\vec{y}(x) \in \mathbb{R}^m$, that is, the case treated above with $n = 1$. Then the general Euler–Lagrange equations become

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i'} = 0 , \qquad i = 1, \dots, m . \tag{2.20}$$

2.3 Propagation of light

Equipped with the Euler–Lagrange equations (2.20), we consider the propagation of light. Fermat formulated the principle named after him, stating that light travels from point to point along the path which requires the shortest time. For such a transition, the total time is

$$T = \int dt = \int \frac{ds}{v(\vec{x})} , \tag{2.21}$$

where ds is the distance element and $v(\vec{x})$ is the speed of light at the point $\vec{x}$. This is usually normalised to the speed of light in vacuum, c,

$$v(\vec{x}) = \frac{c}{n(\vec{x})} , \tag{2.22}$$

where $n(\vec{x})$ is the refractive index. Fermat's principle requires that

$$\begin{aligned} T &= \frac{1}{c} \int ds\, n(\vec{x}) = \frac{1}{c} \int \sqrt{dx^2 + dy^2 + dz^2}\, n(\vec{x}) \\ &= \frac{1}{c} \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, n(\vec{x}) \end{aligned} \tag{2.23}$$

is a minimum. Strictly speaking, Fermat's principle requires the travel time to be *stationary*, $\delta T = 0$, not necessarily minimal: for reflection from a concave mirror the actual path can be a maximum of T among neighbouring paths (see e.g. Born & Wolf, *Principles of Optics*, Ch. 3). The Euler–Lagrange equations only express stationarity.

Let us assume that the index of refraction is a constant,

$$n(\vec{x}) = n = \text{const.}$$

Then

$$T = \frac{n}{c}\int dx\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}. \tag{2.24}$$

Formally, we can write this as

$$T[y,z] = \int dx\, F\left(y'(x), z'(x)\right) \tag{2.25}$$

with

$$F\left(y'(x), z'(x)\right) = \frac{n}{c}\sqrt{1+y'^2(x)+z'^2(x)}\,. \tag{2.26}$$

Note that F does not depend explicitly on y and z.

The Euler–Lagrange equations read

$$\begin{aligned}\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} &= 0\\ \frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} &= 0\,.\end{aligned} \tag{2.27}$$

Computing the derivatives,

$$\begin{aligned}\frac{\partial F}{\partial y} &= 0\,, & \frac{\partial F}{\partial y'} &= \frac{n}{c}\frac{y'}{\sqrt{1+y'^2+z'^2}}\\ \frac{\partial F}{\partial z} &= 0\,, & \frac{\partial F}{\partial z'} &= \frac{n}{c}\frac{z'}{\sqrt{1+y'^2+z'^2}},\end{aligned} \tag{2.28}$$

we find that the Euler–Lagrange equations become

$$\begin{aligned}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2+z'^2}} &= 0\\ \frac{d}{dx}\frac{z'}{\sqrt{1+y'^2+z'^2}} &= 0\,,\end{aligned} \tag{2.29}$$

which can be trivially integrated:

$$\begin{aligned}\frac{y'}{\sqrt{1+y'^2+z'^2}} &= c_1\\ \frac{z'}{\sqrt{1+y'^2+z'^2}} &= c_2\,,\end{aligned} \tag{2.30}$$

where c_1 and c_2 are constants. The solution of these equations is a straight line.

To see this, consider first the simpler case of a light ray in the x-y plane. Then the Euler–Lagrange equation is

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0, \tag{2.31}$$

which is integrated to

$$\frac{y'}{\sqrt{1+y'^2}} = c, \tag{2.32}$$

where c is a constant. This can be solved for y',

$$y' = \frac{c}{\sqrt{1-c^2}} \equiv a, \tag{2.33}$$

where a is a constant. Integrating it, we get the equation of a straight line,

$$y(x) = ax + b\,. \tag{2.34}$$

Likewise, the coupled system (2.30) can be solved to find

$$\begin{aligned} y(x) &= a_1 x + b_1 \\ z(x) &= a_2 x + b_2\,, \end{aligned} \tag{2.35}$$

which are the parametric equations of a straight line.

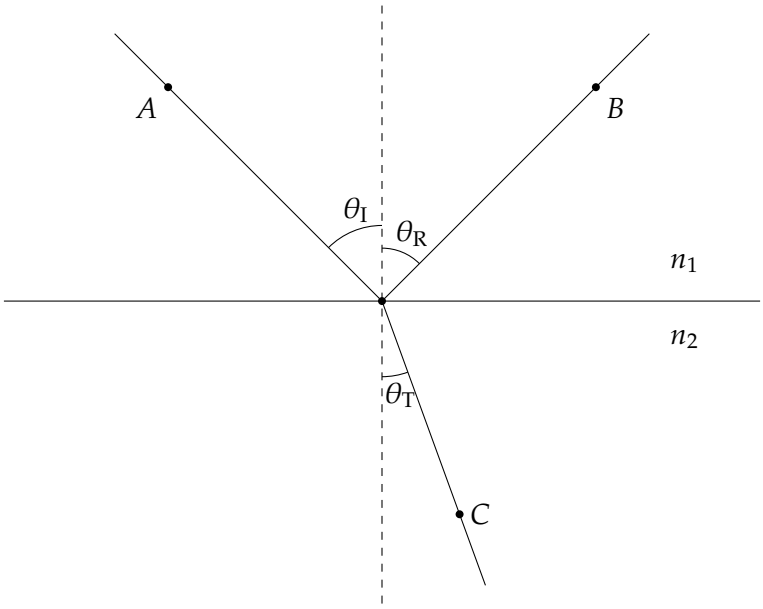


Figure 2.2. The transmission and reflection of light at a boundary separating media with different refractive indices.

The principle of least time yields the correct laws of geometrical optics for the reflection and refraction of light, stating that the angles of incidence and reflection are equal,

$$\theta_I = \theta_R, \tag{2.36}$$

and that the direction of the light transmitted from a medium with refractive index n_1 to a medium with refractive index n_2 is determined by Snell's law,

$$n_1 \sin\theta_I = n_2 \sin\theta_T. \tag{2.37}$$

Indeed, consider a light ray which passes through the point $A(x_a, y_a)$ in Fig. 2.2 and arrives at a point $B(x_b, y_b)$ after it is reflected at a boundary separating two different media at the point $O(0,0)$. The time required for this transition is

$$T_{A\to B} = \frac{\sqrt{x_a^2 + y_a^2}}{v_1} + \frac{\sqrt{x_b^2 + y_b^2}}{v_1}, \tag{2.38}$$

where $v_1 = \frac{c}{n_1}$ is the speed of light in the medium at the side of the boundary where the incident and reflected rays lie. If $L = x_b - x_a$ is the horizontal distance between the two points, then we write

$$cT_{A\to B} = n_1\sqrt{x_a^2 + y_a^2} + n_1\sqrt{(L + x_a)^2 + y_b^2}. \tag{2.39}$$

Requiring the above to be a minimum (at fixed L), we find that the horizontal position x_a must satisfy

$$\begin{aligned} & \frac{\partial T_{A\to B}}{\partial x_a} = 0 \\ \rightsquigarrow \quad & \frac{x_a}{\sqrt{x_a^2 + y_a^2}} + \frac{(L + x_a)}{\sqrt{(L + x_a)^2 + y_b^2}} = 0 \\ \rightsquigarrow \quad & -\sin\theta_I + \sin\theta_R = 0\,, \end{aligned} \tag{2.40}$$

which shows that the angles of the incident and reflected light are equal,

$$\theta_I = \theta_R. \tag{2.41}$$

Repeating the same steps for a transition from the point A inside the medium with refractive index n_1 to a point C inside the medium with refractive index n_2,

$$\begin{aligned} & \frac{\partial T_{A\to C}}{\partial x_a} = 0 \\ \rightsquigarrow \quad & n_1\frac{x_a}{\sqrt{x_a^2 + y_a^2}} + n_2\frac{(L + x_a)}{\sqrt{(L + x_a)^2 + y_c^2}} = 0 \\ \rightsquigarrow \quad & -n_1\sin\theta_I + n_2\sin\theta_T = 0\,, \end{aligned} \tag{2.42}$$

we find Snell's law,

$$n_1\sin\theta_I = n_2\sin\theta_T. \tag{2.43}$$

Interesting phenomena occur when the index of refraction is not uniform. For example, the density of the air in the atmosphere changes with temperature. The non-uniformity of the density is responsible for phenomena such as mirage images in deserts or on the surface of a hot road in the summer and the twinkling of stars. In such media, the trajectory of the light is not a straight line.

2.4 Solution of the brachistochrone problem

We can now return to the problem of Section 2.1. The functional T of Eq. (2.4) can be written as

$$T[x] = \int_{z_b}^{z_a} dz\, F(x'(z)) \tag{2.44}$$

with

$$F = \sqrt{\frac{1 + x'^2(z)}{2g(z_a - z)}}. \tag{2.45}$$

Note that F does not depend explicitly on x.

Requiring that the functional (2.44) be a minimum, we can write the Euler–Lagrange equation (2.14),

$$\frac{\partial F}{\partial x} - \frac{d}{dz}\frac{\partial F}{\partial x'} = 0. \tag{2.46}$$

Performing the differentiations,

$$\frac{\partial F}{\partial x} = 0\,, \qquad \frac{\partial F}{\partial x'} = \frac{x'}{\sqrt{2g(z_a - z)(1 + x'^2(z))}}\,, \tag{2.47}$$

we obtain the second-order differential equation

$$\frac{d}{dz}\frac{x'}{\sqrt{(z_a - z)(1 + x'^2)}} = 0, \tag{2.48}$$

which, after a trivial first integration, gives

$$\frac{x'}{\sqrt{(z_a - z)(1 + x'^2)}} = -C, \tag{2.49}$$

with C a constant to be fixed by our boundary conditions. Solving for x', we obtain

$$x'^2 = \frac{C^2(z_a - z)}{1 - C^2(z_a - z)}. \tag{2.50}$$

For the right-hand side to be non-negative, we must have either

$$z \geq z_a \text{ and } C^2(z_a - z) \geq 1,$$

or

$$z \leq z_a \text{ and } C^2(z_a - z) \leq 1.$$

The first possibility is not physically allowed, since it is not satisfied by the final point (x_b, z_b) which lies lower than (x_a, z_a).

Let us now change variables,

$$C^2(z_a - z) = \sin^2\phi, \quad \phi \in \left[0, \frac{\pi}{2}\right], \tag{2.51}$$

with

$$\frac{dz}{d\phi} = -\frac{1}{C^2}\sin(2\phi). \tag{2.52}$$

The differential equation for $x(\phi)$ becomes

$$\frac{dx}{d\phi} = 2\frac{C}{|C|^3}\sin^2\phi. \tag{2.53}$$

It is now straightforward to perform the integrations over ϕ and compute $x = x(\phi), z = z(\phi)$. We can choose $\phi = 0$ to correspond to the first point (x_a, z_a) of the curve. This gives the solutions

$$x(\phi) = x_a + \frac{C}{2|C|^3}\left[2\phi - \sin(2\phi)\right], \qquad z(\phi) = z_a - \frac{1}{2C^2}\left[1 - \cos(2\phi)\right]. \tag{2.54}$$

Setting $\phi = \theta/2$ with $\theta \in [0, \pi]$, we have

$$x(\theta) = x_a + \frac{C}{2|C|^3}\left[\theta - \sin\theta\right], \qquad z(\theta) = z_a - \frac{1}{2C^2}\left[1 - \cos\theta\right]. \tag{2.55}$$

Choosing a positive value for C, we obtain solutions where the curve lies to the right of the starting point (as in Fig. 2.1), while for $C < 0$ we obtain solutions where the curve lies to the left of the starting point. For $C > 0$ we have

$$x(\theta) = x_a + \frac{1}{2C^2}\left[\theta - \sin\theta\right], \qquad z(\theta) = z_a - \frac{1}{2C^2}\left[1 - \cos\theta\right]. \tag{2.56}$$

These are the equations of a cycloid. To determine the constant C^2 we require that the final point (x_b, z_b) belongs to the curve, i.e. there exists a value $\theta = \theta_b$ such that

$$x_b = x_a + \frac{1}{2C^2}\left[\theta_b - \sin\theta_b\right], \tag{2.57}$$

$$z_b = z_a - \frac{1}{2C^2}\left[1 - \cos\theta_b\right]. \tag{2.58}$$

For any specific values of the pairs (x_a, z_a) and (x_b, z_b), one can use Eqs. (2.57) and (2.58) to determine numerically θ_b and C, and from these the radius of the circle generating the required cycloid,

$$r = \frac{1}{2C^2}. \tag{2.59}$$

Exercise 2.1. Determine C and r for $(x_a, z_a) = (0, 1)$ and $(x_b, z_b) = (2, 0)$.

Remark: for these endpoints $\theta_b \simeq 3.508 > \pi$, i.e. the fastest curve dips below z_b and rises again to B. This lies outside the range $\phi \in [0, \pi/2]$ ($\theta \in [0, \pi]$) used above, where $x(z)$ is single-valued; the parametric cycloid solution remains valid for $\theta \in [0, 2\pi)$ (one finds $C \simeq 0.983$, $r \simeq 0.517$).

A nice demonstration of the brachistochrone problem can be found in this video.

3

Hamilton's principle of least action

Having seen that the behaviour of light can be understood by means of a "principle of least time" and variational techniques, we may ask whether variational methods can also be used more generally for the study of physical systems.

The main result of variational methods is the Euler–Lagrange differential equations. The laws of classical physics, Newton's laws and the Maxwell equations of electrodynamics, were first discovered in the form of differential equations. The question we are posing here is a reverse-engineering problem. If we know the Euler–Lagrange equations (physical laws in the form of differential equations), can we find the integral functional from which they arise? Hamilton's principle of least action states that this is possible.

Consider an isolated system of many particles $i = 1, \ldots, N$ for which we would like to know the trajectories, i.e. their positions at all times. While it is common to use Cartesian coordinates, $\vec{r}_i(t)$, it is often useful to use other variables such as angles, radii, or even more exotic variables to describe the trajectory of a particle. We call these generic variables which describe a trajectory *generalised coordinates*.

Assume now that we know at an initial time t_i the values of a set of **independent** generalised coordinates

$$q_1(t_i), \ldots, q_m(t_i),$$

as well as the initial values of their time derivatives (generalised velocities),

$$\dot{q}_1(t_i), \ldots, \dot{q}_m(t_i).$$

Hamilton's principle of least action states that the generalised coordinates and generalised velocities of the system are determined at a later time t_f by requiring that the *action*,

$$S \equiv \int_{t_i}^{t_f} dt\, L\left[\{q_j(t), \dot{q}_j(t)\}\right], \tag{3.1}$$

is a minimum:

$$\delta S = 0. \tag{3.2}$$

Strictly speaking, the condition $\delta S = 0$ only states that the action is *stationary*. The physical path is a true minimum only for sufficiently short time intervals (before the first kinetic focus); in general it can be a saddle point (Landau–Lifshitz vol. 1, §2, footnote; Gray & Taylor, Am. J. Phys. 75 (2007) 434). "Principle of stationary action" is the precise name. The integrand L of the action is called the *Lagrangian*. The Euler–Lagrange equations derived from the principle of least action for physical systems are called *equations of motion* and read

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \tag{3.3}$$

We note that the Lagrangian of a system is not uniquely defined. Consider two Lagrangians L, L' which differ by a total derivative,

$$L' = L + \frac{df}{dt}, \tag{3.4}$$

where f is an arbitrary function of the coordinates and time only, $f = f(q,t)$ (Landau–Lifshitz vol. 1, §2). Indeed, the variations satisfy $\delta q(t_i) = \delta q(t_f) = 0$ but in general $\delta\dot{q}(t_{i,f}) \neq 0$, so $\delta f(t_{i,f})$ vanishes only if f does not depend on the velocities. The corresponding actions,

$$S' = \int_{t_i}^{t_f} dt\, L' = \int_{t_i}^{t_f} dt\, L + \int_{t_i}^{t_f} dt\, \frac{df}{dt} = S + f(t_f) - f(t_i), \tag{3.5}$$

differ by boundary terms whose variation vanishes, since the generalised coordinates are kept fixed at the endpoints:

$$\delta S' = \delta S + \delta\left(f(t_f) - f(t_i)\right) = \delta S. \tag{3.6}$$

Therefore, Lagrangians which differ by a total derivative yield the same equations of motion.

3.1 The Lagrangian of a free particle in an inertial frame

We will now find Lagrangians which govern known physical systems, starting from the simplest case: a free isolated particle (one which does not interact with other particles), observed in an inertial frame. The equation of motion of the particle is

$$\vec{a} = 0, \tag{3.7}$$

where the acceleration of the free particle is

$$\vec{a} \equiv \ddot{\vec{r}} = \left(\ddot{r}_1, \ddot{r}_2, \ddot{r}_3\right). \tag{3.8}$$

The Lagrangian

$$L = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\sum_{i=1}^{3}\dot{r}_i^2, \tag{3.9}$$

yields the correct equation of motion,

$$0 = \frac{d}{dt}\frac{\partial L}{\partial\,\dot{r}_i} - \frac{\partial L}{\partial r_i} = m\,\ddot{r}_i \rightsquigarrow \ddot{r}_i = 0. \tag{3.10}$$

Could we have guessed a different Lagrangian which also yields a vanishing acceleration for a freely moving particle? For example, the Lagrangian

$$L = (\vec{v}^2)^n \tag{3.11}$$

yields the equations of motion

$$\left(\dot{\vec{r}}^2\right)^{n-2}\left[\dot{\vec{r}}^2\,\ddot{r}_i + 2(n-1)\left(\dot{\vec{r}}\cdot\ddot{\vec{r}}\right)\dot{r}_i\right] = 0. \tag{3.12}$$

For $n \neq \frac{1}{2}$, in a frame where the particle is moving at a certain moment with a velocity $\dot{\vec{r}} \neq 0$, we find that the acceleration must be zero. Nevertheless, such equations of motion do not conform with Galileo's principle of relativity, which requires that under, for example, a boost

$$\vec{r} \to \vec{r}' = \vec{r} + \vec{v}t, \tag{3.13}$$

Eq. (3.12) maintains the same form, i.e.

$$\left(\dot{\vec{r}}'^2\right)^{n-2}\left[\dot{\vec{r}}'^2\,\ddot{r}_i' + 2(n-1)\left(\dot{\vec{r}}'\cdot\ddot{\vec{r}}'\right)\dot{r}_i'\right] = 0. \tag{3.14}$$

It does not, except for $n = 1$! Instead, one finds a frame-dependent form:

$$\left(\vec{u}^{\,2}\right)^{n-2}\left[\vec{u}^{\,2}\,\ddot{r}_i' + 2(n-1)\left(\vec{u}\cdot\ddot{\vec{r}}'\right)u_i\right] = 0\,, \qquad \vec{u} \equiv \dot{\vec{r}}' - \vec{v}\,. \tag{3.15}$$

Remark: the solutions of the transformed equation are still $\ddot{\vec{r}}' = 0$ (for $\dot{\vec{r}}' \neq \vec{v}$), so the observable free motion is the same in both frames; what fails is the form invariance. The sharp criterion (Landau–Lifshitz vol. 1, §4) is that under a boost the Lagrangian may change only by a total time derivative; $(\vec{v}^{\,2})^n$ satisfies this only for $n = 1$, since $\frac{m}{2}(\vec{v}+\vec{V})^2 = \frac{m}{2}\vec{v}^{\,2} + \frac{d}{dt}\left(m\vec{V}\cdot\vec{r} + \frac{m}{2}\vec{V}^2 t\right)$.

A comment is appropriate on the constant factor $\frac{m}{2}$ in the Lagrangian of the free particle of Eq. (3.9). Obviously, this factor is immaterial in deriving the equations of motion. However, we will soon extend the Lagrangian to account for interactions with other particles. When this happens, we will associate m with the physical mass of the particle. The principle of least action tells us that the physical mass is a positive quantity, $m > 0$. Indeed, the general solution of the equations of motion is

$$\ddot{\vec{r}} = 0 \rightsquigarrow \vec{r}(t) = \vec{A} + \vec{B}t. \tag{3.16}$$

We can fix the constants $\vec{A}, \vec{B}$ for a given transition from a space-time point $(t_i, \vec{r}_i)$ to a space-time point $(t_f, \vec{r}_f)$. Then, the physical trajectory is given by

$$\vec{r}_{\rm phys}(t) = \vec{r}_i + \frac{t - t_i}{t_f - t_i}\left(\vec{r}_f - \vec{r}_i\right). \tag{3.17}$$

The value of the action for the physical trajectory is

$$S\left[\vec{r}_{\rm phys}(t)\right] = \int_{t_i}^{t_f} dt\, L[\vec{r}_{\rm phys}(t)] = \frac{1}{2}m\frac{\left(\vec{r}_f - \vec{r}_i\right)^2}{t_f - t_i}. \tag{3.18}$$

According to the principle of least action, any trajectory other than the physical one that we can think of to join the space-time points $(t_i, \vec{r}_i)$ and $(t_f, \vec{r}_f)$ should yield a greater value of the action. Consider, for example, the non-physical trajectory

$$\vec{r}_{\rm non-phys}(t) = \vec{r}_i + \left(\frac{t - t_i}{t_f - t_i}\right)^2\left(\vec{r}_f - \vec{r}_i\right). \tag{3.19}$$

We must have

$$S\left[\vec{r}_{\rm non-phys}(t)\right] - S\left[\vec{r}_{\rm phys}(t)\right] > 0. \tag{3.20}$$

An explicit calculation shows that

$$S\left[\vec{r}_{\rm non-phys}(t)\right] = \int_{t_i}^{t_f} dt\, L[\vec{r}_{\rm non-phys}(t)] = \frac{2}{3}m\frac{\left(\vec{r}_f - \vec{r}_i\right)^2}{t_f - t_i} = \frac{4}{3}S\left[\vec{r}_{\rm phys}(t)\right]. \tag{3.21}$$

Eq. (3.20) then implies that the value of the action for the physical trajectory must be positive,

$$S\left[\vec{r}_{\text{phys}}(t)\right] = \frac{1}{2}m\frac{\left(\vec{r}_f - \vec{r}_i\right)^2}{t_f - t_i} > 0, \tag{3.22}$$

and, since $t_f > t_i$, the mass must be positive, $m > 0$.

3.2 Lagrangian of a particle in a homogeneous force field

Consider a particle subjected to a constant force $\vec{F}$. According to Newton's law, the equation of motion is

$$m\ddot{\vec{r}} - \vec{F} = 0, \tag{3.23}$$

or, in components,

$$m\,\ddot{r}_i - F_i = 0. \tag{3.24}$$

As for the earlier case of the free particle, the first term of the above equation can be written as

$$m\,\ddot{r}_i = \frac{d}{dt}\frac{\partial T}{\partial\,\dot{r}_i}, \tag{3.25}$$

where

$$T = \frac{1}{2}m\dot{\vec{r}}^2, \tag{3.26}$$

is the kinetic energy of the particle. We can write the second term as a derivative,

$$F_i = \frac{\partial}{\partial r_i}\sum_{j=1}^{3} F_j r_j = -\frac{\partial(-\vec{F}\cdot\vec{r})}{\partial r_i}. \tag{3.27}$$

The dot product $-\vec{F}\cdot\vec{r}$ is the work done against the force to bring the particle from the origin of our reference frame to the position $\vec{r}$, or in other words the potential energy of the particle,

$$U(\vec{r}) = -\vec{F}\cdot\vec{r}. \tag{3.28}$$

Therefore, Newton's law for a constant force takes the form

$$\frac{d}{dt}\frac{\partial T}{\partial\,\dot{r}_i} - \frac{\partial(-U)}{\partial r_i} = 0. \tag{3.29}$$

Given that the potential energy U does not depend on the velocities $\dot{r}_i$ and that the kinetic energy T does not depend on the positions, we can rewrite the above equation as

$$\frac{d}{dt}\frac{\partial(T-U)}{\partial\,\dot{r}_i} - \frac{\partial(T-U)}{\partial r_i} = 0. \tag{3.30}$$

This has the form of the Euler–Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial\,\dot{r}_i} - \frac{\partial L}{\partial r_i} = 0, \tag{3.31}$$

where the Lagrangian

$$L = T - U \tag{3.32}$$

is the difference between the kinetic and the potential energy of the particle. While we have derived this result for a constant force $\vec{F}$, we shall see that it is more general.

3.3 Lagrangian of a system of particles with conservative forces

Consider an isolated system of N particles, i.e. particles which interact only with each other. We will postulate that the Lagrangian of the system is the difference of the kinetic energy of the particles and the potential energy of the system:

$$L = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^2 - U(\vec{r}_1, \ldots, \vec{r}_N). \tag{3.33}$$

The Euler–Lagrange equations for this Lagrangian read

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{a,i}} = \frac{\partial L}{\partial r_{a,i}}, \tag{3.34}$$

where $r_{a,i}$ denotes the i-th component of the vector $\vec{r}_a$, the position of the a-th particle. Performing the differentiations, we obtain

$$m_a \ddot{r}_{a,i} = -\frac{\partial U}{\partial r_{a,i}} \tag{3.35}$$

or, in vector notation,

$$m_a \ddot{\vec{r}}_a = -\vec{\nabla}_a U, \tag{3.36}$$

with $\vec{\nabla}_a \equiv (\partial/\partial r_{a,1}, \partial/\partial r_{a,2}, \partial/\partial r_{a,3})$. This is the familiar equation of motion according to Newton's law for a conservative force, i.e. a force which can be written as minus the gradient of a potential,

$$\vec{F}_a = -\vec{\nabla}_a U. \tag{3.37}$$

As an example, consider an isolated system of N particles with masses m_a which interact among themselves gravitationally. The Lagrangian of the system is

$$L = T - U, \tag{3.38}$$

where

$$T = \sum_{a=1}^{N} \frac{1}{2} m_a \dot{\vec{r}}_a^2 \tag{3.39}$$

and

$$U = -\frac{1}{2} \sum_a \sum_{b \neq a} G \frac{m_a m_b}{|\vec{r}_a - \vec{r}_b|}. \tag{3.40}$$

The Euler–Lagrange equations are

$$m_a \ddot{\vec{r}}_a = -\vec{\nabla}_a U, \tag{3.41}$$

where the subscript in $\vec{\nabla}_a$ denotes that the derivatives are taken with respect to the coordinates of the a-th particle.

Exercise 3.1. To carry out the differentiations on the right-hand side, prove first that

$$\vec{\nabla}_a \frac{1}{|\vec{r}_a - \vec{r}_b|} = -\frac{\vec{r}_a - \vec{r}_b}{|\vec{r}_a - \vec{r}_b|^3}. \tag{3.42}$$

We then obtain the equations of motion

$$m_a \ddot{\vec{r}}_a = -\sum_{b \neq a} G m_a m_b \frac{\vec{r}_a - \vec{r}_b}{|\vec{r}_a - \vec{r}_b|^3}, \tag{3.43}$$

in agreement with Newton's law of gravitation.

3.4 Lagrangian of a charge in an electromagnetic field

We now turn our attention to the force experienced by an electric charge which is in the vicinity of other electric charges. We can sum up the effects of all other charges into two vectors:

- $\vec{E}(t,\vec{r})$, the electric field and
- $\vec{B}(t,\vec{r})$, the magnetic field.

The force is then given by

$$m\ddot{\vec{r}} = q\left(\vec{E} + \dot{\vec{r}} \times \vec{B}\right), \tag{3.44}$$

where m is the mass of the charge q. The electric and magnetic fields are determined by the **Maxwell equations**:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \tag{3.45}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{3.46}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{3.47}$$

$$\vec{\nabla}\times\vec{B} = \frac{\vec{j}}{c^2\epsilon_0} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}, \tag{3.48}$$

where $\rho(t,\vec{r})$ is the electric charge density and $\vec{j}(t,\vec{r})$ is the electric current density. The constant ϵ_0 is the so-called vacuum permittivity, with the value

$$\epsilon_0 = 8.854\,187\,8\ldots\times 10^{-12}\,\frac{\mathrm{A\,s}}{\mathrm{V\,m}}, \tag{3.49}$$

and c is the speed of light,

$$c = 2.997\,924\,58\times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}} \tag{3.50}$$

(an exact value, by the definition of the metre). We will not attempt here to find explicit solutions of Maxwell equations for the $\vec{E}$ and $\vec{B}$ fields, but we will assume that such solutions exist and are known to us. Can we obtain the equation of motion (3.44) from Hamilton's variational principle? Unlike the examples that we have seen so far, the electromagnetic force depends not only on the position of the charged particle but also on its velocity. Nevertheless, we will be able to find a Lagrangian which gives the correct equation of motion.

To achieve that, we introduce first the scalar and vector potentials, $\phi(t,\vec{r})$ and $\vec{A}(t,\vec{r})$ respectively, defined by

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi, \tag{3.51}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{3.52}$$

It is easy to verify that the second and third Maxwell equations are automatically satisfied if we express the electric and magnetic fields in terms of the scalar and vector potentials. Consider now the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}}\cdot\vec{A}. \tag{3.53}$$

The Euler–Lagrange equations take the form

$$m\,\ddot{r}_i + q\frac{dA_i}{dt} = -q\partial_i\phi + q\sum_j \dot{r}_j\,\partial_i A_j, \tag{3.54}$$

where we use the shorthand notation

$$\partial_i \equiv \frac{\partial}{\partial r_i}.$$

The total time derivative of the vector potential is

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_j \dot{r}_j\,(\partial_j A_i). \tag{3.55}$$

Then the Euler–Lagrange equations become

$$m\,\ddot{r}_i \;=\; q\left[-\frac{\partial A_i}{\partial t} - \partial_i\phi\right] + q\sum_j\left[\dot{r}_j\,\partial_i A_j - \dot{r}_j\,(\partial_j A_i)\right]. \tag{3.56}$$

In the first bracket, we recognise the i-th component of the electric field,

$$-\frac{\partial A_i}{\partial t} - \partial_i\phi = E_i. \tag{3.57}$$

The second bracket is not as easy to decipher, but we can prove it to be the i-th component of the cross product of the velocity and the magnetic field. Indeed,

$$\begin{aligned}
(\dot{\vec{r}}\times\vec{B})_i &= \left[\dot{\vec{r}}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_i \\
&= \sum_{jklm}\epsilon_{ijk}\,\dot{r}_j\,(\epsilon_{klm}\partial_l A_m) \\
&= \sum_{jlm}\left(\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}\right)\dot{r}_j\,\partial_l A_m \\
&= \sum_j\left[\dot{r}_j\,\partial_i A_j - \dot{r}_j\,(\partial_j A_i)\right].
\end{aligned} \tag{3.58}$$

Thus, we have proven that the Euler–Lagrange equations of the Lagrangian of Eq. (3.53) give the familiar Lorentz force law for a charge in an electromagnetic field,

$$m\,\ddot{r}_i = qE_i + q\left(\dot{\vec{r}}\times\vec{B}\right)_i. \tag{3.59}$$

4

Dynamics of constrained systems

In the last section, we showed the equivalence of traditional Newtonian mechanics and Hamilton's variational principle for a variety of forces acting on systems of particles. In this section, we will show that the principle of least action can also be applied to dynamical systems for which we do not know all the forces acting on the particles a priori; for some of them we only know their effect in limiting the allowed motion of the particles. For example, the particles which make up a ball are kept together by electromagnetic forces. It is difficult to account for these microscopic electromagnetic interactions in our Lagrangian when our aim is simply to describe the motion of the ball in the gravitational field. However, in a variety of problems with constraints, an explicit description of the forces responsible for the constraints can be avoided and we can implement their effect directly.

4.1 Constraints

The configuration of a system of N particles P_j is given by the positions $\vec{r}_j$ and the velocities $\dot{\vec{r}}_j$ of the N particles. **Remark:** in the standard terminology the *configuration* is given by the positions $\vec{r}_j$ alone, while positions together with velocities define the *state* of the system (a point in phase space rather than in configuration space); see Landau–Lifshitz vol. 1, §1, and Goldstein, Sec. 1.4.

If all possible configurations are allowed, the system is *free*. If there are limitations to the possible configurations, the motion of the system is constrained. The geometric or kinematic restrictions on the positions (and velocities) of the particles of the system are called *constraints*.

We will study a class of constraints which limit the positions and/or the velocities of the particles, through equations of the form

$$f(\vec{r}_1, \ldots, \vec{r}_N, \dot{\vec{r}}_1, \ldots, \dot{\vec{r}}_N, t) = 0\,. \tag{4.1}$$

We call these constraints *differential*. If a differential constraint can be cast in a form which does not depend explicitly on the velocities, it is called *holonomic*,

$$f(\vec{r}_1, \ldots, \vec{r}_N, t) = 0\,, \tag{4.2}$$

and it limits only the positions of the particles. If, in particular, a holonomic constraint, Eq. (4.2), does not depend explicitly on time,

$$\frac{\partial f}{\partial t} = 0\,, \tag{4.3}$$

it is called *stationary*. Constraints which are not holonomic are called *non-holonomic*.

Examples of holonomic constraints are:

1. A rigid body may be thought of as a collection of N particles whose mutual distances L_{ij} define stationary geometric constraints, similar to the pendulum constraint of Eq. (4.10) below,
$$(\vec{r_i} - \vec{r_j})^2 - L_{ij}^2 = 0\,, \qquad i,j = 1,\ldots,N\,. \tag{4.4}$$
This is a stationary holonomic constraint. The rigid body may be in motion; therefore, the coordinates $\vec{r}_i = \vec{r}_i(t)$, $\vec{r}_j = \vec{r}_j(t)$ change with time. However, the time dependence of Eq. (4.4) is implicit (through the coordinates) and not explicit; thus we classify the constraint as stationary.

2. A particle is constrained to move on a fixed surface,
$$f(\vec{r}) = f(x,y,z) = 0\,. \tag{4.5}$$
This is also a stationary holonomic constraint.

3. A particle is constrained to move on a surface which is itself in motion,
$$f(\vec{r},t) = 0\,. \tag{4.6}$$
This is a holonomic constraint, but it depends explicitly on time. Therefore, the constraint is not stationary.

4. An ideal moving fluid (i.e. without viscosity) may be thought of as a collection of N particles whose mutual distances $L_{ij}(t)$ change with time and which define holonomic constraints which are not stationary:
$$(\vec{r_i} - \vec{r_j})^2 - L_{ij}^2(t) = 0\,, \qquad i,j = 1,\ldots,N\,. \tag{4.7}$$
This example is only heuristic: in a real fluid the mutual distances of the particles are not prescribed functions of time but are themselves dynamical; the genuine constraint of an ideal incompressible fluid is the conservation of volume. A cleaner example of a non-stationary holonomic constraint is a pendulum whose string length is changed by a prescribed law $L(t)$, $x^2 + z^2 - L^2(t) = 0$ (cf. Goldstein, Sec. 1.3).

Differential constraints may appear, at first sight, to depend on velocities. We may then rush to conclude that they are not holonomic. However, sometimes the apparent velocity dependence can be eliminated upon integration and we can, after all, cast the constraint in the form of Eq. (4.2), which is manifestly holonomic. To see that such a possibility exists, it suffices to observe that we can always produce a constraint with an apparent velocity dependence starting from a holonomic constraint. Let us start from a constraint of the form of Eq. (4.2) and take a total time derivative:

$$f = 0 \quad\longrightarrow\quad \frac{df}{dt} = 0 \quad\longrightarrow\quad \sum_{j=1}^{N} \vec{\nabla}_j f \cdot \dot{\vec{r}}_j + \frac{\partial f}{\partial t} = 0\,, \tag{4.8}$$

which in Cartesian coordinates $\vec{r}_j \equiv (x_j, y_j, z_j)$ is written as

$$\sum_{j=1}^{N} \left(\frac{\partial f}{\partial x_j}\dot{x}_j + \frac{\partial f}{\partial y_j}\dot{y}_j + \frac{\partial f}{\partial z_j}\dot{z}_j \right) + \frac{\partial f}{\partial t} = 0\,. \tag{4.9}$$

In the form of Eq. (4.9), the constraint has an apparent velocity dependence. As a concrete example, consider a pendulum which moves in the x-z plane and is attached to a string of length R. This yields a holonomic stationary constraint, given by the equation

$$\vec{r}^2 - R^2 = x^2 + z^2 - R^2 = 0\,, \tag{4.10}$$

which states that the distance of the pendulum from the centre is fixed. We can give the constraint an apparent velocity dependence. Differentiating Eq. (4.10) with respect to time, we obtain

$$\frac{d}{dt}(x^2 + z^2 - R^2) = 2x\dot{x} + 2z\dot{z} = 0\,, \quad \longrightarrow \quad \vec{r}\cdot\dot{\vec{r}} = 0\,. \tag{4.11}$$

In this form, the constraint states that the motion of the pendulum is orthogonal to the string. Had we been given the constraint in this second form, we would not have been able to classify it immediately as holonomic. Of course, once we realise that the differential constraint of Eq. (4.11) is a total differential, it can be integrated back, making the constraint manifestly holonomic.

Differential constraints which are total time derivatives, Eq. (4.8), can be integrated and we obtain

$$f(\vec{r}_1, \ldots, \vec{r}_N, t) = c\,, \tag{4.12}$$

with c an arbitrary constant. Differential constraints of the form of Eq. (4.8) are said to be *integrable* and they are holonomic.

We shall consider only the simplest class of differential non-integrable constraints: the ones which depend linearly on the velocities $\dot{\vec{r}}_j$ of the N particles,

$$\sum_{j=1}^{N} \vec{l}_j \cdot \dot{\vec{r}}_j + D = 0\,, \tag{4.13}$$

where the vectors $\vec{l}_j$ and the scalar D are functions of the positions $\vec{r}_j$ and of time t, $\vec{l}_j = \vec{l}_j(\vec{r}_1, \ldots, \vec{r}_N, t)$ and $D = D(\vec{r}_1, \ldots, \vec{r}_N, t)$. In Cartesian coordinates, Eq. (4.13) reads

$$\sum_{j=1}^{N} \left(A_j\dot{x}_j + B_j\dot{y}_j + C_j\dot{z}_j\right) + D = 0\,. \tag{4.14}$$

As an example, consider an ice skate, which we represent by means of two particles connected by a rod of length L, as in Fig. 4.1. The rod moves in the x-y plane (the ice rink). The point of contact with the ice, which we take to be the middle point of the rod, moves in the direction of the rod (the blade cannot slide sideways). So, in addition to the fixed length of the rod, which yields the constraint

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L^2\,, \tag{4.15}$$

we must require that the position vector of the rod and the velocity vector of the middle point have the same direction, say at an angle α,

$$\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\dot{y}_1 + \dot{y}_2}{\dot{x}_1 + \dot{x}_2}\,. \tag{4.16}$$

Eq. (4.16) yields the differential constraint

$$\frac{\dot{x}_1 + \dot{x}_2}{x_1 - x_2} = \frac{\dot{y}_1 + \dot{y}_2}{y_1 - y_2}\,, \tag{4.17}$$

which is non-integrable. Thus, the system defined by the constraints (4.15) and (4.17) is non-holonomic. Likewise, any rigid body whose velocity (at the contact points) is constrained to point along a direction fixed in the body (like a car which is moving or parking without skidding) defines a non-holonomic system.

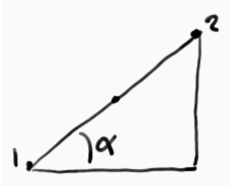


Figure 4.1. An ice skate represented as a rod of length L.

4.2 Possible and virtual displacements

Let us take a system of N particles P_j, with $j = 1, \ldots, N$, subject to d manifestly holonomic constraints,

$$f_i(\vec{r}_1, \ldots, \vec{r}_N, t) = 0\,, \qquad i = 1, \ldots, d\,, \tag{4.18}$$

and to g linear differential constraints,

$$\sum_{j=1}^{N} \vec{l}_{kj} \cdot \dot{\vec{r}}_j + D_k = 0\,, \qquad k = 1, \ldots, g\,. \tag{4.19}$$

By taking the total time derivative of the manifestly holonomic constraints, as in Eq. (4.8),

$$\sum_{j=1}^{N} \vec{\nabla}_j f_i \cdot \dot{\vec{r}}_j + \frac{\partial f_i}{\partial t} = 0\,, \qquad i = 1, \ldots, d\,, \tag{4.20}$$

we replace them by linear differential constraints as well. We see that the constraints of Eqs. (4.19) and (4.20) limit the allowed velocities $\dot{\vec{r}}_j$ of the particles. The velocities $\dot{\vec{r}}_j$ which are allowed by these constraints are called *possible*. At a given time t and positions $\vec{r}_j$ of the particles P_j, there are infinitely many sets of possible velocities, since we have not yet considered the other forces (not responsible for constraints) which are acting on the system. Once we do so, only one set of possible particle velocities is actually realised in the motion of the system.

Multiplying Eqs. (4.19) and (4.20) by the time interval dt,

$$\begin{aligned} &\sum_{j=1}^{N} \vec{\nabla}_j f_i \cdot d\vec{r}_j + \frac{\partial f_i}{\partial t} dt = 0\,, \qquad i = 1, \ldots, d\,, \\ &\sum_{j=1}^{N} \vec{l}_{kj} \cdot d\vec{r}_j + D_k dt = 0\,, \qquad k = 1, \ldots, g\,, \end{aligned} \tag{4.21}$$

we write the linear differential constraints in terms of the *possible displacements*, $d\vec{r}_j = \dot{\vec{r}}_j dt$.

For two possible displacements,

$$d\vec{r}_j = \dot{\vec{r}}_j dt\,, \qquad d\vec{r}'_j = \dot{\vec{r}}'_j dt\,, \tag{4.22}$$

satisfying the constraints of Eq. (4.21), the difference

$$\delta\vec{r}_j = d\vec{r}_j - d\vec{r}'_j\,, \tag{4.23}$$

fulfils the constraints

$$\begin{aligned}
&\sum_{j=1}^{N} \vec{\nabla}_j f_i \cdot \delta\vec{r}_j = 0\,, \qquad i = 1, \ldots, d\,,\\
&\sum_{j=1}^{N} \vec{l}_{kj} \cdot \delta\vec{r}_j = 0\,, \qquad k = 1, \ldots, g\,.
\end{aligned} \tag{4.24}$$

In Cartesian coordinates the above can be written explicitly as

$$\begin{aligned}
&\sum_{j=1}^{N} \left(\frac{\partial f_i}{\partial x_j}\delta x_j + \frac{\partial f_i}{\partial y_j}\delta y_j + \frac{\partial f_i}{\partial z_j}\delta z_j \right) = 0\,, \qquad i = 1, \ldots, d\,,\\
&\sum_{j=1}^{N} \left(A_{kj}\delta x_j + B_{kj}\delta y_j + C_{kj}\delta z_j \right) = 0\,, \qquad k = 1, \ldots, g\,.
\end{aligned} \tag{4.25}$$

The displacements of Eq. (4.23) are called *virtual*. We can think of virtual displacements as the ones which take a possible configuration of the system at a time t to another one infinitely close to it, at the same time t. For stationary constraints, which have a vanishing time derivative, the set of virtual displacements coincides with that of the possible displacements.

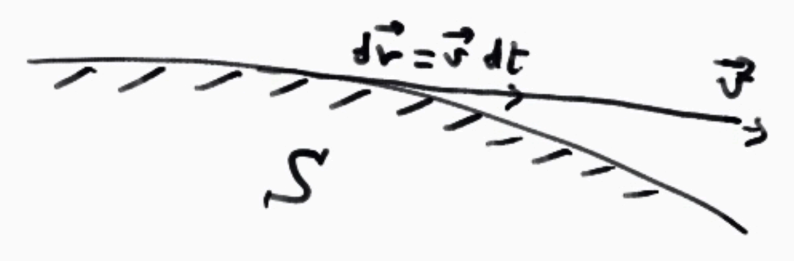


Figure 4.2. A particle in motion on a fixed surface S.

As an example of possible and virtual displacements, consider a particle P which is in motion on a fixed surface S, as in Fig. 4.2. The constraint is $f(\vec{r}) = 0$. Any vector $\vec{v}$ tangent to S is a possible velocity and the differential elements $d\vec{r} = \vec{v}\,dt$ are possible displacements. The difference $\delta\vec{r} = d\vec{r} - d\vec{r}'$ of two tangent vectors is also a tangent vector. In this case, the sets of virtual and possible displacements coincide, as expected, since the constraint is stationary.

Consider now a particle P which is in motion on a surface S moving with velocity $\vec{u}$, as in Fig. 4.3. We denote by $\vec{w}$ the possible velocities of the particle measured with respect to the moving surface. Then the velocity of the particle is $\vec{v} = \vec{w} + \vec{u}$, where $\vec{w}$ is tangent to the surface S. Note that in general the velocity $\vec{v}$ is not tangent to S. Likewise, the possible displacement,

$$d\vec{r} = \vec{v}\,dt = \vec{w}\,dt + \vec{u}\,dt\,, \tag{4.26}$$

is a vector which is not necessarily tangent to S. Take now another possible displacement,

$$d\vec{r}' = \vec{w}' \, dt + \vec{u} \, dt \,, \tag{4.27}$$

and form the difference

$$\delta\vec{r} = d\vec{r} - d\vec{r}' = (\vec{w} - \vec{w}') \, dt \,. \tag{4.28}$$

This is a virtual displacement, and a vector tangent to S. Note that virtual and possible displacements do not coincide since, as we know from Eq. (4.6), the constraint of a moving surface is not stationary.



Figure 4.3. A particle in motion on a moving surface S.

4.3 Smooth constraints

From Eq. (4.25), we know that for a system of N particles P_j, with $j = 1, \ldots, N$, we have $3N$ virtual displacements constrained by d geometric constraints (4.18) and g differential constraints (4.19). Thus there are $3N - d - g$ independent virtual displacements. We can say that our system of N particles has $3N - d - g$ *degrees of freedom*. The constraints imply some restrictions on the accelerations of the particles. In fact, taking the total time derivative of Eqs. (4.20) and (4.19), we obtain

$$\begin{aligned} &\sum_{j=1}^{N} \vec{\nabla}_j f_i \cdot \ddot{\vec{r}}_j + \sum_{j=1}^{N} \left(\frac{d}{dt} \vec{\nabla}_j f_i \right) \cdot \dot{\vec{r}}_j + \frac{d}{dt} \frac{\partial f_i}{\partial t} = 0 \,, \qquad i = 1, \ldots, d \,, \\ &\sum_{j=1}^{N} \vec{l}_{kj} \cdot \ddot{\vec{r}}_j + \sum_{j=1}^{N} \frac{d\vec{l}_{kj}}{dt} \cdot \dot{\vec{r}}_j + \frac{dD_k}{dt} = 0 \,, \qquad k = 1, \ldots, g \,. \end{aligned} \tag{4.29}$$

Let us imagine for a moment the system without constraints. Forces are applied to the system,

$$\vec{F}_j = \vec{F}_j(\vec{r}_i, \dot{\vec{r}}_i, t) \,, \qquad j = 1, \ldots, N \,, \tag{4.30}$$

which are known functions of the positions $\vec{r}_i$ and the velocities $\dot{\vec{r}}_i$ of the particles. Without constraints, the forces would induce accelerations,

$$\vec{F}_j = m_j \ddot{\vec{r}}_j \,, \qquad j = 1, \ldots, N \,, \tag{4.31}$$

where m_j are the masses of the particles. However, if the system is constrained, the accelerations $\ddot{\vec{r}}_j$ are also due to additional forces $\vec{R}_j$, called *reaction forces*, which are

responsible for the constraints. The accelerations, which must be consistent with Eq. (4.29), satisfy

$$m_j \ddot{\vec{r}}_j = \vec{F}_j + \vec{R}_j \,, \qquad j = 1, \ldots, N \,. \tag{4.32}$$

The general problem of the dynamics of a constrained system is as follows: *given the forces $\vec{F}_j$ and the initial positions $\vec{r}_{0,i}$ and velocities $\dot{\vec{r}}_{0,i}$ of the particles, we need to determine the trajectories of the particles and the reaction forces $\vec{R}_j$ of the constraints:*

$$\vec{r}_i(t) \,, \ \vec{R}_j \,, \qquad j = 1, \ldots, N \,. \tag{4.33}$$

Whenever $d + g < 3N$, the known equations of motion and constraints,

$$\begin{aligned} m_j \ddot{\vec{r}}_j &= \vec{F}_j + \vec{R}_j \,, & j &= 1, \ldots, N \,, \\ f_i(\vec{r}_j, t) &= 0 \,, & i &= 1, \ldots, d \,, \\ \sum_{j=1}^{N} \vec{l}_{kj} \cdot \dot{\vec{r}}_j + D_k &= 0 \,, & k &= 1, \ldots, g \,. \end{aligned} \tag{4.34}$$

do not suffice to determine all unknowns, leaving $n = 3N - d - g$ degrees of freedom undetermined. In order to determine the motion of the system, we need $n = 3N - d - g$ additional independent relations among the variables (4.33). We will find the necessary additional equations by examining the properties of the constraints and making an assumption for the reaction forces which generate them.

There is a large class of constraints, called *smooth*, for which the work of the reaction forces vanishes over the virtual displacements,

$$\sum_{j=1}^{N} \vec{R}_j \cdot \delta\vec{r}_j = 0 \,, \tag{4.35}$$

which in Cartesian coordinates is

$$\sum_{j=1}^{N} \left(R_{j,x} \delta x_j + R_{j,y} \delta y_j + R_{j,z} \delta z_j \right) = 0 \,. \tag{4.36}$$

Consider the following examples.

Example 4.1. Consider a particle P which moves on a fixed smooth surface S, as in Fig. 4.4. As we have seen, the virtual and possible displacements coincide in that case; they are vectors tangent to S. Because S is smooth (there is no friction), the reaction force $\vec{R}$ is orthogonal to S. Then $\vec{R} \cdot d\vec{r} = 0$ and $\vec{R} \cdot \delta\vec{r} = 0$. Therefore, the constraint is smooth.

Example 4.2. We now consider a pendulum, for which we have cast the constraint, Eq. (4.11), in the differential form $\vec{r} \cdot \dot{\vec{r}} = 0$. Because the virtual and possible displacements coincide, it is also true that $\vec{r} \cdot \delta\vec{r} = 0$, where the direction of the string, $\hat{r}$, is also the direction of the reaction force $\vec{R}$. Thus, we have that $\vec{R} \cdot \delta\vec{r} = 0$ and the constraint is therefore smooth.

Example 4.3. A point P moves on a smooth surface S, which moves with velocity $\vec{u}$, as in Fig. 4.5. Because S is smooth, the reaction force $\vec{R}$ is orthogonal to S. As we have already seen, in this example the virtual displacement $\delta\vec{r}$ is tangent to S, while the possible displacement $d\vec{r}$ is generally not. Then $\vec{R} \cdot \delta\vec{r} = 0$, while $\vec{R} \cdot d\vec{r} \neq 0$. Albeit not stationary, this constraint is smooth.

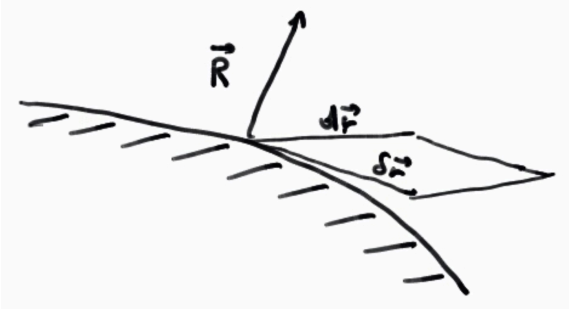


Figure 4.4. The reaction force $\vec{R}$ on a fixed surface S.

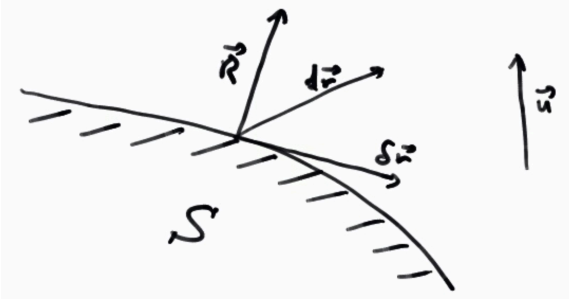


Figure 4.5. The reaction force $\vec{R}$ on a moving surface S.

From Eq. (4.24) for the virtual displacements, we know that among the $3N$ virtual displacements there are $n = 3N - d - g$ independent ones, which define an n-dimensional vector space $\mathbb{R}^n$. We can write any displacement as a function of the n independent ones,

$$\delta\vec{r}_j = \sum_{k=1}^{n} \delta\alpha_k \vec{e}_{k,j} \,, \qquad j = 1, \ldots, N \,, \tag{4.37}$$

where $\delta\alpha_k$ are the independent displacements and the $\vec{e}_{k,j}$ form a basis of the space of virtual displacements. Substituting this into Eq. (4.35), we obtain

$$\sum_{j=1}^{N} \vec{R}_j \cdot \sum_{k=1}^{n} \delta\alpha_k \vec{e}_{k,j} = \sum_{k=1}^{n} \delta\alpha_k \left(\sum_{j=1}^{N} \vec{R}_j \cdot \vec{e}_{k,j} \right) = 0 \,, \tag{4.38}$$

which must be true for arbitrary independent virtual displacements $\delta\alpha_k$. This requirement gives n additional equations,

$$\sum_{j=1}^{N} \vec{R}_j \cdot \vec{e}_{k,j} = 0 \,, \qquad k = 1, \ldots, n \,, \tag{4.39}$$

which are the n missing conditions needed in order to determine the motion of the system completely.

4.4 The general equation of the dynamics

In the previous section, we have established the equations of motion of a constrained system,

$$m_j \ddot{\vec{r}}_j = \vec{F}_j + \vec{R}_j \,, \qquad j = 1, \ldots, N \,, \tag{4.40}$$

with m_j the masses of the particles, $\vec{F}_j$ the forces applied on the system and $\vec{R}_j$ the reactions due to the constraints. In addition, we require that the constraints are smooth, Eq. (4.35),

$$\sum_{j=1}^{N} \vec{R}_j \cdot \delta\vec{r}_j = 0\,. \tag{4.41}$$

Substituting the reactions $\vec{R}_j$ as obtained from Eq. (4.40) into Eq. (4.41), we obtain

$$\sum_{j=1}^{N} \left(\vec{F}_j - m_j \ddot{\vec{r}}_j\right) \cdot \delta\vec{r}_j = 0\,, \tag{4.42}$$

which is the *general equation of the dynamics*. It states that *the sum of the work of the applied forces $\vec{F}_j$ and of the inertial forces $-m_j\ddot{\vec{r}}_j$ vanishes for any virtual displacement*.

4.5 Independent generalised coordinates

In what follows, we will limit ourselves to a system of N particles, with d constraints,

$$f_i(\vec{r}_j, t) = 0\,, \qquad i = 1, \ldots, d\,, \tag{4.43}$$

written in a manifestly holonomic form.

We know from Eq. (4.24) that among the $3N$ virtual displacements there are $n = 3N - d$ independent ones. Thus we can solve the d equations (4.43) in terms of n independent coordinates q_i, with $i = 1, \ldots, n$, and of the time t: the positions of the particles can be taken as functions of the n independent coordinates,

$$\vec{r}_j = \vec{r}_j(q_1(t), \ldots, q_n(t), t)\,, \qquad j = 1, \ldots, N\,. \tag{4.44}$$

In the specific case of stationary constraints, Eq. (4.3), the positions of the particles $\vec{r}_j$ do not depend explicitly on the time t but only through $q_i(t)$.

The essential information about the degrees of freedom being reduced from $3N$ to n is encoded into the n independent coordinates of Eq. (4.44): if we substitute Eq. (4.44) into Eq. (4.43), the constraints become identities. The n independent coordinates q_i are called *independent generalised coordinates*. Consider the following examples.

Example 4.4. A pendulum is moving in the x-z plane and is attached to a string of length R. Its position is fixed by an angle ϕ: the pendulum has only one degree of freedom. The stationary holonomic constraint is given by the equation

$$x^2 + z^2 - R^2 = 0\,. \tag{4.45}$$

If we take ϕ as the independent coordinate, then Eqs. (4.44) become

$$\begin{aligned} x &= R\cos\phi\,, \\ z &= R\sin\phi\,. \end{aligned} \tag{4.46}$$

If we substitute Eqs. (4.46) into Eq. (4.45), we obtain the identity

$$\cos^2\phi + \sin^2\phi - 1 = 0\,. \tag{4.47}$$

Example 4.5. A particle is constrained to move on a sphere of radius R: it has two degrees of freedom. As independent coordinates, we can take the longitude ϕ and the latitude θ. The holonomic stationary constraint is given by the equation

$$x^2 + y^2 + z^2 - R^2 = 0\,. \tag{4.48}$$

In terms of the angles ϕ and θ, Eqs. (4.44) become

$$\begin{aligned} x &= R\cos\phi\cos\theta\,, \\ y &= R\sin\phi\cos\theta\,, \\ z &= R\sin\theta\,. \end{aligned} \tag{4.49}$$

If we substitute Eqs. (4.50) into the constraint of Eq. (4.48), we obtain the identity

$$(\cos^2\phi + \sin^2\phi)\cos^2\theta + \sin^2\theta - 1 = 0\,. \tag{4.50}$$

4.6 Euler–Lagrange equations for systems with smooth constraints and potential forces

We may now express the $3N$ virtual displacements in terms of the $n = 3N - d$ independent coordinates of Eq. (4.44),

$$\delta\vec{r}_j = \sum_{i=1}^{n} \frac{\partial\vec{r}_j}{\partial q_i}\delta q_i\,, \qquad j = 1,\ldots,N\,, \tag{4.51}$$

and substitute them into the general equation of the dynamics, Eq. (4.42),

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\left(\vec{F}_j - m_j\ddot{\vec{r}}_j\right)\cdot\frac{\partial\vec{r}_j}{\partial q_i}\delta q_i = 0\,, \tag{4.52}$$

where we have interchanged the order of the sums. Because the coordinates q_i are independent, so are their virtual displacements δq_i. Thus, their coefficients must vanish identically, yielding

$$\sum_{j=1}^{N}\left(\vec{F}_j - m_j\ddot{\vec{r}}_j\right)\cdot\frac{\partial\vec{r}_j}{\partial q_i} = 0\,, \qquad i = 1,\ldots,n\,. \tag{4.53}$$

The first term of Eq. (4.53),

$$\sum_{j=1}^{N}\vec{F}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i} \equiv Q_i\,, \qquad i = 1,\ldots,n\,, \tag{4.54}$$

is called the *generalised force*. The generalised force has the dimension of a force only if the independent coordinates have the dimension of a length. Now let us analyse the second term of Eqs. (4.53),

$$\sum_{j=1}^{N} m_j\ddot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i}\,, \qquad i = 1,\ldots,n\,, \tag{4.55}$$

which we rewrite as

$$\sum_{j=1}^{N} m_j \frac{d\dot{\vec{r}}_j}{dt} \cdot \frac{\partial \vec{r}_j}{\partial q_i} = \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \dot{\vec{r}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i}\right) - \sum_{j=1}^{N} m_j \dot{\vec{r}}_j \cdot \frac{d}{dt}\frac{\partial \vec{r}_j}{\partial q_i} . \tag{4.56}$$

Next, we derive a couple of useful identities. We start from the velocities in terms of independent coordinates,

$$\dot{\vec{r}}_j = \sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_j}{\partial t} , \tag{4.57}$$

from which we immediately obtain that

$$\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_i} = \frac{\partial \vec{r}_j}{\partial q_i} . \tag{4.58}$$

Then we differentiate Eq. (4.57) with respect to the independent coordinates,

$$\frac{\partial \dot{\vec{r}}_j}{\partial q_i} = \sum_{k=1}^{n} \frac{\partial^2 \vec{r}_j}{\partial q_i \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_j}{\partial q_i \partial t} = \frac{d}{dt}\frac{\partial \vec{r}_j}{\partial q_i} . \tag{4.59}$$

We substitute the identities (4.58) and (4.59) into Eq. (4.56) and obtain

$$\sum_{j=1}^{N} m_j \ddot{\vec{r}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} = \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \dot{\vec{r}}_j \cdot \frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_i}\right) - \sum_{j=1}^{N} m_j \dot{\vec{r}}_j \cdot \frac{\partial \dot{\vec{r}}_j}{\partial q_i} . \tag{4.60}$$

Since the kinetic energy of the holonomic system of N particles is

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j \dot{\vec{r}}_j^{\,2} , \tag{4.61}$$

we can rewrite Eq. (4.60) as

$$\sum_{j=1}^{N} m_j \ddot{\vec{r}}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} , \qquad i = 1, \ldots, n . \tag{4.62}$$

Substituting this into Eq. (4.53) and recalling the definition of the generalised force, Eq. (4.54), we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i , \qquad i = 1, \ldots, n . \tag{4.63}$$

These equations form a system of n second-order differential equations for n unknown variables q_i. Therefore, the motion of the holonomic system with n degrees of freedom is fully determined if the values of q_i and $\dot{q}_i$ are given at an initial time t_0.

Note that the reaction forces $\vec{R}_j$ do not appear explicitly in Eqs. (4.63). They can be obtained by solving Eqs. (4.63) first, and then using Eq. (4.40).

Now let us suppose that some forces can be derived from a potential energy $U = U(\vec{r}_1, \ldots, \vec{r}_N, t)$,

$$\vec{F}_j = -\frac{\partial U}{\partial \vec{r}_j} , \tag{4.64}$$

such that the generalised forces become

$$Q_i = \sum_{j=1}^{N} \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} = -\sum_{j=1}^{N} \frac{\partial U}{\partial \vec{r}_j} \cdot \frac{\partial \vec{r}_j}{\partial q_i} = -\frac{\partial U}{\partial q_i} , \qquad i = 1, \dots, n , \tag{4.65}$$

and that in addition there are non-potential generalised forces,

$$\tilde{Q}_i = \tilde{Q}_i(q_k, \dot{q}_k, t) . \tag{4.66}$$

Eqs. (4.63) become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \tilde{Q}_i , \qquad i = 1, \dots, n . \tag{4.67}$$

Recalling the definition of the Lagrangian,

$$L = T - U , \tag{4.68}$$

we can cast Eq. (4.67) in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tilde{Q}_i , \qquad i = 1, \dots, n . \tag{4.69}$$

In the absence of non-potential forces, Eqs. (4.69) become the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 , \qquad i = 1, \dots, n . \tag{4.70}$$

We have therefore proved that Hamilton's principle of least action, which yields the same Euler–Lagrange equations, also holds for constrained systems with smooth holonomic constraints and potential forces.

5

Application of the Euler–Lagrange equations: the simple pendulum

We have seen that we can obtain the equations of motion for a system with holonomic constraints from a Lagrangian $\mathcal{L}$ by requiring that the action integral is a minimum. This gives rise to the Euler–Lagrange equations (the equations of motion),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0. \tag{5.1}$$

In the above, all generalised coordinates q_i are independent, and we have first used the holonomic constraints to eliminate additional variables which are not independent.

Eqs. (5.1) are of course equivalent to (and have also been derived from) Newton's law. It is instructive to take a simple physics problem and use it as a testbed to compare the two methods. Consider a simple pendulum, as in Fig. 5.1. According to Newton's law,

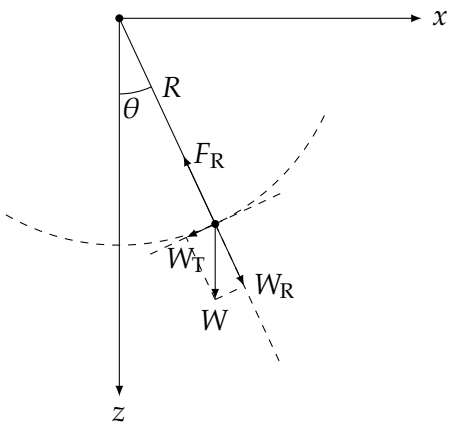


Figure 5.1. A simple pendulum consists of a particle of mass m which is attached via a (massless) rope of length R to a fixed point.

$$m\ddot{\vec{r}} = \vec{F}_R + \vec{W}, \tag{5.2}$$

where $\vec{W}$ is the weight force and $\vec{F}_R$ is the reaction force from the rope. The acceleration is

$$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}. \tag{5.3}$$

We will need to change to polar coordinates, with the following change of basis for the

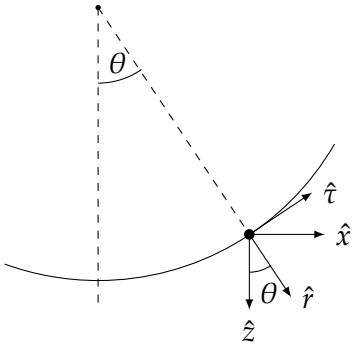


Figure 5.2. The unit vectors defining the Cartesian and polar coordinates.

unit vectors (see Fig. 5.2):

$$(\hat{x}, \hat{y}, \hat{z}) \longrightarrow (\hat{r}, \hat{y}, \hat{\tau}), \tag{5.4}$$

where $\hat{r}$ is the radial unit vector and $\hat{\tau}$ the unit vector tangential to the motion of the pendulum. We have

$$\left.\begin{aligned} \hat{z} &= (\hat{z}\cdot\hat{r})\hat{r} + (\hat{z}\cdot\hat{\tau})\hat{\tau} \\ \hat{x} &= (\hat{x}\cdot\hat{r})\hat{r} + (\hat{x}\cdot\hat{\tau})\hat{\tau} \end{aligned}\right\} \Longrightarrow \begin{cases} \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\tau} \\ \hat{x} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\tau} \end{cases} \tag{5.5}$$

or, in matrix form,

$$\begin{pmatrix} \hat{z} \\ \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\tau} \\ \hat{y} \end{pmatrix} \tag{5.6}$$

As expected, the change from Cartesian to polar coordinates is a rotation by an angle θ around the $\hat{y}$-axis. Expressing the acceleration in the polar basis and using that the length of the pendulum is fixed, $r = R$, we have

$$\ddot{\vec{r}} = \hat{r}(-R\dot{\theta}^2) + \hat{\tau}(R\ddot{\theta}). \tag{5.7}$$

The force acting on the pendulum is decomposed as

$$\vec{F} = \vec{W} + \vec{F}_R = -(mg\sin\theta)\hat{\tau} + (mg\cos\theta - F_R)\hat{r}.$$

Newton's law then gives two equations: the equation of motion determining the evolution of the angle θ,

$$mR\ddot{\theta} = -mg\sin\theta \rightsquigarrow \boxed{\ddot{\theta} + \frac{g}{R}\sin\theta = 0}, \tag{5.8}$$

and an equation which determines the reaction force,

$$-mR\dot{\theta}^2 = mg\cos\theta - F_R \rightsquigarrow \boxed{F_R = m(R\dot{\theta}^2 + g\cos\theta)} \tag{5.9}$$

Let us now use the Euler–Lagrange method. The Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 - mg(z_0 - z).$$

We have

$$\vec{r} = \hat{z}r\cos\theta + \hat{x}r\sin\theta + \hat{y}y.$$

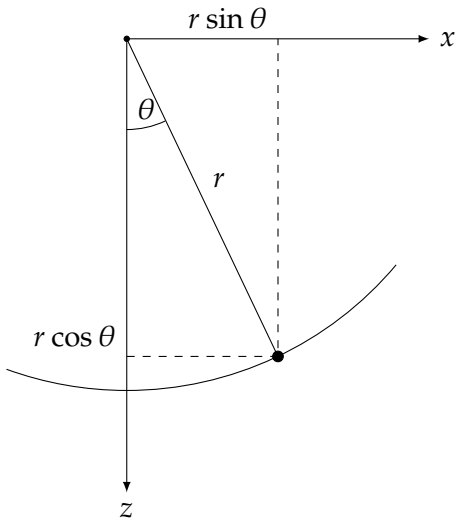


Figure 5.3. Visualisation of the constraints.

We can use the constraints (see Fig. 5.3)

$$r = R = \text{const}, \quad y = \text{const}.$$

Then,

$$\dot{\vec{r}} = R(-\hat{z}\sin\theta + \hat{x}\cos\theta)\dot{\theta} \rightsquigarrow \dot{\vec{r}}^2 = R^2\dot{\theta}^2.$$

Thus,

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta \underbrace{-mgz_0}_{\text{unimportant constant}}.$$

The equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \rightsquigarrow mR^2\ddot{\theta} + mgR\sin\theta = 0 \rightsquigarrow \boxed{\ddot{\theta} + \frac{g}{R}\sin\theta = 0},$$

which is the same result as with Newton's law, Eq. (5.8).

Notice that in neither of the two methods have we said anything about how to solve the equation of motion. In this sense, which of the two methods we use in order to derive Eq. (5.8) is a matter of taste. Nevertheless, we can make a few comparisons:

- The Euler–Lagrange method requires only scalar quantities. There is no need to perform vector rotations.
- The Euler–Lagrange method does not make an explicit reference to the reaction forces. This is an advantage if we are not interested in them, but a disadvantage if we care about them. For example, an engineer building a pendulum would like to know how large the reaction force is, in order to use a sufficiently strong rope.

6

Minimisation with Lagrange multipliers

The mathematical problem that we are facing in classical mechanics is one of minimisation of a functional (the action). Strictly speaking, Hamilton's principle requires only that the action be *stationary* ($\delta S = 0$); for long enough time intervals the physical path is in general a saddle point, not a minimum (e.g. a harmonic oscillator beyond half a period). See Landau–Lifshitz vol. 1, §2 (footnote), and Goldstein, Sec. 2.1. The same remark applies to "minimum" below.

This problem becomes more complicated in the presence of constraints, which make the variables of the functional (the generalised coordinates) dependent on each other. We have seen that for simple (holonomic) constraints it may be possible to eliminate the dependent variables. When this is not possible, we may use the more sophisticated technique of Lagrange multipliers. We will first recall the salient features of this method for the minimisation of functions with constraints, and then we will see how to apply it to functionals, and to the action specifically.

6.1 Minimisation of a multivariate function with Lagrange multipliers

Consider a function

$$u = f(x_1, \ldots, x_N),$$

where $x_1, \ldots, x_N$ are all independent variables. We are interested in finding the extrema (minima or maxima) of the function. These are given by the condition that the total differential of the function vanishes:

$$\mathrm{d}u = 0 \Rightarrow \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \mathrm{d}x_i = 0.$$

Since we have taken the variables x_i to be independent, the coefficients of $\mathrm{d}x_i$ must all vanish. Thus, we have that an extremum occurs if

$$\frac{\partial f}{\partial x_i} = 0 \quad \forall i = 1, \ldots, N.$$

Let us now assume that the variables x_i are not all independent, due to, for example, a constraint

$$\phi(x_1, \ldots, x_N) = 0.$$

The extrema of the function are given in this case too by the condition

$$\mathrm{d}u = 0,$$

but, unlike before, one of the variables, let us say x_N, is no longer independent. A straightforward way to find the minimum would be to solve the constraint

$$\phi(x_1, \ldots, x_N) = 0 \rightsquigarrow x_N = x_N(x_1, \ldots, x_{N-1})$$

and eliminate this variable from the function,

$$u = f(x_1, \ldots, x_N(x_1, \ldots, x_{N-1})) = g(x_1, \ldots, x_{N-1}).$$

Then we can minimise $g(x_1, \ldots, x_{N-1})$ as before, since the remaining variables are all independent. However, there is an alternative way where we do not need to solve the constraint explicitly. We start from

$$\begin{aligned} \mathrm{d}u = 0 &\rightsquigarrow \sum_{i=1}^{N-1} \left(\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_N} \frac{\partial x_N}{\partial x_i} \right) \mathrm{d}x_i = 0 \\ &\rightsquigarrow \sum_{i=1}^{N-1} \frac{\partial f}{\partial x_i} \mathrm{d}x_i + \frac{\partial f}{\partial x_N} \sum_{i=1}^{N-1} \frac{\partial x_N}{\partial x_i} \mathrm{d}x_i = 0. \end{aligned} \tag{6.1}$$

The sum in the last term is

$$\sum_{i=1}^{N-1} \frac{\partial x_N}{\partial x_i} \mathrm{d}x_i = \mathrm{d}x_N.$$

We therefore find again the same condition

$$\mathrm{d}u = 0 \rightsquigarrow \frac{\partial f}{\partial x_1} \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_{N-1}} \mathrm{d}x_{N-1} + \frac{\partial f}{\partial x_N} \mathrm{d}x_N = 0, \tag{6.2}$$

although now the $\mathrm{d}x_i$ are not all independent and we cannot demand that their coefficients vanish separately. Consider now the constraint:

$$\begin{aligned} \phi(x_1, \ldots, x_N) = 0 &\rightsquigarrow \mathrm{d}\phi(x_1, \ldots, x_N) = 0 \\ &\Rightarrow \frac{\partial \phi}{\partial x_1} \mathrm{d}x_1 + \cdots + \frac{\partial \phi}{\partial x_N} \mathrm{d}x_N = 0. \end{aligned} \tag{6.3}$$

We can multiply Eq. (6.3) by a constant λ, the so-called *Lagrange multiplier*, and subtract it from Eq. (6.2):

$$\boxed{\sum_{i=1}^{N} \left[\frac{\partial f}{\partial x_i} - \lambda \frac{\partial \phi}{\partial x_i} \right] \mathrm{d}x_i = 0}$$

It can be proven (see any analysis course) that we can use the extra parameter λ in order to make all terms in the brackets vanish,

$$\frac{\partial f}{\partial x_i} - \lambda \frac{\partial \phi}{\partial x_i} = 0, \quad i = 1, \ldots, N.$$

Solving this system of equations, together with the constraint, determines the extremum.

Therefore, the procedure to find the extremum in the presence of a constraint is identical to the one without constraints, replacing the original function $f(x_1, \ldots, x_N)$ with a function which includes the constraint,

$$f_\lambda(x_1, \ldots, x_N) \equiv f(x_1, \ldots, x_N) - \lambda \phi(x_1, \ldots, x_N).$$

Let us see how this works in a couple of examples.

Example 6.1. Find the minimum distance of a straight line from the origin. We will solve this problem in two ways.

1. Eliminating dependent variables.
   The square of the distance of a point (x, y) from the origin is

   $$f(x, y) \equiv d(x, y)^2 = x^2 + y^2.$$

   The constraint demands that the point belongs to a line,

   $$y = ax + b, \quad a, b \in \mathbb{R}.$$

   Eliminating the variable y, we have

   $$f(x, y) = f(x, y(x)) = x^2 + (ax + b)^2 \equiv g(x).$$

   We now have one independent variable x. We can minimise by demanding

   $$\frac{\partial g(x)}{\partial x} = 0 \rightsquigarrow 2x + 2a(ax + b) = 0 \Rightarrow x = \frac{-ab}{1 + a^2}.$$

   Then

   $$y = a\left(\frac{-ab}{1 + a^2}\right) + b = \frac{b}{1 + a^2}.$$

   The point of the line closest to the origin is therefore

   $$(x, y) = \left(\frac{-ab}{1 + a^2}, \frac{b}{1 + a^2}\right),$$

   and the minimum distance is $|b|/\sqrt{1 + a^2}$.

2. Using Lagrange multipliers.
   We first define the function

   $$f_\lambda(x, y) = x^2 + y^2 - \lambda(y - ax - b),$$

   which contains the square of the distance of the point (x, y) as well as a Lagrange multiplier times the constraint that the point must belong to the line, $y - ax - b = 0$. We require that the partial derivatives of this function with respect to all variables vanish:

   $$\left.\begin{array}{l} \frac{\partial}{\partial x} f_\lambda(x, y) = 0 \\ \frac{\partial}{\partial y} f_\lambda(x, y) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} 2x + \lambda a = 0 \\ 2y - \lambda = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = -\frac{\lambda}{2} a \\ y = \frac{\lambda}{2} \end{array}$$

   We can now use the constraint,

   $$y - ax - b = 0 \Rightarrow \frac{\lambda}{2}(1 + a^2) = b \Rightarrow \lambda = \frac{2b}{1 + a^2}.$$

   Thus the minimum is at

   $$(x, y) = \left(\frac{-a}{1 + a^2} b, \frac{b}{1 + a^2}\right),$$

   as expected.

You may wonder what the advantage of Lagrange multipliers is, since in our first example it was trivial to solve the problem by eliminating the dependent variable. However, in other problems it is easier (or the only possibility) to solve for a Lagrange multiplier instead. We demonstrate this in the following, more complicated example.

Example 6.2. Find the minimum and maximum distance from the origin on an ellipse (see Fig. 6.1),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We introduce a Lagrange multiplier for the constraint and extremise the function

$$f_\lambda(x, y) = x^2 + y^2 - \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

by demanding that

$$\frac{\partial f_\lambda}{\partial x} = 0 \rightsquigarrow x\left(1 - \frac{\lambda}{a^2}\right) = 0$$
$$\frac{\partial f_\lambda}{\partial y} = 0 \rightsquigarrow y\left(1 - \frac{\lambda}{b^2}\right) = 0$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The solutions of the three equations are

$$x = 0, \lambda = b^2, y = \pm b$$
$$\text{or}$$
$$y = 0, \lambda = a^2, x = \pm a$$

The extrema are then at

$$(x, y) \in \{(0, \pm b), (\pm a, 0)\}.$$

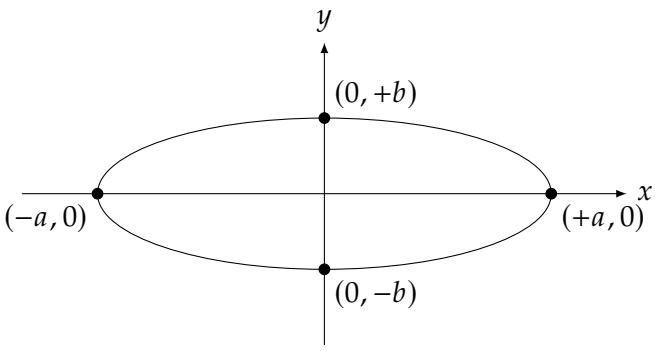


Figure 6.1. An ellipse with semi-axes a and b.

6.2 Minimising the action with Lagrange multipliers

We have seen that the principle of least action (Hamilton's principle) can be applied to physics problems with constraints. It is natural to use the method of Lagrange multipliers in order to impose these constraints. We can achieve this as follows. Consider a system with independent coordinates $\{q_i\}$, described by the action

$$S = \int \mathrm{d}t\, L[q_i, \dot{q}_i].$$

Minimising the action gives

$$\delta S = 0 \rightsquigarrow \int \mathrm{d}t \sum_i \delta q_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right] = 0.$$

Since the δq_i are all independent of each other, their coefficients must vanish, yielding the familiar Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

In the presence of a constraint,

$$f(q_i, \dot{q}_i) = 0,$$

the δq_i are no longer independent. **Remark:** the multiplier must in general be allowed to depend on time, $\lambda = \lambda(t)$; varying S_λ then gives $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda \frac{\partial f}{\partial \dot{q}_i}\right) - \lambda \frac{\partial f}{\partial q_i}$, which reduces to the form used below only for $\dot{\lambda} = 0$ or $\partial f / \partial \dot{q}_i = 0$. Moreover, for genuinely velocity-dependent (non-holonomic) constraints, adding λf to the Lagrangian yields the so-called vakonomic dynamics, which in general does *not* agree with Newton's laws with smooth constraints (d'Alembert's principle, Sec. 4.4); there one must impose $\sum_i \frac{\partial f}{\partial \dot{q}_i} \delta q_i = 0$ on the virtual displacements instead. For holonomic constraints $f(q_i, t) = 0$, as in the example below, the method is correct. See Arnold, Kozlov, Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, Sec. 1.2; Goldstein, Sec. 2.4.

From the constraint, we deduce that the integral

$$\int \mathrm{d}t\, \lambda f(q_i, \dot{q}_i) = 0$$

vanishes, and thus

$$\delta S_\lambda = 0 \qquad \text{with} \qquad S_\lambda = \int \mathrm{d}t\, L_\lambda(q_i, \dot{q}_i)$$

and

$$L_\lambda(q_i, \dot{q}_i) = \underbrace{L(q_i, \dot{q}_i)}_{\substack{\text{original} \\ \text{Lagrangian}}} - \underbrace{\lambda}_{\substack{\text{Lagrange} \\ \text{multiplier}}} \underbrace{f(q_i, \dot{q}_i)}_{\text{constraint}}$$

gives us

$$\delta S_\lambda = 0 \rightsquigarrow \int \mathrm{d}t \sum_i \delta q_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_\lambda}{\partial \dot{q}_i} - \frac{\partial L_\lambda}{\partial q_i} \right] = 0.$$

The presence of the parameter λ allows us to set all the coefficients of δq_i to zero,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\lambda}{\partial \dot{q}_i} - \frac{\partial L_\lambda}{\partial q_i} = 0,$$

despite the fact that the δq_i are not independent.

Example 6.3. Let us solve the simple pendulum problem once again, this time using Lagrange multipliers. The Lagrangian with a Lagrange multiplier for the constraint is

$$L_\lambda = T - V - \lambda(r - \underbrace{R}_{\text{constant}})$$

with

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 \qquad \text{and} \qquad V = -mgr\cos\theta.$$

The corresponding Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\lambda}{\partial \dot{r}} - \frac{\partial L_\lambda}{\partial r} = 0 \rightsquigarrow m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\lambda}{\partial \dot{\theta}} - \frac{\partial L_\lambda}{\partial \theta} = 0 \rightsquigarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0.$$

Using in addition the constraint

$$\boxed{r = R = \text{const.}}$$

we have

$$\boxed{\ddot{\theta} + \frac{g}{R}\sin\theta = 0} \tag{6.4}$$

$$\boxed{mg\cos\theta + mR\dot{\theta}^2 = \lambda} \tag{6.5}$$

Eq. (6.4) is the equation of motion of the pendulum. Eq. (6.5) is very interesting. On the left-hand side we recognise the reaction force F_R of the rope (see Fig. 6.2). We see then that the Lagrange multiplier has a physical meaning: it is the reaction force. This is not an accident. On the one hand, we have

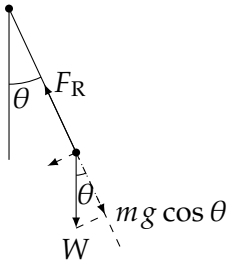


Figure 6.2. The simple pendulum with the reaction force F_R.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\lambda}{\partial \dot{q}_i} - \frac{\partial L_\lambda}{\partial q_i} = 0$$
$$\rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i}\right).$$

On the other hand, we have proven in general that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \underbrace{\tilde{Q}_i}_{\substack{\uparrow \\ \text{generalised} \\ \text{forces}}} = \sum_{j=1}^{N} \underbrace{\vec{F}_{R,j}}_{\substack{\uparrow \\ \text{non-potential} \\ \text{forces}}} \cdot \frac{\partial \vec{r}_j}{\partial q_i}.$$

Thus,

$$\lambda \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i} \right) = \tilde{Q}_i.$$

In our case $f = r - R$ and the left-hand side is

$$\lambda \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial (r-R)}{\partial \dot{r}} - \frac{\partial (r-R)}{\partial r} \right) = -\lambda = \tilde{Q}_r.$$

7

Conservation laws

Observing the time evolution of many physical systems, we often find that certain quantities remain unchanged at all times. A classic example is the energy of mechanical systems without friction. The Lagrangian formalism will allow us to gain a deeper understanding of the conservation of such quantities. We will realise that conservation and symmetry are interconnected.

7.1 Conservation of energy

Let us take a look at systems with a Lagrangian

$$L[q_i(t), \dot{q}_i(t), t] = L[q_i(t), \dot{q}_i(t)].$$

The lack of explicit time dependence of the Lagrangian means that there is no special time. Whenever we let the system evolve, it will do so in exactly the same way, irrespective of the starting time t_0.

The total time derivative of the Lagrangian is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L &= \sum_i \left[\frac{\partial L}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}\dot{q}_i}{\mathrm{d}t}\right] + \underbrace{\frac{\partial L}{\partial t}}_{=0} \\ &= \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right].\end{aligned}$$

Let us now recall the Euler–Lagrange equations,

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}.$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_i \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i.\end{aligned}$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right] = 0$$

and therefore

$$\boxed{\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \equiv E = \text{const.}} \tag{7.1}$$

We have found a conserved quantity, which we call energy. The reason for its conservation is the fact that the behaviour of the system does not change if we let it evolve from a different starting time.

We will now justify why the expression (7.1) is the familiar energy. Consider a particle with potential energy $U(\vec{r})$. The Lagrangian is

$$\begin{aligned} L &= T - U(\vec{r}) \\ &= \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r}). \end{aligned}$$

According to Eq. (7.1), the energy is

$$\begin{aligned} E &= \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{y}}\dot{y} + \frac{\partial L}{\partial \dot{z}}\dot{z} - L \\ &= m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \frac{1}{2}m\dot{\vec{r}}^2 + U(\vec{r}) \\ &= m\dot{\vec{r}}^2 - \frac{1}{2}m\dot{\vec{r}}^2 + U(\vec{r}) \\ &= \frac{1}{2}m\dot{\vec{r}}^2 + U(\vec{r}) = \underbrace{T}_{\text{kinetic}} + \underbrace{U}_{\text{potential}}. \end{aligned}$$

This is indeed the sum of the kinetic and potential energies.

7.2 Conservation of momentum

Let us now trace the origin of the conservation of momentum. We will find that momentum is conserved for physical systems for which we can shift the positions of all of their constituents without changing their physical behaviour. For example, if a physical system can be studied in an experiment in Zürich but also in an independent experiment in New York, and the two experiments are expected to find the same results, then the total momentum of the system is conserved. Formally, we assume that a translation of the positions of all particles,

$$\vec{r}_i \longrightarrow \vec{r}_i + \vec{\epsilon},$$

leaves the equations of motion invariant. Let us see how the Lagrangian changes if $\vec{\epsilon}$ is taken to be infinitesimal:

$$\begin{aligned} \Delta L &= L[\vec{r}_i + \vec{\epsilon}, \dot{\vec{r}}_i + \dot{\vec{\epsilon}}, t] - L[\vec{r}_i, \dot{\vec{r}}_i, t] \\ &= \sum_{i=1}^{N}\left[\frac{\partial L}{\partial r_{i,x}}\epsilon_x + \frac{\partial L}{\partial r_{i,y}}\epsilon_y + \frac{\partial L}{\partial r_{i,z}}\epsilon_z\right] \\ &= \sum_{i=1}^{N}\left(\vec{\nabla}_i L \cdot \vec{\epsilon}\right). \end{aligned}$$

The equations of motion are invariant under this displacement if the change in the Lagrangian is zero:

$$\Delta L = 0 \rightsquigarrow \boxed{\vec{\epsilon} \cdot \sum_{i=1}^{N} \vec{\nabla}_i L = 0}.$$

The direction of $\vec{\epsilon}$ is arbitrary. Choosing for example

$$\vec{\epsilon} = (1,0,0) \qquad \text{or} \qquad \vec{\epsilon} = (0,1,0) \qquad \text{or} \qquad \vec{\epsilon} = (0,0,1),$$

we obtain

$$\sum_{i=1}^{N} \frac{\partial L}{\partial x_i} = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \frac{\partial L}{\partial y_i} = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \frac{\partial L}{\partial z_i} = 0, \tag{7.2}$$

where $(x_i, y_i, z_i) \equiv \vec{r}_i$ is the position vector of the i-th particle. For a Lagrangian of the form $L = T - U$, Eq. (7.2) gives

$$\sum_{i=1}^{N} \vec{\nabla}_i U = 0.$$

However, $\vec{F}_i = -\vec{\nabla}_i U$ is the force acting on the i-th particle. Thus we conclude that

$$\boxed{\sum_{i=1}^{N} \vec{F}_i = 0},$$

where N is the number of particles in the system. In the special case of only two particles, $N = 2$, we have

$$\vec{F}_1 = -\vec{F}_2,$$

which is Newton's "action and reaction" law.

Let us now recall the Euler–Lagrange equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} &= \frac{\partial L}{\partial x_i} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}_i} &= \frac{\partial L}{\partial y_i} \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}_i} &= \frac{\partial L}{\partial z_i}. \end{aligned}$$

Then we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{y}_i}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{z}_i}\right) = 0.$$

Therefore, the quantities

$$\begin{aligned} P_x &\equiv \sum_i \frac{\partial L}{\partial \dot{x}_i} = \text{const.} \\ P_y &\equiv \sum_i \frac{\partial L}{\partial \dot{y}_i} = \text{const.} \\ P_z &\equiv \sum_i \frac{\partial L}{\partial \dot{z}_i} = \text{const.} \end{aligned}$$

are constant and the vector

$$\vec{P} = (P_x, P_y, P_z)$$

is conserved. We can associate $\vec{P}$ with the total momentum of the system.

Exercise 7.1. For a Lagrangian

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 - V(\vec{r}_1, \ldots, \vec{r}_N), \tag{7.3}$$

verify that

$$\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i, \quad \frac{\partial L}{\partial \dot{y}_i} = m_i \dot{y}_i, \quad \frac{\partial L}{\partial \dot{z}_i} = m_i \dot{z}_i. \tag{7.4}$$

We have therefore found that

$$\vec{P} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i = \sum_{i=1}^{N} \vec{P}_i$$

is indeed the total momentum of all particles in the system and that it is conserved.

7.3 Angular momentum

Consider a system of particles with masses m_i which are at positions $\vec{r}_i(t)$ at a time t. Now assume that we can rotate all vectors $\vec{r}_i$ by an angle $\delta\phi$ around a certain axis $\hat{n}$, as in Fig. 7.1. Let us define the vector

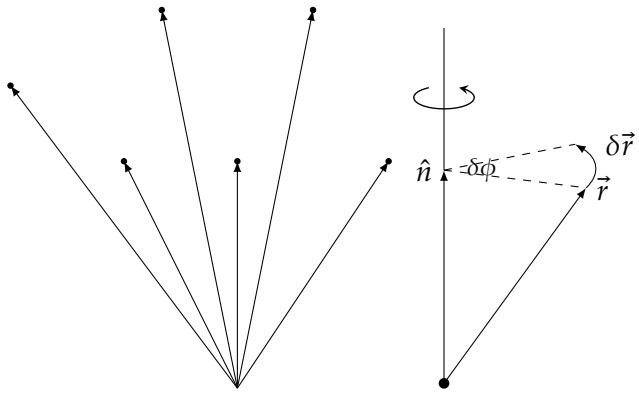


Figure 7.1. Rotation of all position vectors $\vec{r}_i$ in a system of particles around an axis $\hat{n}$ by an angle $\delta\phi$.

$$\delta\vec{\phi} = \hat{n}\delta\phi.$$

A position vector $\vec{r}$ after a rotation becomes

$$\vec{r} \to \vec{r} + \delta\vec{r} + O(\delta\phi^2)$$

with $\delta\vec{r} = \delta\vec{\phi} \times \vec{r}$. For illustration, we can verify this for a rotation around the z-axis:

$$\begin{aligned}
\vec{r} \to \vec{r}' &= \mathbf{R} \cdot \vec{r} \\
&= \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & -\delta\phi & 0 \\ \delta\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + O(\delta\phi^2) \\
&= \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \delta\phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \vec{r} + \delta\phi \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.
\end{aligned}$$

Consider

$$\begin{aligned}
(\hat{n} \times \vec{r})_i &= \sum_{j,k} \epsilon_{ijk} \hat{n}_j r_k \qquad \hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \sum_k \epsilon_{i3k} r_k.
\end{aligned}$$

Thus

$$\begin{aligned}
(\hat{n} \times \vec{r})_1 &= \epsilon_{132} r_2 = -1y = -y \\
(\hat{n} \times \vec{r})_2 &= \epsilon_{231} r_1 = +1x = x \\
(\hat{n} \times \vec{r})_3 &= \epsilon_{33k} r_k = 0,
\end{aligned}$$

and

$$\begin{aligned}
\vec{r} \to \vec{r}' &= \vec{r} + \delta\vec{r} \\
&= \vec{r} + \delta\vec{\phi} \times \vec{r} + O(\delta\phi^2).
\end{aligned}$$

Now we can find the conserved quantities associated with rotational symmetry by repeating the same steps as for space translations in the previous section. Under a rotation the Lagrangian changes by

$$L \to L' = L + \delta L$$

with

$$\begin{aligned}
\delta L &\equiv L[\vec{r} + \delta\vec{\phi} \times \vec{r}, \dot{\vec{r}} + \delta\vec{\phi} \times \dot{\vec{r}}] - L[\vec{r}, \dot{\vec{r}}] \\
&= \sum_i \left[\frac{\partial L}{\partial x_i}(\delta r_i)_x + \frac{\partial L}{\partial y_i}(\delta r_i)_y + \frac{\partial L}{\partial z_i}(\delta r_i)_z \right. \\
&\quad \left. + \frac{\partial L}{\partial \dot{x}_i}(\delta \dot{r}_i)_x + \frac{\partial L}{\partial \dot{y}_i}(\delta \dot{r}_i)_y + \frac{\partial L}{\partial \dot{z}_i}(\delta \dot{r}_i)_z \right] \\
&= \sum_i \left[\dot{P}_{x,i}(\delta r_i)_x + \dot{P}_{y,i}(\delta r_i)_y + \dot{P}_{z,i}(\delta r_i)_z \right. \\
&\quad \left. + P_{x,i}(\delta \dot{r}_i)_x + P_{y,i}(\delta \dot{r}_i)_y + P_{z,i}(\delta \dot{r}_i)_z \right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\phi} \cdot \left[\sum_i \vec{r}_i \times \vec{P}_i \right] \\
&= \delta\vec{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{r}_i \times \vec{P}_i .
\end{aligned}$$

If this change of the Lagrangian is zero, then the rotation has no effect on the physical system. This implies

$$\delta L = 0 \rightsquigarrow \delta\vec{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{r}_i \times \vec{P}_i = 0.$$

Since $\delta\vec{\phi}$ is an arbitrary vector (we can choose the rotation axis at will), we must have

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{r}_i \times \vec{P}_i = 0 \Rightarrow \boxed{\vec{M} = \sum_i \vec{r}_i \times \vec{P}_i = \text{const.}}$$

We have therefore found that the total angular momentum is conserved.

7.4 Summary of conservation laws and symmetries

We have realised that symmetry plays an important role in conservation laws. In the following table we summarise our findings for typical conserved quantities and their symmetry origin.

Symmetry	*Conserved quantity*
Time translation	Energy: $E = T + U$
Space translation	Momentum: $\vec{P} = \sum_i \partial L / \partial \dot{\vec{r}}_i = \sum_i m_i \dot{\vec{r}}_i$
Rotations	Angular momentum: $\vec{M} = \sum_i \vec{r}_i \times \vec{P}_i$

These symmetries are profound and we expect them to hold for many systems.

7.5 Conservation of generalised momenta

In analogy to "Cartesian momenta", if the Lagrangian is expressed in terms of generalised coordinates, we can define the generalised momenta

$$P_i = \frac{\partial L}{\partial \dot{q}_i}$$

and the generalised forces

$$\dot{P}_i = \frac{\partial L}{\partial q_i}.$$

For a Lagrangian which depends on $\dot{q}_i$ but not on q_i, we have

$$\tilde{Q}_i \equiv \dot{P}_i \equiv \frac{\partial L}{\partial q_i} = 0$$

(the generalised force vanishes). Then, the generalised momentum P_i is conserved:

$$0 = \frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}P_i \rightsquigarrow \boxed{P_i = \text{const.}}$$

7.6 Noether's theorem

If the equations of motion are invariant under symmetry transformations, then there are quantities which are conserved. This is a general statement known as *Noether's theorem*.

Assume that an infinitesimal transformation of the generalised coordinates,

$$q_i(t) \longrightarrow q_i(t) + \Delta q_i(t), \tag{7.5}$$

is a symmetry transformation, i.e. it leaves the equations of motion invariant. The change that this transformation induces in the Lagrangian is

$$\Delta L = \sum_i \left[\frac{\partial L}{\partial q_i}\Delta q_i + \frac{\partial L}{\partial \dot{q}_i}\Delta \dot{q}_i\right].$$

Using the Euler–Lagrange equations, we substitute

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}.$$

Then

$$\begin{aligned}\Delta L &= \sum_i \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\Delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta q_i)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i \\ \Rightarrow \boxed{\Delta L &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i}.\end{aligned}$$

If $\Delta L = 0$, then the Lagrangian is invariant and the action is also invariant, leading to the same equations of motion. But we should be more careful! More generally, the equations of motion are invariant if the Lagrangian changes under the symmetry transformation by a total time derivative, $\Delta L = dF/dt$. The symmetry condition $\Delta L = \frac{\mathrm{d}F}{\mathrm{d}t}$ must hold identically, for *arbitrary* paths $q_i(t)$ (off-shell), not only for solutions; only afterwards are the Euler–Lagrange equations used to identify ΔL with $\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i$. Otherwise every

transformation would trivially be "a symmetry" on the solutions; see José–Saletan, Sec. 3.2, or Goldstein, Sec. 13.7. Then

$$\Delta L = \frac{\mathrm{d}F}{\mathrm{d}t} \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i = \frac{\mathrm{d}F}{\mathrm{d}t} \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i - F\right) = 0.$$

Hence the quantity

$$\boxed{J = \sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i - F}$$

is conserved.

Example 7.1. Consider a particle in a homogeneous gravitational field,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz.$$

Since the field is uniform along the z-axis, we expect translations

$$z \to z + a$$

to be a symmetry. Under this transformation, the Lagrangian changes by

$$L \to L - mga \Rightarrow \Delta L = -mga = -a\frac{\mathrm{d}}{\mathrm{d}t}(mgt).$$

We have been able to cast the change as a total derivative. Then, according to Noether's theorem,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\Delta z\right) &= \Delta L \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\Delta z\right) &= -a\frac{\mathrm{d}}{\mathrm{d}t}(mgt) \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}a) &= -a\frac{\mathrm{d}}{\mathrm{d}t}(mgt) \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\dot{z} + gt) &= 0 \\
\Rightarrow \boxed{\dot{z} + gt = \text{const.}}
\end{aligned}$$

We have thus found a conserved quantity. We can verify that this is correct directly from the equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \Rightarrow m\ddot{z} + mg = 0 \overset{\int \mathrm{d}t}{\Rightarrow} \boxed{\dot{z} + gt = \text{const.}}$$

The other possible translations, along the x and y directions, induce no change in the Lagrangian,

$$x \to x + a \Rightarrow \Delta L = 0.$$

These are symmetry transformations. According to Noether's theorem,

$$\frac{\partial L}{\partial \dot{x}}\Delta x = \text{const.} \Rightarrow m\dot{x}a = \text{const.} \Rightarrow \dot{x} = \text{const.}$$

The equations of motion give

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \rightsquigarrow m\ddot{x} = 0 \rightsquigarrow \boxed{\dot{x} = \text{const}},$$

confirming the result of Noether's theorem; similarly, $y \to y + a$ leads to $\dot{y} = \text{const.}$

7.7 Noether's theorem and time symmetries[1]

In the action integral,

$$S = \int_{t_i}^{t_f} dt\, L[q_i(t), \dot{q}_i(t)], \tag{7.6}$$

time is an integration variable, while the generalised coordinates q_i and their corresponding velocities $\dot{q}_i$ are functions of it. In Eq. (7.5) we examined a symmetry transformation which acts on the coordinates. However, as we have seen in Sec. 7.1, a time symmetry (time translation), which affects the integration variable t of the action integral, also leads to a conservation law. Our proof of Noether's theorem in the previous section does not cover this case. We will see here that Noether's theorem can be extended to symmetries of the integration variable as well. This proof will be particularly useful in the future, when we study continuous systems and fields. For convenience, in this section we will use Einstein's summation convention: $A_i B_i \equiv \sum_i A_i B_i$.

Consider a symmetry transformation,

$$t \to t' = t'(t) = t + \delta t,$$

which leaves the action invariant:

$$\begin{aligned}
\int_{t_i}^{t_f} \mathrm{d}t\, L[q_i(t), \dot{q}_i(t)] &= \int_{t_i'}^{t_f'} \mathrm{d}t'\, L[q_i'(t'), \dot{q}_i'(t')] \\
&= \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right) L[q'(t'), \dot{q}_i'(t')] \\
&= \int_{t_i}^{t_f} \mathrm{d}t \left[1 + \frac{\mathrm{d}}{\mathrm{d}t}\delta t\right] L[q_i(t) + \Delta q_i'(t'), \dot{q}_i(t) + \Delta \dot{q}_i'(t')].
\end{aligned} \tag{7.7}$$

We have assumed that the coordinates also change by a certain amount when we perform this transformation,

$$q_i(t) \to q_i'(t') = q_i(t) + \Delta q_i'(t').$$

We can now write

$$L[q(t) + \Delta q_i'(t'), \dot{q}_i(t) + \Delta \dot{q}_i'(t')] = L[q_i(t), \dot{q}_i(t)] + \frac{\partial L}{\partial q_i}\Delta q_i'(t') + \frac{\partial L}{\partial \dot{q}_i}\Delta \dot{q}_i'(t').$$

Then Eq. (7.7) gives

$$\int_{t_i}^{t_f} \mathrm{d}t \left[L\frac{\mathrm{d}\delta t}{\mathrm{d}t} + \frac{\partial L}{\partial q_i}\Delta q_i'(t') + \frac{\partial L}{\partial \dot{q}_i}\Delta \dot{q}_i'(t') + \frac{\partial L}{\partial t}\delta t\right] = 0. \tag{7.8}$$

where the last term vanishes, since L has no explicit time dependence; we drop it from now on. We decompose the change in the coordinates as follows:

$$\begin{aligned}
\Delta q_i'(t') &\equiv q_i'(t') - q_i(t) \\
&= \underbrace{q_i'(t') - q_i(t')}_{\delta_* q_i} + q_i(t') - q_i(t) \\
&= \delta_* q_i + q_i(t) + \frac{\mathrm{d}}{\mathrm{d}t} q_i(t)\delta t - q_i(t) \\
&= \delta_* q_i + \dot{q}_i \delta t \\
\rightsquigarrow \Delta q_i'(t') &= q_i'(t') - q_i(t') + \dot{q}_i \delta t,
\end{aligned}$$

[1] This section can be omitted on a first reading.

where we separate the change due to moving the argument of the coordinate function from one time instant t to another, t', from the change of the function itself, from q_i to q_i', at the same time t'. Similarly,

$$\begin{aligned}\Delta\dot{q}_i'(t') &= \dot{q}_i'(t') - \dot{q}_i(t') + \dot{q}_i(t') - \dot{q}_i(t)\\ \rightsquigarrow \Delta\dot{q}_i' &= \frac{\mathrm{d}}{\mathrm{d}t}(\delta_* q_i) + \ddot{q}_i \delta t.\end{aligned}$$

Substituting into Eq. (7.8), we obtain

$$\begin{aligned}0 &= \int_{t_i}^{t_f} \mathrm{d}t \left[L\frac{\mathrm{d}}{\mathrm{d}t}\delta t + \frac{\partial L}{\partial q_i}(\delta_* q_i + \dot{q}_i \delta t) + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta_* q_i + \ddot{q}_i \delta t\right)\right]\\ &= \int_{t_i}^{t_f} \mathrm{d}t \left\{\left[L\frac{\mathrm{d}}{\mathrm{d}t}\delta t + \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right)\delta t\right] + \left(\frac{\partial L}{\partial q_i}\delta_* q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta_* q_i\right)\right\}.\end{aligned}$$

The term in the square bracket is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\delta t)L[q_i, \dot{q}_i]\right] = L\frac{\mathrm{d}}{\mathrm{d}t}(\delta t) + \delta t\left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right).$$

Thus, using also the Euler–Lagrange equations,

$$\begin{aligned}0 &= \int \mathrm{d}t \left[\frac{\mathrm{d}}{\mathrm{d}t}(L\delta t) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\delta_* q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta_* q_i\right]\\ &= \int \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left[L\delta t + \frac{\partial L}{\partial \dot{q}_i}\delta_* q_i\right].\end{aligned}$$

The above should hold for arbitrary boundaries. Thus the integrand must vanish, i.e.

$$L\delta t + \frac{\partial L}{\partial \dot{q}_i}\delta_* q_i = \text{const.}$$

We can now write

$$\delta_* q_i = \Delta q_i - \dot{q}_i \delta t \qquad \text{with} \qquad \Delta q_i \equiv q_i'(t') - q_i(t).$$

We then have

$$\boxed{\left(L - \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)\delta t + \frac{\partial L}{\partial \dot{q}_i}\Delta q_i = \text{const.}}$$

For the special (but very common) case $\Delta q_i = 0$, we recover the result of Section 7.1 that the energy is conserved,

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = E = \text{constant}. \tag{7.9}$$

7.8 Continuous symmetry transformations

As we have seen, symmetry plays an important role in the conservation of physical quantities. In this section, we will review some of the mathematical properties of continuous symmetry transformations.

Consider a set of generalised coordinates

$$\begin{pmatrix} q_1 \\ \vdots \\ q_M \end{pmatrix}$$

describing a system with a Lagrangian

$$L[q_i(t), \dot{q}_i(t), t].$$

Assume that the system is invariant under a transformation

$$q_i \to q_i' = T_{ij}(\theta_a) q_j \quad \text{where } a = 1, \ldots, N \quad \text{and} \quad i, j = 1, \ldots, M.$$

We take the parameters θ_a to be continuous. For example, if a system is described by two Cartesian coordinates in a plane

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

and is symmetric under rotations

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(see Fig. 7.2), the angle θ is the continuous parameter which parametrises the transformations. The symmetry transformations form a *Lie group* and the product of two such

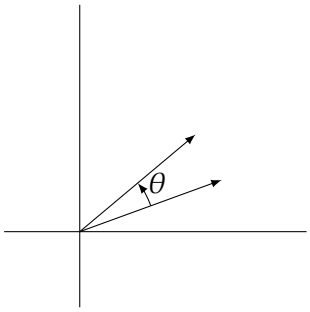


Figure 7.2. Rotation symmetry in a plane.

transformations is also a symmetry transformation belonging to the group:

$$T(\theta_1^a)\, T(\theta_2^a) = T(\theta_3^a) \quad \text{where} \quad \theta_3^a = f^a(\theta_1^a, \theta_2^a). \tag{7.10}$$

We arrange for the unit element of the symmetry group to correspond to all parameters being zero:

$$T(\theta^a = 0) = 1. \tag{7.11}$$

Then

$$\begin{aligned} T(f^a(0, \theta^a)) &= T(0) T(\theta^a) \\ &= 1\, T(\theta^a) = T(\theta^a) \\ \rightsquigarrow\ f^a(0, \theta^a) &= \theta^a. \end{aligned} \tag{7.12}$$

Similarly,

$$f^a(\theta^a, 0) = \theta^a. \tag{7.13}$$

Let us now perform a Taylor expansion around $\theta^a = 0$:

$$\theta_1^a = f^a(\theta_1^a, \cancelto{0}{\theta_2^a}) = f^a(0,0) + \frac{\partial f^a}{\partial \theta_1^b}\theta_1^b + \cancel{\frac{\partial f^a}{\partial \theta_2^b}\theta_2^b} + \frac{1}{2}\frac{\partial^2 f^a}{\partial\theta_1^b \partial\theta_1^c}\theta_1^b\theta_1^c \cancelto{0}{} + \cancel{\frac{1}{2}\frac{\partial^2 f^a}{\partial\theta_2^b \partial\theta_2^c}\theta_2^b\theta_2^c} + \cancel{\frac{\partial^2 f^a}{\partial\theta_1^b \partial\theta_2^c}\theta_1^b\theta_2^c} + \dots$$

Imposing Eqs. (7.12) and (7.13), we have

$$f^a(\theta_1^a, \theta_2^a) = \theta_1^a + \theta_2^a + \sum_{b,c} f_{bc}^a \theta_1^b \theta_2^c + \dots$$

For small parameters θ^a, we can expand

$$T(\theta^a) = 1 + \mathrm{i}\theta^a t_a + \frac{1}{2}\sum_{b,c} \theta^b\theta^c t_{bc} + \dots$$

where 1, t_a, t_{bc} are matrices of the same dimension as $T(\theta^a)$. In addition,

$$t_{bc} = t_{cb} \quad \text{(symmetric)}.$$

Making a Taylor expansion of Eq. (7.10), we have

$$\begin{aligned}
&\left(1 + \mathrm{i}\theta_1^a t_a + \frac{1}{2}\theta_1^b\theta_1^c t_{bc}\right)\left(1 + \mathrm{i}\theta_2^d t_d + \frac{1}{2}\theta_2^d\theta_2^e t_{de}\right) \\
&\quad = 1 + \mathrm{i}\left(\theta_1^a + \theta_2^a + f_{bc}^a\theta_1^b\theta_2^c + \dots\right) t_a \\
&\qquad + \frac{1}{2}\left(\theta_1^b + \theta_2^b + \dots\right)\left(\theta_1^c + \theta_2^c + \dots\right) t_{bc} + \dots
\end{aligned}$$

Matching the terms of the expansion, we have

$$t_{bc} = -t_b t_c - \mathrm{i} f_{bc}^a t_a.$$

From the symmetry $t_{bc} = t_{cb}$, we obtain

$$\begin{aligned}
t_b t_c + \mathrm{i} f_{bc}^a t_a &= t_c t_b + \mathrm{i} f_{cb}^a t_a \\
\rightsquigarrow t_b t_c - t_c t_b &= \mathrm{i}(f_{cb}^a - f_{bc}^a) t_a.
\end{aligned}$$

For reasons which will become clear later, the matrices t_a are called the *generators* of the symmetry transformations. We also define the commutator of two such generators as

$$[t_a, t_b] \equiv t_a t_b - t_b t_a.$$

The combinations

$$C_{bc}^a \equiv f_{cb}^a - f_{bc}^a$$

are called the *structure constants* and they are antisymmetric:

$$\boxed{C^a_{bc} = -C^a_{cb}}.$$

The equation

$$[t_b, t_c] = \mathrm{i} C^a_{bc} t_a \tag{7.14}$$

defines the *Lie algebra* of the group. Notice that

$$[t_a, [t_b, t_c]] + [t_b, [t_c, t_a]] + [t_c, [t_a, t_b]] = 0. \tag{7.15}$$

This is the so-called *Jacobi identity*, and it is satisfied by the generators. It gives an identity for the structure constants. Using the Lie algebra, we have

$$\begin{aligned} &[t_a, \mathrm{i} C^d_{bc} t_d] + [t_b, \mathrm{i} C^d_{ca} t_d] + [t_c, \mathrm{i} C^d_{ab} t_d] = 0 \\ &\rightsquigarrow \boxed{C^e_{ad} C^d_{bc} + C^e_{bd} C^d_{ca} + C^e_{cd} C^d_{ab} = 0}. \end{aligned}$$

Let us now define

$$\tilde{t}^b_{ac} = \mathrm{i} C^c_{ab}.$$

Then the above equation becomes

$$[\tilde{t}^a, \tilde{t}^b] = \mathrm{i} C^d_{ab} \tilde{t}^d,$$

i.e. the structure constants furnish another *representation* of the Lie algebra. We call this the *adjoint representation*.

It is now time to justify why we call the t_a matrices generators. Using the product rule of Eq. (7.10) an infinite number of times, we can write for a symmetry transformation with parameters θ^a

$$\begin{aligned} T(\theta^a) &= \lim_{N\to\infty} T\left(\frac{\theta^a}{N}\right)^N \quad \text{(product rule)} \\ &= \lim_{N\to\infty} \left(1 + \mathrm{i}\frac{\theta^a}{N} t_a\right)^N \\ &= \exp(\mathrm{i}\theta^a t_a). \end{aligned}$$

In other words, the generators t_a, together with the product rule, are sufficient to produce any "large" transformation from infinitesimal ones by means of a simple exponentiation:

$$T(\theta^a) = \exp(\mathrm{i}\theta^a t_a). \tag{7.16}$$

Example 7.2. For rotations in a plane, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{T(\theta)} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For a small angle θ,

$$\begin{aligned} T(\theta) &\approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O(\theta^2) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{i}\theta \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} + O(\theta^2). \end{aligned}$$

The generator is

$$t \equiv \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$$

(there is only one). Notice that

$$t^2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Thus,

$$\begin{aligned} \exp(\mathrm{i}\theta t) &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n \theta^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^{2n} \theta^{2n} (t^2)^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{\mathrm{i}^{2n+1} \theta^{2n+1} t^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n}}{(2n)!} + t \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n+1}}{(2n+1)!}. \end{aligned}$$

But

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta} &= \cos\theta + \mathrm{i}\sin\theta \\ &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\mathrm{i}\theta)^{2n+1}}{(2n+1)!}. \end{aligned}$$

Thus,

$$\begin{aligned} \exp(\mathrm{i}\theta t) &= \cos\theta\, 1 + \mathrm{i}t \sin\theta \\ &= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin\theta \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \checkmark. \end{aligned}$$

Exercise 7.2. Find the generators and structure constants for rotations in three dimensions (the group SO(3)).

8

Motion in one dimension

The knowledge of conservation laws such as energy conservation can be helpful for solving the equations of motion of a system and determining the trajectories of its particles. We can demonstrate this in the simplest possible example, a one-particle system which is specified by a single generalised coordinate.

Let us consider a Lagrangian of the form

$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q),$$

with q a generalised coordinate. This Lagrangian does not depend explicitly on time. Thus, the energy of the system is conserved:

$$E = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = \frac{1}{2}a(q)\dot{q}^2 + V(q) = \text{const.}$$

It now becomes easy to solve the equation of motion, without even writing down the Euler–Lagrange equation. From energy conservation, we have

$$\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 = \frac{2(E - V(q))}{a(q)}$$

$$\rightsquigarrow \int \mathrm{d}t = \int \mathrm{d}q \frac{a(q)^{\frac{1}{2}}}{\sqrt{2(E - V(q))}}$$

$$\Rightarrow \boxed{t = \int \mathrm{d}q \sqrt{\frac{a(q)}{2(E - V(q))}} + \text{const.}}$$

For a particle moving in one dimension,

$$q = x, \quad a(q) = m, \qquad L = \frac{1}{2}m\dot{x}^2 - V(x),$$

we have

$$t - t_0 = \sqrt{\frac{m}{2}} \int \frac{\mathrm{d}x}{\sqrt{E - V(x)}}.$$

The kinetic energy of the particle can never be negative,

$$\frac{1}{2}m\dot{x}^2 = E - V(x) \geq 0.$$

This condition may limit the interval in which the particle can move: the solution of the inequality determines the range of the motion. If the motion is bounded by two points, as happens for the potential illustrated in Fig. 8.1, the motion is oscillatory. The period of the oscillation is then

$$T = 2\sqrt{\frac{m}{2}} \int_{x_a}^{x_b} \frac{\mathrm{d}x}{\sqrt{E - V(x)}}.$$

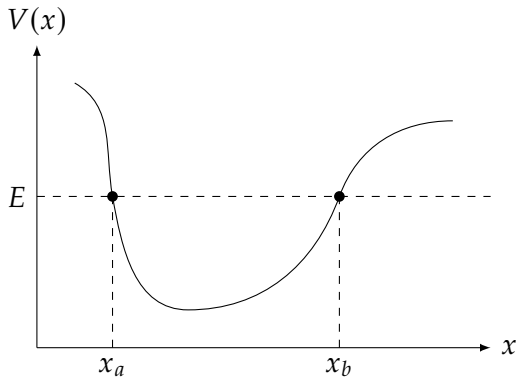


Figure 8.1. A potential $V(x)$ with turning points x_a, x_b, where $E = V(x_a) = V(x_b)$.

9

Two-body problem

In this section we will discuss the interaction of two particles with masses m_1 and m_2, assuming that the force which they exert on each other depends only on their distance. The Lagrangian which describes such a system is

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|).$$

We need to find the trajectories $\vec{r}_1 = \vec{r}_1(t)$ and $\vec{r}_2 = \vec{r}_2(t)$ of the two particles. We shall see that the symmetry of the problem simplifies this task.

A first simplification we can make is to use a clever frame of reference. This is the so-called *centre-of-mass* frame, in which the total momentum of all particles in a system is zero. To determine this frame, we first find the mass-weighted average position of the particles,

$$\vec{R} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}. \tag{9.1}$$

We also define the relative position vector of the two particles. Assuming that $m_2 > m_1$, we define, conventionally,

$$\vec{r} = \vec{r}_1 - \vec{r}_2. \tag{9.2}$$

We can express the position vectors of the two particles in terms of $\vec{r}$ and $\vec{R}$. After a little algebra, we find

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}, \tag{9.3}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}. \tag{9.4}$$

In terms of the variables $(\vec{r}, \vec{R})$, the Lagrangian becomes

$$L = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r), \tag{9.5}$$

where $r \equiv |\vec{r}|$ and μ is the *reduced mass*,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{9.6}$$

The name is more transparent if we cast Eq. (9.6) in the form

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \tag{9.7}$$

from which we can easily see that

$$\frac{1}{\mu} > \frac{1}{m_1}, \frac{1}{m_2} \rightsquigarrow \mu < m_1, m_2. \tag{9.8}$$

As the name suggests, the reduced mass is smaller than either of the masses of the two particles. In Fig. 9.1, we plot the vectors $\vec{r}$, $\vec{R}$, $\vec{r}_1$ and $\vec{r}_2$. An interesting configuration is

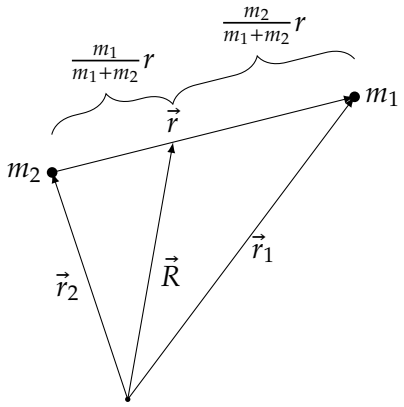


Figure 9.1. The position vectors of the two particles, their relative position and the position of the centre of mass.

when the mass m_2 is much larger than m_1. This corresponds, for example, to a typical *star-planet* system like the Sun and the Earth. For $m_2 \gg m_1 \rightsquigarrow \frac{m_1}{m_2} \to 0$, we have

$$\vec{r}_2 \approx \vec{R}, \quad \vec{r}_1 \approx \vec{R} + \vec{r}, \tag{9.9}$$

while the reduced mass is

$$\mu \approx m_1, \tag{9.10}$$

which is the mass of the small object.

Let us now examine the structure of the Lagrangian of Eq. (9.5). We notice that the coordinates of the centre-of-mass vector $\vec{R} \equiv (R_x, R_y, R_z)$ do not appear in the Lagrangian, but only the velocities $\dot{\vec{R}} = (\dot{R}_x, \dot{R}_y, \dot{R}_z)$ do. Therefore, the velocity of the centre of mass must be constant:

$$\dot{\vec{R}} = \text{const.} \tag{9.11}$$

Indeed, we recall that

$$\frac{\partial L}{\partial R_x} = 0 \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{R}_x} = 0 \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}[(m_1 + m_2)\dot{R}_x] = 0 \rightsquigarrow \dot{R}_x = \text{const},$$

and similarly for $\dot{R}_y$, $\dot{R}_z$. We can now exercise our freedom to choose our frame of reference such that the centre of mass is at rest:

$$\dot{\vec{R}} = 0. \tag{9.12}$$

In this frame, the Lagrangian takes the form

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r). \tag{9.13}$$

We have turned the two-body problem into an equivalent problem of a single particle with a "mass" equal to the reduced mass and a "position" equal to the relative position vector of the two particles.

The potential in Eq. (9.13) does not depend on the direction of $\vec{r}$ but only on its magnitude. Therefore, we expect the Lagrangian to be symmetric under rotations:

$$\vec{r} \to \vec{r}' = \vec{r} + \delta\vec{\phi} \times \vec{r}. \tag{9.14}$$

Then the angular momentum,

$$\vec{M} = \vec{r} \times \vec{P}, \tag{9.15}$$

with

$$\vec{P} = \left(\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{y}}, \frac{\partial L}{\partial \dot{z}}\right) = \mu(\dot{x}, \dot{y}, \dot{z}) = \mu\dot{\vec{r}}, \tag{9.16}$$

is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{M} = 0 \rightsquigarrow \vec{r} \times \vec{P} = \mu(\vec{r} \times \dot{\vec{r}}) = \text{const.} \tag{9.17}$$

The position $\vec{r}$ and the velocity $\dot{\vec{r}}$ are always perpendicular to the angular momentum. Since the latter is constant, always pointing in the same direction, the vector $\vec{r}$ always lies in the same plane (see Fig. 9.2). We can therefore use polar coordinates in this plane, with

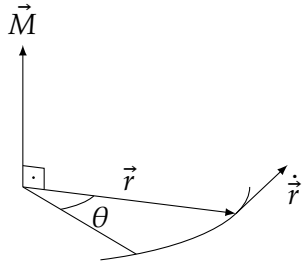


Figure 9.2. The angular momentum $\vec{M}$ is perpendicular to the plane of the motion.

the z-axis in the direction of the angular momentum,

$$\vec{M} = M\hat{z}.$$

The Lagrangian then takes the form

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{9.18}$$

The conservation of angular momentum is manifest in the Lagrangian of Eq. (9.18), since it does not depend explicitly on the angle θ, but only on the angular velocity $\dot{\theta}$. Indeed, the corresponding Euler–Lagrange equation gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \underbrace{\frac{\partial L}{\partial \theta}}_{=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mu r^2\dot{\theta}) \rightsquigarrow \boxed{\mu r^2\dot{\theta} = M = \text{const.}} \tag{9.19}$$

As usual, the total energy is also conserved:

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{const.} \tag{9.20}$$

Equivalently, eliminating $\dot{\theta}$ using Eq. (9.19), we write

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{M^2}{2\mu r^2}. \tag{9.21}$$

Exercise 9.1. Verify that by taking the time derivative of Eq. (9.20) or (9.21),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0,$$

you can obtain the second equation of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0. \tag{9.22}$$

We can interpret Eq. (9.21) as the energy of a particle with mass μ moving in only one dimension in a potential

$$V_{\text{eff}}(r) = V(r) + \underbrace{\frac{M^2}{2\mu r^2}}_{\substack{\text{“centrifugal} \\ \text{energy”}}}. \tag{9.23}$$

Let us rewrite Eq. (9.21) in the form

$$\frac{1}{2}\mu\dot{r}^2 = E - V_{\text{eff}}(r) = E - \frac{M^2}{2\mu r^2} - V(r). \tag{9.24}$$

The kinetic energy on the left-hand side cannot be negative:

$$\frac{1}{2}\mu\dot{r}^2 \geq 0 \rightsquigarrow E - V_{\text{eff}}(r) \geq 0. \tag{9.25}$$

The motion is therefore restricted to distances for which the total energy exceeds the effective potential energy V_{eff}. The turning points of the motion are the solutions of the equation

$$E = V_{\text{eff}}(r). \tag{9.26}$$

This can yield different types of trajectories.

Example 9.1. In Fig. 9.3, the motion is restricted to a finite range of distances, $[r_{\min}, r_{\max}]$.

Example 9.2. In Fig. 9.4, the energy is larger than $V_{\text{eff}}(r)$ up to infinite distances.

Let us now look at the condition

$$\begin{aligned} E - V_{\text{eff}}(r) &\geq 0 \\ \rightsquigarrow E - V(r) - \frac{M^2}{2\mu r^2} &\geq 0 \\ \rightsquigarrow \boxed{2\mu E r^2 - 2\mu r^2 V(r) - M^2 \geq 0}&. \end{aligned}$$

Taking the limit $r \to 0$, we have

$$-\lim_{r\to 0}[r^2 V(r)] \geq \frac{M^2}{2\mu}, \tag{9.27}$$

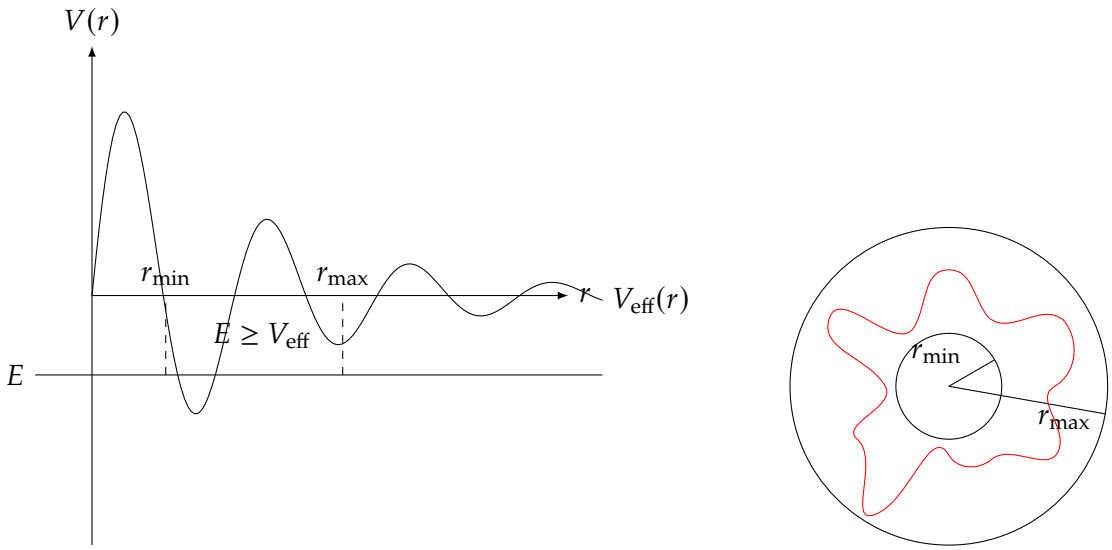


Figure 9.3. Bounded motion between $r_{\min}$ and $r_{\max}$.

which is a condition on the potential for the distance r to be able to become zero. We clearly need an attractive potential ($V(r) < 0$) in order for Eq. (9.27) to have a chance of being satisfied. But even then, this is not guaranteed if the angular momentum is not zero. For example, for a gravitational or an attractive electrostatic potential,

$$V(r) = -\frac{a}{r}, \quad a > 0, \tag{9.28}$$

Eq. (9.27) is not satisfied. This explains, for example, why the Moon does not fall onto the Earth. We may rush to conclude that this is also the reason for the stability of atoms, explaining the orbits of electrons in analogy to planetary motions in a solar system. However, this analogy is not correct. In classical electrodynamics, electrons in orbit around a nucleus must emit electromagnetic radiation, thus continuously losing energy and angular momentum. Eventually, the electrons of classical physics would collapse onto the nucleus. To explain the stability of atoms we need quantum mechanics.

Angular momentum and energy conservation are sufficient to determine the trajectory $r(t)$ fully, from

$$\begin{aligned} E &= \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \\ \rightsquigarrow \frac{\mathrm{d}r}{\mathrm{d}t} &= \sqrt{\frac{2}{\mu}}\,(E - V_{\text{eff}}(r))^{\frac{1}{2}} \\ \rightsquigarrow \mathrm{d}t &= \sqrt{\frac{\mu}{2}}\frac{\mathrm{d}r}{\sqrt{E - V_{\text{eff}}(r)}} \qquad (9.29) \\ \rightsquigarrow t &= \sqrt{\frac{\mu}{2}}\int \frac{\mathrm{d}r}{\sqrt{E - V_{\text{eff}}(r)}} + \text{const.} \qquad (9.30) \end{aligned}$$

For the angle $\theta(t)$ we have

$$M = \mu r^2\dot{\theta} \rightsquigarrow \int \mathrm{d}\theta = \int \mathrm{d}t\,\frac{M}{\mu r^2}.$$

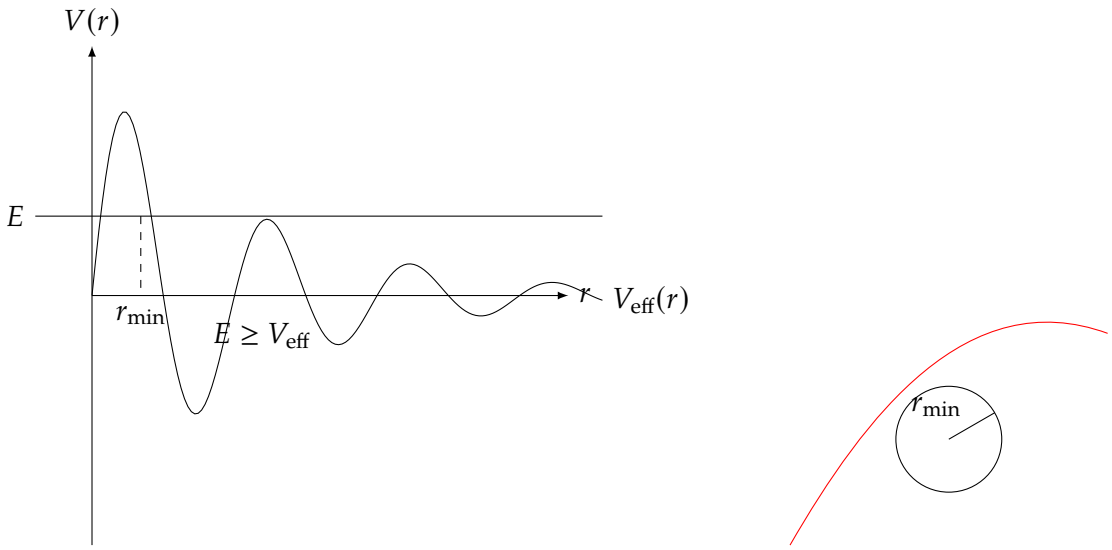


Figure 9.4. Unbounded motion.

Using Eq. (9.29) to trade $\mathrm{d}t$ for $\mathrm{d}r$, we have

$$\int \mathrm{d}\theta = \sqrt{\frac{\mu}{2}} \int \frac{\mathrm{d}r}{\sqrt{E - V_{\text{eff}}(r)}} \frac{M}{\mu r^2}$$

$$\Rightarrow \boxed{\theta = \frac{M}{\sqrt{2\mu}} \int \frac{\mathrm{d}r}{r^2} \frac{1}{\sqrt{E - V_{\text{eff}}}} + \text{const.}} \tag{9.31}$$

9.1 The $1/r$ potential

Let us now consider a potential of the form

$$V(r) = -\frac{a}{r}, \quad a > 0. \tag{9.32}$$

This is the gravitational potential due to a massive object, or an attractive electrostatic potential due to a point charge. We will first study the case $a > 0$, but our results can easily be used for repulsive potentials with $a < 0$.

The effective potential, which takes into account the centrifugal energy, is

$$V_{\text{eff}} = \frac{M^2}{2\mu r^2} - \frac{a}{r}. \tag{9.33}$$

We notice that the centrifugal energy is positive and behaves as $\frac{1}{r^2}$ while the potential energy $V(r)$ is negative and behaves as $\frac{1}{r}$. At small distances, $r \to 0$, the centrifugal energy dominates and V_{eff} tends asymptotically to plus infinity. In this region V_{eff} is repulsive:

$$\lim_{r \to 0} -\frac{\partial V_{\text{eff}}}{\partial r} > 0.$$

At large distances, $r \to \infty$, the $\frac{1}{r}$ term is larger than the $\frac{1}{r^2}$ term and the effective potential is dominated by $V(r)$. In this limit the force is attractive:

$$\lim_{r \to \infty} -\frac{\partial V_{\text{eff}}}{\partial r} < 0.$$

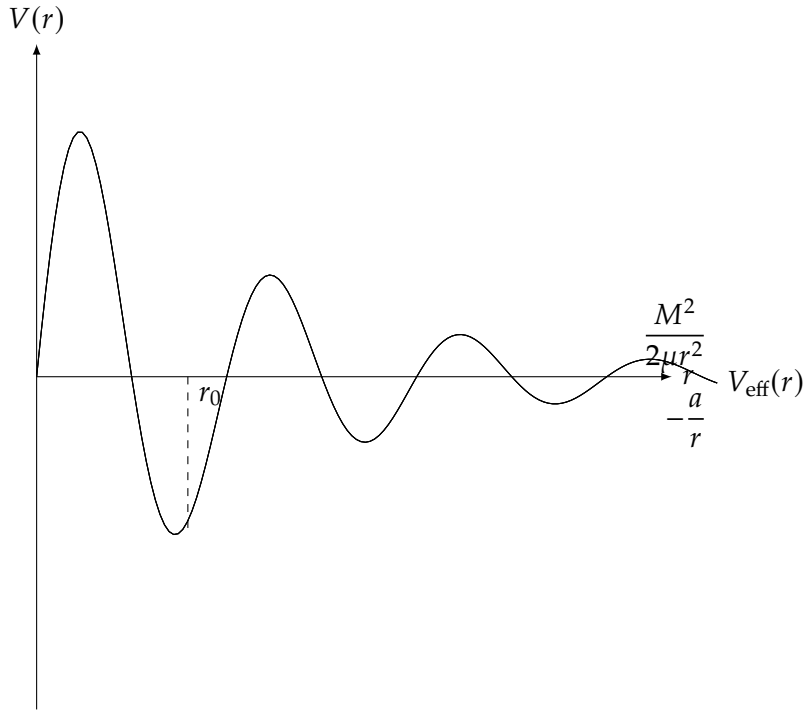


Figure 9.5. The effective potential for $V(r) = -a/r$.

In the intermediate region the potential has a minimum:

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_{r=r_0} = 0 \rightsquigarrow r_0 = \frac{M^2}{\mu a}. \tag{9.34}$$

The shape of the potential is shown in Fig. 9.5. To calculate the trajectory, we need to evaluate the integral of Eq. (9.31) for V_{eff} given by Eq. (9.33):

$$\theta(r) = \frac{M}{\sqrt{2\mu}} \int \frac{\mathrm{d}r}{r^2} \frac{1}{\sqrt{E - \frac{M^2}{2\mu r^2} + \frac{a}{r}}} + \text{const.} \tag{9.35}$$

We can cast the above formula as

$$\theta(r) = \int \frac{\mathrm{d}r}{r^2} \frac{1}{\sqrt{-\left(\frac{1}{r} - \frac{1}{p}\right)^2 + \frac{e^2}{p^2}}} + \text{const}, \tag{9.36}$$

where

$$p \equiv \frac{M^2}{\mu a} \tag{9.37}$$

and

$$e \equiv \sqrt{1 + \frac{2EM^2}{\mu a^2}}. \tag{9.38}$$

With the change of variables

$$\frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos\omega, \tag{9.39}$$

the integral of Eq. (9.36) becomes

$$\theta = \int \mathrm{d}\omega + \text{const.}$$

which leads to

$$\theta = \omega + \theta_0. \tag{9.40}$$

Thus, from Eq. (9.39) we find the trajectory

$$\frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos(\theta - \theta_0)$$

or, equivalently,

$$\boxed{\frac{p}{r} = 1 + e\cos(\theta - \theta_0)}. \tag{9.41}$$

Eq. (9.41) describes the trajectory $r = r(\theta)$ for the distance of the two particles. As we can see, the trajectory depends on the values of the two parameters (e, p), which in turn depend on the energy E and the angular momentum M. We will find three types of trajectories, listed in Table 9.1. For the minimum value of the energy,

$e > 1$	$E > 0$	hyperbola
$e = 1$	$E = 0$	parabola
$e < 1$	$E < 0$	ellipse

Table 9.1. Types of trajectories in the $1/r$ potential with $a > 0$.

$$E = -\frac{\mu a^2}{2M^2},$$

we have $e = 0$ and the trajectory is a perfect circle of radius p. Let us check that Eq. (9.41) indeed describes these types of trajectories. Choosing $\theta_0 = 0$ and substituting

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad r = \sqrt{x^2 + y^2}$$

into Eq. (9.41), we have

$$\begin{aligned} p &= \sqrt{x^2 + y^2} + ex \\ \rightsquigarrow\ x^2 + y^2 &= (p - ex)^2 \\ \rightsquigarrow\ x^2(1 - e^2) + 2epx + y^2 &= p^2. \end{aligned} \tag{9.42}$$

For $e \neq 1$ we have

$$\begin{aligned} x^2 + \frac{2epx}{1-e^2} + \frac{y^2}{1-e^2} &= \frac{p^2}{1-e^2} \\ \rightsquigarrow \left(x + \frac{ep}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} &= \frac{p^2}{1-e^2} + \frac{e^2p^2}{(1-e^2)^2} \\ \Rightarrow \left[x + \frac{ep}{1-e^2}\right]^2 + \frac{y^2}{1-e^2} &= \frac{p^2}{(1-e^2)^2}. \end{aligned}$$

For $p \neq 0$, we can write

$$\frac{\left[x + \frac{ep}{1-e^2}\right]^2}{\frac{p^2}{(1-e^2)^2}} + \frac{y^2}{\frac{p^2}{1-e^2}} = 1. \tag{9.43}$$

Let us now take $e < 1$. Then Eq. (9.43) is indeed the equation of an ellipse,

$$\frac{(x-x_0)^2}{A^2} + \frac{(y-y_0)^2}{B^2} = 1,$$

where

$$x_0 = -\frac{ep}{1-e^2}, \quad y_0 = 0 \tag{9.44}$$

and

$$A = \frac{p}{1-e^2} \tag{9.45}$$

$$B = \frac{p}{\sqrt{1-e^2}} = \sqrt{1-e^2}A \leq A \tag{9.46}$$

are the semi-major and semi-minor axes of the ellipse (see Fig. 9.6). We observe that

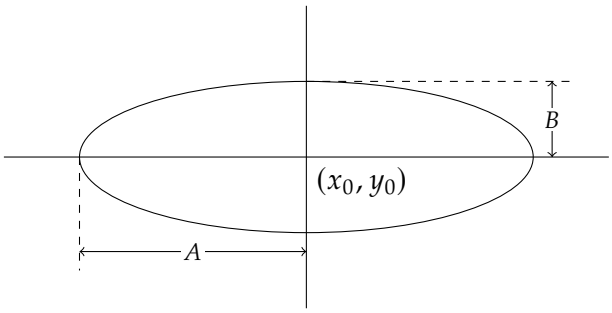


Figure 9.6. An ellipse with semi-major axis A and semi-minor axis B.

$$A = \frac{p}{1-e^2} = -\frac{a}{2E} :$$

the semi-major axis of the ellipse depends only on the energy E, while the semi-minor axis

$$B = A\sqrt{1-e^2} = A\sqrt{-\frac{2EM^2}{\mu a^2}} = \sqrt{-\frac{M^2}{2\mu E}}$$

depends on both energy and angular momentum.

Let us now return to Eq. (9.43) and consider the case $E > 0$, corresponding to $e > 1$. Then we can map Eq. (9.43) to the equation of a hyperbola (see Fig. 9.7),

$$\frac{(x-x_0)^2}{A^2} - \frac{(y-y_0)^2}{B^2} = 1 \tag{9.47}$$

with

$$x_0 = \frac{ep}{e^2-1}, \quad y_0 = 0$$

and

$$A = \frac{p}{e^2 - 1} \tag{9.48}$$

$$B = \frac{p}{\sqrt{e^2 - 1}} = \sqrt{e^2 - 1}A \tag{9.49}$$

We leave the investigation of the special case $e = 1$, i.e. $E = 0$, as an exercise.

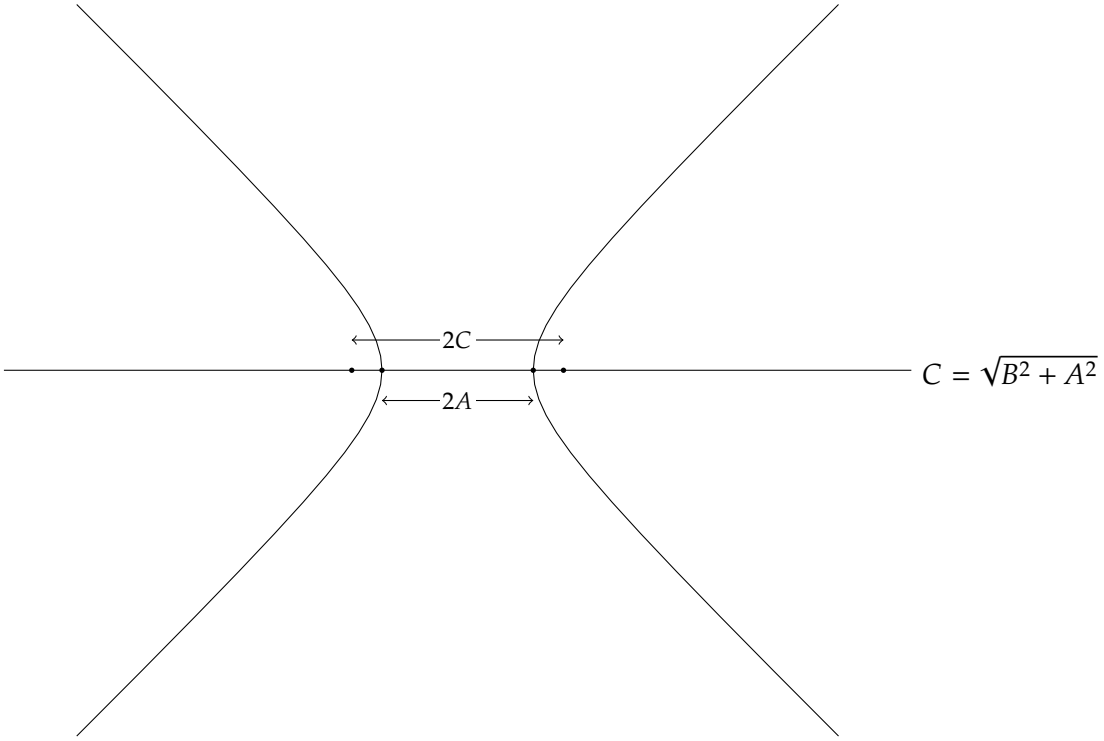


Figure 9.7. A hyperbolic trajectory.

9.2 Laplace–Runge–Lenz vector

Consider the Lagrangian of a particle in a central potential $-\frac{a}{r}$,

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + \frac{a}{r}. \tag{9.50}$$

The Euler–Lagrange equations yield Newton's law,

$$m\ddot{\vec{r}} = -a\frac{\vec{r}}{r^3}. \tag{9.51}$$

Now let us consider the cross product of the momentum and the angular momentum,

$$\vec{P} \times \vec{M}.$$

Differentiating with respect to time, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{P}\times\overrightarrow{M}) &= \frac{\mathrm{d}\vec{P}}{\mathrm{d}t}\times\overrightarrow{M}\\ &= -\frac{a}{r^3}(\vec{r}\times\overrightarrow{M})\\ &= -\frac{a}{r^3}\left[\vec{r}\times(\vec{r}\times\vec{P})\right]\\ &= -\frac{ma}{r^3}\left[\vec{r}\times(\vec{r}\times\dot{\vec{r}})\right]. \end{aligned} \tag{9.52}$$

Exercise 9.2. Prove the identity

$$\vec{r}\times(\vec{r}\times\dot{\vec{r}}) = \vec{r}(\vec{r}\cdot\dot{\vec{r}}) - r^2\dot{\vec{r}},$$

and use

$$2r\dot{r} = \frac{\mathrm{d}r^2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\dot{\vec{r}} \rightsquigarrow \vec{r}\cdot\dot{\vec{r}} = r\dot{r}.$$

Thus,

$$\vec{r}\times(\vec{r}\times\dot{\vec{r}}) = r^3\left(\frac{\vec{r}\,\dot{r}}{r^2} - \frac{\dot{\vec{r}}}{r}\right) = -r^3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{r}}{r}\right).$$

Returning to Eq. (9.52), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}\times\overrightarrow{M} &= ma\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vec{r}}{r}\right)\\ \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{P}\times\overrightarrow{M} - am\frac{\vec{r}}{r}\right] &= 0.\end{aligned}$$

Thus, the vector

$$\vec{A} \equiv \vec{P}\times\overrightarrow{M} - \frac{\vec{r}}{r}ma = \text{const.} \tag{9.53}$$

is conserved. This vector is known as the *Laplace–Runge–Lenz (LRL) vector*.

It is easy to verify that the LRL vector is perpendicular to the angular momentum,

$$\vec{A}\cdot\overrightarrow{M} = 0, \tag{9.54}$$

and, thus, it must lie in the plane of the motion. We will see below that the LRL vector is conserved due to a special symmetry, which is owed to the special ($\frac{1}{r}$) form of the potential.

Exercise 9.3. Find the size and the orientation of the LRL vector for

(i) an elliptical orbit

(ii) a hyperbolic orbit.

It is important to note that any variation of the LRL vector indicates a possible deviation from Newton's law of gravity. Thus, measurements of $\vec{A}(t)$ as a function of time are stringent tests of Newton's law. Such deviations are expected, for example, in Einstein's theory of General Relativity. Strictly speaking, $\dot{\vec{A}} \neq 0$ signals any deviation of the force from a pure $1/r^2$ central force acting on an isolated two-body system, not only a deviation from Newton's law. For Mercury the observed perihelion advance (about 575'' per century)

is dominated by Newtonian perturbations from the other planets (about 532″); only the residual 43″ is due to General Relativity (Goldstein, 3rd ed., Sec. 3.9).

What is the symmetry giving rise to the conservation of the LRL vector? It is the transformation

$$r_i \longrightarrow r_i + \delta_k r_i,$$

with

$$\delta_k r_i = \epsilon\mu \left(\dot{r}_i r_k - \frac{1}{2} r_i \dot{r}_k - \delta_{ik} \left(\frac{1}{2} \vec{r} \cdot \dot{\vec{r}} \right) \right) \tag{9.55}$$

where we change all three r_i simultaneously. From Eq. (9.55), we obtain

$$\delta_k \dot{r}_i = \frac{\epsilon\mu}{2} \left[\dot{r}_i \dot{r}_k - \delta_{ik} \dot{r}^2 - \frac{a}{\mu} \frac{r_i r_k}{r^3} + \delta_{ik} \frac{a}{\mu} \frac{1}{r} \right]. \tag{9.56}$$

The Lagrangian changes by

$$\delta L = \mu \dot{\vec{r}} \cdot \delta \dot{\vec{r}} - a\vec{r} \cdot \frac{\delta \vec{r}}{r^3}$$
$$\overset{(9.55),(9.56)}{\Rightarrow} \delta_k L = \epsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\mu a \frac{r_k}{r} \right).$$

Then we can use Noether's theorem:

$$\frac{\partial L}{\partial \dot{r}_i} \delta_k r_i - F = \text{const.}$$
$$\Rightarrow \vec{A} = \vec{P} \times \vec{M} - \mu a \frac{\vec{r}}{r} = \text{const.}$$

Remark: Eq. (9.56) is obtained using the equation of motion $\mu \ddot{\vec{r}} = -a\vec{r}/r^3$, so $\delta_k L = \epsilon \frac{\mathrm{d}}{\mathrm{d}t}(\mu a r_k / r)$ holds only on-shell. Off-shell (as Noether's theorem requires) one finds $\delta_k L = \epsilon \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2}\mu a \frac{r_k}{r} + \frac{1}{2}\mu^2 \left(\dot{\vec{r}}^2 r_k - (\vec{r} \cdot \dot{\vec{r}}) \dot{r}_k \right) \right]$, and the Noether charge is then $\frac{1}{2}A_k$: the same conserved vector (cf. Goldstein, 3rd ed., Ch. 9, on the symmetry of the Kepler problem).

Exercise 9.4. Fill in the gaps of the above derivation.

9.3 Scattering

Consider a particle which scatters off a fixed target at the centre of our coordinate system. We will assume that the interaction of the particle and the target is described by a central potential $V(r)$, and that the particle comes from very far away ($r \to \infty$) and scatters off the target (see Fig. 9.8). The trajectory of the particle is as in Fig. 9.9; it is symmetric about the

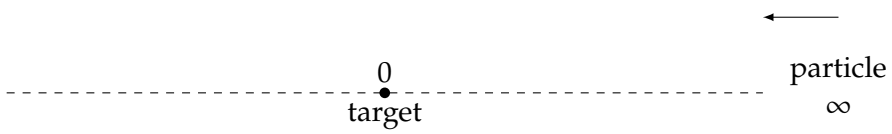


Figure 9.8. A particle coming from very far away towards a target at the centre.

line joining the centre to the point of closest approach. Therefore the deflection angle χ of the particle is

$$\chi = |\pi - 2\phi_0|,$$

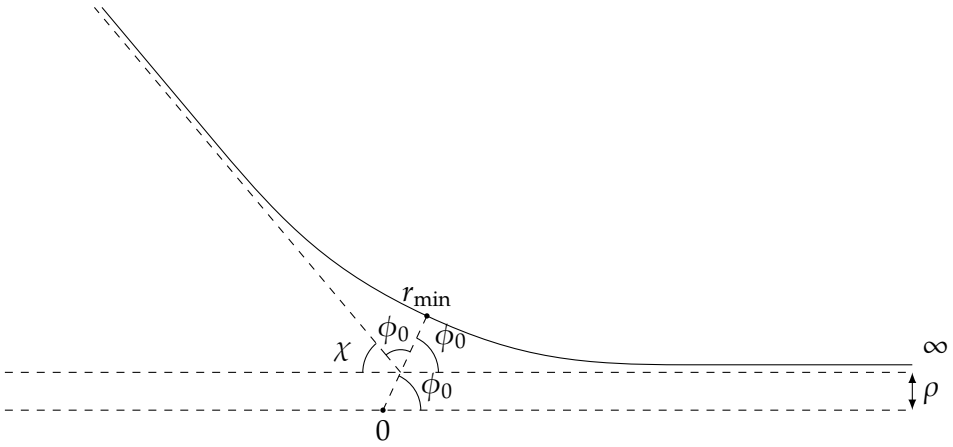


Figure 9.9. The trajectory of the scattered particle.

where

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\frac{M}{r^2}\,\mathrm{d}r}{\sqrt{2m[E - V(r)] - \frac{M^2}{r^2}}}. \tag{9.57}$$

To compute $r_{\min}$, we note that

$$\frac{1}{2}m\dot{r}^2 = E - V(r) - \frac{M^2}{2mr^2}, \tag{9.58}$$

and $\dot{r} = 0$ for $r = r_{\min}$ leads to

$$E - V(r_{\min}) - \frac{M^2}{2mr_{\min}^2} = 0. \tag{9.59}$$

Let us now express the energy and the angular momentum in terms of the velocity of the particle at infinity, v_∞, and the *impact parameter* ρ. We have

$$E = \frac{1}{2}mv_\infty^2 \tag{9.60}$$

$$M = m\rho v_\infty. \tag{9.61}$$

Thus,

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\frac{\rho}{r^2}\,\mathrm{d}r}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2V}{mv_\infty^2}}}. \tag{9.62}$$

Let us now consider a beam of particles with the same velocity at infinity which scatter off the target. The beam particles have different impact parameters; thus, they scatter at different angles. We denote by

$$\mathrm{d}N \equiv \frac{\#\text{ of scattered particles in between } \chi \text{ and } \chi + \mathrm{d}\chi}{\text{unit time}} \tag{9.63}$$

the rate of particles scattered into this range of angles. This rate depends on the flux of particles in the incoming beam. We define the flux as

$$n \equiv \frac{\#\text{ of incoming particles}}{\text{time} \cdot \text{area}}.$$

We can measure experimentally the ratio

$$\mathrm{d}\sigma = \frac{\mathrm{d}N}{n}. \tag{9.64}$$

This ratio is called the *effective scattering cross section*. It has units of area and it characterises the physical properties of the scattering process and the potential. Let us assume that particles with impact parameters between $\rho(\chi)$ and $\rho(\chi) + \mathrm{d}\rho$ scatter at angles between χ and $\chi + \mathrm{d}\chi$ (see Fig. 9.10). Then

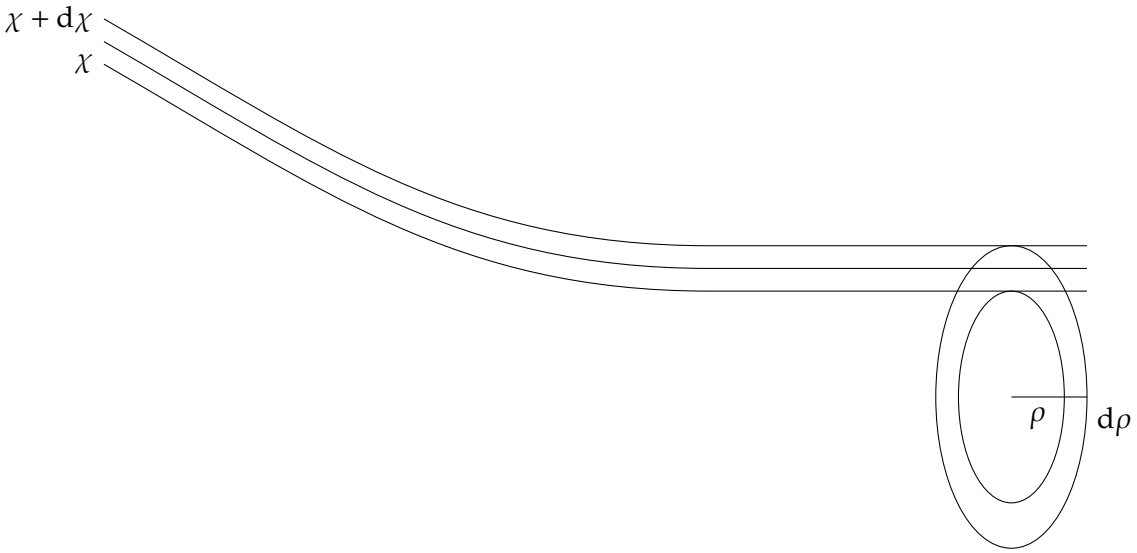


Figure 9.10. Particles with impact parameters between $\rho(\chi)$ and $\rho(\chi) + \mathrm{d}\rho$ scatter at angles between χ and $\chi + \mathrm{d}\chi$.

$$\mathrm{d}N = \underbrace{(2\pi\rho\,\mathrm{d}\rho)}_{\text{area}} \cdot \underbrace{n}_{\frac{\text{particles}}{\text{time·area}}}.$$

The cross section is then

$$\mathrm{d}\sigma = \frac{\mathrm{d}N}{n} = 2\pi\rho\,\mathrm{d}\rho, \tag{9.65}$$

and, changing variables,

$$\begin{aligned}\mathrm{d}\sigma &= 2\pi\rho \left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right| \mathrm{d}\chi \\ &= \underbrace{(2\pi \sin\chi\,\mathrm{d}\chi)}_{\text{solid angle } \mathrm{d}\Omega} \frac{\rho}{\sin\chi}\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right|.\end{aligned}$$

9.3.1 Rutherford's formula

For a potential $V(r) = \frac{a}{r}$, we can compute

$$\phi = \cdots = \arccos \frac{\frac{a}{mv_\infty^2\rho}}{\sqrt{1 + \left(\frac{a}{mv_\infty^2\rho}\right)^2}}$$

Exercise 9.5. Verify this result.

$$\rightsquigarrow \rho^2 = \frac{a^2}{m^2 v_\infty^4} \cot^2 \frac{\chi}{2} \quad \left(\phi_0 = \frac{\pi - \chi}{2}\right).$$

Then $\mathrm{d}\rho^2 = 2\rho\,\mathrm{d}\rho$ is easy to compute, and we obtain Rutherford's formula,

$$\mathrm{d}\sigma = \left(\frac{a}{2mv_\infty^2}\right)^2 \frac{\mathrm{d}\Omega}{\sin^4 \frac{\chi}{2}}.$$

10

Virial theorem

Consider N particles interacting via Newton's law of gravitation. We will show that, if their positions remain bounded, the time averages of the kinetic and potential energies satisfy

$$\langle T \rangle = -\frac{\langle V \rangle}{2}.$$

To this end, consider

$$\begin{aligned} G &= \sum_i \vec{P}_i \cdot \vec{r}_i \\ \rightsquigarrow \frac{\mathrm{d}G}{\mathrm{d}t} &= \sum_i \frac{\mathrm{d}\vec{P}_i}{\mathrm{d}t} \cdot \vec{r}_i + \sum_i \vec{P}_i \cdot \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} \\ &= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T \\ \rightsquigarrow \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{P}_i \cdot \vec{r}_i &= 2T + \sum_i \vec{F}_i \cdot \vec{r}_i. \end{aligned}$$

The time average of a function $F(t)$ over an interval τ is

$$\langle F \rangle = \frac{1}{\tau} \int_0^\tau F(t)\,\mathrm{d}t.$$

Thus,

$$\begin{aligned} &\frac{1}{\tau} \int_0^\tau \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{P}_i \cdot \vec{r}_i = 2\langle T \rangle + \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle \\ &\Rightarrow 2\langle T \rangle + \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle = \frac{G(\tau) - G(0)}{\tau}. \end{aligned}$$

Assume that the motion is bounded (in positions and momenta). Then $G(\tau)$ is bounded as well. If we take $\tau \to \infty$, we have

$$\frac{G(\tau) - G(0)}{\tau} \xrightarrow{\tau \to \infty} 0.$$

Then

$$\boxed{\langle T \rangle = -\frac{1}{2} \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle}.$$

This is known as the *virial theorem*.

For a single particle in a potential $V(\vec{r})$,

$$\begin{aligned}\langle T\rangle &= -\frac{1}{2}\left\langle \sum_{i=1}^{1} \vec{F}_i \cdot \vec{r}_i \right\rangle \\ &= -\frac{1}{2}\left\langle (-\vec{\nabla}V)\cdot\vec{r}\right\rangle .\end{aligned}$$

For a central potential, $V = V(r)$, we get

$$\langle T\rangle = \frac{1}{2}\left\langle \frac{\partial V}{\partial r} r\right\rangle .$$

If $V = ar^n$, we have $r\dfrac{\partial V}{\partial r} = nar^n = nV$, thus

$$\langle T\rangle = \frac{n}{2}\langle V\rangle .$$

For a gravitational or electrostatic potential, we have

$$n = -1 \rightsquigarrow \boxed{\langle T\rangle = -\frac{\langle V\rangle}{2}}.$$

For many particles interacting through a pairwise $\frac{1}{r}$ potential,

$$V = \frac{1}{2}\sum_{k\neq l}\frac{a}{|\vec{r}_k - \vec{r}_l|}$$

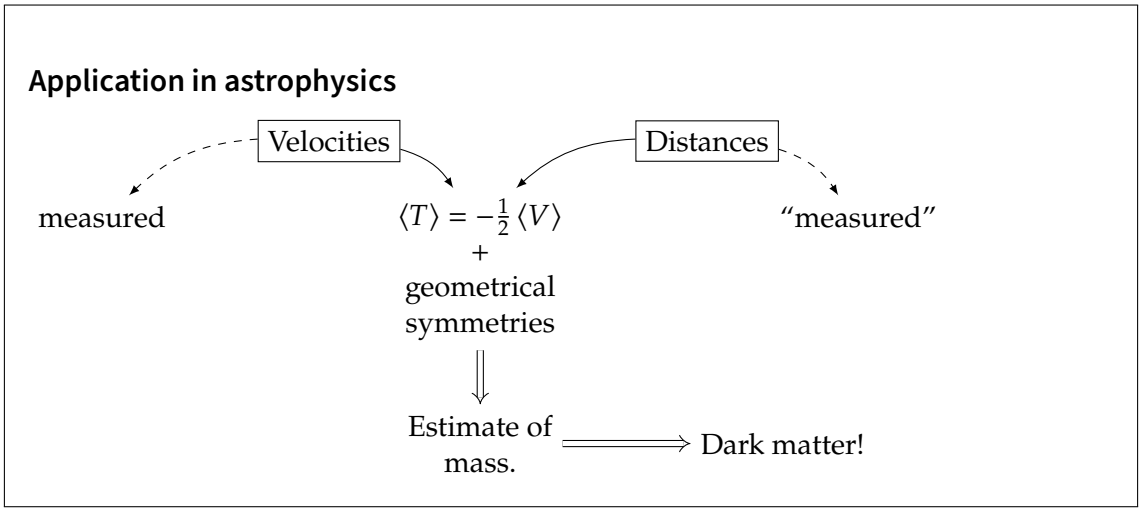
(with $a < 0$ for gravity), we have

$$\begin{aligned}-\sum_i (\vec{\nabla}_i V)\cdot\vec{r}_i &= -\sum_i \vec{r}_i \cdot \vec{\nabla}_i \frac{1}{2}\sum_{k\neq l}\frac{a}{|\vec{r}_k - \vec{r}_l|} \\ &= +2\frac{1}{2}\sum_i \vec{r}_i \cdot \sum_{l\neq i}\frac{a(\vec{r}_i - \vec{r}_l)}{|\vec{r}_i - \vec{r}_l|^3} \\ &= +\sum_{l<i}\frac{a(\vec{r}_i - \vec{r}_l)\cdot\vec{r}_i}{|\vec{r}_i - \vec{r}_l|^3} + \sum_{i<l}\frac{a(\vec{r}_i - \vec{r}_l)\cdot\vec{r}_i}{|\vec{r}_i - \vec{r}_l|^3} \\ &= +\sum_{l<i}\frac{a(\vec{r}_i - \vec{r}_l)\cdot\vec{r}_i}{|\vec{r}_i - \vec{r}_l|^3} + \sum_{l<i}\frac{a(\vec{r}_l - \vec{r}_i)\cdot\vec{r}_l}{|\vec{r}_i - \vec{r}_l|^3} \\ &= +\sum_{l<i}\frac{a\left[(\vec{r}_i - \vec{r}_l)\cdot\vec{r}_i - (\vec{r}_i - \vec{r}_l)\cdot\vec{r}_l\right]}{|\vec{r}_i - \vec{r}_l|^3} \\ &= +\sum_{l<i}\frac{a(\vec{r}_i - \vec{r}_l)^2}{|\vec{r}_i - \vec{r}_l|^3} \\ &= +\sum_{l<i}\frac{a}{|\vec{r}_i - \vec{r}_l|} = V.\end{aligned}$$

Thus we have proven that

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle .$$

The following diagram summarises how the virial theorem is used in astrophysics: measured velocities and distances of, e.g., the galaxies in a cluster yield an estimate of its mass, which turns out to be much larger than the visible mass.



11

Oscillations

11.1 Small oscillations

Oscillations can occur in potentials with points of stable equilibrium. For systems with one degree of freedom, this means that for a potential $V(q)$ there exists a q_0 with

$$V'(q_0) = 0 \quad \text{and} \quad V''(q_0) > 0.$$

The force then always pushes the particle back towards q_0. The Taylor expansion of the potential around q_0 is

$$V(q) = V(q_0) + \cancelto{0}{V'(q_0)}(q - q_0) + \frac{1}{2}V''(q_0)(q - q_0)^2 + O\left[(q - q_0)^3\right].$$

If we set $V(q_0) = 0$, which amounts to shifting the potential by a constant, we find

$$V(q) \approx \frac{1}{2}k(q - q_0)^2 \quad \text{with} \quad k \equiv V''(q_0) > 0.$$

For a single particle, with $x \equiv q - q_0$ and q a Cartesian coordinate, we have

$$L \approx \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

which gives us the following Euler–Lagrange equation:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \\ &\rightsquigarrow m\ddot{x} + kx = 0 \\ &\rightsquigarrow \ddot{x} + \frac{k}{m}x = 0 \\ &\boxed{\ddot{x} + \omega^2 x = 0} \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}. \end{aligned}$$

The general solution is

$$x = c_1 \cos \omega t + c_2 \sin \omega t.$$

Indeed,

$$\begin{aligned} \dot{x} &= \omega\left[-c_1 \sin \omega t + c_2 \cos \omega t\right] \\ \ddot{x} &= \omega^2\left[-c_1 \cos \omega t - c_2 \sin \omega t\right] = -\omega^2 x \quad \checkmark. \end{aligned}$$

We can rewrite this as

$$x = \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos\omega t + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin\omega t\right].$$

Set

$$\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \equiv \cos\theta_0 \qquad -\frac{c_2}{\sqrt{c_1^2 + c_2^2}} \equiv \sin\theta_0 \qquad A \equiv \sqrt{c_1^2 + c_2^2}$$

to obtain

$$x = A\left(\cos\theta_0\cos\omega t - \sin\theta_0\sin\omega t\right)$$
$$\rightsquigarrow \boxed{x = A\cos(\omega t + \theta_0)}.$$

The motion is shown in Fig. 11.1. The frequency depends only on the characteristics of

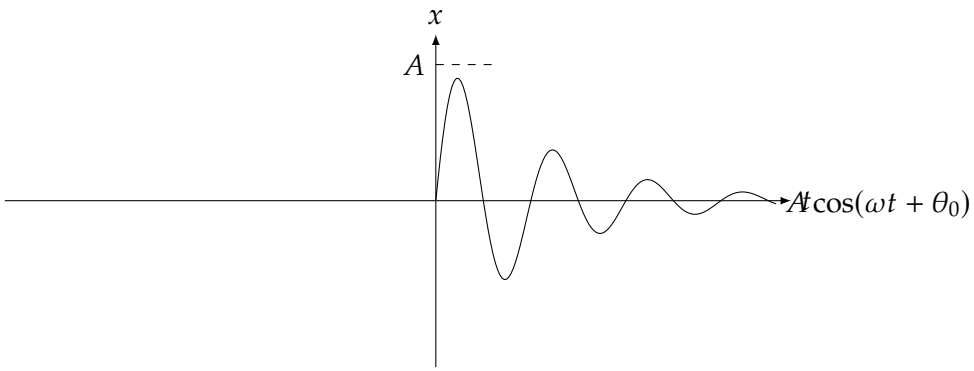


Figure 11.1. Harmonic oscillation with amplitude A.

the system (m, k) and not on the initial conditions (for small oscillations only).

The energy of the system is

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\left[-A\omega\sin(\omega t + \theta_0)\right]^2 + \frac{m}{2}\omega^2\left[A\cos(\omega t + \theta_0)\right]^2 \\ \rightsquigarrow &\boxed{E = \frac{1}{2}m\omega^2A^2}. \end{aligned}$$

Thus,

$$x = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t + \theta_0).$$

The amplitude A of the oscillation depends on the energy, as expected.

It is often convenient to "complexify" the amplitude:

$$A = |A|e^{i\theta_0}.$$

Then,

$$x = \mathrm{Re}\left(Ae^{i\omega t}\right).$$

Indeed,

$$\begin{aligned} x &= \mathrm{Re}\left(|A|\mathrm{e}^{\mathrm{i}(\omega t+\theta_0)}\right) \\ &= |A|\cos(\omega t+\theta_0). \end{aligned}$$

11.2 Forced oscillations

Let us now exert an external time-dependent force on an oscillatory system. Then

$$\begin{aligned} V &\approx \frac{1}{2}kx^2 + V_{\mathrm{ex}}(x,t) \\ &\approx \frac{1}{2}kx^2 + x\left[\frac{\partial V_{\mathrm{ex}}}{\partial x}\right]_{x=0} + \underbrace{V_{\mathrm{ex}}(0,t)}_{\substack{\text{can be written} \\ \text{as a time derivative}}}, \end{aligned}$$

where the last term can be dropped from the Lagrangian, since a function of time alone is the total time derivative of its integral. Thus,

$$\begin{aligned} V(x,t) \approx \frac{1}{2}kx^2 - xF(t), \qquad F(t) &\equiv -\left[\frac{\partial V_{\mathrm{ex}}}{\partial x}\right]_{x=0}, \\ L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t) & \end{aligned} \tag{11.1}$$

which yields the following equation of motion:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \rightsquigarrow m\ddot{x} &= -kx + F(t) \\ \rightsquigarrow \boxed{\ddot{x} + \omega^2 x = \frac{F(t)}{m}}. & \end{aligned} \tag{11.2}$$

To find the general solution, we set

$$\begin{aligned} \zeta &= \dot{x} + \mathrm{i}\omega x \\ \dot{\zeta} &= \ddot{x} + \mathrm{i}\omega\dot{x} \\ -\mathrm{i}\omega\zeta &= -\mathrm{i}\omega\dot{x} + \omega^2 x \end{aligned} \tag{11.3}$$

which gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta - \mathrm{i}\omega\zeta = \frac{F(t)}{m}. \tag{11.4}$$

Setting now $\zeta = \mathrm{e}^{\mathrm{i}\omega t}y$, we find

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\omega t}\dot{y} + \mathrm{i}\omega\zeta - \mathrm{i}\omega\zeta &= \frac{F(t)}{m} \\ \Rightarrow \dot{y} &= \frac{\mathrm{e}^{-\mathrm{i}\omega t}F(t)}{m} \\ \Rightarrow y &= \int \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{F(t)}{m} + y_0 \\ \Rightarrow \boxed{\zeta = \mathrm{e}^{\mathrm{i}\omega t}\left[y_0 + \int \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\frac{F(t)}{m}\right]}. & \end{aligned} \tag{11.5}$$

The energy of the system is

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m|\zeta|^2 \\ &= \frac{1}{2}m(\dot{x} - \mathrm{i}\omega x)(\dot{x} + \mathrm{i}\omega x) \\ \Rightarrow &\boxed{E = \frac{1}{2}m\,|\zeta(t)|^2}. \end{aligned} \tag{11.6}$$

The energy is not conserved. Assuming that all the energy is "pumped" into the system by $F(t)$, we have for the energy of the system after infinite time

$$\begin{aligned} E &= \frac{1}{2}m\,|\zeta(+\infty)|^2 \\ &= \frac{1}{2}m\left|\int_{-\infty}^{\infty} \mathrm{d}t\,\frac{F(t)}{m}\mathrm{e}^{-\mathrm{i}\omega t} + y_0\right|^2, \end{aligned} \tag{11.7}$$

where the integral can be recognised as the Fourier transform of the force.

Let us now work out explicitly the trajectory for

$$F(t) = f\cos(\gamma t + \beta). \tag{11.8}$$

Then

$$\begin{aligned} \zeta(t) &= \mathrm{e}^{+\mathrm{i}\omega t}\left[\int \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\cos(\gamma t + \beta)\frac{f}{m} + y_0\right] \\ \Rightarrow \zeta(t) &= \mathrm{e}^{\mathrm{i}\omega t}y_0 + \frac{f\gamma}{m(\gamma^2 - \omega^2)}\sin(\gamma t + \beta) - \mathrm{i}\frac{f\omega}{m(\gamma^2 - \omega^2)}\cos(\gamma t + \beta). \end{aligned} \tag{11.9}$$

But

$$\begin{aligned} \zeta(t) &= \dot{x} + \mathrm{i}\omega x \\ \Rightarrow \mathrm{e}^{+\mathrm{i}\omega t}\zeta(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mathrm{i}\omega t}x) \\ \Rightarrow x &= \mathrm{e}^{-\mathrm{i}\omega t}\left(c + \int \mathrm{d}t\,\mathrm{e}^{+\mathrm{i}\omega t}\zeta(t)\right). \end{aligned} \tag{11.10}$$

Performing the integrations, we have

$$x(t) = -\frac{f/m}{\gamma^2 - \omega^2}\cos(\gamma t + \beta) + A\mathrm{e}^{-\mathrm{i}\omega t}.$$

The real part of the solution is

$$\boxed{x(t) = |A|\cos(\omega t + \theta_0) + \frac{f/m}{\omega^2 - \gamma^2}\cos(\gamma t + \beta)}. \tag{11.11}$$

The first term is the solution of the homogeneous equation; it is the same as in the absence of the external force. The second term is new. It is also an oscillatory function of time, but with the frequency of the external force.

The amplitude of the oscillation increases as the frequency of the external force γ approaches the value of the characteristic frequency of the system ω. This phenomenon is called *resonance*. To calculate the motion at exactly $\omega = \gamma$, we write (with a redefined constant B)

$$x(t) = |B|\cos(\omega t + \theta_0) + \frac{f}{m(\omega^2 - \gamma^2)}\left[\cos(\gamma t + \beta) - \cos(\omega t + \beta)\right]. \tag{11.12}$$

At $\gamma \to \omega$, we have

$$x(t) = |B|\cos(\omega t + \theta_0) + \frac{f}{2m\omega} t \sin(\omega t + \beta). \tag{11.13}$$

Notice that the amplitude increases linearly with time. Since the Lagrangian of Eq. (11.1) is only valid for small x, the above formula ceases to be valid after some time, when the amplitude of the oscillation has become large.

Let us now complexify Eq. (11.11):

$$\begin{aligned} x &= A_1 e^{i\omega t} + A_2 e^{i\gamma t} \\ &= \left[A_1 + A_2 e^{i(\gamma-\omega)t}\right] e^{i\omega t}. \end{aligned} \tag{11.14}$$

For $\gamma = \omega + \epsilon$, close to the resonance, with ϵ small, we can write

$$\begin{aligned} x(t) &= \left[A_1 + A_2 e^{i\epsilon t}\right] e^{i\omega t} \\ &= C(t) e^{i\omega t}, \end{aligned} \tag{11.15}$$

where the "amplitude" $C(t)$ varies slowly in time. Setting

$$A_1 = |A_1| e^{i\theta_1} \quad \text{and} \quad A_2 = |A_2| e^{i\theta_2},$$

we obtain

$$\begin{aligned} |C(t)|^2 &= |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\theta_2 - \theta_1 + \epsilon t) \\ \Rightarrow (|A_1| - |A_2|)^2 &\le |C(t)|^2 \le (|A_1| + |A_2|)^2 \\ \Rightarrow ||A_1| - |A_2|| &\le |C(t)| \le |A_1| + |A_2|. \end{aligned} \tag{11.16}$$

The amplitude varies periodically between two values. This phenomenon is known as *beats*.

11.3 Oscillations of systems with many degrees of freedom

We will now study oscillatory behaviour in systems with many degrees of freedom. A Lagrangian describing such a system is of the form

$$L = \frac{1}{2}\sum_{i,j}(m_{ij}\dot{x}_i\dot{x}_j - k_{ij}x_i x_j), \tag{11.17}$$

where

$$m_{ij} = m_{ji} \tag{11.18}$$

$$k_{ij} = k_{ji}. \tag{11.19}$$

To compute the equations of motion, we first need

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_j m_{ij}\dot{x}_j,$$
$$\frac{\partial L}{\partial x_i} = -\sum_j k_{ij}x_j.$$

The Euler–Lagrange equations then give

$$\sum_j (m_{ij}\ddot{x}_j + k_{ij}x_j) = 0, \quad i = 1, \ldots, N. \tag{11.20}$$

This is a system of N differential equations for N unknowns. We can set

$$x_j = A_j e^{i\omega t} \tag{11.21}$$

(and only need to remember to keep the real part of this expression at the end of the calculation). Substituting into Eq. (11.20), we obtain

$$\sum_j (k_{ij} - \omega^2 m_{ij})A_j = 0 \tag{11.22}$$

or, in matrix form,

$$\begin{pmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & \ldots \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{11.23}$$

The condition for the system to have non-trivial solutions of the form of Eq. (11.21) is then

$$\det(k_{ij} - \omega^2 m_{ij}) = 0. \tag{11.24}$$

The solutions ω^2 of Eq. (11.24) are real and non-negative. Indeed,

$$\sum_j (k_{ij} - m_{ij}\omega^2)A_j = 0$$
$$\rightsquigarrow \sum_{i,j} (k_{ij} - m_{ij}\omega^2)A_i^\star A_j = 0$$
$$\Rightarrow \omega^2 = \frac{\sum_{i,j} k_{ij}A_i^\star A_j}{\sum_{i,j} m_{ij}A_i^\star A_j},$$

and, using the symmetry of k_{ij} and m_{ij},

$$\omega^2 = \frac{\sum_{i,j} k_{ji}A_i^\star A_j}{\sum_{i,j} m_{ji}A_i^\star A_j} = \frac{\sum_{i,j} k_{ij}A_i A_j^\star}{\sum_{i,j} m_{ij}A_i A_j^\star} = (\omega^2)^\star \qquad \rightsquigarrow \omega^2 \text{ is real.}$$

Notice also that, writing $A_j = a_j + ib_j$,

$$\sum_{i,j} k_{ij}A_i A_j^\star = \sum_{i,j} k_{ij}(a_i + ib_i)(a_j - ib_j)$$
$$= \sum_{i,j} k_{ij}(a_i a_j + b_i b_j) + i\sum_{i,j} \underbrace{k_{ij}}_{\text{symmetric}} \cancelto{0}{(b_i a_j - a_i b_j)}$$
$$= \sum_{i,j} k_{ij}(a_i a_j + b_i b_j) \geq 0.$$

The last inequality is not automatic: it holds because k_{ij} is positive (semi-)definite at a stable minimum of the potential, and likewise $\sum m_{ij} A_i^\star A_j > 0$ because the kinetic energy is positive definite. Only with both properties is ω^2 real and non-negative (Landau–Lifshitz Vol. 1, Sec. 23; Goldstein, 3rd ed., Sec. 6.2). The roots ω_a^2 of Eq. (11.24) give the characteristic frequencies (eigenfrequencies) of the system. Constructing the solutions $x_j(t)$ after this step is a simple exercise in linear algebra: we need to find the eigenvectors corresponding to each ω_a. Let us assume, for simplicity, that there is no degeneracy. Then the solutions are

$$x_j(t) = \sum_a \Delta_{ja}\,\theta_a(t), \tag{11.25}$$

with Δ_{ja} the minors of the determinant (evaluated at $\omega = \omega_a$) and

$$\theta_a(t) = \mathrm{Re}\left[C_a \exp(\mathrm{i}\omega_a t)\right]. \tag{11.26}$$

We can regard the equation

$$x_j = \sum_a \Delta_{ja}\,\theta_a$$

as a change from one set of generalised coordinates to another. The coordinates θ_a satisfy the differential equations

$$\ddot{\theta}_a + \omega_a^2 \theta_a = 0 \tag{11.27}$$

and are called *normal coordinates*. They describe independent oscillators. In normal coordinates, the Lagrangian takes the form

$$L = \sum_a \frac{1}{2} m_a (\dot{\theta}_a^2 - \omega_a^2 \theta_a^2),$$

with suitable constants $m_a > 0$.

Exercise 11.1. Consider the Lagrangian

$$L = \frac{1}{2}\left[\dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2 - 3x_1^2 - 4x_2^2 + 2x_1x_2\right].$$

Find the eigenfrequencies and normal coordinates.

Remark: as written, the kinetic matrix $m_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is singular, since the kinetic term is $\frac{1}{2}(\dot{x}_1 + \dot{x}_2)^2$. Then $\det(k - \omega^2 m) = 11 - 9\omega^2$ is only linear in ω^2: there is a single frequency $\omega^2 = 11/9$ for $u = x_1 + x_2$, while $x_1 - x_2$ carries no kinetic energy and is fixed by the constraint $4x_1 = 5x_2$. A standard version of the exercise needs a positive-definite m_{ij}, e.g. a coefficient smaller than 2 in front of $\dot{x}_1\dot{x}_2$.

11.4 Damped oscillations

We now consider oscillatory systems with friction losses. The equation of motion for the simplest one-dimensional oscillator is

$$\boxed{m\ddot{x} + kx + \alpha\dot{x} = 0}$$

or, equivalently,

$$\ddot{x} + \omega_0^2 x + 2\lambda\dot{x} = 0$$

with

$$\omega_0^2 = \frac{k}{m} \quad \text{and} \quad 2\lambda = \frac{\alpha}{m}.$$

We look for solutions of the form

$$x(t) = \mathrm{e}^{kt};$$

then

$$\begin{aligned} k^2 + \omega_0^2 + 2\lambda k &= 0 \\ \rightsquigarrow (k+\lambda)^2 &= \lambda^2 - \omega_0^2 \\ \Rightarrow k &= -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}. \end{aligned}$$

Case 1: $\lambda < \omega_0$.

Then we have two complex solutions,

$$k_\pm = -\lambda \pm \mathrm{i}\sqrt{\omega_0^2 - \lambda^2}$$

and the general solution is (see Fig. 11.2)

$$\begin{aligned} x &= \mathrm{Re}\left[A\mathrm{e}^{-\lambda t + \mathrm{i}\sqrt{\omega_0^2-\lambda^2}t}\right] \\ &= \underbrace{|A|\mathrm{e}^{-\lambda t}}_{C(t)} \cos(\omega t + \theta_0), \quad \omega \equiv \sqrt{\omega_0^2 - \lambda^2}. \end{aligned}$$

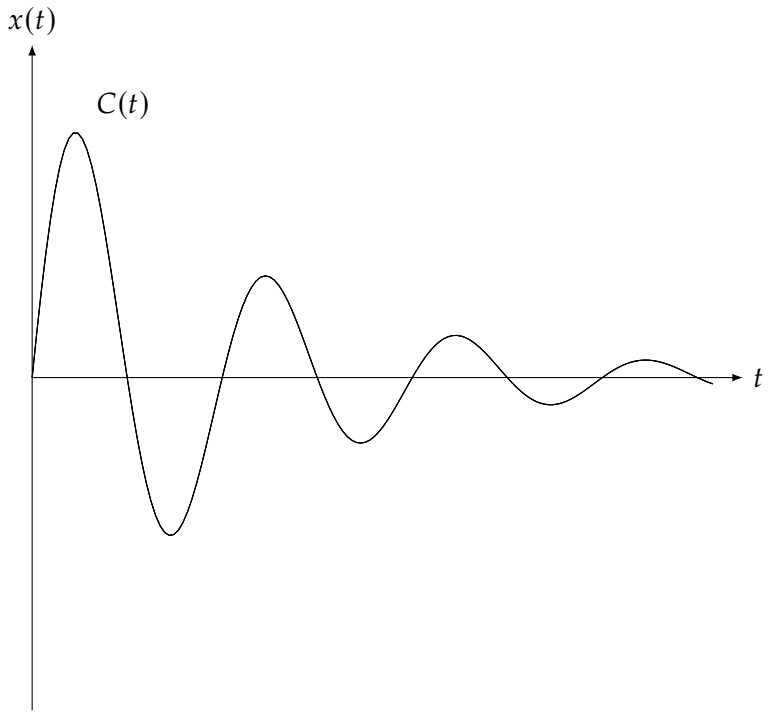


Figure 11.2. Damped oscillation with time-varying amplitude $C(t)$.

Case 2: $\lambda > \omega_0$.

We have two real solutions and no oscillation; this is called *aperiodic damping*:

$$x = \mathrm{e}^{-\lambda t}\left[c_1 \mathrm{e}^{-\sqrt{\lambda^2-\omega_0^2}t} + c_2 \mathrm{e}^{+\sqrt{\lambda^2-\omega_0^2}t}\right].$$

For many degrees of freedom, the frictional forces take the form

$$f_i = -\sum_j \alpha_{ij}\dot{x}_j.$$

Then, we can write

$$f_i = -\frac{\partial F}{\partial \dot{x}_i} \quad \text{with} \quad F = \frac{1}{2}\sum_{i,j}\alpha_{ij}\dot{x}_i\dot{x}_j.$$

The equations of motion can then be written as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = -\frac{\partial F}{\partial \dot{x}_i}}.$$

The function F is known as the *dissipative function*. Let us now compute the rate of change of the energy (with summation over i implied):

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - L\right)\\
&= \ddot{x}_i\frac{\partial L}{\partial \dot{x}_i} - \frac{\mathrm{d}L}{\mathrm{d}t} + \dot{x}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i}\\
&= \ddot{x}_i\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i}\dot{x}_i - \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i + \dot{x}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i}\\
&= \dot{x}_i\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i}\right]\\
&= -\dot{x}_i\frac{\partial F}{\partial \dot{x}_i}\\
&= -2F.
\end{aligned}$$

For the system to lose energy, we require that

$$F > 0.$$

12

Rigid body

So far, we have studied the motion of point-like particles. We now move on to study the motion of macroscopic objects which do not change shape during their motion. We call these objects *rigid bodies*. A rigid body is a special case of a system of many point-like particles, with the constraint that the distances among the constituent particles are fixed.

Consider a reference point O on the rigid body. We can determine the position of every other point A of the rigid body with respect to the reference point O with a vector (see Fig. 12.1)

$$\overrightarrow{OA} = \vec{l}_a.$$

The magnitude of $\vec{l}_a$ is fixed, since it connects two points of the rigid body. As the object

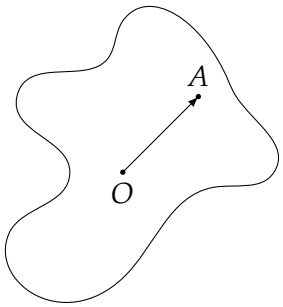


Figure 12.1. Rigid body with origin O and point A.

moves, only the orientation of $\overrightarrow{OA}$ can change, while the length of the vector stays fixed. The motion of $\vec{l}_a$ is therefore a rotation. The reference point O is of course also moving. The position of the point A in the body with respect to a fixed point F outside the body is

$$\overrightarrow{FA} = \overrightarrow{FO} + \overrightarrow{OA}$$

or, equivalently (see Fig. 12.2),

$$\vec{r}_a(t) = \vec{R}(t) + \vec{l}_a(t). \tag{12.1}$$

In a small time interval $\mathrm{d}t$, the displacement of the point A is

$$\mathrm{d}\vec{r}_a = \mathrm{d}\vec{R} + \mathrm{d}\vec{l}_a. \tag{12.2}$$

We can write the change $\mathrm{d}\vec{l}_a$ as an infinitesimal rotation,

$$\mathrm{d}\vec{l}_a = \mathrm{d}\vec{\phi} \times \vec{l}_a. \tag{12.3}$$

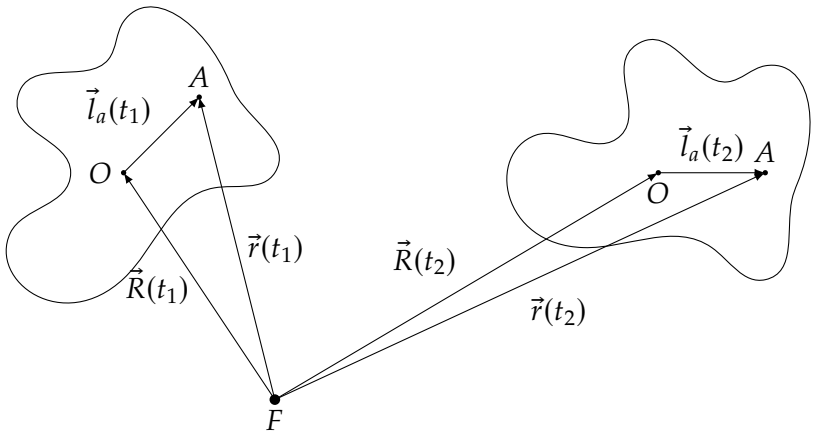


Figure 12.2. The same body with a different orientation.

Thus, we have that

$$\boxed{\frac{\mathrm{d}\vec{r}_a}{\mathrm{d}t} = \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} + \frac{\mathrm{d}\vec{\phi}}{\mathrm{d}t} \times \vec{l}_a}. \tag{12.4}$$

This equation relates the velocity of a particle A in the rigid body to the velocity of the reference point O and the angular velocity

$$\vec{\omega} = \frac{\mathrm{d}\vec{\phi}}{\mathrm{d}t} \tag{12.5}$$

of the point A with respect to the point O. All points in a rigid body rotate together, as they are required to keep fixed distances. Therefore, the angular velocity $\vec{\omega}$ is common to all particles.

Let us call

$$\vec{V} \equiv \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} \tag{12.6}$$

the velocity of the reference point O. We will assume that a body is composed of $a = 1, \ldots, N$ particles with masses m_a. The kinetic energy of the body is then

$$\begin{aligned} T &= \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a^2 \\ &= \sum_a \frac{1}{2} m_a \left(\vec{V} + \vec{\omega} \times \vec{l}_a\right)^2 \\ &= \frac{1}{2}\left(\sum_a m_a\right) V^2 + \frac{1}{2}\sum_a m_a \left(\vec{\omega} \times \vec{l}_a\right)^2 + \vec{V} \cdot \left(\vec{\omega} \times \sum_a m_a \vec{l}_a\right). \end{aligned} \tag{12.7}$$

We will now choose our reference point O to be the centre of mass,

$$O : \sum_a m_a \vec{l}_a = 0. \tag{12.8}$$

Then the last term of Eq. (12.7) vanishes, and the kinetic energy is

$$T = \frac{1}{2} m V^2 + \frac{1}{2}\sum_a m_a \left(\vec{\omega} \times \vec{l}_a\right)^2, \tag{12.9}$$

with

$$m \equiv \sum_a m_a. \tag{12.10}$$

We note that the centre-of-mass frame is not necessarily an inertial frame of reference. We will now prove the identity

$$\left(\vec{a} \times \vec{b}\right)^2 = \vec{a}^2\vec{b}^2 - \left(\vec{a} \cdot \vec{b}\right)^2. \tag{12.11}$$

Proof.

$$\begin{aligned}
\left(\vec{a} \times \vec{b}\right)^2 &= \left(\epsilon_{ijk} a_j b_k\right)^2 \\
&= \left(\epsilon_{ijk} a_j b_k\right)\left(\epsilon_{ilm} a_l b_m\right) \\
&= \epsilon_{ijk}\epsilon_{ilm} a_j b_k a_l b_m \\
&= \begin{vmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{vmatrix} a_j b_k a_l b_m \\
&= \left(\delta_{jl}\delta_{km} - \delta_{kl}\delta_{jm}\right) a_j b_k a_l b_m \\
&= a_j^2 b_k^2 - \left(a_j b_j\right)\left(a_k b_k\right) \\
&= \vec{a}^2\vec{b}^2 - \left(\vec{a} \cdot \vec{b}\right)^2.
\end{aligned}$$

□

Therefore, the "rotational" term of the kinetic energy in Eq. (12.9) becomes

$$\frac{1}{2}\sum_a m_a \left(\vec{\omega} \times \vec{l}_a\right)^2 = \frac{1}{2}\sum_a m_a \left(\omega^2 l_a^2 - \left(\vec{\omega} \cdot \vec{l}_a\right)^2\right). \tag{12.12}$$

Let us now write the above in terms of the Cartesian components of the position vectors,

$$\vec{l}_a \equiv (l_{a1}, l_{a2}, l_{a3}) \equiv (l_{ai}).$$

We have

$$\begin{aligned}
\omega^2 l_a^2 - \left(\vec{\omega} \cdot \vec{l}_a\right)^2 &= \left(\omega_i \omega_i\right) l_a^2 - \left(\omega_i l_{ai}\right)^2 \\
&= \omega_i \omega_j \delta_{ij} l_a^2 - \omega_i \omega_j l_{ai} l_{aj} \\
&= \left(\delta_{ij} l_a^2 - l_{ai} l_{aj}\right) \omega_i \omega_j,
\end{aligned} \tag{12.13}$$

with summation over repeated indices implied. Therefore, the kinetic energy takes the form

$$T = \frac{1}{2} m V^2 + \frac{1}{2} I_{ij} \omega_i \omega_j, \tag{12.14}$$

where $m \equiv \sum_a m_a$ is the total mass of the rigid body and I_{ij} is the so-called *tensor of inertia*.

12.1 Tensor of inertia

The tensor of inertia is characteristic of the geometry and mass distribution of the rigid body. From its definition,

$$I_{ij} \equiv \sum_a m_a \left(l_a^2 \delta_{ij} - l_{ai} l_{aj}\right) \tag{12.15}$$

we see that it is a symmetric tensor:

$$I_{jk} = I_{kj}.$$

It therefore has six independent components. In matrix notation, the tensor of inertia is

$$I_{jk} = \sum_a \begin{pmatrix} l_{a2}^2 + l_{a3}^2 & -l_{a1}l_{a2} & -l_{a1}l_{a3} \\ -l_{a1}l_{a2} & l_{a1}^2 + l_{a3}^2 & -l_{a2}l_{a3} \\ -l_{a1}l_{a3} & -l_{a2}l_{a3} & l_{a1}^2 + l_{a2}^2 \end{pmatrix} m_a. \tag{12.16}$$

Using standard linear algebra methods, we can diagonalise this matrix. The diagonalisation corresponds to a change of axes for the vectors $\vec{l}_a$. The axes for which the tensor of inertia is diagonal are called *principal axes*.

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \overset{\text{Rotation}}{\underset{\text{of axes}}{\longrightarrow}} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{12.17}$$

The eigenvalues I_1, I_2, I_3 are called the *principal moments of inertia*.

Exercise 12.1. Demonstrate that principal axes always exist.

We can prove the following inequalities:

$$\begin{aligned} I_1 + I_2 &\geq I_3 \\ I_2 + I_3 &\geq I_1 \\ I_3 + I_1 &\geq I_2. \end{aligned}$$

Indeed,

$$\begin{aligned} I_1 + I_2 &= \sum_a m_a \left(l_{a2}^2 + l_{a3}^2 + l_{a1}^2 + l_{a3}^2 \right) \\ &\geq \sum_a m_a \left(l_{a1}^2 + l_{a2}^2 \right) \\ &= I_3. \end{aligned}$$

In the definition of the tensor of inertia of Eq. (12.16), the position vectors $\vec{l}_a$ of the points of the rigid body are measured with respect to the centre of mass. Let us now consider the analogous expression

$$I'_{ij} \equiv \sum_a m_a \left(r_a^2 \delta_{ij} - r_{ai} r_{aj} \right) \tag{12.18}$$

where the vectors $\vec{r}_a \equiv \vec{l}_a + \vec{c}$ are measured with respect to a different point. Then,

$$\begin{aligned} I'_{ij} &= \sum_a m_a \left(\left(\vec{l}_a + \vec{c} \right)^2 \delta_{ij} - \left(l_{ai} + c_i \right) \left(l_{aj} + c_j \right) \right) \\ &= \sum_a m_a \left(\vec{l}_a^2 \delta_{ij} - l_{ai} l_{aj} \right) + \left(\sum_a m_a \right) \left(\vec{c}^2 \delta_{ij} - c_i c_j \right) \\ &\quad + 2 \left(\sum_a m_a \vec{l}_a \right) \cdot \vec{c} \delta_{ij} - \left(\sum_a m_a l_{ai} \right) c_j - \left(\sum_a m_a l_{aj} \right) c_i. \end{aligned}$$

However, $\sum_a m_a \vec{l}_a = 0$. Therefore, we obtain

$$I'_{ij} = I_{ij} + m\left(\vec{c}^2\delta_{ij} - c_i c_j\right). \tag{12.19}$$

Eq. (12.19) is convenient when the sum over all particles is easier to perform for position vectors with an origin other than the centre of mass.

Finally, we remark that in the case of a continuous solid body, the sum over all particles must be replaced by an integral over a continuous mass density:

$$I_{ij} = \int \mathrm{d}^3\vec{l}\,\rho(\vec{l})\left[\vec{l}^2\delta_{ij} - l_i l_j\right]. \tag{12.20}$$

We can classify rigid objects according to their principal moments of inertia, as in Table 12.1.

$I_1 \neq I_2 \neq I_3$	asymmetric top
$I_1 = I_2 \neq I_3$	symmetric top
$I_1 = I_2 = I_3$	spherical top

Table 12.1. Classification of rigid objects.

12.1.1 Examples of moments of inertia

Example 12.1. Consider a system of particles on a line, as in Fig. 12.3. The tensor of inertia

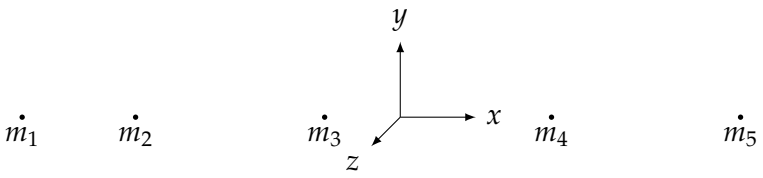


Figure 12.3. A system of particles on a line.

is

$$I_{jk} = \sum_a m_a\left(r_a^2\delta_{jk} - r_{ak}r_{aj}\right),$$

but $r_{ai} = r_a\delta_{ix}$, thus

$$\begin{aligned} I_{jk} &= \sum_a m_a\left(r_a^2\delta_{jk} - r_a^2\delta_{kx}\delta_{jx}\right) \\ &= \sum_a m_a r_a^2\left(\delta_{jk} - \delta_{jx}\delta_{kx}\right). \end{aligned}$$

This gives us

$$
\begin{aligned}
I_{xx} &= 0 \\
I_{yy} &= \sum_a m_a r_a^2 \\
I_{zz} &= \sum_a m_a r_a^2 \\
I_{xy} &= \sum_a m_a r_a^2 \left(\cancelto{0}{\delta_{xy}} - \delta_{xx} \cancelto{0}{\delta_{yx}} \right) = 0 \\
I_{xz} &= \sum_a m_a r_a^2 \left(\cancelto{0}{\delta_{xz}} - \delta_{xx} \cancelto{0}{\delta_{zx}} \right) = 0 \\
I_{yz} &= \sum_a m_a r_a^2 \left(\cancelto{0}{\delta_{yz}} - \cancelto{0}{\delta_{xy}} \cancelto{0}{\delta_{zx}} \right) = 0.
\end{aligned}
$$

For a molecule of two atoms (Fig. 12.4),

$$I_2 = I_3 = m_1 r_1^2 + m_2 r_2^2.$$

We want to express this in terms of the distance d between the atoms. We start with the

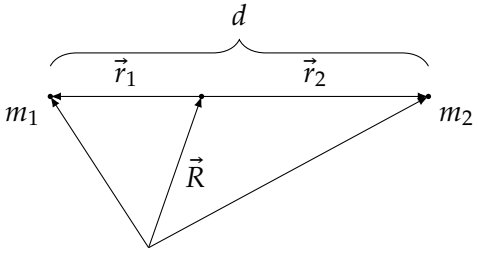


Figure 12.4. Two atoms with masses m_1 and m_2 at a distance d.

centre-of-mass condition:

$$
\begin{aligned}
0 &= m_1 \vec{r}_1 + m_2 \vec{r}_2 \\
\Rightarrow \vec{r}_2 &= -\frac{m_1}{m_2} \vec{r}_1.
\end{aligned}
$$

We obtain for the distance

$$
\begin{aligned}
\vec{d} &= \vec{r}_1 - \vec{r}_2 \\
&= \vec{r}_1 \left(1 + \frac{m_1}{m_2} \right) \\
&= \vec{r}_1 \frac{m_1 + m_2}{m_2},
\end{aligned}
$$

and thus

$$\boxed{\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{d}}$$

$$\boxed{\vec{r}_2 = -\frac{m_1}{m_1 + m_2} \vec{d}}.$$

Then, the moments of inertia are

$$\begin{aligned} I_{2,3} &= m_1 r_1^2 + m_2 r_2^2 \\ &= \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \right] d^2 \\ &= \frac{m_1 + m_2}{(m_1 + m_2)^2} \cdot m_1 m_2 d^2 \\ \Rightarrow \boxed{I_{yy} = I_{zz} = \frac{m_1 m_2}{m_1 + m_2} d^2} \end{aligned}$$

Now, recall

$$I_{jk} = \sum_a m_a \left(r_a^2 \delta_{jk} - r_{aj} r_{ak} \right)$$

and let us shift

$$\vec{r}_a \longrightarrow \vec{r}_a + \vec{c};$$

then

$$I'_{jk} = \sum_a m_a \left[(\vec{r}_a + \vec{c})^2 \delta_{jk} - (r_{aj} + c_j)(r_{ak} + c_k) \right].$$

The linear term is

$$\sum_a m_a \left[(2\vec{r}_a \cdot \vec{c}) \delta_{jk} - \left(r_{aj} c_k + r_{ak} c_j \right) \right],$$

but it vanishes since

$$\sum_a m_a \vec{r}_a = 0$$

and therefore

$$\begin{aligned} I'_{jk} &= \sum_a m_a \left[(\vec{r}_a + \vec{c})^2 \delta_{jk} - (\vec{r}_a + \vec{c})_j (\vec{r}_a + \vec{c})_k \right] \\ &= I_{jk} + \left(\sum_a m_a \right) \left[\vec{c}^2 \delta_{jk} - c_j c_k \right]. \end{aligned}$$

Example 12.2. Consider two points with the same mass m_1, at a distance a from each other and shifted by λ from the x-axis, and another point with mass m_2, as in Fig. 12.5. Then the centre-of-mass condition gives us a relation between λ and h:

$$\begin{aligned} (h - \lambda) m_2 &= 2 m_1 \lambda \\ \Rightarrow \lambda &= \frac{m_2}{2m_1 + m_2} h. \end{aligned}$$

The moments of inertia are

$$\begin{aligned} I_{yy} &= \sum_a m_a \left(r_a^2 \delta_{yy} - r_{ay}^2 \right) \\ &= \sum_a m_a r_{ax}^2 \\ &= 2 m_1 \left(\frac{a}{2} \right)^2 \\ &= \frac{1}{2} m_1 a^2, \end{aligned}$$

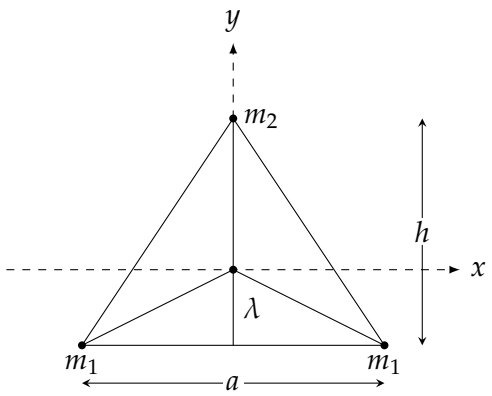


Figure 12.5. Two masses m_1 at the same distance from the centre of mass and one mass m_2 at a y-distance h from them.

$$\begin{aligned} I_{xx} &= \sum_a m_a r_{ay}^2 \\ &= m_2 (h-\lambda)^2 + 2m_1\lambda^2 \\ &= \left[2m_1 \left(\frac{m_2}{m}\right)^2 + m_2 \left(\frac{2m_1}{m}\right)^2 \right] h^2 \\ &= \frac{2m_1 m_2}{m^2} h^2 (m_2 + 2m_1) \\ &= \frac{2m_1 m_2}{m} h^2, \end{aligned}$$

$$\begin{aligned} I_{zz} &= \sum_a m_a \left(r_a^2 \delta_{zz} - r_{az}^2 \right) \\ &= \sum_a m_a \left(r_{ax}^2 + r_{ay}^2 \right) \\ &= I_{yy} + I_{xx}. \end{aligned}$$

Exercise 12.2. What are the moments of inertia for a cone?

Example 12.3. What is the frequency of a compound pendulum (Fig. 12.6)? Let α, β, γ

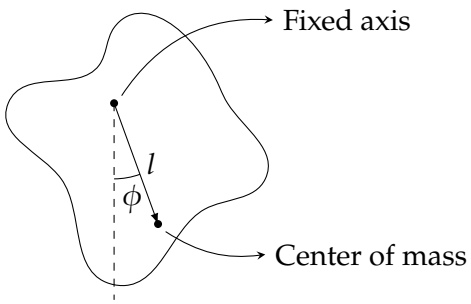


Figure 12.6. A compound pendulum.

be the angles between the principal axes and the axis of rotation, and l the distance of

the centre of mass from this axis. The velocity of the centre of mass is $v = l\dot{\phi}$. Then the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}mv^2 + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) - V \\ &= \frac{1}{2}ml^2\dot{\phi}^2 + \frac{1}{2}\left(I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right)\dot{\phi}^2 - V(\phi), \end{aligned}$$

where

$$\begin{aligned} V(\phi) &= \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})gz \\ &= mgz \\ &= -mgl\cos\phi \\ &\approx -mgl + \frac{1}{2}mgl\phi^2 + \ldots, \end{aligned}$$

and thus

$$L = \frac{1}{2}\left[ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma\right]\dot{\phi}^2 - \frac{1}{2}mgl\phi^2 + \ldots.$$

The frequency of the pendulum is

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{mgl}{ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma}}. \end{aligned}$$

12.2 Angular momentum

The angular momentum of a rigid body (Fig. 12.7) is

$$\begin{aligned} \vec{M} &= \sum_a m_a\left(\vec{r}_a \times \dot{\vec{r}}_a\right) \\ &= \sum_a m_a\left(\left(\vec{R} + \vec{l}_a\right) \times \left(\dot{\vec{R}} + \vec{\omega} \times \vec{l}_a\right)\right). \end{aligned}$$

In the centre-of-mass frame,

$$\sum_a m_a\vec{l}_a = 0,$$

the angular momentum is

$$\vec{M} = \sum_a m_a\vec{l}_a \times \left(\vec{\omega} \times \vec{l}_a\right). \tag{12.21}$$

We have

$$\begin{aligned} \left[\vec{a} \times \left(\vec{b} \times \vec{a}\right)\right]_i &= \epsilon_{ijk}a_j\epsilon_{klm}b_la_m \\ &= \epsilon_{kij}\epsilon_{klm}a_ja_mb_l \\ &= \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} a_jb_la_m \\ &= b_ia^2 - a_i\left(\vec{a}\cdot\vec{b}\right) \\ \Rightarrow \vec{a} \times \left(\vec{b} \times \vec{a}\right) &= \vec{b}a^2 - \vec{a}\left(\vec{a}\cdot\vec{b}\right). \end{aligned} \tag{12.22}$$

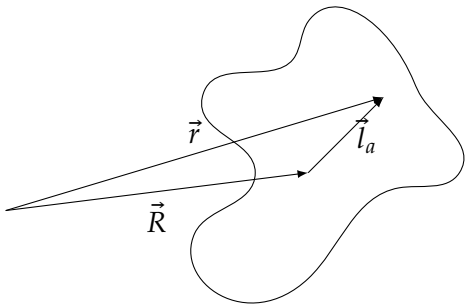


Figure 12.7. Position vectors of a particle of a rigid body.

Thus,

$$\begin{aligned}
\overrightarrow{M} &= \sum_a m_a \left[l_a^2 \vec{\omega} - \vec{l}_a \left(\vec{\omega} \cdot \vec{l}_a \right) \right] \\
M_i &= \sum_a m_a \left[l_a^2 \omega_i - l_{ai} \left(l_{aj} \omega_j \right) \right] \\
&= \left(\sum_a m_a \left[l_a^2 \delta_{ij} - l_{ai} l_{aj} \right] \right) \omega_j \\
\Rightarrow & \boxed{M_i = I_{ij} \omega_j} .
\end{aligned} \tag{12.23}$$

Changing our axes to the principal axes of inertia, we have

$$M_i = I_i \omega_i . \tag{12.24}$$

We observe that the angular momentum is not necessarily aligned with the angular velocity. For a general body, $\overrightarrow{M}$ is parallel to $\vec{\omega}$ only when $\vec{\omega}$ points along a principal axis (Landau–Lifshitz vol. 1, §32). For a spherical top,

$$I_1 = I_2 = I_3 = I,$$

they are always parallel:

$$\overrightarrow{M} = I \vec{\omega} . \tag{12.25}$$

12.2.1 Precession

We now examine the motion of a symmetrical top,

$$I_1 = I_2 \neq I_3,$$

which moves freely, without the influence of external forces (Fig. 12.8). In the centre-of-mass frame, the motion of the symmetrical top is purely rotational. We can determine it completely by using angular momentum conservation. One principal axis is the axis of symmetry $\hat{x}_3$, as shown in Fig. 12.9. The other two principal axes are perpendicular to it. Due to the symmetry, we have the freedom to choose their orientation. We choose the $\hat{x}_2$-axis to be perpendicular to the plane defined by $\hat{x}_3$ and the angular momentum $\overrightarrow{M}$. Then,

$$\overrightarrow{M} \perp \hat{x}_2 \Rightarrow M_2 \equiv \overrightarrow{M} \cdot \hat{x}_2 = 0,$$

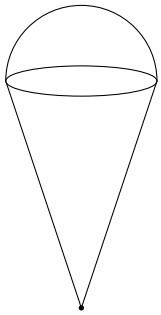


Figure 12.8. A symmetrical top.

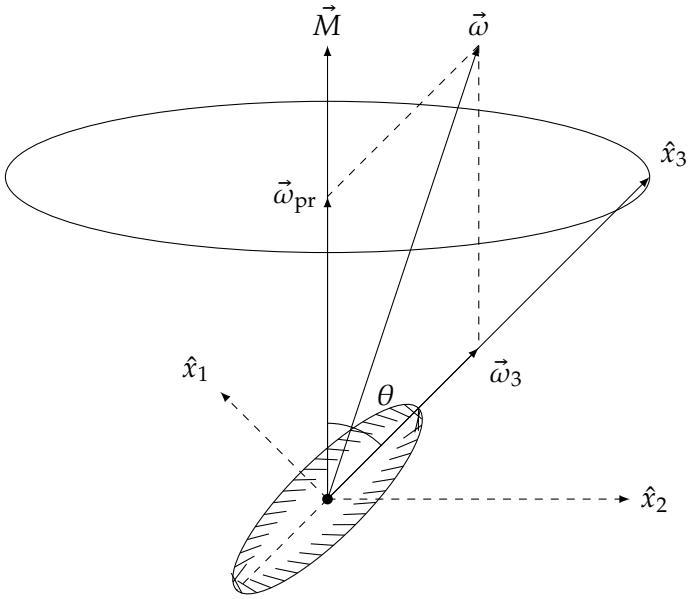


Figure 12.9. Rigid body with angular momentum $\vec{M}$ perpendicular to the principal axis $\hat{x}_2$ and angle θ between the angular momentum and the symmetry axis $\hat{x}_3$ of the body.

but

$$M_2 = I_2\omega_2 \Rightarrow \boxed{\omega_2 = 0},$$

thus $\vec{M}$, $\vec{\omega}$, $\hat{x}_3$ all lie in the same plane. Consider now the points of the top along the symmetry axis $\hat{x}_3$. Their velocity is

$$\vec{V} = \vec{\omega} \times \vec{r} \qquad \vec{r} \parallel \hat{x}_3$$
$$\Rightarrow \vec{V} \perp \left(\vec{\omega}, \vec{M}, \hat{x}_3\right).$$

The velocity of the points of the $\hat{x}_3$-axis is perpendicular to the plane of $(\vec{\omega}, \vec{M}, \hat{x}_3)$. We have

$$\begin{aligned}\vec{V} &= (\vec{\omega} \times \hat{x}_3)\, x_3 \\ &= \left[\left(\omega_1 \hat{x}_1 + \cancelto{0}{\omega_2} \hat{x}_2 + \omega_3 \hat{x}_3\right) \times \hat{x}_3\right] x_3 \\ &= \left[\omega_1 \left(\hat{x}_1 \times \hat{x}_3\right)\right] x_3 \\ &= -\frac{M_1}{I_1} x_3 \hat{x}_2.\end{aligned}$$

Since $\vec{M}$ is conserved,

$$\frac{M_1}{I_1} \equiv \omega_1$$

is constant. Therefore, the axis of the top *rotates uniformly around the direction of* $\vec{M}$, describing a circular cone. We have therefore identified two rotations:

1. Rotation around $\vec{M}$. This is called *regular precession*. The frequency of this rotation is

$$\begin{aligned}\omega_{\text{pr}} &= \frac{\omega_1}{\sin\theta} \\ &= \frac{M_1}{I_1 \sin\theta} \\ &= \frac{M \sin\theta}{I_1 \sin\theta} \\ \Rightarrow \boxed{\omega_{\text{pr}} &= \frac{M}{I_1}}.\end{aligned}$$

2. Rotation around the symmetry axis. The frequency of this rotation is

$$\boxed{\omega_3 = \frac{M_3}{I_3} = \frac{M \cos\theta}{I_3}}.$$

Remark: ω_3 is the component of $\vec{\omega}$ along the symmetry axis (as in Landau–Lifshitz §33). In the decomposition $\vec{\omega} = \omega_{\text{pr}} \hat{M} + \omega_{\text{s}} \hat{x}_3$ the spin rate relative to the precessing plane $(\vec{M}, \hat{x}_3)$ is $\omega_{\text{s}} = \omega_3 - \omega_{\text{pr}} \cos\theta = M \cos\theta\,(1/I_3 - 1/I_1) = \Omega$, the frequency found from Euler's equations below.

12.3 Equations of motion of a rigid body

We need to determine $\vec{R}(t)$ and $\vec{\omega}(t)$. Then the geometry of the rigid body will give us the positions $\vec{l}_a(t)$ relative to a reference point in the rigid body. We have

$$\begin{aligned}
\vec{f}_a &= \frac{\mathrm{d}\vec{P}_a}{\mathrm{d}t} \quad \forall \text{ particles } a \\
\Rightarrow \sum_a \vec{f}_a &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \vec{P}_a \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \sum_a m_a \left(\dot{\vec{R}} + \vec{\omega} \times \vec{l}_a \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(\sum_a m_a \right) \vec{V} \right] + \frac{\mathrm{d}}{\mathrm{d}t} \left[\cancelto{0}{\left(\sum_a m_a \vec{l}_a \right)} \times \vec{\omega} \right] \\
&= m \frac{\mathrm{d}}{\mathrm{d}t} \vec{V}.
\end{aligned}$$

The sum of the forces

$$\vec{F} = \sum_a \vec{f}_a$$

includes only external forces, because internal forces cancel out. Indeed, in the absence of external forces, internal forces cannot give momentum to the body; thus

$$\sum_a \vec{f}_a^{(\mathrm{int})} = 0.$$

Notice that the Lagrangian describing a rigid body is

$$L = \frac{1}{2} m \vec{V}^2 + \frac{1}{2} I_{ij} \omega_i \omega_j - V(\vec{R}, \phi).$$

Then the Euler–Lagrange equations are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \vec{V}} - \frac{\partial L}{\partial \vec{R}} &= 0 \\
\Rightarrow m \dot{\vec{V}} &= -\frac{\partial V}{\partial \vec{R}} \equiv \vec{F},
\end{aligned}$$

where we have used the shorthand notation

$$\frac{\partial}{\partial \vec{x}} = \vec{\nabla}_{\vec{x}} \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right).$$

Now let us look at the rate of change of the total angular momentum. Let us choose the reference frame such that the centre of mass does not move,

$$\vec{V} = 0.$$

This is not necessarily an inertial frame. Then

$$\begin{aligned}\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_a \vec{l}_a \times \vec{P}_a\right)\\ &= \sum_a \frac{\mathrm{d}}{\mathrm{d}t}\vec{l}_a \times \vec{P}_a + \sum_a \vec{l}_a \times \frac{\mathrm{d}\vec{P}_a}{\mathrm{d}t}\\ &= \sum_a \cancel{\frac{\mathrm{d}}{\mathrm{d}t}\vec{l}_a \times \left(m_a \frac{\mathrm{d}\vec{l}_a}{\mathrm{d}t}\right)}^{0} + \sum_a \vec{l}_a \times \frac{\mathrm{d}\vec{P}_a}{\mathrm{d}t}\\ &= \sum_a \vec{l}_a \times \vec{f}_a.\end{aligned}$$

In the (generally accelerated) centre-of-mass frame each particle also feels the inertial force $-m_a\dot{\vec{V}}$; its torque $-\left(\sum_a m_a \vec{l}_a\right) \times \dot{\vec{V}}$ vanishes because $\sum_a m_a \vec{l}_a = 0$, which is why $\vec{f}_a$ alone appears here (Landau–Lifshitz §34). The sum

$$\vec{K} \equiv \sum_a \vec{l}_a \times \vec{f}_a$$

is the total torque. The above sum includes only external forces, since the internal forces alone cannot produce angular momentum. The equation

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \vec{K}$$

can be derived from the Lagrangian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{\omega}} - \frac{\partial L}{\partial \vec{\phi}} = 0,$$

where

$$\begin{aligned}\frac{\partial L}{\partial \omega_i} &= \frac{\partial}{\partial \omega_i}\left(\frac{1}{2}\omega_i \sum_j I_{ij}\omega_j\right) = \sum_j I_{ij}\omega_j = M_i\\ \frac{\partial L}{\partial \phi_i} &= -\frac{\partial V}{\partial \phi_i}.\end{aligned}$$

Since

$$\begin{aligned}\partial V &= -\sum_a \underbrace{\vec{f}_a \cdot \partial\vec{l}_a}_{\text{work}}\\ &= -\sum_a \vec{f}_a \cdot \left(\partial\vec{\phi} \times \vec{l}_a\right)\\ &= -\partial\vec{\phi} \cdot \underbrace{\sum_a \left(\vec{l}_a \times \vec{f}_a\right)}_{=\vec{K}}\\ &= -\partial\vec{\phi} \cdot \vec{K},\end{aligned}$$

we obtain

$$-\frac{\partial V}{\partial \vec{\phi}} = \vec{K}.$$

The equations of motion

$$\frac{\mathrm{d}\vec{P}}{\mathrm{d}t} = \vec{F} \qquad \text{(total force)}$$
$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \vec{K} \qquad \text{(total torque)}$$

have been derived for a reference frame where the axes have a fixed orientation. However, it is often convenient to use a reference frame with rotating axes. Typically, these are the principal axes of inertia. Consider two coordinate systems, where $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ are fixed and $(\hat{y}_1(t), \hat{y}_2(t), \hat{y}_3(t))$ are rotating. If we express a vector $\vec{A}$ in terms of the rotating system, we have

$$\vec{A} = \sum_i A_i \hat{y}_i(t).$$

The total derivative is

$$\begin{aligned}\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} &= \sum_i \frac{\mathrm{d}A_i}{\mathrm{d}t}\hat{y}_i(t) + \sum_i A_i(t)\frac{\mathrm{d}\hat{y}_i(t)}{\mathrm{d}t} \\ &= \sum_i \frac{\mathrm{d}A_i}{\mathrm{d}t}\hat{y}_i(t) + \sum_i A_i(t)\vec{\omega}\times\hat{y}_i(t) \\ \Rightarrow \boxed{\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} &= \left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \vec{\omega}\times\vec{A}}. \end{aligned} \tag{12.26}$$

In Eq. (12.26), $\left(\frac{\mathrm{d}\vec{A}}{\mathrm{d}t}\right)_{\mathrm{rot}}$ is the rate of change of the vector $\vec{A}$ as it is perceived in the rotating frame.

Therefore, in a rotating reference frame with a fixed (or freely moving) centre the equations of motion take the form

$$\left(\frac{\mathrm{d}\vec{P}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \vec{\omega}\times\vec{P} = \vec{F}$$
$$\left(\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \vec{\omega}\times\vec{M} = \vec{K},$$

where $\vec{\omega}$ is the angular velocity of the rotating frame. In terms of components, the first equation reads

$$m\left(\frac{\mathrm{d}V_1}{\mathrm{d}t} + \omega_2 V_3 - \omega_3 V_2\right) = F_1$$
$$m\left(\frac{\mathrm{d}V_2}{\mathrm{d}t} + \omega_3 V_1 - \omega_1 V_3\right) = F_2$$
$$m\left(\frac{\mathrm{d}V_3}{\mathrm{d}t} + \omega_1 V_2 - \omega_2 V_1\right) = F_3.$$

We can choose the axes of the rotating system to be the principal axes of inertia. Then, using

$$M_1 = I_1\omega_1 \qquad M_2 = I_2\omega_2 \qquad M_3 = I_3\omega_3$$

and substituting into

$$\left(\frac{\mathrm{d}\vec{M}}{\mathrm{d}t}\right)_{\text{rot}} + \vec{\omega}\times\vec{M} = \vec{K},$$

we have

$$\left.\begin{aligned} I_1\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (I_3 - I_2)\omega_2\omega_3 &= K_1 \\ I_2\frac{\mathrm{d}\omega_2}{\mathrm{d}t} + (I_1 - I_3)\omega_3\omega_1 &= K_2 \\ I_3\frac{\mathrm{d}\omega_3}{\mathrm{d}t} + (I_2 - I_1)\omega_1\omega_2 &= K_3. \end{aligned}\right\} \text{ Euler's equations}$$

12.3.1 Applications: free rotational motion

Example 12.4 (Symmetrical top)**.** In this case, we have

$$I_1 = I_2 \neq I_3, \qquad \vec{K} = 0,$$

and from the third Euler equation

$$\frac{\mathrm{d}\omega_3}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \omega_3 = \text{const.}$$

The other two give

$$\begin{aligned} \frac{\mathrm{d}\omega_1}{\mathrm{d}t} + \Omega\omega_2 &= 0, \\ \frac{\mathrm{d}\omega_2}{\mathrm{d}t} - \Omega\omega_1 &= 0, \end{aligned}$$

with

$$\Omega \equiv \omega_3\frac{I_3 - I_1}{I_1}.$$

This leads to

$$\begin{aligned} \dot{\omega}_1 &= -\Omega\omega_2 \\ \dot{\omega}_2 &= \Omega\omega_1 \end{aligned}$$

and we can decouple this system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\omega_1 + \mathrm{i}\omega_2) &= \Omega(-\omega_2 + \mathrm{i}\omega_1) \\ &= \mathrm{i}\Omega(\omega_1 + \mathrm{i}\omega_2) \\ \Rightarrow \omega_1 + \mathrm{i}\omega_2 &= A\mathrm{e}^{\mathrm{i}\Omega t} \\ &= A(\cos\Omega t + \mathrm{i}\sin\Omega t) \\ \Rightarrow \boxed{\omega_1 = A\cos\Omega t \qquad \omega_2 = A\sin\Omega t}&. \end{aligned}$$

Notice that

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = A^2 + \omega_3^2 = \text{const.} \quad \Rightarrow \quad |\vec{\omega}| = \text{const},$$

i.e. the magnitude of the angular velocity is constant. Since ω_3 is constant too, $\vec{\omega}$ rotates around the symmetry axis of the top with angular velocity Ω.

Example 12.5 (Asymmetrical top)**.** Let us now consider the general case of an asymmetric top:

$$I_3 > I_2 > I_1.$$

Energy and angular momentum conservation give (in the principal axes)

$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = E$$
$$\Rightarrow \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E$$

and

$$M_1^2 + M_2^2 + M_3^2 = M^2.$$

The vector $\vec{M}$ therefore lies on the intersection of *a sphere and an ellipsoid*. We can easily prove the inequalities

$$\boxed{2EI_1 \le M^2 \le 2EI_3}.$$

If $M^2 = 2EI_1$ (smallest value), we find

$$M_2 = M_3 = 0.$$

For $M^2 = 2EI_3$ (largest value),

$$\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E$$
$$M_1^2 + M_2^2 + M_3^2 = M^2.$$

Subtracting the second equation divided by I_3 from the first gives $M_1^2(1/I_1 - 1/I_3) + M_2^2(1/I_2 - 1/I_3) = 0$, hence $M_1 = M_2 = 0$: the body rotates about $\hat{x}_3$ (Landau–Lifshitz §37). For $M^2 = 2EI_2$,

$$\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E$$
$$\frac{M_1^2}{I_2} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_2} = 2E$$
$$\Rightarrow M_1^2\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + M_3^2\left(\frac{1}{I_3} - \frac{1}{I_2}\right) = 0$$
$$\Rightarrow M_1^2 = \frac{I_1(I_3 - I_2)}{I_3(I_2 - I_1)}M_3^2.$$

12.3.2 Euler angles

Euler's equations give the angular velocity $\vec{\omega}(t)$ in a rotating system whose axes are the principal axes of inertia. What, then, is the motion of this rotating system as we observe it from a system with fixed axes? To relate a fixed and a rotating system with a common centre, we can use the so-called *Euler angles* (Fig. 12.10).

Here (x, y, z) is a system with fixed axes and (x_1, x_2, x_3) is a rotating one. To go from (x, y, z) to (x_1, x_2, x_3) we perform a series of rotations.

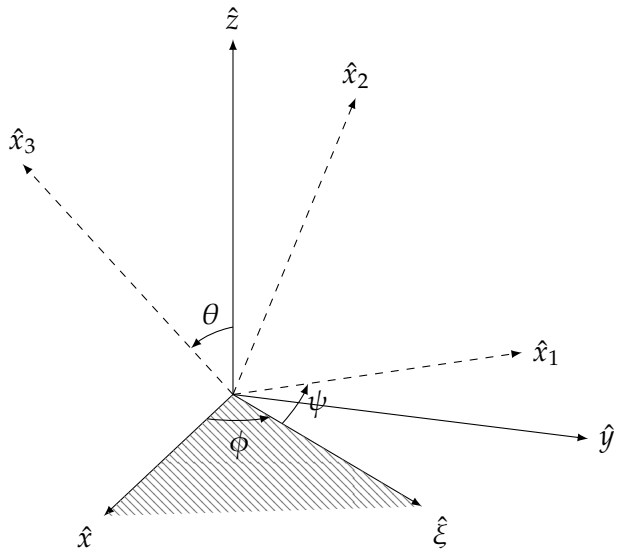


Figure 12.10. Rotation of the coordinate system $(\hat{x}, \hat{y}, \hat{z})$ by the Euler angles ϕ, θ, ψ to the system $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$.

1. Consider the intersection $\overrightarrow{ON}$ of the (x_1, x_2)-plane and the (x, y)-plane. Align the x-axis with $\overrightarrow{ON}$ by rotating anticlockwise by an angle ϕ around the z-axis. We then go to coordinates

$$(x, y, z) \to (\xi, \eta, z)$$
$$\begin{pmatrix} \xi \\ \eta \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

2. Then we rotate anticlockwise around the $\xi (\equiv \overrightarrow{ON})$-axis by θ:

$$\begin{pmatrix} \xi \\ \eta' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ z \end{pmatrix}.$$

3. Finally, we rotate around the z'-axis by ψ:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta' \\ z' \end{pmatrix}.$$

Collectively, we can write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We can now express the angular velocity

$$\vec{\omega} = (\omega_1, \omega_2, \omega_3),$$

with components in the (x_1, x_2, x_3)-frame, in terms of the Euler angles and their time derivatives. For the angular velocities of the three Euler rotations we have

$$\begin{aligned}\dot{\vec{\phi}} &= \dot{\phi}\hat{z}\\ \dot{\vec{\theta}} &= \dot{\theta}\hat{\xi}\\ \dot{\vec{\psi}} &= \dot{\psi}\hat{z}'.\end{aligned}$$

We want to express these rotation axes in terms of $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$:

$$\begin{aligned}\hat{\xi} &= (\hat{\xi}\cdot\hat{x}_1)\cdot\hat{x}_1 + (\hat{\xi}\cdot\hat{x}_2)\cdot\hat{x}_2 + \cancelto{0}{(\hat{\xi}\cdot\hat{x}_3)}\cdot\hat{x}_3\\ &= \cos\psi\hat{x}_1 + \cos\left(\psi+\frac{\pi}{2}\right)\hat{x}_2\\ &= \cos\psi\hat{x}_1 - \sin\psi\hat{x}_2\end{aligned}$$

$$\begin{aligned}\hat{z} &= \cos\theta\hat{x}_3 + \sin\theta\cos\left(\frac{\pi}{2}-\psi\right)\hat{x}_1 + \sin\theta\cos\psi\hat{x}_2\\ &= \cos\theta\hat{x}_3 + \sin\theta\sin\psi\hat{x}_1 + \sin\theta\cos\psi\hat{x}_2\end{aligned}$$

$$\hat{z}' = \hat{x}_3,$$

then

$$\begin{aligned}\vec{\omega} &= \dot{\vec{\theta}} + \dot{\vec{\phi}} + \dot{\vec{\psi}}\\ &= (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{x}_1 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\hat{x}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{x}_3.\end{aligned}$$

12.4 The heavy symmetric top

We now have all the tools to study the classic problem of rigid-body dynamics: a symmetric top, $I_1 = I_2 \neq I_3$, which spins in the gravitational field of the Earth while one point O of its symmetry axis, the tip, stays fixed on the floor. The centre of mass lies on the symmetry axis $\hat{x}_3$ at a distance l from O. The free top of the previous sections precesses uniformly. We will see that gravity adds a richer motion of the symmetry axis, the *nutation*, and that a fast top can stand upright, "sleeping", on its tip.

12.4.1 Lagrangian in Euler angles

We take the fixed point O as the common origin of the fixed frame (x, y, z), with $\hat{z}$ pointing vertically upwards, and of the body frame (x_1, x_2, x_3) of principal axes. Since O does not move, the kinetic energy of Eq. (12.7) with $\vec{V} = 0$ is purely rotational,

$$T = \frac{1}{2}I_1\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_3\omega_3^2,$$

where now the moments of inertia refer to O and not to the centre of mass. By Eq. (12.19) with $\vec{c} = l\hat{x}_3$, $I_1 = I_1^{\mathrm{cm}} + ml^2$, while $I_3 = I_3^{\mathrm{cm}}$ is unchanged. Notice that $I_1 = I_2$ still holds, so

we may use the results for a symmetric body. Substituting the components of $\vec{\omega}$ in terms of the Euler angles, we find

$$\begin{aligned}\omega_1^2 + \omega_2^2 &= \left(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi\right)^2 + \left(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi\right)^2 \\ &= \dot\theta^2 + \dot\phi^2 \sin^2\theta, \\ \omega_3 &= \dot\psi + \dot\phi \cos\theta.\end{aligned}$$

The angle ψ has dropped out of $\omega_1^2 + \omega_2^2$, as necessitated by the symmetry of the body around $\hat{x}_3$. The angle θ is the angle between the vertical $\hat{z}$ and the symmetry axis, so the height of the centre of mass is $l\cos\theta$ and the potential energy is $V = mgl\cos\theta$. The Lagrangian of the heavy symmetric top is

$$\boxed{L = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2 - mgl\cos\theta}. \tag{12.27}$$

12.4.2 Integrals of motion and the effective potential

The angles ϕ and ψ are cyclic. Their conjugate momenta are therefore conserved:

$$\begin{aligned}p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_3\left(\dot\psi + \dot\phi\cos\theta\right) = I_3\omega_3 \equiv M_3 = \text{const}, \\ p_\phi &= \frac{\partial L}{\partial \dot\phi} = I_1\dot\phi\sin^2\theta + I_3\omega_3\cos\theta \equiv M_z = \text{const}.\end{aligned}$$

Their meaning is easy to find. $M_3 = I_3\omega_3$ is the component of the angular momentum $\vec{M} = I_1\omega_1\hat{x}_1 + I_1\omega_2\hat{x}_2 + I_3\omega_3\hat{x}_3$ along the symmetry axis. Using the components of $\hat{z}$ in the body frame, $\hat{z} = \sin\theta\sin\psi\,\hat{x}_1 + \sin\theta\cos\psi\,\hat{x}_2 + \cos\theta\,\hat{x}_3$, we obtain

$$\begin{aligned}\vec{M}\cdot\hat{z} &= I_1\sin\theta\left(\omega_1\sin\psi + \omega_2\cos\psi\right) + I_3\omega_3\cos\theta \\ &= I_1\dot\phi\sin^2\theta + I_3\omega_3\cos\theta = p_\phi,\end{aligned}$$

i.e. p_ϕ is the vertical component M_z of the angular momentum. Both conservation laws can be understood from the torque of gravity, $\vec{K} = l\hat{x}_3 \times \left(-mg\hat{z}\right)$, which is perpendicular to both $\hat{z}$ and $\hat{x}_3$. The conservation of ω_3 is also the third Euler equation, $I_3\dot\omega_3 = K_3 = 0$, for $I_1 = I_2$.

The Lagrangian does not depend explicitly on time, therefore the energy

$$E = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{I_3}{2}\omega_3^2 + mgl\cos\theta$$

is conserved too. We now eliminate $\dot\phi$ and $\dot\psi$ with the help of the two momenta,

$$\dot\phi = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta}, \qquad \dot\psi = \frac{M_3}{I_3} - \dot\phi\cos\theta. \tag{12.28}$$

Substituting in E and moving the constant $M_3^2/(2I_3)$ to the left-hand side, we obtain a one-dimensional problem for θ:

$$E' \equiv E - \frac{M_3^2}{2I_3} = \frac{I_1}{2}\dot\theta^2 + V_{\text{eff}}(\theta), \qquad V_{\text{eff}}(\theta) = \frac{\left(M_z - M_3\cos\theta\right)^2}{2I_1\sin^2\theta} + mgl\cos\theta. \tag{12.29}$$

This is the same trick as for the motion in a central potential, where the conserved angular momentum produced a centrifugal barrier. Here $V_{\text{eff}} \to \infty$ for $\theta \to 0, \pi$ (unless $M_z = \pm M_3$), so θ oscillates between two turning points $\theta_1 \le \theta \le \theta_2$ where $V_{\text{eff}}(\theta) = E'$.

The algebra becomes polynomial with the variable $u = \cos\theta$, for which $\dot{u} = -\sin\theta\,\dot{\theta}$. Let us introduce the constants

$$\alpha = \frac{2E'}{I_1}, \qquad \beta = \frac{2mgl}{I_1} > 0, \qquad a = \frac{M_3}{I_1}, \qquad b = \frac{M_z}{I_1}.$$

Multiplying Eq. (12.29) by $2\sin^2\theta/I_1$ we find

$$\boxed{\dot{u}^2 = f(u) \equiv \left(1-u^2\right)\left(\alpha - \beta u\right) - (b - au)^2}, \qquad \dot{\phi} = \frac{b - au}{1-u^2}. \tag{12.30}$$

The solution $t = \int \mathrm{d}u/\sqrt{f(u)}$ is an elliptic integral, but the qualitative features of the motion can be read off directly from the cubic polynomial $f(u)$:

- For $u \to +\infty$, $f(u) \simeq \beta u^3 \to +\infty$.
- At $u = \pm 1$, $f(\pm 1) = -(b \mp a)^2 \le 0$.
- A physical motion needs $\dot{u}^2 = f(u) \ge 0$ for some $-1 \le u \le 1$.

Therefore f has two roots $u_1 \le u_2$ in $[-1, 1]$ and the third root u_3 is larger than 1, i.e. unphysical. The figure axis oscillates between the two circles $\cos\theta = u_1$ and $\cos\theta = u_2$ on the unit sphere. This up-and-down oscillation of the axis is called *nutation*.

12.4.3 Precession and nutation of the figure axis

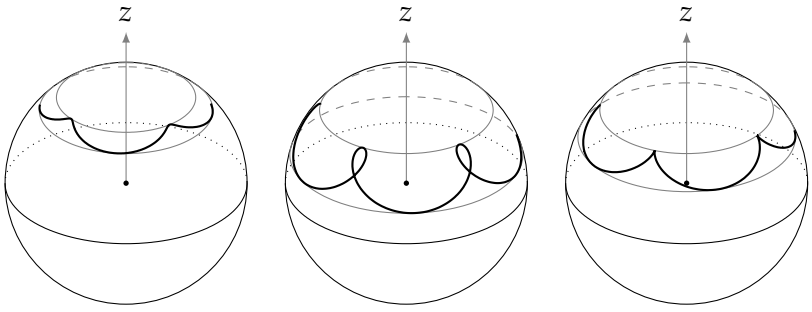
While θ oscillates between the two turning points, the azimuth of the figure axis advances with the rate $\dot{\phi}$ of Eq. (12.30), the *precession*. The shape of the path that $\hat{x}_3$ traces on the unit sphere depends on where the angular velocity $\dot{\phi}$ vanishes, $u' = b/a$. There are three cases, shown in Fig. 12.11:

(a) u' lies outside $[u_1, u_2]$. Then $\dot{\phi}$ has always the same sign and the axis undulates between the two circles.

(b) $u_1 < u' < u_2$. Then $\dot{\phi}$ changes sign during every nutation period and the axis draws loops.

(c) $u' = u_2$. Then $\dot{\phi}$ and $\dot{\theta}$ vanish together at the upper turning circle and the path has cusps.

Why can a cusp appear only on the upper circle? At u' the velocity of the axis vanishes, $\dot{\theta} = \dot{\phi} = 0$, and the energy is $E' = mglu'$. Since $E' = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mglu \ge mglu$, we have $u \le u'$ during the whole motion. Case (c) is the most common in practice: we spin the top and release its axis at rest at some angle θ_0. The axis first falls, then rises back to θ_0, and so on.



(a) undulating (b) loops (c) cusps

Figure 12.11. Path of the figure axis $\hat{x}_3$ of a heavy symmetric top on the unit sphere, from a numerical integration of the equations of motion of Eq. (12.27). The grey circles are the turning points $\cos\theta = u_{1,2}$. The three cases differ only by the initial precession rate $\dot{\phi}$.

Uniform precession. The figure axis precesses uniformly when $\theta = \theta_0$ stays constant, i.e. when there is no nutation. This requires $u_1 = u_2$, a double root of f, or equivalently a minimum of V_{eff}. It is simpler to use the Euler–Lagrange equation for θ,

$$I_1\ddot{\theta} = I_1\dot{\phi}^2 \sin\theta\cos\theta - I_3\omega_3\dot{\phi}\sin\theta + mgl\sin\theta.$$

Setting $\ddot{\theta} = 0$ and $\dot{\phi} = \Omega$, we find for $\sin\theta_0 \neq 0$

$$I_1\cos\theta_0\,\Omega^2 - I_3\omega_3\,\Omega + mgl = 0 \quad\Rightarrow\quad \Omega_\pm = \frac{I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1 mgl\cos\theta_0}}{2I_1\cos\theta_0}. \tag{12.31}$$

For a top above the horizontal plane, $\cos\theta_0 > 0$, uniform precession is possible only if the top spins fast enough, $I_3^2\omega_3^2 \geq 4I_1 mgl\cos\theta_0$. For a *fast top*, $I_3^2\omega_3^2 \gg I_1 mgl$, we can expand the square root and obtain two very different frequencies:

$$\boxed{\Omega_- \simeq \frac{mgl}{I_3\omega_3}} \quad \text{(slow precession)}, \qquad \Omega_+ \simeq \frac{I_3\omega_3}{I_1\cos\theta_0} \quad \text{(fast precession)}. \tag{12.32}$$

The fast precession is essentially the free precession that we found for the force-free top, slightly modified by gravity. The slow precession is the gyroscopic effect. It does not depend on θ_0 and it can be understood without any calculation. For a fast top $\vec{M} \simeq I_3\omega_3\hat{x}_3$, and the torque $\vec{K}$ of gravity, of magnitude $mgl\sin\theta$, is horizontal and perpendicular to $\vec{M}$. Then $\mathrm{d}\vec{M}/\mathrm{d}t = \vec{K}$ rotates $\vec{M}$ around $\hat{z}$ with the angular velocity $mgl\sin\theta/(I_3\omega_3\sin\theta) = mgl/(I_3\omega_3)$. Instead of falling, the top turns sideways!

Fast top released at rest. What happens if we do not prepare the top carefully with $\dot{\phi} = \Omega_-$, but simply release its axis at rest at θ_0? Then $\dot{\theta} = \dot{\phi} = 0$ at $u_0 = \cos\theta_0$, so that $b = au_0$ and $\alpha = \beta u_0$, and

$$f(u) = (u_0 - u)\left[\beta\left(1 - u^2\right) - a^2(u_0 - u)\right].$$

For a fast top $a = I_3\omega_3/I_1$ is large and $x \equiv u_0 - u$ stays small. Keeping the leading terms, $\dot{x}^2 = x\left(\beta\sin^2\theta_0 - a^2x\right)$, which, after differentiating, becomes a harmonic oscillator $\ddot{x} = \frac{\beta}{2}\sin^2\theta_0 - a^2x$. With $x(0) = \dot{x}(0) = 0$,

$$x(t) = \frac{\beta\sin^2\theta_0}{2a^2}(1 - \cos at), \qquad \dot{\phi} = \frac{ax}{\sin^2\theta_0} = \frac{\beta}{2a}(1 - \cos at).$$

The axis nutates with the frequency $a = I_3\omega_3/I_1$ and with the angular amplitude $\Delta\theta = \beta \sin\theta_0/a^2 = 2I_1 mgl \sin\theta_0/(I_3\omega_3)^2$, while it precesses with the average rate $\beta/(2a) = mgl/(I_3\omega_3)$, the slow precession frequency of Eq. (12.32). This is the cusped path (c) of Fig. 12.11. For a fast top the nutation is so fast and so small that the eye sees only the slow precession ("pseudo-regular precession"). Friction then damps the nutation.

12.4.4 The sleeping top

A top which spins with its axis vertical, $\theta = 0$, looks as if it does not move at all; it is called a *sleeping top*. Is this motion stable? For $\theta = 0$ the axes $\hat{x}_3$ and $\hat{z}$ coincide, so that $M_z = M_3$. The effective potential becomes

$$V_{\text{eff}}(\theta) = \frac{M_3^2\,(1-\cos\theta)^2}{2I_1 \sin^2\theta} + mgl\cos\theta = \frac{M_3^2}{2I_1}\tan^2\frac{\theta}{2} + mgl\cos\theta$$
$$\simeq mgl + \left(\frac{M_3^2}{8I_1} - \frac{mgl}{2}\right)\theta^2 + \ldots$$

The vertical position is a minimum of V_{eff}, and therefore stable, if $M_3^2 > 4I_1 mgl$, i.e.

$$\boxed{\omega_3^2 > \frac{4I_1 mgl}{I_3^2}}\,. \tag{12.33}$$

We can also see this from the cubic. With $a = b$ and $E' = mgl$ ($\alpha = \beta$),

$$f(u) = (1-u)^2\left[\beta\,(1+u) - a^2\right].$$

Besides the double root $u = 1$, the third root is $u_3 = a^2/\beta - 1$. If Eq. (12.33) holds, $u_3 > 1$ and $u = 1$ is the only physical point: a small disturbance leads only to a small wobble. If Eq. (12.33) is violated, $u_3 < 1$ and the slightest disturbance makes the axis nutate between $\theta = 0$ and $\theta = \arccos u_3$. Friction slowly reduces ω_3. Once ω_3 drops below the bound Eq. (12.33), the top "wakes up", starts to wobble, and finally falls.

Exercise 12.3. Show that the condition $V'_{\text{eff}}(\theta_0) = 0$, with $M_z = I_1\Omega\sin^2\theta_0 + M_3\cos\theta_0$, reproduces the quadratic equation (12.31) for the rate Ω of uniform precession. Show that a top hanging below its fixed point, $\cos\theta_0 < 0$, can precess uniformly for any ω_3, and that then one of the two rates $\Omega_\pm$ is negative. Expand $\Omega_\pm$ for large ω_3 to derive Eq. (12.32).

Exercise 12.4. A toy gyroscope has a rotor of mass $m = 0.15\,\text{kg}$, which we treat as a uniform disk of radius $r = 3\,\text{cm}$. Its centre is at $l = 4\,\text{cm}$ from the fixed tip. It spins with 6000 revolutions per minute. Compute $I_3 = \frac{1}{2}mr^2$ and $I_1 = \frac{1}{4}mr^2 + ml^2$. Find the period of the slow precession, the frequency and the amplitude of the nutation if it is released at rest at $\theta_0 = 60°$, and the minimum number of revolutions per minute for which it can sleep. (Answers: $2\pi/\Omega_- \approx 4.5\,\text{s}$; $I_3\omega_3/I_1 \approx 155\,\text{s}^{-1}$, i.e. about 25 Hz; $\Delta\theta \approx 0.9°$; about 1140 rpm.)

13

Hamiltonian mechanics

As we established in Sec. 4.6, the motion of a holonomic system with n degrees of freedom is determined by the Lagrange equations, that is, a set of n second-order differential equations for n unknown variables q_i, with initial conditions given by the values of q_i and $\dot{q}_i$ at a given time t.

It has been known since Lagrange's time that higher-order differential equations can be reduced to systems of first-order differential equations. Thus, we expect that a system of n second-order differential equations can be reduced to a system of $2n$ first-order differential equations. In general, it is more convenient to deal with a system of first-order differential equations, because an approximate solution may be found even when an exact solution is not known. To illustrate this point, let us consider a point whose motion is described by the first-order differential equation

$$\dot{\vec{r}}(t) = \vec{f}\,(\vec{r}(t); t)\ . \tag{13.1}$$

Given an initial condition at a time t_0, Eq. (13.1) can be integrated to

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{f}\,(\vec{r}(\tau); \tau)\, \mathrm{d}\tau\ . \tag{13.2}$$

The integral equation (13.2) is more convenient than the differential one (13.1) because it can be approximated. In fact, given the initial condition $\vec{r}(t_0)$, we can build a series of successive approximations through the iterative formula

$$\vec{r}_{k+1}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{f}\,(\vec{r}_k(\tau); \tau)\, \mathrm{d}\tau\ , \qquad k = 0, 1, 2, \ldots\ . \tag{13.3}$$

For example, let us take the non-linear differential equation in one dimension

$$\dot{x}(t) = x^2 + t^2\ , \tag{13.4}$$

with $t_0 = 0$ and initial condition $x(t_0) = 0$, and start from $x_0(t) = 0$. Then

$$\begin{aligned} x_1(t) &= \int_0^t \left[x_0^2(\tau) + \tau^2\right] \mathrm{d}\tau = \int_0^t \tau^2 \mathrm{d}\tau = \frac{t^3}{3}\ , \\ x_2(t) &= \int_0^t \left[x_1^2(\tau) + \tau^2\right] \mathrm{d}\tau = \int_0^t \left(\frac{\tau^6}{9} + \tau^2\right) \mathrm{d}\tau = \frac{t^3}{3} + \frac{t^7}{63}\ , \end{aligned} \tag{13.5}$$

and so on. One can continue like this until one solves Eq. (13.2) to the desired accuracy. Thus, a first-order differential equation can be transformed into an integral equation which can be solved to the desired accuracy.

Another reason to go to first-order differential equations is that they have a straightforward geometric interpretation. To see this, let us consider the motion of a particle in the x-y plane, described by a set of first-order differential equations,

$$\dot{\vec{r}}(t) = \vec{v}(x, y; t)\,, \tag{13.6}$$

which can be written in Cartesian coordinates as

$$\begin{aligned} \dot{x} &= f(x, y; t)\,, \\ \dot{y} &= g(x, y; t)\,. \end{aligned} \tag{13.7}$$

Then finding $\vec{r}$ means looking for the curve which at every point has $\dot{\vec{r}}$ as its tangent vector, as in Fig. 13.1.

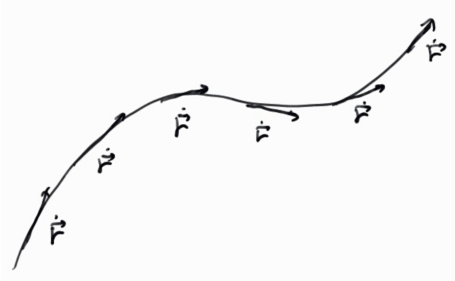


Figure 13.1. A curve in the x-y plane and the vectors tangent to it.

Conversely, if we have a system of second-order differential equations,

$$\begin{aligned} \ddot{x} &= f(x, y, \dot{x}, \dot{y}; t)\,, \\ \ddot{y} &= g(x, y, \dot{x}, \dot{y}; t)\,, \end{aligned} \tag{13.8}$$

then, in order to associate a vector $\ddot{\vec{r}} = \vec{a}(\vec{r}, \dot{\vec{r}}; t)$ to any point $\vec{r}$, we must introduce a four-dimensional space with coordinates $(x, y, \dot{x}, \dot{y})$, whose tangent vector is $(\dot{x}, \dot{y}, \ddot{x}, \ddot{y})$. Thus a geometrical interpretation is much less obvious.

13.1 Hamilton equations

In 1834, William Rowan Hamilton transformed the Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \qquad i = 1, \ldots, n\,, \tag{13.9}$$

into a system of $2n$ first-order differential equations. To do so, he introduced the momenta p_i *conjugate* to the generalised coordinates q_i,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,, \qquad i = 1, \ldots, n\,, \tag{13.10}$$

and proposed to use q_i, p_i and t as the canonical variables, rather than the variables q_i, $\dot{q}_i$ and t used by Lagrange. If the Jacobian of p_i in Eq. (13.10) is non-vanishing,

$$\left|\frac{\partial p_i}{\partial \dot{q}_j}\right| = \left|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right| \neq 0\,, \tag{13.11}$$

then we can invert Eq. (13.10) and write

$$\dot{q}_i = \dot{q}_i(q_k, p_k, t)\,, \tag{13.12}$$

that is, we can express the Lagrange variables in terms of the Hamilton variables and vice versa. In particular, for natural systems, for which the Lagrangian is a quadratic function of the generalised velocities, Eq. (13.10) expresses a linear relation between conjugate momenta and generalised velocities.

Hamilton introduced a function $H(q_k, p_k, t)$, called the *Hamiltonian*, which is defined through the *Legendre transformation*,

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L\,, \tag{13.13}$$

and showed that one can transform the Lagrange equations into a system of first-order differential equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad i = 1, \ldots, n\,, \tag{13.14}$$

which are called the *Hamilton equations*.

In order to prove the Hamilton equations, we shall use *Donkin's theorem*, which states that given a function of n variables, $X(x_1, \ldots, x_n)$, with a non-vanishing Hessian,

$$\left| \frac{\partial^2 X}{\partial x_i \partial x_j} \right| \neq 0\,, \tag{13.15}$$

and given a set of auxiliary variables

$$y_i = \frac{\partial X}{\partial x_i}\,, \qquad i = 1, \ldots, n\,, \tag{13.16}$$

there is a transformation which is the inverse of Eq. (13.16) and is generated by a function $Y(y_1, \ldots, y_n)$, such that

$$x_i = \frac{\partial Y}{\partial y_i}\,, \qquad i = 1, \ldots, n\,. \tag{13.17}$$

The function Y generating the inverse transformation is related to the function X by the Legendre transformation,

$$Y = \sum_{i=1}^{n} x_i y_i - X\,. \tag{13.18}$$

Furthermore, if X depends on m more variables, $\alpha_1, \ldots, \alpha_m$, then Y also depends on these variables,

$$X = X(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m), \qquad Y = Y(y_1, \ldots, y_n; \alpha_1, \ldots, \alpha_m),$$

with the relation

$$\frac{\partial X}{\partial \alpha_k} = -\frac{\partial Y}{\partial \alpha_k}\,, \qquad k = 1, \ldots, m\,. \tag{13.19}$$

Proof of Donkin's theorem. Because the Hessian of X, which is the Jacobian of the y_i, is non-vanishing,

$$\left| \frac{\partial^2 X}{\partial x_i \partial x_j} \right| = \left| \frac{\partial y_i}{\partial x_j} \right| \neq 0\,, \tag{13.20}$$

Eq. (13.16) can be inverted, and we can write $x_i = x_i(y_1, \ldots, y_n)$. Let us take a function $Y(y_1, \ldots, y_n)$ to be given by the Legendre transformation (13.18). Then

$$\begin{aligned} \frac{\partial Y}{\partial y_i} &= \frac{\partial}{\partial y_i}\left(\sum_{k=1}^{n} x_k y_k - X\right) \\ &= \sum_k \frac{\partial x_k}{\partial y_i} y_k + x_i - \frac{\partial X}{\partial y_i} \\ &= \sum_k \frac{\partial x_k}{\partial y_i} y_k + x_i - \sum_k \frac{\partial X}{\partial x_k}\frac{\partial x_k}{\partial y_i} \\ &= x_i \,, \end{aligned} \tag{13.21}$$

where in the last step we used the chain rule and Eq. (13.16).

If X depends on m more variables, $X = X(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m)$, those variables appear in Eq. (13.16) and in the inverse transformation as well,

$$x_i = x_i(y_1, \ldots, y_n; \alpha_1, \ldots, \alpha_m) \,. \tag{13.22}$$

The function Y is given by the Legendre transformation (13.18) with the variables x_i given by the inverse transformation (13.22), so that

$$\begin{aligned} \left.\frac{\partial Y}{\partial \alpha_i}\right|_{y_j} &= \frac{\partial}{\partial \alpha_i}\left(\sum_{k=1}^{n} x_k y_k - X\right) \\ &= \sum_k \frac{\partial x_k}{\partial \alpha_i} y_k - \sum_k \frac{\partial X}{\partial x_k}\frac{\partial x_k}{\partial \alpha_i} - \frac{\partial X}{\partial \alpha_i} \\ &= -\frac{\partial X}{\partial \alpha_i} \,, \end{aligned} \tag{13.23}$$

which completes the proof of Donkin's theorem.

In order to obtain the Hamilton equations, we identify

$$\begin{aligned} X &\equiv L \,, \\ x_i &\equiv \dot{q}_i \,, \\ y_i &\equiv p_i \,, \\ \alpha_i &\equiv q_i, t \,, \end{aligned} \tag{13.24}$$

with $i = 1, \ldots, n$. Then Donkin's theorem implies that given a function $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ and a set of auxiliary variables (13.10), there is a function $H(q_1, \ldots, q_n; p_1, \ldots, p_n; t)$ which generates the inverse transformation of Eq. (13.10),

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \,, \qquad i = 1, \ldots, n \,. \tag{13.25}$$

The Hamiltonian H is related to the Lagrangian L by the Legendre transformation (13.18),

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \,, \tag{13.26}$$

with

$$\begin{aligned} \frac{\partial L}{\partial q_i} &= -\frac{\partial H}{\partial q_i}\,, \\ \frac{\partial L}{\partial t} &= -\frac{\partial H}{\partial t}\,. \end{aligned} \tag{13.27}$$

The Lagrange equations (13.9) and Eq. (13.10) imply that

$$\dot{p}_i = \frac{\partial L}{\partial q_i}\,, \tag{13.28}$$

and using Eq. (13.27), we get

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{13.29}$$

Eqs. (13.25) and (13.29) form a system of $2n$ first-order differential equations, the Hamilton equations (13.14).

The Hamilton equations imply that the total time derivative of the Hamiltonian H equals the partial time derivative,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \left(\frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial q_i}\dot{q}_i \right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}\,. \tag{13.30}$$

From Eq. (7.1) and from the Legendre transformation (13.13), we realise that for a conservative system the Hamiltonian is the total energy of the system. But for a conservative system the total energy is conserved, so

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = 0\,, \tag{13.31}$$

that is, the Hamiltonian $H = H(q_1, \ldots, q_n; p_1, \ldots, p_n)$ is a constant of the motion.

For elementary systems, solving a system of Hamilton equations is not usually simpler than solving the analogous system of Lagrange equations. However, the strength of the Hamilton equations may be appreciated when dealing with complex dynamical systems, like the planetary orbits in celestial mechanics, which can be treated through approximate methods of perturbation theory using Hamiltonian mechanics. Furthermore, the Hamilton equations naturally incorporate the concept of symmetry: if a coordinate q is associated with a symmetry, its conjugate momentum p is conserved and the Hamiltonian does not depend on q, effectively reducing the system to $n-1$ degrees of freedom. That is not apparent with the Lagrange equations, which in that case still depend on n velocities $\dot{q}_i$. Also, although the Hamilton equations cannot be used as such in quantum mechanics, their structure inspires the equations of quantum mechanics. Finally, Hamiltonian mechanics has a natural geometric structure in terms of symplectic manifolds, which we will touch upon through the symplectic notation introduced below. In the following sections, we will briefly explore some of the benefits of Hamiltonian mechanics listed above.

13.2 The Hamiltonian as a conserved quantity and/or the total energy

The simplest Hamiltonian we can consider is that of a particle moving in the x-y plane under the action of a potential. The Lagrangian is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x, y).$$

The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x},$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y},$$

which we invert to

$$\dot{x} = \frac{p_x}{m}, \qquad \dot{y} = \frac{p_y}{m},$$

such that the Lagrangian becomes

$$L = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - V(x, y).$$

The Hamiltonian is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{1}{m}\left(p_x^2 + p_y^2\right) - \frac{1}{2m}\left(p_x^2 + p_y^2\right) + V(x, y) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + V(x, y). \end{aligned}$$

Note that the Hamiltonian is the total energy of the particle, and it is also a conserved quantity, since

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = 0.$$

Let us examine these concepts in more detail. For a conservative system, the kinetic and potential energies do not depend on time. Then neither does the Lagrangian,

$$\frac{\partial L}{\partial t} = 0,$$

and thus

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

is a conserved quantity, which we identify with the Hamiltonian

$$H = \sum_i p_i\dot{q}_i - L.$$

We just saw that for a particle in a potential, H is also the total energy E, and that is in fact true for any conservative system.

One can show that for a conservative system, defined as a holonomic system with

- only stationary constraints,
- a potential energy which does not depend explicitly on time,
- no non-potential forces,

the total energy is conserved:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$

Let us examine the concepts of *total energy* and *conserved quantity* in the Hamiltonian language. We know that if

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = 0,$$

H is a conserved quantity. Under what conditions is H the total energy of the system?

In the Lagrangian formalism, we established that for a holonomic system the Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tilde{Q}_i$$

with $L = T - U$ and the non-potential force $\tilde{Q}_i(q_j, \dot{q}_j, t)$. Let us suppose that the non-potential force $\tilde{Q}_i$ (so called because it does not come from an ordinary potential $U(q_i, t)$) actually comes from a generalised potential $V(q_i, \dot{q}_i, t)$, through the equation

$$\tilde{Q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{q}_i} - \frac{\partial V}{\partial q_i}. \tag{13.32}$$

Then we can still write the Euler–Lagrange equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

if we take $L = T - V$ (with V now including U). Eq. (13.32) implies that

$$\tilde{Q}_i = \sum_n \frac{\partial^2 V}{\partial \dot{q}_i \partial \dot{q}_n}\ddot{q}_n + \dots.$$

However, the generalised force does not depend explicitly on the accelerations $\ddot{q}_n$, since

$$\tilde{Q}_i = \tilde{Q}_i(q_n, \dot{q}_n, t).$$

Thus

$$\frac{\partial^2 V}{\partial \dot{q}_i \partial \dot{q}_n} = 0,$$

so the potential V can be at most linear in the velocities:

$$\begin{aligned} V &= \sum_i a_i \dot{q}_i + U \\ &= V_1 + U, \end{aligned}$$

where U is the ordinary potential. We have already seen an example of such a potential: a particle with charge e in an electromagnetic field has the potential

$$V = e\phi - e\,\dot{\vec{r}} \cdot \vec{A}$$

(cf. Eq. (3.53)).

Next, we restrict ourselves to "natural" systems, i.e. systems for which the kinetic energy, and thus the Lagrangian, are at most quadratic in the velocities (so far we have only met "natural" systems, but there is an important "non-natural" system: a relativistic

particle). However, apart from this restriction, the following discussion is general. The kinetic energy of a natural system is

$$T = T_2 + T_1 + T_0$$

with

$$T_2 = \sum_{i,n} a_{in}\dot{q}_i\dot{q}_n \qquad T_1 = \sum_i b_i\dot{q}_i$$

and the Lagrangian is

$$L = T - V = L_2 + L_1 + L_0$$

with

$$L_2 = T_2 \qquad L_1 = T_1 - V_1 \qquad L_0 = T_0 - U.$$

The Hamiltonian is

$$H = \sum_i p_i\dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L.$$

Then we use Euler's theorem on homogeneous functions,

$$\sum_i \frac{\partial L_2}{\partial \dot{q}_i}\dot{q}_i = 2L_2 \qquad \sum_i \frac{\partial L_1}{\partial \dot{q}_i}\dot{q}_i = L_1,$$

thus

$$\begin{aligned} H &= 2L_2 + L_1 - (L_2 + L_1 + L_0) \\ &= L_2 - L_0, \end{aligned}$$

which is independent of whether a velocity-dependent potential is present or not. In particular,

$$\boxed{H = T_2 - T_0 + U}.$$

For a system with constraints which do not depend explicitly on time, $T_0 = T_1 = 0$ and $T = T_2$; then

$$\boxed{H = T + U}.$$

Thus we have established that the most general condition for a natural system to have the Hamiltonian as the total energy is that the constraints do not depend explicitly on time.

Example 13.1. For a particle in an electromagnetic field, the Lagrangian is

$$L = \frac{m\dot{r}^2}{2} - e\phi + e\,\dot{\vec{r}}\cdot\vec{A}.$$

The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + eA_x \quad \text{and likewise for } y, z.$$

The Hamiltonian is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\ &= \frac{m\dot{r}^2}{2} + e\phi \\ &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi, \end{aligned}$$

i.e. the total energy of the particle.

Now the difference between the Hamiltonian being a *conserved quantity* and being the *total energy* is clear. A system might have a Hamiltonian

$$H = H(q_i, p_i)$$

which does not depend explicitly on time; then

$$\frac{\partial H}{\partial t} = 0$$

and H is a conserved quantity. However, the constraints might be non-stationary,

$$\vec{r}_m = \vec{r}_m(q_1, \ldots, q_n, t);$$

then

$$H = T_2 - T_0 + U$$

is not the total energy.

13.3 Cyclic coordinates

When studying conservation laws, we saw that if a coordinate q_n is *cyclic*, i.e. such that

$$\frac{\partial L}{\partial q_n} = 0,$$

the Euler–Lagrange equations imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_n} = 0.$$

Because the conjugate momentum is

$$p_n = \frac{\partial L}{\partial \dot{q}_n},$$

this means that p_n is conserved, $p_n = c_n$.

In the Hamiltonian formalism, this means that H depends on $2(n-1)$ variables and a constant c_n, to be fixed by the initial conditions,

$$H = H(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}; c_n; t),$$

since q_n does not occur in the Legendre transform

$$H = \sum_i p_i \dot{q}_i - L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t)$$

and

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} = 0 \quad \Rightarrow \quad p_n = c_n.$$

Effectively, the system has gone from n to $n-1$ degrees of freedom (DOF). In contrast, in the Lagrangian formalism the system still depends on n DOF:

$$L = L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n; t).$$

So a Hamiltonian formulation is particularly convenient when cyclic coordinates are present.

Example 13.2. Consider the motion of a particle in a central potential $V(r)$, in polar coordinates r, θ.

The Lagrangian is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).$$

Because the system is conservative, the Hamiltonian is

$$H = T + V(r).$$

The conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r},$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

which can be inverted,

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}.$$

Thus, the Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r).$$

The Hamilton equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$
$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \qquad \text{(the radial equation of motion)}$$
$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad \text{(angular momentum conservation).}$$

Note that θ is cyclic and $p_\theta = M$ is conserved. The Hamiltonian depends on one DOF and one constant, the angular momentum M:

$$H = H(r; p_r; M).$$

13.4 Hamilton's equations and Hamilton's principle

Hamilton's principle requires the action

$$S = \int_{t_i}^{t_f} \mathrm{d}t\, L\left[\left\{q_j(t)\right\}, \left\{\dot{q}_j(t)\right\}; t\right]$$

to be stationary,

$$\delta S = 0,$$

where $\{q_i\}$ is shorthand for $q_1, \ldots, q_n$. In the Hamiltonian formulation, we require that

$$\delta S = \delta \int_{t_i}^{t_f} \mathrm{d}t \left(\sum_i p_i\dot{q}_i - H\left[\left\{p_j\right\}, \left\{q_j\right\}, t\right]\right) = 0$$

for independent variations of the paths $q_i(t)$, $p_i(t)$ in the $2n$-dimensional phase space. The integrand depends on the $4n$ quantities $q, p, \dot{q}, \dot{p}$, treated as independent. Since it contains no $\dot{p}_i$, the variations δp_i need not vanish at the end points (Goldstein, Sect. 8.5).

This is a problem of the calculus of variations: we must extremise the integral

$$I\left[\{q\},\{p\}\right] = \int_{t_i}^{t_f} \mathrm{d}t\, f\left(\{q\},\{p\},\{\dot{q}\},\{\dot{p}\},t\right)$$

where $f = \sum_i p_i\dot{q}_i - H$. Then $\delta I = 0$ leads to the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i} = 0 \tag{13.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{p}_i} - \frac{\partial f}{\partial p_i} = 0 \tag{13.34}$$

which, for f as given above, yield

$$(13.33) \quad \Rightarrow \quad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$

$$(13.34) \quad \Rightarrow \quad -\dot{q}_i + \frac{\partial H}{\partial p_i} = 0.$$

So, in a very concise way, we have obtained Hamilton's equations!

13.5 Canonical transformations

The motion of a particle in a plane under the action of a central potential can be described using Cartesian coordinates x, y or polar coordinates r, θ as generalised coordinates. The advantage of polar coordinates is that θ is cyclic.

In general, given a Hamiltonian

$$H(q_1,\ldots,q_n;p_1,\ldots,p_n;t),$$

the coordinates $q_1,\ldots,q_n$ might not be cyclic, but there might be another set of coordinates $Q_1,\ldots,Q_n$ some or all of which are cyclic. It is then important to have a procedure to go from one set $\left(\{q\},\{p\}\right)$ in the $2n$-dimensional space to another:

$$Q_i = Q_i\left(\{q\};\{p\};t\right),$$
$$P_i = P_i\left(\{q\};\{p\};t\right).$$

The coordinates Q_i, P_i must also be *canonical*, i.e. there must be a function

$$K(Q_1,\ldots,Q_n;P_1,\ldots,P_n;t)$$

such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

So they must satisfy Hamilton's principle,

$$\delta\int_{t_i}^{t_f} \mathrm{d}t\left(\sum_i P_i\dot{Q}_i - K\left(\{Q\};\{P\};t\right)\right) = 0,$$

just like the old coordinates,

$$\delta \int_{t_i}^{t_f} \mathrm{d}t \left(\sum_i p_i \dot{q}_i - H\left(\{q\}; \{p\}; t\right) \right) = 0.$$

Then the two integrands must be the same, up to a total time derivative, whose integral depends only on the end points and has vanishing variation:

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{13.35}$$

In order to read off the transformation, it is convenient to take F to be a function of $2n$ independent variables, half of them old and half of them new, for example $F = F\left(\{q\}, \{Q\}; t\right)$. Then Eq. (13.35) implies that

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i \right) + \frac{\partial F}{\partial t}.$$

Since q_i and Q_i are independent, so are their time derivatives $\dot{q}_i$ and $\dot{Q}_i$, whose coefficients must vanish. Thus, we obtain the transformation equations

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i},$$

and

$$K = H + \frac{\partial F}{\partial t}.$$

The function F is called the *generating function* of the canonical transformation above. Adding $\mathrm{d}F/\mathrm{d}t$ for any function $F(\{q\}, \{p\}, \{Q\}, \{P\}, t)$ would leave the variational principle unchanged; the choice of $2n$ independent arguments, half old and half new, is what allows us to equate the coefficients of the independent $\dot{q}_i$, $\dot{Q}_i$ (Goldstein, Sect. 9.1). More generally, the two integrands may also differ by a constant factor (extended canonical transformations); we do not consider this case here. If we set, in particular,

$$F\left(\{q\}, \{Q\}\right) = \sum_{k=1}^{n} q_k Q_k,$$

then the transformation equations are

$$p_i = \frac{\partial F}{\partial q_i} = Q_i,$$
$$P_i = -\frac{\partial F}{\partial Q_i} = -q_i,$$

i.e. this canonical transformation interchanges momenta and coordinates. In the Hamiltonian formalism, momenta and space coordinates are coordinates of the $2n$-dimensional phase space on an equal footing. There is no memory of q being a space coordinate and p a mass times a velocity.

If p_i cannot be taken as a function of q_j and Q_j, it is suitable to take a function of the type $F = F\left(\{q\}, \{P\}; t\right)$ as a generating function. But since $\dot{P}_i$ does not occur in Eq. (13.35), it is more convenient to set

$$F = G\left(\{q\}, \{P\}; t\right) - \sum_k Q_k P_k$$

such that Eq. (13.35) becomes

$$\sum_i p_i \dot{q}_i - H = -\sum_i Q_i \dot{P}_i - K + \frac{\mathrm{d}G}{\mathrm{d}t}. \tag{13.36}$$

Making the total derivative explicit,

$$\sum_i p_i \dot{q}_i - H = -\sum_i Q_i \dot{P}_i - K + \sum_i \left(\frac{\partial G}{\partial q_i} \dot{q}_i + \frac{\partial G}{\partial P_i} \dot{P}_i \right) + \frac{\partial G}{\partial t},$$

and considering that q_i and P_i are independent, we obtain the transformation equations

$$p_i = \frac{\partial G}{\partial q_i}, \qquad Q_i = \frac{\partial G}{\partial P_i},$$

and

$$K = H + \frac{\partial G}{\partial t}.$$

Note that if we take, in particular,

$$G\left(\{q\}, \{P\}\right) = \sum_{k=1}^{n} q_k P_k, \tag{13.37}$$

then the transformation equations are

$$p_i = \frac{\partial G}{\partial q_i} = P_i,$$
$$Q_i = \frac{\partial G}{\partial P_i} = q_i,$$

and

$$K = H,$$

i.e. the new coordinates equal the old ones: the function (13.37) generates the *identity transformation*.

One can also consider

$$F = F\left(\{p\}, \{Q\}, t\right)$$
$$F = F\left(\{p\}, \{P\}, t\right)$$

as generating functions (a classification of, and relations among, the various F is given in Goldstein's book [2]), and any combination of them. The classification above is not exhaustive: there might be generating functions which do not conform to any of the types given above.

13.6 A canonical transformation for the harmonic oscillator

Consider a harmonic oscillator in one dimension, with force constant k. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

where one can set $k = m\omega^2$, with ω the angular frequency of the oscillation. Because H is a sum of two squares, if we find a canonical transformation

$$\begin{aligned} p &= f(P)\cos Q \\ q &= \frac{f(P)}{m\omega}\sin Q \end{aligned} \tag{13.38}$$

then

$$\begin{aligned} K = H &= \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right) \\ &= \frac{f^2(P)}{2m} \end{aligned}$$

so that Q is cyclic. We must find a function $f(P)$ which makes the transformation canonical. As a generating function, we try

$$F = \frac{m\omega q^2}{2}\cot Q,$$

then the transformation equations are

$$\begin{aligned} p &= \frac{\partial F}{\partial q} = m\omega q\cot Q, \\ P &= -\frac{\partial F}{\partial Q} = -\frac{m\omega q^2}{2}\left(-1 - \frac{\cos^2 Q}{\sin^2 Q}\right) = \frac{m\omega q^2}{2\sin^2 Q} \end{aligned}$$

which we can solve for q and p,

$$\begin{aligned} p &= \sqrt{2m\omega P}\cos Q \\ q &= \sqrt{\frac{2P}{m\omega}}\sin Q. \end{aligned}$$

Comparing with Eq. (13.38), we see that the transformation is canonical if we choose

$$f(P) = \sqrt{2m\omega P}.$$

Then the Hamiltonian is

$$H = \frac{2m\omega P}{2m} = \omega P.$$

The harmonic oscillator is a conservative system; thus the Hamiltonian is the total energy. Since Q is cyclic, the momentum P is constant:

$$P = \frac{E}{\omega}.$$

Hamilton's equation for $\dot{Q}$ is

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega,$$

which is trivially integrated:

$$Q(t) = \omega t + \alpha.$$

Returning to the original variables gives us

$$q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin(\omega t + \alpha),$$
$$p(t) = \sqrt{2mE}\,\cos(\omega t + \alpha).$$

Accordingly, the phase-space plots also look different for the old variables q, p and the new variables Q, P (see Fig. 13.2). Since

$$\frac{q^2}{\frac{2E}{m\omega^2}} + \frac{p^2}{2mE} = 1,$$

the (q, p) plot is an ellipse with

$$\text{semi-axis along } q: \qquad A = \sqrt{\frac{2E}{m\omega^2}},$$
$$\text{semi-axis along } p: \qquad B = \sqrt{2mE},$$

and the (Q, P) plot is a straight line: $P = E/\omega$ is constant while $Q = \omega t + \alpha$ grows

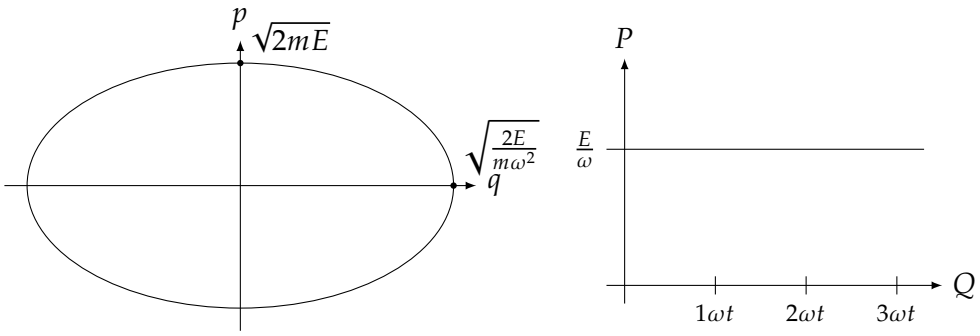


Figure 13.2. Trajectories of the harmonic oscillator in the phase space of (q, p) (ellipse) and of (Q, P) (straight line).

linearly, so the (Q, P) trajectory is a line parallel to the Q axis. (Since q and p have different dimensions, the two semi-axes A and B of the ellipse cannot be compared with each other.)

13.7 Canonical transformations in symplectic notation

For a system of n degrees of freedom, we can cast Hamilton's equations in a compact and useful notation with a deep mathematical meaning.

Let us write the coordinates q_i and the momenta p_i as a vector with $2n$ elements,

$$\omega_i = q_i, \qquad \omega_{i+n} = p_i, \qquad i = 1, \ldots, n.$$

Likewise, we define

$$\left(\frac{\partial H}{\partial \vec{\omega}}\right)_i = \frac{\partial H}{\partial q_i}, \qquad \left(\frac{\partial H}{\partial \vec{\omega}}\right)_{i+n} = \frac{\partial H}{\partial p_i}.$$

Then we introduce the $2n \times 2n$ *symplectic* matrix

$$\mathbf{J} = \left(\begin{array}{c|c} 0 & \mathbf{1}_n \\ \hline -\mathbf{1}_n & 0 \end{array}\right) \qquad \text{with } \mathbf{1}_n \text{ the } n \times n \text{ identity matrix.}$$

Note that the transpose matrix equals the inverse,

$$\mathbf{J}^T = \mathbf{J}^{-1} = \left(\begin{array}{c|c} 0 & -\mathbf{1}_n \\ \hline \mathbf{1}_n & 0 \end{array}\right) = -\mathbf{J};$$

thus

$$\mathbf{J}\mathbf{J}^T = \mathbf{J}^T\mathbf{J} = \left(\begin{array}{c|c} \mathbf{1}_n & 0 \\ \hline 0 & \mathbf{1}_n \end{array}\right) = \mathbf{1}_{2n}$$

and

$$\mathbf{J}^2 = -\mathbf{1}_{2n}.$$

Then Hamilton's equations can be written as

$$\dot{\vec{\omega}} = \mathbf{J}\frac{\partial H}{\partial \vec{\omega}}$$

or, in components,

$$\left(\begin{array}{c} \dot{q}_i \\ \hline \dot{p}_i \end{array}\right) = \left(\begin{array}{c|c} 0 & \mathbf{1}_n \\ \hline -\mathbf{1}_n & 0 \end{array}\right)\left(\begin{array}{c} \frac{\partial H}{\partial q_i} \\ \hline \frac{\partial H}{\partial p_i} \end{array}\right) \quad \Rightarrow \quad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} .$$

Now let us consider a *restricted* canonical transformation (i.e. one with no explicit dependence on time):

$$\begin{aligned} Q_i &= Q_i\left(\{q\},\{p\}\right) \\ P_i &= P_i\left(\{q\},\{p\}\right). \end{aligned} \tag{13.39}$$

Because there is no dependence on time, the new Hamiltonian equals the old one,

$$K = H.$$

The time derivative of Q_i is

$$\begin{aligned} \dot{Q}_i &= \sum_j \left(\frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j\right) \\ &= \sum_j \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right). \end{aligned} \tag{13.40}$$

We can also invert Eq. (13.39),

$$\begin{aligned} q_i &= q_i\left(\{Q\},\{P\}\right), \\ p_i &= p_i\left(\{Q\},\{P\}\right), \end{aligned}$$

and write the Hamiltonian as a function of Q and P. Thus, we can write

$$\frac{\partial H}{\partial P_i} = \sum_j \left[\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i}\right]$$

and, equating it to Eq. (13.40), we obtain

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}, \tag{13.41}$$

where the subscripts remind us that on the left-hand sides Q_i is a function of q, p, and on the right-hand sides q_j, p_j are functions of Q, P. A comparison of $\dot{P}_i$ with $\dfrac{\partial H}{\partial Q_i}$ yields

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}. \tag{13.42}$$

Eqs. (13.41) and (13.42) characterise a (restricted) canonical transformation, and are equivalent to the method of the generating function introduced before. Thus, a canonical transformation can be obtained by finding a suitable generating function or through Eqs. (13.41) and (13.42). Although the procedure above has been displayed for restricted canonical transformations, it can be extended to canonical transformations in general.

Recall that in symplectic notation Hamilton's equations are

$$\dot{\vec{\omega}} = \mathbf{J}\frac{\partial H}{\partial \vec{\omega}} \quad \text{with } \mathbf{J} = \left(\begin{array}{c|c} 0 & \mathbf{1} \\ \hline -\mathbf{1} & 0 \end{array}\right).$$

The restricted canonical transformation can be written as

$$\vec{\Omega} = \vec{\Omega}(\vec{\omega}).$$

Then, taking the time derivative, we have

$$\dot{\vec{\Omega}} = \mathbf{M}\dot{\vec{\omega}}$$

or, in components,

$$\dot{\Omega}_i = \frac{\partial \Omega_i}{\partial \omega_j}\dot{\omega}_j \quad \text{with } M_{ij} = \frac{\partial \Omega_i}{\partial \omega_j}.$$

Using Hamilton's equations, we write

$$\dot{\vec{\Omega}} = \mathbf{M}\mathbf{J}\frac{\partial H}{\partial \vec{\omega}}.$$

Using the inverse transformation, we can regard H as a function of the Ω_i and write

$$\frac{\partial H}{\partial \omega_i} = \frac{\partial H}{\partial \Omega_j}\frac{\partial \Omega_j}{\partial \omega_i} \quad \text{or} \quad \frac{\partial H}{\partial \vec{\omega}} = \mathbf{M}^T\frac{\partial H}{\partial \vec{\Omega}},$$

then

$$\dot{\vec{\Omega}} = \mathbf{M}\mathbf{J}\mathbf{M}^T\frac{\partial H}{\partial \vec{\Omega}}. \tag{13.43}$$

But the Hamiltonian is the same in the old and the new variables, and Hamilton's equations do not change form, so

$$\dot{\vec{\Omega}} = \mathbf{J}\frac{\partial H}{\partial \vec{\Omega}}$$

and, equating with Eq. (13.43), we obtain the condition

$$\boxed{\mathbf{M}\mathbf{J}\mathbf{M}^T = \mathbf{J}}$$

which defines the symplectic matrices, and is called the *symplectic condition* for a canonical transformation.

13.8 Infinitesimal canonical transformations

The canonical transformation

$$\vec{\Omega} = \vec{\Omega}(\vec{\omega}, t)$$

with

$$\omega_i = q_i \qquad \omega_{i+n} = p_i \qquad \Omega_i = Q_i \qquad \Omega_{i+n} = P_i$$

changes continuously with time. If the transformation $\vec{\omega} \to \vec{\Omega}(t)$ is canonical, and so is the transformation $\vec{\omega} \to \vec{\Omega}(t_0)$, then $\vec{\Omega}(t_0) \to \vec{\Omega}(t)$ should be canonical as well. To show this, let us consider an infinitesimal canonical transformation (ICT)

$$\vec{\Omega} = \vec{\omega} + \delta\vec{\omega},$$

or, in components,

$$Q_i = q_i + \delta q_i, \qquad P_i = p_i + \delta p_i.$$

Let us take a generating function of the type

$$G\left(\{q\}, \{P\}, t\right)$$

as a continuous deformation of the identity transformation:

$$G = \sum_k q_k P_k + \epsilon \tilde{G}\left(\{q\}, \{P\}, t\right) \quad \text{with } \epsilon \ll 1.$$

Then, the transformation equations are

$$p_i = \frac{\partial G}{\partial q_i} = P_i + \epsilon \frac{\partial \tilde{G}}{\partial q_i},$$
$$Q_i = \frac{\partial G}{\partial P_i} = q_i + \epsilon \frac{\partial \tilde{G}}{\partial P_i},$$

so the deformations of the canonical variables are

$$\delta p_i = P_i - p_i = -\epsilon \frac{\partial \tilde{G}}{\partial q_i},$$
$$\delta q_i = Q_i - q_i = \epsilon \frac{\partial \tilde{G}}{\partial P_i} = \epsilon \frac{\partial \tilde{G}}{\partial p_i},$$

where in the equation for δq_i we replaced P_i by p_i because the difference is $O(\epsilon)$. Likewise, the generating function of the ICT, $\tilde{G}\left(\{q\}, \{P\}, t\right)$, can be replaced by $\tilde{G}\left(\{q\}, \{p\}, t\right)$. In symplectic notation,

$$\delta\vec{\omega} = \epsilon \mathbf{J} \frac{\partial \tilde{G}}{\partial \vec{\omega}}. \tag{13.44}$$

Now, as an ICT consider

$$\vec{\Omega}(t) = \vec{\Omega}(t_0 + \delta t).$$

The matrix $\mathbf{M}$ of this transformation is

$$M_{ij} = \frac{\partial \Omega_i}{\partial \omega_j} = \delta_{ij} + \frac{\partial \delta\omega_i}{\partial \omega_j}$$

or, using Eq. (13.44),

$$\begin{aligned}\mathbf{M} &= \mathbf{1} + \frac{\partial \delta\vec{\omega}}{\partial\vec{\omega}} \\ &= \mathbf{1} + \epsilon\mathbf{J}\frac{\partial^2\tilde{G}}{\partial\vec{\omega}\partial\vec{\omega}}\end{aligned}$$

or, in components,

$$M_{ij} = \delta_{ij} + \epsilon J_{ik}\frac{\partial^2\tilde{G}}{\partial\omega_k\partial\omega_j}$$

with

$$\frac{\partial^2\tilde{G}}{\partial\omega_k\partial\omega_j} = \frac{\partial^2\tilde{G}}{\partial\omega_j\partial\omega_k}.$$

But

$$\mathbf{J}^T = -\mathbf{J} \quad \Rightarrow \quad J_{ik} = -J_{ki};$$

then

$$\left(M_{ji}\right)^T = M_{ij} = \delta_{ij} - \epsilon\frac{\partial^2\tilde{G}}{\partial\omega_j\partial\omega_k}J_{ki}$$

i.e.

$$\mathbf{M}^T = \mathbf{1} - \epsilon\frac{\partial^2\tilde{G}}{\partial\vec{\omega}\partial\vec{\omega}}\mathbf{J}$$

so

$$\begin{aligned}\mathbf{M}\mathbf{J}\mathbf{M}^T &= \left(\mathbf{1} + \epsilon\mathbf{J}\frac{\partial^2\tilde{G}}{\partial\vec{\omega}\partial\vec{\omega}}\right)\mathbf{J}\left(\mathbf{1} - \epsilon\frac{\partial^2\tilde{G}}{\partial\vec{\omega}\partial\vec{\omega}}\mathbf{J}\right) \\ &= \mathbf{J} + O\left(\epsilon^2\right)\end{aligned}$$

which is the symplectic condition. So the ICT $\vec{\Omega}(t) = \vec{\Omega}\left(t_0 + \delta t\right)$ is indeed canonical, and we conclude that any canonical transformation, with or without time dependence, fulfils the symplectic condition.

We have seen that:

- we can continuously transform the canonical variables through a canonical transformation,
- the identity transformation is canonical.

Now, the inverse of a canonical transformation is also a canonical transformation; the composition of two canonical transformations is also a canonical transformation, and that composition is associative:

$$a(bc) = (ab)c.$$

Therefore, the canonical transformations form a (continuous) group.

13.9 Integrals of the equations of motion, Poisson brackets and canonical invariants

The primary goal of a canonical transformation is to find a set of canonical variables with as many cyclic coordinates as possible.

More generally, suppose that $f\left(\{q\},\{p\},t\right)$ is an integral of the equations of motion (IEM),

$$\frac{\mathrm{d}f}{\mathrm{d}t}=0,$$

i.e. f is a conserved quantity:

$$f\left(\{q\},\{p\},t\right)=c.$$

For example, we know that for a conservative system the Hamiltonian is a conserved quantity, or IEM, and that for a system with cyclic coordinates q_i, the conjugate momenta p_i are conserved quantities, or IEM.

Suppose that we have k IEM, $f_1,\ldots,f_k$, with $k<2n$. Then any arbitrary function of those k IEM is also an IEM. Thus, it is clear that it suffices to consider only the independent IEM.

If we know $2n$ independent IEM,

$$f_k\left(\{q\},\{p\},t\right)=c_k,\quad k=1,\ldots,2n,$$

then the solutions of the equations of motion can be written as

$$\begin{aligned}q_i&=q_i\left(c_1,\ldots,c_{2n},t\right),\\p_i&=p_i\left(c_1,\ldots,c_{2n},t\right),\end{aligned}$$

i.e. the system is completely integrable. The broader goal is to find as many independent IEM as possible.

If $f\left(\{q\},\{p\},t\right)$ is an IEM, then

$$\frac{\mathrm{d}f}{\mathrm{d}t}=0\quad\Rightarrow\quad\sum_i\left(\frac{\partial f}{\partial q_i}\dot{q}_i+\frac{\partial f}{\partial p_i}\dot{p}_i\right)+\frac{\partial f}{\partial t}=0,$$

using Hamilton's equations

$$\begin{aligned}\frac{\partial f}{\partial t}+\sum_i\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)&=0\\\text{or}\quad\frac{\partial f}{\partial t}+\left[f,H\right]_{q,p}&=0,\end{aligned}$$

where we have introduced the *Poisson bracket*

$$\left[\phi,\psi\right]_{q,p}=\sum_i\left(\frac{\partial\phi}{\partial q_i}\frac{\partial\psi}{\partial p_i}-\frac{\partial\phi}{\partial p_i}\frac{\partial\psi}{\partial q_i}\right).$$

Thus, saying that f is an IEM is equivalent to requiring that

$$\frac{\partial f}{\partial t}+\left[f,H\right]_{q,p}=0.$$

This offers a criterion to test candidate IEM: in order for f to be an IEM, it must fulfil

$$\frac{\partial f}{\partial t}=\left[H,f\right]_{q,p}$$

and, if a particular f does not depend explicitly on time, it must fulfil

$$\left[H,f\right]_{q,p}=0.$$

The Poisson bracket $\left[\phi, \psi\right]_{q,p}$ has a symplectic structure. In fact,

$$\left[\phi, \psi\right]_{\vec{\omega}} = \left(\frac{\partial \phi}{\partial \vec{\omega}}\right)^T \mathbf{J} \frac{\partial \psi}{\partial \vec{\omega}} \quad \text{for} \quad \omega_i = q_i \qquad \omega_{i+n} = p_i$$

as can be seen by writing $\left[\phi, \psi\right]_{\vec{\omega}}$ in components (check this!). If ϕ, ψ are taken among the variables q_i, p_i themselves, then from the definition of the Poisson bracket

$$\begin{aligned} \left[q_j, q_k\right]_{q,p} &= \left[p_j, p_k\right]_{q,p} = 0 \\ \left[q_j, p_k\right]_{q,p} &= \delta_{jk}, \end{aligned}$$

thus we can write the antisymmetric matrix $\mathbf{J}$ as the Poisson bracket matrix

$$[\vec{\omega}, \vec{\omega}]_{\vec{\omega}} = \mathbf{J} \quad \text{with} \quad \left([\vec{\omega}, \vec{\omega}]_{\vec{\omega}}\right)_{jk} = \left[\omega_j, \omega_k\right]$$

(again, write it explicitly in components). Now take a set of new canonical variables,

$$\Omega_i = Q_i, \qquad \Omega_{i+n} = P_i;$$

then, by definition,

$$\left[\vec{\Omega}, \vec{\Omega}\right]_{\vec{\omega}} = \frac{\partial \vec{\Omega}}{\partial \vec{\omega}} \mathbf{J} \left(\frac{\partial \vec{\Omega}}{\partial \vec{\omega}}\right)^T,$$

but $\dfrac{\partial \vec{\Omega}}{\partial \vec{\omega}}$ is the transformation matrix $\mathbf{M}$; in components,

$$\frac{\partial \Omega_i}{\partial \omega_j} = M_{ij},$$

then

$$\left[\vec{\Omega}, \vec{\Omega}\right]_{\vec{\omega}} = \mathbf{M}\mathbf{J}\mathbf{M}^T.$$

If the transformation is canonical,

$$\mathbf{M}\mathbf{J}\mathbf{M}^T = \mathbf{J} \quad \Rightarrow \quad \left[\vec{\Omega}, \vec{\Omega}\right]_{\vec{\omega}} = \mathbf{J},$$

i.e. the *fundamental* Poisson brackets (those made out of the canonical variables) are invariant under canonical transformations: they are *canonical invariants*.

Now we show that any Poisson bracket is a canonical invariant. Consider the Poisson bracket

$$\left[\phi, \psi\right]_{\vec{\omega}}.$$

We know that

$$\frac{\partial H}{\partial \vec{\omega}} = \mathbf{M}^T \frac{\partial H}{\partial \vec{\Omega}}.$$

Likewise, we can write

$$\frac{\partial \psi}{\partial \vec{\omega}} = \mathbf{M}^T \frac{\partial \psi}{\partial \vec{\Omega}},$$

and, analogously,

$$\left(\frac{\partial \phi}{\partial \vec{\omega}}\right)^T = \left(\mathbf{M}^T \frac{\partial \phi}{\partial \vec{\Omega}}\right)^T = \left(\frac{\partial \phi}{\partial \vec{\Omega}}\right)^T \mathbf{M},$$

so

$$\begin{aligned}\left[\phi,\psi\right]_{\vec{\omega}} &= \left(\frac{\partial\phi}{\partial\vec{\omega}}\right)^T \mathbf{J}\frac{\partial\psi}{\partial\vec{\omega}} \\ &= \left(\frac{\partial\phi}{\partial\vec{\Omega}}\right)^T \mathbf{M}\mathbf{J}\mathbf{M}^T\frac{\partial\psi}{\partial\vec{\Omega}} \\ &= \left(\frac{\partial\phi}{\partial\vec{\Omega}}\right)^T \mathbf{J}\frac{\partial\psi}{\partial\vec{\Omega}} \\ &= \left[\phi,\psi\right]_{\vec{\Omega}},\end{aligned}$$

thus any Poisson bracket is a canonical invariant. It therefore does not matter which set of canonical variables we use to compute it, and we can drop the subscript of the symbol $[\cdot,\cdot]$. The Poisson brackets have the following properties:

(i) $\left[\phi,\psi\right] = -\left[\psi,\phi\right]$ (antisymmetry)

(ii) $\left[a\phi + b\psi,\chi\right] = a\left[\phi,\chi\right] + b\left[\psi,\chi\right]$, with a, b constants (linearity)

(iii) $\left[\phi\psi,\chi\right] = \left[\phi,\chi\right]\psi + \phi\left[\psi,\chi\right]$

(iv) $\frac{\partial}{\partial t}\left[\phi,\psi\right] = \left[\frac{\partial\phi}{\partial t},\psi\right] + \left[\phi,\frac{\partial\psi}{\partial t}\right]$

(v) $\left[\left[\phi,\psi\right],\chi\right] + \left[\left[\psi,\chi\right],\phi\right] + \left[\left[\chi,\phi\right],\psi\right] = 0.$

All but (v) follow directly from the definition of the Poisson bracket. Property (v) is called the *Jacobi identity*. To prove it, let us write

$$\left[\phi,\psi\right] = \frac{\partial\phi}{\partial\omega_i}J_{ij}\frac{\partial\psi}{\partial\omega_j} \qquad \text{with } J_{ij} = -J_{ji},$$

where sums over repeated (dummy) indices are understood. Then

$$\begin{aligned}\left[\left[\phi,\psi\right],\chi\right] &= \frac{\partial\left[\phi,\psi\right]}{\partial\omega_i}J_{ij}\frac{\partial\chi}{\partial\omega_j} \\ &= \frac{\partial}{\partial\omega_i}\left[\frac{\partial\phi}{\partial\omega_k}J_{kl}\frac{\partial\psi}{\partial\omega_l}\right]J_{ij}\frac{\partial\chi}{\partial\omega_j} \\ &= \left(\frac{\partial^2\phi}{\partial\omega_i\partial\omega_k}J_{kl}\frac{\partial\psi}{\partial\omega_l} + \frac{\partial\phi}{\partial\omega_k}J_{kl}\frac{\partial^2\psi}{\partial\omega_i\partial\omega_l}\right)J_{ij}\frac{\partial\chi}{\partial\omega_j}.\end{aligned}$$

Likewise, we can write $\left[\left[\psi,\chi\right],\phi\right]$ and $\left[\left[\chi,\phi\right],\psi\right]$. In total, we will have six terms, each with a fourfold sum over i, j, k, l. In particular,

$$\left[\left[\chi,\phi\right],\psi\right] = \left(\frac{\partial^2\chi}{\partial\omega_k\partial\omega_j}J_{ji}\frac{\partial\phi}{\partial\omega_i} + \frac{\partial\chi}{\partial\omega_j}J_{ji}\frac{\partial^2\phi}{\partial\omega_k\partial\omega_i}\right)J_{kl}\frac{\partial\psi}{\partial\omega_l}.$$

Let us collect the second derivatives of ϕ:

$$J_{kl}\left(J_{ij} + J_{ji}\right)\frac{\partial^2\phi}{\partial\omega_i\partial\omega_k}\frac{\partial\psi}{\partial\omega_l}\frac{\partial\chi}{\partial\omega_j} = 0$$

where we used that $\partial^2\phi$ is symmetric in its indices and J_{ij} is antisymmetric. Likewise, one shows that the coefficients of $\partial^2\psi$ and $\partial^2\chi$ vanish, which proves the Jacobi identity.

Note that the Poisson bracket is not an associative composition rule. In fact,

$$\begin{aligned}\left[\phi,\left[\psi,\chi\right]\right] &= -\left[\left[\psi,\chi\right],\phi\right] \\ &= \left[\left[\phi,\psi\right],\chi\right] + \left[\left[\chi,\phi\right],\psi\right] \\ &\neq \left[\left[\phi,\psi\right],\chi\right].\end{aligned}$$

The Poisson brackets form a non-associative algebra. In particular, the properties of the Poisson brackets (antisymmetry, linearity, Jacobi identity) define a *Lie algebra*.

We have studied in some depth the properties of the Poisson brackets. We can now prove the *Jacobi–Poisson theorem*, which states that if f and g are IEM then $\left[f,g\right]$ is also an IEM.

Proof. Since f and g are IEM, we can write

$$\frac{\partial f}{\partial t} + \left[f,H\right] = 0, \qquad \frac{\partial g}{\partial t} + \left[g,H\right] = 0,$$

then

$$\begin{aligned}\frac{\partial\left[f,g\right]}{\partial t} &= \left[\frac{\partial f}{\partial t},g\right] + \left[f,\frac{\partial g}{\partial t}\right] \\ &= -\left[\left[f,H\right],g\right] - \left[f,\left[g,H\right]\right] \\ &= \left[\left[H,f\right],g\right] + \left[\left[g,H\right],f\right],\end{aligned}$$

so, thanks to the Jacobi identity,

$$\frac{\partial\left[f,g\right]}{\partial t} + \left[\left[f,g\right],H\right] = 0,$$

so $\left[f,g\right]$ is also an IEM. □

If we have k independent IEM, with $k < 2n$, the repeated application of Poisson brackets between independent IEM may in principle generate the missing IEM.

13.10 Conservation theorems and canonical transformations through the Poisson bracket

In the Lagrangian formalism, we established through Noether's theorem that if the equations of motion are invariant under symmetry transformations, there are corresponding conserved quantities. We saw that if the symmetry transformation is a time shift, the conserved quantity is the energy; if it is a space shift, the conserved quantity is the momentum; if it is a rotation, angular momentum is conserved.

We also saw that continuous symmetry transformations form a group, and that the generators of the transformation form a Lie algebra.

Now we look at the conservation laws through the Hamiltonian formalism and the Poisson brackets.

Recall that a continuous deformation of the identity transformation is generated by the function

$$G = \sum_k q_k P_k + \epsilon\tilde{G}\left(\left\{q\right\},\left\{p\right\},t\right).$$

The ICT is

$$\vec{\Omega} = \vec{\omega} + \delta\vec{\omega} \quad \text{with } \delta\vec{\omega} = \epsilon \mathbf{J} \frac{\partial \tilde{G}}{\partial \vec{\omega}}.$$

Recall that the Poisson bracket is defined as

$$\left[\phi, \psi\right] = \left(\frac{\partial \phi}{\partial \vec{\omega}}\right)^T \mathbf{J} \frac{\partial \psi}{\partial \vec{\omega}}.$$

Then, in particular,

$$\left[\vec{\omega}, \psi\right] = \mathbf{J} \frac{\partial \psi}{\partial \vec{\omega}},$$

so we can write the ICT as

$$\delta\vec{\omega} = \epsilon \left[\vec{\omega}, \tilde{G}\right].$$

Now suppose that the continuous parameter of the ICT is the time t, such that $\epsilon = \mathrm{d}t$. Then the change in the Hamiltonian under the ICT is

$$K = H + \frac{\partial G}{\partial t} = H + \epsilon \frac{\partial \tilde{G}}{\partial t} \quad \Rightarrow \quad \delta H = \mathrm{d}t \frac{\partial \tilde{G}}{\partial t}$$

and we can take the Hamiltonian as the generator $\tilde{G}$. Then

$$\delta\vec{\omega} = \mathrm{d}t \left[\vec{\omega}, H\right] = \mathrm{d}t \dot{\vec{\omega}} = \mathrm{d}\vec{\omega},$$

i.e. the motion from a time t to $t + \mathrm{d}t$ is described by an ICT generated by the Hamiltonian. So the Hamiltonian is the generator of the time shifts.

If $\tilde{G}$ is a constant of motion, it generates an ICT which does not change the Hamiltonian. Thus, the constants of motion are the generators of the canonical transformations which leave the Hamiltonian invariant. Here we mean the change of the value of H when the phase-space point is moved by the ICT (active view), $\delta H = H(\vec{\omega} + \delta\vec{\omega}) - H(\vec{\omega}) = \epsilon \left[H, \tilde{G}\right]$, rather than the new Hamiltonian $K = H + \partial G/\partial t$ of the passive view. For $\tilde{G}$ without explicit time dependence, δH vanishes if and only if $\tilde{G}$ is a constant of motion (Goldstein, Sect. 9.6).

We know that if a coordinate q_i is cyclic, the Hamiltonian is independent of q_i, so H is invariant under an ICT that changes q_i only. Now take

$$\tilde{G}\left(\left\{q\right\}, \left\{p\right\}, t\right) = p_i,$$

then the transformation equations are

$$\begin{aligned} \delta q_j &= \epsilon \delta_{ij}, \\ \delta p_j &= 0, \end{aligned}$$

so only q_i changes, as required, and the conjugate momentum p_i is a constant of motion. In symplectic notation, we can write the generator $\tilde{G}$ as

$$\tilde{G}_i = (\mathbf{J}\vec{\omega})_i = J_{ik}\omega_k,$$

then $\delta\vec{\omega} = \epsilon \mathbf{J} \frac{\partial \tilde{G}}{\partial \vec{\omega}}$ implies

$$\begin{aligned} \delta\omega_j &= \epsilon J_{jl} \frac{\partial \tilde{G}_i}{\partial \omega_l} \\ &= \epsilon J_{jl} J_{ik} \delta_{kl} \\ &= \epsilon J_{jk} J_{ik} = \epsilon J_{jk} J^T_{ki} \\ &= \epsilon \delta_{ij}. \end{aligned}$$

So a shift of any canonical variable ω_i is generated by the conjugate variable through $\tilde{G}$ as above. In particular, if $\omega_i = q_i$, $\tilde{G}_i = p_i$; if $\omega_i = p_i$, $\tilde{G}_i = -q_i$. Thus we can say that the momentum is the generator of space shifts or, in full generality, that the conjugate variable is the generator of shifts of a phase-space variable.

Now consider the ICT which yields a rotation by $\delta\theta$ about the z-axis. Under the rotation, the changes in the coordinates are

$$\delta x = -y\delta\theta, \qquad \delta y = x\delta\theta, \qquad \delta z = 0.$$

The conjugate momentum rotates in the same way, so

$$\delta p_x = -p_y\delta\theta, \qquad \delta p_y = p_x\delta\theta, \qquad \delta p_z = 0.$$

The transformation equations are

$$\delta p = -\epsilon\frac{\partial\tilde{G}}{\partial q},$$
$$\delta q = \epsilon\frac{\partial\tilde{G}}{\partial p},$$

and it is easy to check that the rotations above are generated by

$$\tilde{G} = xp_y - yp_x,$$

which is the z-component of angular momentum:

$$\tilde{G} = M_z = \left(\vec{r}\times\vec{p}\right)_z.$$

So the generating function of an infinitesimal rotation about an axis $\vec{n}$ is, in general,

$$\tilde{G} = \overrightarrow{M}\cdot\vec{n}.$$

Let us now consider finite canonical transformations. Let us take the change of a function $\phi\left(\{q\},\{p\},t\right)$ under an ICT, where the trajectory in phase space is parametrised by α:

$$\delta\phi = \phi\left(\vec{\omega}+\delta\vec{\omega}\right) - \phi(\vec{\omega}) = \left(\frac{\partial\phi}{\partial\vec{\omega}}\right)^T\delta\vec{\omega},$$

but $\delta\vec{\omega} = \mathrm{d}\alpha\mathbf{J}\dfrac{\partial\tilde{G}}{\partial\vec{\omega}}$, so

$$\delta\phi = \mathrm{d}\alpha\left(\frac{\partial\phi}{\partial\vec{\omega}}\right)^T\mathbf{J}\frac{\partial\tilde{G}}{\partial\vec{\omega}}.$$

Recalling the definition of the Poisson bracket, we can write

$$\delta\phi = \mathrm{d}\alpha\left[\phi,\tilde{G}\right] \quad \text{or} \quad \frac{\partial\phi}{\partial\alpha} = \left[\phi,\tilde{G}\right].$$

We can take the variation of $\left[\phi,\tilde{G}\right]$ and likewise find

$$\frac{\partial^2\phi}{\partial\alpha^2} = \frac{\partial\left[\phi,\tilde{G}\right]}{\partial\alpha} = \left[\left[\phi,\tilde{G}\right],\tilde{G}\right].$$

Then, if we Taylor-expand the function ϕ about the initial value $\alpha = 0$, we get

$$\begin{aligned}\phi(\alpha) &= \phi(0) + \left.\frac{\partial\phi}{\partial\alpha}\right|_{\alpha=0} \alpha + \left.\frac{\partial^2\phi}{\partial\alpha^2}\right|_{\alpha=0} \frac{\alpha^2}{2} + \dots \\ &= \phi(0) + \left[\phi, \tilde{G}\right] \alpha + \left[\left[\phi, \tilde{G}\right], \tilde{G}\right] \frac{\alpha^2}{2} + \dots\end{aligned}$$

which can be formally exponentiated,

$$\phi(\alpha) = \phi(0)\, \mathrm{e}^{\hat{G}\alpha}\Big|_{\alpha=0},$$

where the exponential is just shorthand for the power series of nested Poisson brackets above. The power series can be used in practice, though.

Example 13.3. Take the ICT of a rotation about the z-axis. Then

$$\delta\vec{\omega} = \mathrm{d}\theta\, [\vec{\omega}, M_z],$$

with

$$\delta x = -y\, \mathrm{d}\theta, \qquad \delta y = x\, \mathrm{d}\theta, \qquad M_z = xp_y - yp_x,$$

and it is immediate to check that

$$[x, M_z] = -y, \qquad \left[y, M_z\right] = x.$$

Then we can build the nested Poisson brackets

$$\begin{aligned}[[x, M_z], M_z] &= -\left[y, M_z\right] = -x \\ [[[x, M_z], M_z], M_z] &= -[x, M_z] = y \\ [[[[x, M_z], M_z], M_z], M_z] &= \left[y, M_z\right] = x \quad \text{and so on,}\end{aligned}$$

and the power series for the rotated variable is

$$\begin{aligned}X &= x - y\theta - x\frac{\theta^2}{2} + y\frac{\theta^3}{3!} + x\frac{\theta^4}{4!} + \dots \\ &= x\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots\right) - y\left(\theta - \frac{\theta^3}{3!} + \dots\right) \\ &= x\cos\theta - y\sin\theta\end{aligned}$$

which corresponds to a finite rotation about the z-axis.

Finally, if the generator of the ICT is the Hamiltonian and the parameter is the time t, the formal power series becomes

$$\phi(t) = \phi(0) + \left[\phi, H\right] t + \left[\left[\phi, H\right], H\right] \frac{t^2}{2} + \left[\left[\left[\phi, H\right], H\right], H\right] \frac{t^3}{3!} + \dots$$

which is formally exponentiated as

$$\phi(t) = \phi(0)\, \mathrm{e}^{\hat{H}t}\Big|_{t=0}.$$

This has formal analogies with the Heisenberg picture of quantum mechanics.

13.11 More on angular momentum and Poisson brackets

Let us take the ICT of a rotation by an angle $\mathrm{d}\theta$ about an axis $\vec{n}$. We can write

$$\delta\vec{\omega} = \mathrm{d}\theta \left[\vec{\omega}, \overrightarrow{M} \cdot \vec{n}\right]$$

where $\overrightarrow{M}$ is the angular momentum. However, we know that under a rotation by $\mathrm{d}\theta$ about $\vec{n}$, a generic vector $\vec{v}$ changes as

$$\delta\vec{v} = \mathrm{d}\theta\, \vec{n} \times \vec{v};$$

then, taking the change in $\vec{v}$ as the ICT, we obtain

$$\left[\vec{v}, \overrightarrow{M} \cdot \vec{n}\right] = \vec{n} \times \vec{v},$$

or, in components,

$$\begin{aligned} \left[v_i, M_j n_j\right] &= \epsilon_{ijk} n_j v_k, \\ \text{i.e. } \left[v_i, M_j\right] &= \epsilon_{ijk} v_k. \end{aligned}$$

In particular, if we take $\vec{v} = \overrightarrow{M}$ we have the relations

$$\left[\overrightarrow{M}, \overrightarrow{M} \cdot \vec{n}\right] = \vec{n} \times \overrightarrow{M} \tag{13.45}$$

or, in components,

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k. \tag{13.46}$$

Eq. (13.45) implies that

$$\left[M^2, \overrightarrow{M} \cdot \vec{n}\right] = 0,$$

and Eq. (13.46) is formally equivalent to the relation for the generators of the Lie algebra of SO(3), the group of rotations in three dimensions (recall that for the Lie algebra generators, $[\cdot,\cdot]$ is a commutator of the matrices which represent the generators, while here $[\cdot,\cdot]$ is a Poisson bracket; they share the same formal properties). More on this later.

Note that if M_x and M_y are constants of motion, the Jacobi–Poisson theorem says that M_z is also a constant of motion, since

$$\left[M_x, M_y\right] = M_z.$$

In addition, if we take the generic vector $\vec{v}$ to be the momentum $\vec{p}$, then

$$\left[\vec{p}, \overrightarrow{M} \cdot \vec{n}\right] = \vec{n} \times \vec{p},$$

or, in components,

$$\left[p_i, M_j\right] = \epsilon_{ijk} p_k,$$

so if p_z is conserved, then

$$\begin{aligned} \left[p_z, M_x\right] &= p_y, \\ \left[p_z, M_y\right] &= -p_x, \end{aligned}$$

and we see that p_x and p_y are also conserved. So if we started from M_x, M_y, p_z as constants of motion, or IEM, then the Jacobi–Poisson theorem says that M_z, p_x, p_y are also IEM; thus $\overrightarrow{M}$ and $\vec{p}$ are IEM.

However, note that had we started from p_x, p_y, M_z as IEM, we would have found no new IEM through the Jacobi–Poisson theorem, so one must rely on the kindness of Nature for the IEM that are given to begin with!

Exercise 13.1. Check the statement above.

Also note that for two canonical momenta

$$\left[p_i, p_j\right] = 0.$$

But

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k,$$

so two components of the angular momentum cannot both be canonical momenta. However, the magnitude of the total angular momentum,

$$\vec{M} = \sum_n \vec{r}_n \times \vec{p}_n,$$

and one of its components can be, since

$$\left[M^2, \vec{M} \cdot \vec{n}\right] = 0.$$

This also has a formal analogue in quantum mechanics, where the operators $\hat{M}_i$, $\hat{M}_j$ $(i \neq j)$ do not commute and cannot be simultaneously sharp, while $\hat{M}^2$ and one $\hat{M}_i$ commute and can be diagonalised simultaneously. Note that $[M^2, M_z] = 0$ is necessary but not sufficient for $|\vec{M}|$ and M_z to be canonical momenta; that they are is shown e.g. by the action-angle (Delaunay) variables of the Kepler problem (Goldstein, Ch. 10).

13.12 Kepler problem in the Hamiltonian formalism

We saw that the angular momentum $\vec{M}$ is the generator of rotations, and that it satisfies the relation

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k.$$

We also saw previously that the generators of the Lie algebra of SO(3), the group of rotations in three dimensions, are obtained from the rotation of a vector,

$$\delta\vec{v} = \delta\theta\, \vec{n} \times \vec{v},$$

and fulfil the commutation relation

$$\left[J_i, J_j\right] = \mathrm{i}\epsilon_{ijk} J_k.$$

The generators have the matrix representation

$$(J_i)_{jk} = -\mathrm{i}\epsilon_{ijk},$$

$$J_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We noted the formal analogy between the Poisson bracket of the components of the angular momentum,

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k,$$

and the commutator of the matrices

$$\left[J_i, J_j\right] = \mathrm{i}\epsilon_{ijk} J_k.$$

Now let us consider again the Kepler problem, with the central potential

$$V(r) = -\frac{a}{r}.$$

The angular momentum $\vec{M}$ is conserved, but we know that another quantity, the Laplace–Runge–Lenz (LRL) vector,

$$\vec{A} = \vec{p} \times \vec{M} - \frac{ma\vec{r}}{r},$$

is also conserved. Because $\vec{A}$ is a vector, we can take the relation

$$\left[v_i, M_j\right] = \epsilon_{ijk} v_k$$

and immediately establish that

$$\left[A_i, M_j\right] = \epsilon_{ijk} A_k.$$

With some more work, it can be shown that

$$[A_1, A_2] = -\left(p^2 - \frac{2ma}{r}\right) M_3.$$

Exercise 13.2. Check this relation.

Note that the energy E is

$$E = T + V = \frac{p^2}{2m} - \frac{a}{r}$$

so

$$[A_1, A_2] = -2mEM_3,$$

where $E < 0$ for a bound orbit. We can rescale the LRL vector,

$$\vec{D} \equiv \frac{\vec{A}}{\sqrt{-2mE}},$$

thus we have

$$[D_1, D_2] = M_3.$$

Cyclically permuting the indices, we obtain

$$\left[D_i, D_j\right] = \epsilon_{ijk} M_k,$$

so the complete set of Poisson brackets for Kepler bound orbits is

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k, \qquad \left[D_i, D_j\right] = \epsilon_{ijk} M_k, \qquad \left[D_i, M_j\right] = \epsilon_{ijk} D_k.$$

The same relations are fulfilled by the generators of the Lie algebra of SO(4), the group of rotations in four dimensions:

$$\left[J_i, J_j\right] = \mathrm{i}\epsilon_{ijk} J_k, \qquad \left[N_i, N_j\right] = \mathrm{i}\epsilon_{ijk} J_k, \qquad \left[N_i, J_j\right] = \mathrm{i}\epsilon_{ijk} N_k.$$

To see this, let us add a zeroth row and column to the J_i:

$$J_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J_3 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The generators N_i of rotations in the x_0-x_i plane are

$$N_1 = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_3 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that they indeed satisfy the commutation relations above. For unbound orbits, the energy E is positive and we rescale the LRL vector as

$$\vec{C} \equiv \frac{\vec{A}}{\sqrt{2mE}};$$

the complete set of Poisson brackets is then

$$\left[M_i, M_j\right] = \epsilon_{ijk} M_k, \qquad \left[C_i, M_j\right] = \epsilon_{ijk} C_k, \qquad \left[C_i, C_j\right] = -\epsilon_{ijk} M_k.$$

The same relations are fulfilled by the generators of the Lie algebra of the restricted Lorentz group. However, the analogy is purely formal: the Kepler problem belongs to non-relativistic Newtonian mechanics and has nothing to do with special relativity.

13.13 Poincaré (Cartan) integral invariants

The Euler–Lagrange equations are the necessary and sufficient condition for the variation of the action

$$\delta S = \delta \int_{t_i}^{t_f} \mathrm{d}t\, L\left(\{q_i\}, \{\dot{q}_i\}, t\right) = 0$$

of a holonomic system to vanish. We also know that the Hamilton equations follow from the variation of the action in phase space,

$$\delta S = \delta \int_{t_i}^{t_f} \mathrm{d}t \left(\sum_i p_i \dot{q}_i - H\left(\{p_i\}, \{q_i\}, t\right) \right) = 0.$$

In all those variational manipulations, we always required that the initial and final times are fixed. Let us relax this condition, and consider the $(2n+1)$-dimensional space, called the *extended phase space*, with coordinates $\left(\{q\}, \{p\}, t\right)$, and suppose that we can parametrise them through a parameter ζ:

$$q_k = q_k(\zeta), \qquad p_k = p_k(\zeta), \qquad t = t(\zeta).$$

Then the initial and final times also depend on ζ:

$$t_i = t_i(\zeta), \qquad t_f = t_f(\zeta).$$

It can be shown that the variation of the action

$$S = \int_{t_i(\zeta)}^{t_f(\zeta)} \mathrm{d}t\, L\left(\{q\}, \{\dot{q}\}, t\right)$$

equals

$$\delta S = \left| \sum_k p_k \delta q_k - H \delta t \right|_{t_i(\zeta)}^{t_f(\zeta)} + \int_{t_i(\zeta)}^{t_f(\zeta)} \mathrm{d}t \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right].$$

On the trajectories in the extended phase space where the Hamilton equations are fulfilled, the integral term vanishes, and we have

$$\delta S = \left| \sum_k p_k \delta q_k - H \delta t \right|_{t_i(\zeta)}^{t_f(\zeta)}$$

(if, in particular, t_i and t_f are fixed, then $\delta S = 0$). Now let us take an arbitrary closed curve C_i, given by the equations

$$q_k = q_k^i(\zeta), \qquad p_k = p_k^i(\zeta), \qquad t^i = t^i(\zeta), \quad \text{with } 0 \le \zeta \le T,$$

and

$$q_k^i(0) = q_k^i(T), \qquad p_k^i(0) = p_k^i(T), \qquad t^i(0) = t^i(T).$$

Through every point of the contour C_i, we can draw a trajectory using Hamilton's equations. Thus we obtain a cylinder of trajectories (Fig. 13.3). Let us take another arbitrary closed curve C_f on this cylinder, with

$$q_k^f(0) = q_k^f(T), \qquad p_k^f(0) = p_k^f(T), \qquad t^f(0) = t^f(T).$$

Next, we integrate the variation of the action

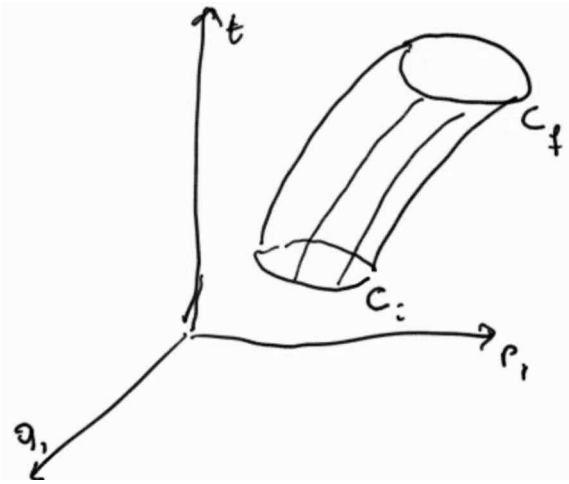


Figure 13.3. A cylinder of trajectories in the extended phase space, with two closed curves C_i and C_f encircling it.

$$\delta S = \frac{\partial S}{\partial \zeta} \delta \zeta = \left| \sum_k p_k \delta q_k - H \delta t \right|_{t_i(\zeta)}^{t_f(\zeta)}$$

between 0 and T:

$$\begin{aligned} S(T) - S(0) &= \int_0^T \left| \sum_k p_k \delta q_k - H \delta t \right|_{t_i(\zeta)}^{t_f(\zeta)} \\ &= \int_0^T \left(\sum_k p_k^f \delta q_k^f - H_f \delta t_f \right) - \int_0^T \left(\sum_k p_k^i \delta q_k^i - H_i \delta t_i \right) \\ &= \oint_{C_f} \left(\sum_k p_k \delta q_k - H \delta t \right) - \oint_{C_i} \left(\sum_k p_k \delta q_k - H \delta t \right), \end{aligned}$$

but $S(T) = S(0)$, since the curves are closed, so

$$\Rightarrow \oint_{C_f} \left(\sum_k p_k \delta q_k - H \delta t \right) = \oint_{C_i} \left(\sum_k p_k \delta q_k - H \delta t \right).$$

Thus we have established that

$$I = \oint \left(\sum_k p_k \delta q_k - H \delta t \right)$$

is an integral invariant. It is called the *Poincaré–Cartan integral invariant* (and in some, more geometric, formulations it is considered to be the fundamental object of mechanics). It is possible to show that the canonical transformations can be obtained from the invariance of the Poincaré–Cartan integral. Several theorems on flows in hydrodynamics also follow from the Poincaré–Cartan integral.

If, in particular, we consider contours C of simultaneous states, i.e. we take the hyperplane of constant time in the cylinder above, then $\delta t = 0$ and we obtain

$$I_1 = \oint \sum_k p_k \delta q_k$$

which is called the *Poincaré integral invariant*.

I_1 is invariant for any Hamiltonian system. Conversely, if for a system given by the equations

$$\dot{q}_k = Q_k \left(\{q\}, \{p\}, t \right), \qquad \dot{p}_k = P_k \left(\{q\}, \{p\}, t \right),$$

I_1 is invariant, it follows that the system is Hamiltonian. Indeed, using $\dfrac{\mathrm{d}}{\mathrm{d}t} \delta q = \delta \dot{q}$ and integrating by parts along the closed contour, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_1 = 0 &= \oint \sum_k \left(\dot{p}_k \delta q_k + p_k \frac{\mathrm{d}}{\mathrm{d}t} \delta q_k \right) \\ &= \oint \sum_k \left(\dot{p}_k \delta q_k + p_k \delta \dot{q}_k \right) \\ &= \oint \sum_k \left(\dot{p}_k \delta q_k - \dot{q}_k \delta p_k \right), \end{aligned}$$

thus

$$\oint \sum_k \left(P_k \delta q_k - Q_k \delta p_k \right) = 0.$$

Since the contour is arbitrary, the integrand must be the total differential of some function. If we call that function

$$-H \left(\{q\}, \{p\}, t \right),$$

then

$$\sum_k \left(P_k \delta q_k - Q_k \delta p_k \right) = -\delta H = -\sum_k \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k \right).$$

It follows that

$$Q_k = \frac{\partial H}{\partial p_k}, \qquad P_k = -\frac{\partial H}{\partial q_k},$$

which implies that the system is Hamiltonian.

Now think of Stokes' theorem in three dimensions,

$$\oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{l} = \iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot \mathrm{d}\vec{S}$$

where ∂S is a contour C which is the boundary of a surface S. In particular, in two dimensions,

$$\oint_{\partial S} A_x \,\mathrm{d}x + A_y \,\mathrm{d}y = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \mathrm{d}x \,\mathrm{d}y$$

which is also called *Green's theorem*.

Stokes' and Green's theorems (as well as Gauss' theorem) are all particular cases of the generalised Stokes theorem for *differential forms*, which states that

$$\oint_{C=\partial S} \omega = \iint_S \mathrm{d}\omega,$$

i.e. the integral of a *form* ω along a closed contour C equals the integral of the form $\mathrm{d}\omega$ over the surface S bounded by the contour C. In a nutshell, consider the differential form in two dimensions

$$\omega = \vec{A} \cdot \mathrm{d}\vec{l} = A_x \,\mathrm{d}x + A_y \,\mathrm{d}y;$$

then the derivative

$$\mathrm{d}\omega = \mathrm{d}\vec{A} \wedge \mathrm{d}\vec{l}$$

is defined through the wedge product $\wedge$, which is linear and antisymmetric, just like the cross product:

$$a \wedge b = -b \wedge a.$$

Then

$$\begin{aligned}
\mathrm{d}\vec{A} \wedge \mathrm{d}\vec{l} &= \mathrm{d}A_x \wedge \mathrm{d}x + \mathrm{d}A_y \wedge \mathrm{d}y \\
&= \frac{\partial A_x}{\partial x} \cancelto{0}{\mathrm{d}x \wedge \mathrm{d}x} + \frac{\partial A_x}{\partial y} \mathrm{d}y \wedge \mathrm{d}x + \frac{\partial A_y}{\partial x} \mathrm{d}x \wedge \mathrm{d}y + \cancelto{0}{\frac{\partial A_y}{\partial y} \mathrm{d}y \wedge \mathrm{d}y} \\
&= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

so

$$\oint \vec{A} \cdot \mathrm{d}\vec{l} = \iint \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y,$$

where $\mathrm{d}x \wedge \mathrm{d}y$ is the (oriented) area element. For our purposes, we can just take it to be $\mathrm{d}x \,\mathrm{d}y$. Thus we have obtained Green's theorem.

Now, take the form

$$\omega = \vec{p} \cdot \mathrm{d}\vec{q} = p_1 \,\mathrm{d}q_1 + \cdots + p_n \,\mathrm{d}q_n$$

in the $2n$-dimensional phase space; then

$$\mathrm{d}\omega = \mathrm{d}\vec{p} \wedge \mathrm{d}\vec{q} = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n.$$

Then Stokes' theorem implies

$$\begin{aligned}
\int_{\partial S=C} \omega &= \iint_S \mathrm{d}\omega \\
\Rightarrow I_1 = \oint_{\partial S=C} \sum_i p_i \,\mathrm{d}q_i &= \iint_S \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i \equiv J_2
\end{aligned}$$

where $\mathrm{d}p_i \wedge \mathrm{d}q_i$ are area elements.

We have related the Poincaré integral invariant I_1 to a two-dimensional integral invariant J_2. In fact, this structure can be iterated, i.e. there is an invariant integral

$$I_3 = \iiint \sum_{i,k} p_i \delta q_i \delta p_k \delta q_k$$

which can be related to

$$J_4 = \iiiint \sum_{i,k} \delta p_i \delta q_i \delta p_k \delta q_k$$

and so on, until one gets to I_{2n-1} and

$$J_{2n} = \int \cdots \int \mathrm{d}q_1 \ldots \mathrm{d}q_n \; \mathrm{d}p_1 \ldots \mathrm{d}p_n$$

which is of particular interest, because it is the volume of a region of phase space. It is particularly simple to see that J_{2n} is an invariant. If

$$\mathrm{d}\vec{\omega} = \mathrm{d}q_1 \ldots \mathrm{d}q_n \; \mathrm{d}p_1 \ldots \mathrm{d}p_n,$$

then under a canonical transformation the volume element becomes

$$\mathrm{d}\vec{\Omega} = \mathrm{d}Q_1 \ldots \mathrm{d}Q_n \; \mathrm{d}P_1 \ldots \mathrm{d}P_n,$$

but $\mathrm{d}\vec{\omega}$ and $\mathrm{d}\vec{\Omega}$ are related by the Jacobian $\det \mathbf{M}$:

$$\mathrm{d}\vec{\Omega} = |\det \mathbf{M}| \, \mathrm{d}\vec{\omega} \quad \text{with } M_{ij} = \frac{\mathrm{d}\Omega_i}{\mathrm{d}\omega_j}$$

and we know that

$$\mathbf{M}\mathbf{J}\mathbf{M}^T = \mathbf{J} \Rightarrow (\det \mathbf{M})^2 = 1,$$

so

$$|\det \mathbf{M}| = 1 \Rightarrow \mathrm{d}\vec{\Omega} = \mathrm{d}\vec{\omega}.$$

The fact that the volume element is invariant under a canonical transformation immediately implies *Liouville's theorem* of statistical mechanics, i.e. that the density of systems around some given system in phase space remains constant in time. In fact, the number of systems in a closed neighbourhood of a given system does not change with time; the closed shape just moves about in phase space (see Fig. 13.4). Canonical variables at different times are related by a canonical transformation, and we know that the volume element J_{2n} is a canonical invariant, so the volume cannot change with time. Therefore, the density of systems is a constant of motion.

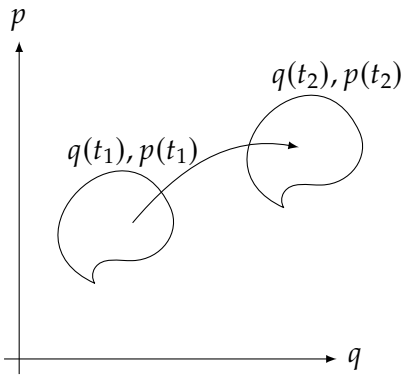


Figure 13.4. A closed region moving with time in a two-dimensional phase space; its area is conserved.

14

Hamilton–Jacobi formalism

14.1 Hamilton–Jacobi equation

Canonical transformations help us to understand the dynamics of a mechanical system, i.e. to solve its equations of motion. They can do that in two ways:

(a) By transforming the Hamiltonian $H\left(q_1,\ldots,q_n;p_1,\ldots,p_n;t\right)$ to a Hamiltonian $K\left(Q_1,\ldots,Q_n;P_1,\ldots,P_n;t\right)$ where all the coordinates Q_i are cyclic. In the new variables, Hamilton's equations are

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Then, because $\dfrac{\partial K}{\partial Q_i} = 0$, $P_i = \text{const.} = c_i$, so the Hamiltonian becomes

$$K = K\left(c_1,\ldots,c_n;t\right).$$

Suppose that the Hamiltonian is conserved,

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\partial K}{\partial t} = 0;$$

then

$$\dot{Q}_i = \frac{\partial K}{\partial c_i} = f_i\left(c_1,\ldots,c_n\right),$$

which can be trivially integrated to

$$Q_i = f_i\left(\{c\}\right)t + \gamma_i,$$

so the system is fully specified in terms of $2n$ constants of motion, c_i and γ_i.

(b) The second way is to transform to a set of $2n$ independent constants of motion (IEM),

$$f_k\left(\{q\},\{p\},t\right) = c_k, \quad k = 1,\ldots,2n,$$

such that

$$q_i = q_i\left(c_1,\ldots,c_{2n},t\right),$$
$$p_i = p_i\left(c_1,\ldots,c_{2n},t\right),$$

so that the system is completely integrable. The $2n$ independent IEM might also be the initial conditions, such that

$$q_i = q_i\left(\{q_0\}, \{p_0\}, t\right)$$
$$p_i = p_i\left(\{q_0\}, \{p_0\}, t\right).$$

Then one looks for a canonical transformation that maps the set of variables $(\{q\}, \{p\})$ to the set $(\{q_0\}, \{p_0\})$.

We shall follow the second avenue. Let us consider a canonical transformation generated by a function $F\left(\{q\}, \{P\}, t\right)$ (the discussion is fully general, and can also be carried out with other generating functions). The transformation equations are

$$p_i = \frac{\partial F}{\partial q_i}, \qquad Q_i = \frac{\partial F}{\partial P_i}, \qquad K = H + \frac{\partial F}{\partial t}.$$

Suppose that we have found a function F, such that the new Hamiltonian vanishes:

$$K = 0 = H + \frac{\partial F\left(\{q\}, \{P\}, t\right)}{\partial t}.$$

Then the new equations of motion can be trivially solved:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \quad \Rightarrow \quad Q_i = \text{const.} = \beta_i$$
$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \quad \Rightarrow \quad P_i = \text{const.} = \alpha_i.$$

The equation $K = 0$ is called the *Hamilton–Jacobi (HJ) equation*:

$$H\left(\{q\}, \left\{\frac{\partial F}{\partial q}\right\}, t\right) + \frac{\partial F}{\partial t} = 0.$$

It is a first-order partial differential equation in the $n+1$ variables $q_1, \ldots, q_n, t$. Its complete solution,

$$F \equiv S\left(q_1, \ldots, q_n; t; \alpha_1, \ldots, \alpha_{n+1}\right),$$

depends on $n+1$ independent constants of integration. S is called *Hamilton's principal function*. Because S appears in the HJ equation only through its derivatives, if S is a solution, $S + \alpha$ is also a solution. So S is determined up to an additive constant, which is irrelevant since the transformation equations are given through derivatives of S. Then

$$S = S\left(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right).$$

Because the generating function is $F\left(\{q\}, \{P\}; t\right)$, we may take the new momenta P_i to be the constants α_i,

$$P_i = \alpha_i, \quad \text{or any combination of them,} \quad P_i = f_i\left(\alpha_1, \ldots, \alpha_n\right).$$

Then the transformation equations are

$$p_i = \frac{\partial S\left(\{q\}, \{\alpha\}, t\right)}{\partial q_i}, \tag{14.1}$$

$$Q_i = \beta_i = \frac{\partial S\left(\{q\}, \{\alpha\}, t\right)}{\partial \alpha_i}. \tag{14.2}$$

At $t = t_0$, Eqs. (14.1) are n equations connecting the α_i to $\{q_0\}$, $\{p_0\}$. The β_i of Eqs. (14.2) can also be obtained from the initial conditions, by computing the right-hand side at $t = t_0$. Solving Eqs. (14.2) for the q_i, one gets

$$q_i = q_i\left(\{\alpha\}, \{\beta\}, t\right)$$

and, through Eq. (14.1), also

$$p_i = p_i\left(\{\alpha\}, \{\beta\}, t\right),$$

which solves the system. To understand the physical meaning of S, let us take the time derivative:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \\ &= \sum_i p_i \dot{q}_i - H = L.\end{aligned}$$

So, up to a constant, S is the action:

$$S = \int L\,\mathrm{d}t + \text{const.}$$

Finally, if the Hamiltonian is conserved,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = 0,$$

then the HJ equation is

$$H\left(\{q\}, \left\{\frac{\partial S}{\partial q}\right\}\right) + \frac{\partial S}{\partial t} = 0$$

and one can separate the variable t from the others,

$$S\left(\{q\}, \{\alpha\}, t\right) = W\left(\{q\}, \{\alpha\}\right) - \alpha_n t,$$

where W is called *Hamilton's characteristic function*. Substituting this into the HJ equation, we get the reduced HJ equation

$$H\left(\{q\}, \left\{\frac{\partial W}{\partial q}\right\}\right) = \alpha_n.$$

For conservative systems, $\alpha_n = E$ is the energy. The physical meaning of W is understood again by taking the time derivative:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \sum_i \frac{\partial W}{\partial q_i}\dot{q}_i = \sum_i p_i \dot{q}_i.$$

Integrating gives us

$$W = \sum_i \int \mathrm{d}t\, p_i \dot{q}_i = \sum_i \int \mathrm{d}q_i\, p_i.$$

Note that W also generates a canonical transformation, different from the one generated by S. The transformation is

$$p_i = \frac{\partial W}{\partial q_i}, \qquad Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i}, \qquad K = H = \alpha_n.$$

Thus W generates a canonical transformation in which all coordinates are cyclic. Hamilton's equations are

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \quad \Rightarrow \quad P_i = \text{const.} = \alpha_i$$
$$\dot{Q}_i = \frac{\partial K}{\partial \alpha_i} = 0 \quad i = 1, \ldots, n-1 \quad \Rightarrow \quad Q_i = \text{const.} = \beta_i$$
$$\dot{Q}_n = \frac{\partial K}{\partial \alpha_n} = 1 \quad \Rightarrow \quad Q_n = t + \beta_n .$$

Note that $W\left(\{q\}, \{\alpha\}\right)$ depends on n constants, but one of them is additive since, in the reduced HJ equation

$$H = H\left(\{q\}, \left\{\frac{\partial W}{\partial q}\right\}\right) = \alpha_n ,$$

W appears only through its derivatives. So W depends on $n-1$ independent non-additive constants, say $\alpha_1, \ldots, \alpha_{n-1}$, which, together with α_n, form a set of n constants, the new momenta P_i. The difference between the canonical transformations generated by S (left column) and by W (right column) is summarised below:

The Hamiltonian is

$H = H\left(\{q\}, \{p\}, t\right)$	an IEM: $H = H\left(\{q\}, \{p\}\right)$

Look for canonical transformations to $(\{Q\}, \{P\})$ such that

$\{Q\}, \{P\}$ are constants	$\{P\}$ are constants

Require that the new Hamiltonian

vanishes: $K = 0$	has $K = H = \alpha_n$, so Q_i are cyclic

Hamilton's equations in the new variables are

$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = v_i\left(\alpha_i\right)$
$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$	$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$

with solutions

$Q_i = \beta_i$	$Q_i = \beta_i + v_i\left(\alpha_i\right) t$
$P_i = \alpha_i$	$P_i = \alpha_i$

The generating function is Hamilton's

principal function $S\left(\{q\}, \{P\}, t\right)$	characteristic function $W\left(\{q\}, \{P\}\right)$

The HJ equation is

$H\left(\{q\}, \left\{\frac{\partial S}{\partial q}\right\}, t\right) + \frac{\partial S}{\partial t} = 0$	$H\left(\{q\}, \left\{\frac{\partial W}{\partial q}\right\}\right) = \alpha_n$

A complete solution has

n non-additive constants $\alpha_1, \ldots, \alpha_n$	$n-1$ non-additive constants $\alpha_1, \ldots, \alpha_{n-1}$ which together with α_n make n independent constants

14.2 Simple harmonic oscillator

The Hamiltonian for the simple harmonic oscillator is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E \qquad \text{with } \omega = \sqrt{\frac{k}{m}}.$$

The Hamilton–Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0.$$

Since $\dfrac{\mathrm{d}H}{\mathrm{d}t} = 0$, we are in the case of the right-hand column of the overview above, so

$$S = W - \alpha t, \qquad \text{with } \alpha = E.$$

The reduced HJ equation is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

which can easily be integrated to

$$W = \sqrt{2m\alpha}\int \mathrm{d}q\,\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} \quad \Rightarrow \quad S = \sqrt{2m\alpha}\int \mathrm{d}q\,\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}} - \alpha t.$$

The transformation equation

$$\beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\int \frac{\mathrm{d}q}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t$$

can also easily be integrated, to

$$\beta + t = \frac{1}{\omega}\arcsin\left(q\sqrt{\frac{m\omega^2}{2\alpha}}\right),$$

that is,

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \gamma) \quad \text{with } \gamma = \beta\omega$$

which is the usual solution for the harmonic oscillator. The other transformation equation yields

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\alpha}\cos(\omega t + \gamma).$$

The constants α and β are related to the initial conditions at $t = t_0$:

$$\alpha = \frac{1}{2m}\left(p_0^2 + m^2\omega^2 q_0^2\right), \qquad \frac{q_0}{p_0} = \frac{1}{m\omega}\tan\left(\omega(t_0 + \beta)\right)$$

So Hamilton's principal function generates a canonical transformation to a new momentum, the energy, and a new coordinate, the phase constant of the oscillation.

14.3 Two-dimensional harmonic oscillator

In two dimensions, the Hamiltonian for the harmonic oscillator is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2\right) \quad \text{with } \omega_x = \sqrt{\frac{k_x}{m}}, \quad \omega_y = \sqrt{\frac{k_y}{m}},$$

and we have again

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \quad \Rightarrow \quad S\left(x, y, \alpha_x, \alpha_y; t\right) = W\left(x, y, \alpha_x, \alpha_y\right) - \alpha t \quad \text{with } \alpha = E.$$

The reduced HJ equation is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2\right] = \alpha$$

which can be separated into

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2\omega_x^2 x^2\right] = \alpha_x$$
$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha_y \quad \text{with } \alpha = \alpha_x + \alpha_y$$

which can easily be solved.

14.4 Separation of variables

In Sec. 14.1, we assumed (very optimistically!) that we know Hamilton's principal function S. But in general it may be very difficult to solve the HJ equation: partial differential equations in many variables are hard to deal with. With the harmonic oscillator, however, we saw that a solution can be found if we can separate the variables. In fact, this is the case in which the HJ equation becomes a useful tool. Suppose that in the HJ equation the coordinate q_1 enters only through the function $H_1\left(q_1, \dfrac{\partial S}{\partial q_1}\right)$, such that

$$H\left(H_1\left(q_1, \frac{\partial S}{\partial q_1}\right); q_2, \ldots, q_n; \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0,$$

then q_1 is called *separable*, and we look for a solution of the type

$$S = S_1\left(q_1; \alpha_1, \ldots, \alpha_n\right) + \overline{S}\left(q_2, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right).$$

If we substitute it into the HJ equation, we get the identity

$$H\left(H_1\left(q_1, \frac{\partial S_1}{\partial q_1}\right); q_2, \ldots, q_n; \frac{\partial \overline{S}}{\partial q_2}, \ldots, \frac{\partial \overline{S}}{\partial q_n}; t\right) + \frac{\partial \overline{S}}{\partial t} = 0.$$

For this to be true for any q_1, it is necessary and sufficient to take

$$H_1 = \text{const.} = \alpha_1.$$

So we get two equations:

$$H_1\left(q_1, \frac{\partial S_1}{\partial q_1}\right) = \alpha_1$$
$$\overline{H}\left(q_2, \ldots, q_n; \frac{\partial \overline{S}}{\partial q_2}, \ldots, \frac{\partial \overline{S}}{\partial q_n}; \alpha_1; t\right) + \frac{\partial \overline{S}}{\partial t} = 0.$$

The first equation is a first-order ordinary differential equation, which is easy to solve. The second equation is still a partial differential equation, but with one less variable. This procedure can be iterated if we have k separable coordinates $q_1, \ldots, q_k$. Then the principal function can be written as

$$S\left(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right) = \sum_{i=1}^{k} S_i\left(q_i; \alpha_1, \ldots, \alpha_n\right) + \overline{S}\left(q_{k+1}, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right)$$

and the HJ equation splits into k first-order ordinary differential equations

$$H_i\left(q_i, \frac{\partial S_i}{\partial q_i}\right) = \alpha_i, \quad i = 1, \ldots, k,$$

and the partial differential equation

$$\overline{H}\left(q_{k+1}, \ldots, q_n; \frac{\partial \overline{S}}{\partial q_{k+1}}, \ldots \frac{\partial \overline{S}}{\partial q_n}; \alpha_1, \ldots, \alpha_k; t\right) + \frac{\partial \overline{S}}{\partial t} = 0.$$

If all the variables q_i and the time t are separable, the principal function can be written as

$$S\left(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right) = \sum_{i=1}^{n} W_i\left(q_i; \alpha_1, \ldots, \alpha_n\right) - \alpha t.$$

The reduced HJ equation is *completely separable* and can be written as

$$H\left(H_1\left(q_1, \frac{\partial W_1}{\partial q_1}\right); \ldots; H_n\left(q_n, \frac{\partial W_n}{\partial q_n}\right)\right) = \alpha.$$

We get n first-order differential equations

$$H_i\left(q_i, \frac{\partial W_i}{\partial q_i}\right) = \alpha_i, \quad i = 1, \ldots, n,$$

and the condition

$$H\left(\alpha_1, \ldots, \alpha_n\right) = \alpha$$

where the α_i are called *separation constants*. Note that if q_1 is separable and cyclic, $\dfrac{\partial H}{\partial q_1} = 0$, then

$$H_1\left(q_1, \frac{\partial S_1}{\partial q_1}\right) = \frac{\partial S_1}{\partial q_1} = \alpha_1 \quad \Rightarrow \quad S_1 = \alpha_1 q_1.$$

So if k variables are cyclic, the principal function becomes

$$S = \sum_{i=1}^{k} \alpha_i q_i + \overline{S}\left(q_{k+1}, \ldots, q_n; \alpha_1, \ldots, \alpha_n; t\right).$$

14.5 Action–angle variables

Consider a conservative system with a periodic motion. For simplicity, let us take a system with one degree of freedom, with phase-space variables (q, p). The Hamiltonian is

$$H(q, p) = \text{const.} = \alpha.$$

As the new (constant) momentum, let us take the Poincaré integral invariant over one period,

$$J = \oint p \, \mathrm{d}q.$$

J is called the *action variable*. Hamilton's characteristic function is $W = W(q, J)$. The coordinate conjugate to J is the *angle variable* w. The transformation equation is

$$w = \frac{\partial W}{\partial J} \qquad \text{and } w \text{ is cyclic.}$$

Hamilton's equation of motion is

$$\dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J)$$

with ν a constant, which is a function of J. As expected, w integrates to

$$w = \nu t + \beta.$$

Consider the change in w over a period,

$$\Delta w = \oint \frac{\partial w}{\partial q} \, \mathrm{d}q = \oint \frac{\partial^2 W}{\partial q \partial J} \, \mathrm{d}q.$$

Because J is a constant, we can take it out of the integral:

$$\Delta w = \frac{\mathrm{d}}{\mathrm{d}J} \oint \frac{\partial W}{\partial q} \, \mathrm{d}q = \frac{\mathrm{d}}{\mathrm{d}J} \oint \underbrace{p \, \mathrm{d}q}_{=J} = 1.$$

So, as q goes through a cycle, w changes by 1. Then

$$\Delta w = \nu \tau = 1,$$

where τ is the period. So

$$\nu = \frac{1}{\tau}$$

is the frequency of the periodic motion, which we have found *without solving the equations of motion of the system*. The discussion can be directly extended to a periodic motion in a $2n$-dimensional phase space, in terms of n action variables

$$J_i = \oint p_i \, \mathrm{d}q_i.$$

This form requires the HJ equation to be completely separable in $q_1, \ldots, q_n$, each (q_i, p_i) performing its own libration or rotation, so that $p_i = \partial W_i(q_i; \vec{\alpha})/\partial q_i$ (Goldstein §10.7). In general (Liouville–Arnold), the action variables are defined as $J_i = \oint_{\gamma_i} \sum_k p_k \, \mathrm{d}q_k$ over n independent cycles γ_i of the invariant torus (Arnold §50); with the normalisation used here (no $1/2\pi$) w_i advances by 1 per cycle and $\nu_i = \partial H/\partial J_i$ are frequencies, not angular frequencies.

15

Special relativity

Special relativity is based on the assumption that the laws of nature are the same for inertial observers whose coordinates are related via Lorentz transformations,

$$x^\mu \longrightarrow x^{\mu\prime} = \Lambda^\mu_{\ \nu} x^\nu + \rho^\mu, \tag{15.1}$$

where

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{15.2}$$

is a "four-vector" comprising the space-time coordinates, with $x^0 = ct$, ρ^μ is a constant four-vector and $\Lambda^\mu_{\ \nu}$ satisfies

$$\Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} = g_{\rho\sigma}. \tag{15.3}$$

The 4×4 matrix $g_{\mu\nu}$ is the so-called *metric*, defined as

$$g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \neq \nu. \end{cases} \tag{15.4}$$

In the above we have used Einstein's summation convention. For example, written out explicitly,

$$\Lambda^\mu_{\ \nu} x^\nu = \Lambda^\mu_{\ 0} x^0 + \Lambda^\mu_{\ 1} x^1 + \Lambda^\mu_{\ 2} x^2 + \Lambda^\mu_{\ 3} x^3. \tag{15.5}$$

This is a convention that we will use extensively from now on.

15.1 Proper time

Lorentz transformations leave "proper-time" intervals invariant. These are defined as

$$\mathrm{d}\tau^2 \equiv c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{15.6}$$

Remark: throughout this section the metric is $(+,-,-,-)$ and τ as defined here has the dimension of a *length*: it is c times the time shown by a clock moving with the particle (in the rest frame $\mathrm{d}\tau = c\,\mathrm{d}t$). This is the $c = 1$ convention of Weinberg, *Gravitation and Cosmology*, ch. 2, with c restored; it is used consistently below (e.g. in f^μ, $p^\mu = mc\,\mathrm{d}x^\mu/\mathrm{d}\tau$ and $g_{\mu\nu}\,\mathrm{d}x^\mu/\mathrm{d}\tau\,\mathrm{d}x^\nu/\mathrm{d}\tau = 1$). Many texts instead define $c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2$, so that $p^\mu = m\,\mathrm{d}x^\mu/\mathrm{d}\tau$. Indeed, in a different reference frame we have, from Eq. (15.1),

$$\mathrm{d}x^{\mu\prime} = \Lambda^\mu_{\ \nu}\,\mathrm{d}x^\nu. \tag{15.7}$$

A proper-time interval in the new frame is

$$\begin{aligned} \mathrm{d}\tau'^2 &= g_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu} \\ &= g_{\mu\nu}\left(\Lambda^{\mu}_{\ \rho}\,\mathrm{d}x^{\rho}\right)\left(\Lambda^{\nu}_{\ \sigma}\,\mathrm{d}x^{\sigma}\right) \\ &= \left(g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\right)\mathrm{d}x^{\rho}\,\mathrm{d}x^{\sigma} \\ &= g_{\rho\sigma}\,\mathrm{d}x^{\rho}\,\mathrm{d}x^{\sigma} = \mathrm{d}\tau^2. \end{aligned} \tag{15.8}$$

As a consequence of the invariance of proper-time intervals, *the speed of light is the same in all inertial frames*. Indeed, for light we have

$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right| = c \rightsquigarrow \mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2 = 0. \tag{15.9}$$

In a new frame,

$$\mathrm{d}\tau'^2 = \mathrm{d}\tau^2 = 0 \rightsquigarrow \left|\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t'}\right| = c. \tag{15.10}$$

Lorentz transformations are the only non-singular transformations which preserve proper-time intervals:

$$\begin{aligned} \mathrm{d}\tau^2 &= \mathrm{d}\tau'^2 \\ \rightsquigarrow g_{\rho\sigma}\,\mathrm{d}x^{\rho}\,\mathrm{d}x^{\sigma} &= g_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu} \\ \rightsquigarrow g_{\rho\sigma}\,\mathrm{d}x^{\rho}\,\mathrm{d}x^{\sigma} &= g_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\,\mathrm{d}x^{\rho}\,\mathrm{d}x^{\sigma}, \end{aligned} \tag{15.11}$$

and we conclude that

$$g_{\rho\sigma} = g_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}. \tag{15.12}$$

Differentiating with respect to x^{ϵ}, we obtain

$$0 = g_{\mu\nu}\left[\frac{\partial^2 x'^{\mu}}{\partial x^{\epsilon}\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\epsilon}\partial x^{\sigma}}\frac{\partial x'^{\nu}}{\partial x^{\rho}}\right]. \tag{15.13}$$

To this, we add the same equation with $\epsilon \leftrightarrow \rho$ and subtract the same equation with $\epsilon \leftrightarrow \sigma$. We obtain

$$0 = 2g_{\mu\nu}\frac{\partial^2 x'^{\mu}}{\partial x^{\epsilon}\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}. \tag{15.14}$$

Assuming that the transformation $x^{\mu} \to x'^{\mu}$ is a well-behaved differentiable function and that the inverse of the transformation also exists,

$$\frac{\partial x'^{\mu}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} = \delta^{\mu}_{\ \nu}, \tag{15.15}$$

we obtain that

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\epsilon}\partial x^{\rho}} = 0. \tag{15.16}$$

Therefore, the transformation $x^{\mu} \to x'^{\mu}$ ought to be linear:

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}x^{\nu} + \rho^{\mu}. \tag{15.17}$$

15.2 Subgroups of Lorentz transformations

The set of all Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + \rho^{\mu}, \quad g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma}, \tag{15.18}$$

forms a group (**exercise:** prove it), which is known as the *inhomogeneous Lorentz group* or the Poincaré group. The subset of transformations with $\rho^{\mu} = 0$ is known as the *homogeneous Lorentz group*.

From

$$g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have

$$(\Lambda^0_{\ 0})^2 - \sum_{i=1}^{3}(\Lambda^i_{\ 0})^2 = 1 \rightsquigarrow (\Lambda^0_{\ 0})^2 \geq 1. \tag{15.19}$$

Also, in matrix form the definition of the Lorentz transformation becomes

$$g = \Lambda^T g \Lambda \rightsquigarrow \det g = \det(\Lambda^T g \Lambda) \rightsquigarrow (\det \Lambda)^2 = 1. \tag{15.20}$$

The subgroup of transformations with

$$\det \Lambda = 1, \ \Lambda^0_{\ 0} \geq 1,$$

which contains the identity $\mathbf{1} = \delta^{\mu}_{\nu}$, is known as the group of *proper* Lorentz transformations. All other transformations are known as *improper Lorentz transformations*. It is impossible with a continuous change of parameters to change

$$\det \Lambda = 1 \rightarrow \det \Lambda = -1 \text{ or } \Lambda^0_{\ 0} \geq 1 \rightarrow \Lambda^0_{\ 0} \leq -1.$$

Improper Lorentz transformations involve either space-reflection ($\det \Lambda = -1, \Lambda^0_{\ 0} \geq 1$) or time-inversion ($\det \Lambda = -1, \Lambda^0_{\ 0} \leq -1$) or both ($\det \Lambda = 1, \Lambda^0_{\ 0} \leq -1$).

Proper homogeneous or inhomogeneous Lorentz transformations have a further subgroup: the group of rotations,

$$\Lambda^0_{\ 0} = 1, \ \Lambda^i_{\ 0} = \Lambda^0_{\ i} = 0, \ \Lambda^i_{\ j} = R_{ij}, \tag{15.21}$$

with

$$\det R = 1, \ R^T R = 1. \tag{15.22}$$

Thus, for rotations and translations ($x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \rho^{\mu}$) Lorentz transformations are no different from Galilean transformations.

A difference from Galilean transformations arises in *boosts*. Assume a reference frame O in which a certain particle appears at rest, and O' a reference frame where the particle appears to move with a velocity $\vec{v}$. Space-time intervals in the two frames are related via

$$\mathrm{d}x'^{\mu} = \Lambda^{\mu}_{\ \nu}\,\mathrm{d}x^{\nu} = \Lambda^{\mu}_{\ 0} c\,\mathrm{d}t, \tag{15.23}$$

given that $\mathrm{d}\vec{x} = 0$ in the frame O. For $\mu = 0$, this equation gives

$$\mathrm{d}t' = \Lambda^0_{\ 0}\,\mathrm{d}t. \tag{15.24}$$

For $\mu = i = 1, 2, 3$ we have

$$\mathrm{d}x'^{i} = \Lambda^i_{\ 0} c\,\mathrm{d}t. \tag{15.25}$$

Dividing the two, we have

$$v^i \equiv \frac{\mathrm{d}x'^i}{\mathrm{d}t'} = c\frac{\Lambda^i{}_0}{\Lambda^0{}_0} \rightsquigarrow \Lambda^i{}_0 = \frac{v^i}{c}\Lambda^0{}_0. \tag{15.26}$$

From

$$g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have

$$\begin{aligned} & (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1 \\ \rightsquigarrow \quad & \Lambda^0{}_0 = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \end{aligned} \tag{15.27}$$

and thus

$$\Lambda^i{}_0 = \gamma\frac{v^i}{c}. \tag{15.28}$$

The remaining components are not determined uniquely by knowing the velocity $\vec{v}$ of the particle. Indeed, two Lorentz transformations

$$\Lambda^\mu_\nu \quad \text{and} \quad \Lambda^\mu_\rho R^\rho_\nu$$

where R is a rotation, boost a particle at rest to the same velocity. For coordinate systems O and O' with parallel axes we find that (**exercise**)

$$\Lambda^i{}_j = \delta^i_j + \frac{v^i v^j}{v^2}(\gamma - 1) \tag{15.29}$$

and

$$\Lambda^0{}_j = \gamma\frac{v^j}{c}. \tag{15.30}$$

15.3 Time dilation

Consider an inertial observer O who looks at a clock at rest. Two ticks of the clock correspond to a space-time interval

$$\mathrm{d}\vec{x} = 0, \quad \mathrm{d}t = \Delta t. \tag{15.31}$$

The proper time interval is

$$\mathrm{d}\tau = (c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2)^{\frac{1}{2}} = c\Delta t. \tag{15.32}$$

A second observer sees the clock moving with velocity $\vec{v}$. Two ticks of the clock define a space-time interval

$$\mathrm{d}t' = \Delta t', \quad \mathrm{d}\vec{x}' = \vec{v}\,\mathrm{d}t'. \tag{15.33}$$

The proper-time interval in the new frame is

$$\mathrm{d}\tau' = (c^2\,\mathrm{d}t'^2 - \mathrm{d}\vec{x}'^2)^{\frac{1}{2}} = c\Delta t'\sqrt{1 - \left|\frac{\mathrm{d}\vec{x}'}{c\mathrm{d}t'}\right|^2} = c\Delta t'\sqrt{1 - \frac{\vec{v}^2}{c^2}}. \tag{15.34}$$

The proper time is invariant under a change of inertial reference frame. Thus we conclude that

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \gamma\Delta t. \tag{15.35}$$

15.4 Doppler effect

Take our clock to be a source of light with frequency

$$\omega = \frac{2\pi}{\Delta t}.$$

For an observer for whom the light source is moving with velocity $\vec{v}$, this time interval is measured to be

$$\mathrm{d}t' = \gamma \Delta t.$$

In the same period, the distance of the observer from the light source increases by

$$v_r \, \mathrm{d}t',$$

where v_r is the component of the velocity of the light source along the line of sight of the observer. The time $\mathrm{d}t_0$ elapsing between the reception of two successive light wave fronts by the observer is given by

$$c \, \mathrm{d}t_0 = c \, \mathrm{d}t' + v_r \, \mathrm{d}t'. \tag{15.36}$$

The frequency measured by the observer is

$$\omega' = \frac{2\pi}{\mathrm{d}t_0} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v_r}{c}} \omega. \tag{15.37}$$

If the light source is moving along the line of sight, $v_r = v$, we have

$$\omega' = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \omega. \tag{15.38}$$

If the light source moves away from the observer, $v_r > 0$, the frequency decreases and the light appears redder (red shift). If the source moves towards the observer, the frequency increases (blue shift).

Exercise 15.1. Calculate the angle of the direction of motion of the light source with respect to the line of sight of the observer for which there is no shift in the frequency.

For an application of the Doppler effect in cosmology, read about Hubble's law.

15.5 Particle dynamics

How can we compute the force on a particle which moves with a relativistic velocity $\vec{v}$? We should expect that the classical formulae of Newtonian mechanics need to be modified. On the other hand, Newtonian expressions for the force should be valid if a particle is at rest. We can always change the reference frame with a Lorentz transformation to bring a particle to rest and calculate the change in its velocity in a small time interval using Newtonian mechanics. However, we will need to perform these changes of reference frame at every small increase of the velocity of the particle during its acceleration due to the force.

In a more elegant solution to the problem, we define a relativistic force acting on a particle as

$$f^\mu = mc^2 \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}, \tag{15.39}$$

where m is the mass of the particle.[1] If the particle is at rest, the proper-time interval $\mathrm{d}\tau$ coincides (up to the factor c) with the ordinary time interval $\mathrm{d}t$,

$$\mathrm{d}\tau = c\,\mathrm{d}t.$$

Therefore, in the rest frame of the particle, the "space" components of the force four-vector become

$$f^i_{\text{rest}} = m\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = F^i_{\text{Newton}}, \quad \text{for } i = 1,2,3, \tag{15.40}$$

where $\vec{F}_{\text{Newton}}$ is the force vector as we know it from Newtonian mechanics. The "time" component of the force four-vector vanishes:

$$f^0_{\text{rest}} = mc\frac{\mathrm{d}^2t}{\mathrm{d}t^2} = 0. \tag{15.41}$$

Under a Lorentz transformation, f^μ transforms as

$$f^\mu = mc^2\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} \to mc^2\frac{\mathrm{d}^2x'^\mu}{\mathrm{d}\tau^2} = mc^2\frac{\mathrm{d}^2\left(\Lambda^\mu_{\ \nu}x^\nu + \rho^\mu\right)}{\mathrm{d}\tau^2} = \Lambda^\mu_{\ \nu}\,mc^2\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}. \tag{15.42}$$

Therefore,

$$f'^\mu = \Lambda^\mu_{\ \nu}f^\nu. \tag{15.43}$$

The components of f^μ transform under Lorentz transformations in exactly the same way as the components of space-time coordinates. It is therefore a four-vector as well.

For a specific transformation from the rest frame of a particle to a frame where the particle moves with a velocity $\vec{v}$, we have

$$f^\mu = \Lambda^\mu_{\ \nu}(\vec{v})f^\nu_{\text{rest}}, \tag{15.44}$$

where we have found that

$$\begin{aligned}\Lambda^0_{\ 0}(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda^i_{\ 0}(\vec{v}) = \Lambda^0_{\ i}(\vec{v}) = \gamma\frac{v^i}{c},\\ \Lambda^i_{\ j}(\vec{v}) = \delta^i_j + \frac{v^iv^j}{v^2}(\gamma - 1).\end{aligned} \tag{15.45}$$

Therefore, the force on a moving particle is

$$\vec{f} = \vec{F}_{\text{Newton}} + (\gamma - 1)\frac{\vec{v}\left(\vec{F}_{\text{Newton}}\cdot\vec{v}\right)}{v^2}, \tag{15.46}$$

and

$$f^0 = \gamma\frac{\vec{v}\cdot\vec{F}_{\text{Newton}}}{c} = \frac{\vec{v}}{c}\cdot\vec{f}. \tag{15.47}$$

In Newtonian mechanics, if the force $\vec{F}$ is given, we can compute the trajectory $\vec{x}(t)$ by solving the second-order differential equation

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \frac{\vec{F}(\vec{x},t)}{m}. \tag{15.48}$$

[1] By mass we mean the mass of a particle as it is measured in its rest frame. We will refrain from using the "relativistic", velocity-dependent, mass.

Similarly, in special relativity, when the relativistic force f^μ is known, the differential equation (15.39) can, in principle, be solved to give the space-time coordinates as functions of the proper time τ,

$$x^\mu = x^\mu(\tau). \tag{15.49}$$

To calculate the trajectory, we then need to calculate the proper-time in terms of the time coordinate by inverting

$$x^0 = x^0(\tau) \rightsquigarrow \tau = \tau(x^0), \tag{15.50}$$

which we can use to cast the space components directly as functions of the time-coordinate.

We should not forget a second constraint that must be satisfied by our solutions $x^\mu(\tau)$, namely

$$\Omega \equiv g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 1. \tag{15.51}$$

For the derivative of Ω with respect to proper time, we have

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = 2g_{\mu\nu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{2}{mc^2}g_{\mu\nu}f^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{15.52}$$

The right-hand side is a Lorentz-invariant quantity. In a new frame,

$$g_{\mu\nu}f'^\mu\frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau} = g_{\mu\nu}\left(\Lambda^\mu{}_\rho f^\rho\right)\frac{(\Lambda^\nu{}_\sigma \mathrm{d}x^\sigma)}{\mathrm{d}\tau} = \left(g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\right)f^\rho\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = g_{\rho\sigma}f^\rho\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}.$$

We are therefore allowed to compute $\dfrac{\mathrm{d}\Omega}{\mathrm{d}\tau}$ in any reference frame we wish. Let us choose the rest frame of the particle, where

$$x^\mu = (ct, \vec{0}), \quad f^\mu = (0, \vec{F}_{\text{Newton}}).$$

We obtain

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\tau} = \frac{2}{mc^2}\left(f^0\frac{\mathrm{d}x^0}{\mathrm{d}\tau} - \vec{f}\cdot\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right) = 0. \tag{15.53}$$

Therefore, the quantity Ω is always a constant:

$$\Omega(\tau) = \text{constant}. \tag{15.54}$$

If for some initial value τ_0 we choose the constant to be one, we will have

$$\Omega(\tau) = \Omega(\tau_0) = 1, \qquad \forall\ \tau. \tag{15.55}$$

Remark: strictly speaking, the argument is partly circular: $f^0_{\text{rest}} = 0$ was obtained using $\mathrm{d}\tau = c\ \mathrm{d}t$ in the rest frame, which already presupposes $\Omega = 1$. Logically, the statement is that the definition (15.39) is consistent only for forces satisfying $g_{\mu\nu}f^\mu\ \mathrm{d}x^\nu/\mathrm{d}\tau = 0$ (i.e. $f^0_{\text{rest}} = 0$), and for such forces Ω is conserved (Weinberg, *Gravitation and Cosmology*, sec. 2.3).

Exercise 15.2. Calculate the trajectory of a particle on which the four-force exerted is $f^\mu = 0$.

15.6 Energy and momentum

We can define a relativistic four-vector analogue of momentum as

$$p^\mu = mc\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{15.56}$$

We have that

$$\mathrm{d}\tau = \left(c^2\,\mathrm{d}t^2 - \mathrm{d}\vec{x}^2\right)^{\frac{1}{2}} = c\,\mathrm{d}t\left[1-\left(\frac{\mathrm{d}\vec{x}}{c\,\mathrm{d}t}\right)^2\right]^{\frac{1}{2}} = c\,\mathrm{d}t\left[1-\frac{\vec{v}^2}{c^2}\right]^{\frac{1}{2}} = \frac{c\,\mathrm{d}t}{\gamma}. \tag{15.57}$$

Thus, for the time-component ($\mu = 0$) of the four-momentum we have

$$p^0 = mc\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = m\gamma c. \tag{15.58}$$

For the space-components ($\mu = i = 1,2,3$) we have

$$p^i = mc\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = m\gamma\frac{\mathrm{d}x^i}{\mathrm{d}t} = m\gamma v^i. \tag{15.59}$$

For small velocities, we can expand the factor γ as

$$\gamma = \left[1-\frac{v^2}{c^2}\right]^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + O\left(\frac{v^4}{c^4}\right). \tag{15.60}$$

Therefore, for small velocities the space-components of the four-momentum become the classical momentum,

$$p^i \approx mv^i + \ldots, \tag{15.61}$$

while the time-component becomes

$$p^0 \approx mc + \frac{1}{2c}mv^2 + \ldots \tag{15.62}$$

In the second term of the above expansion we recognise the kinetic energy $\frac{1}{2}mv^2$ of the particle. We then identify the relativistic energy of a particle with

$$E = cp^0 = m\gamma c^2. \tag{15.63}$$

Eliminating the velocity $\vec{v}$ from Eqs. (15.59) and (15.63), we obtain the relation

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4}. \tag{15.64}$$

15.7 The inverse of a Lorentz transformation

Recall the metric matrix

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1). \tag{15.65}$$

We define its inverse

$$g^{\mu\nu}: \quad g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho, \tag{15.66}$$

where δ^μ_ν is the Kronecker delta. It is easy to verify that the inverse of the metric is the metric itself:

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1). \tag{15.67}$$

Now consider a Lorentz transformation $\Lambda^{\mu}_{\ \nu}$, which satisfies the identity

$$\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}g_{\mu\nu} = g_{\rho\sigma}. \tag{15.68}$$

We can prove that the matrix

$$\Lambda_{\mu}^{\ \nu} \equiv g_{\mu\rho}g^{\nu\sigma}\Lambda^{\rho}_{\ \sigma} \tag{15.69}$$

is the inverse of $\Lambda^{\mu}_{\ \nu}$. Indeed

$$\Lambda^{\mu}_{\ \lambda}\Lambda_{\mu}^{\ \nu} = g_{\mu\rho}g^{\nu\sigma}\Lambda^{\rho}_{\ \sigma}\Lambda^{\mu}_{\ \lambda} = g_{\sigma\lambda}g^{\nu\sigma} = \delta^{\nu}_{\ \lambda}. \tag{15.70}$$

If $\Lambda^{\mu}_{\ \nu}$ is the boost with velocity $\vec{v}$ of Eq. (15.45), then

$$\begin{aligned}\Lambda^{0}_{\ 0}(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda^{0}_{\ i}(\vec{v}) = \Lambda^{i}_{\ 0}(\vec{v}) = -\gamma\frac{v^i}{c},\\ \Lambda^{j}_{\ i}(\vec{v}) = \delta^{j}_{i} + \frac{v^i v^j}{v^2}(\gamma - 1).\end{aligned} \tag{15.71}$$

We therefore have that the inverse of a boost is

$$\Lambda_{\mu}^{\ \nu}(\vec{v}) = \Lambda^{\mu}_{\ \nu}(-\vec{v}), \tag{15.72}$$

as we also expect physically.

15.8 Vectors and tensors

It is now time to give a formal definition of vectors in special relativity. We call any set of four components which transform according to the rule

$$V^{\mu} \to V'^{\mu} = \Lambda^{\mu}_{\ \nu}V^{\nu} \tag{15.73}$$

a *contravariant* vector. Contravariant vectors transform in the same way as space-time coordinates x^{μ} do under homogeneous Lorentz transformations.

Not all vectors transform as contravariant vectors. Consider the derivative $\frac{\partial}{\partial x^{\mu}}$. Under a Lorentz transformation, it transforms as

$$\frac{\partial}{\partial x^{\mu}} \to \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\rho}}. \tag{15.74}$$

We have that

$$\left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) = \delta^{\rho}_{\ \nu} \rightsquigarrow \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\Lambda^{\mu}_{\ \nu} = \delta^{\rho}_{\ \nu}. \tag{15.75}$$

Therefore, $\left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)$ is the inverse of a Lorentz transformation $\Lambda^{\mu}_{\ \nu}$:

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \Lambda_{\mu}^{\ \nu}. \tag{15.76}$$

Substituting into Eq. (15.74), we find

$$\frac{\partial}{\partial x^{\mu}} \to \frac{\partial}{\partial x'^{\mu}} = \Lambda_{\mu}^{\ \rho}\frac{\partial}{\partial x^{\rho}}. \tag{15.77}$$

We have found that the derivative does not transform according to the Lorentz transformation but according to its inverse. All vectors which transform with the inverse Lorentz transformation,

$$U_\mu \to U'_\mu = \Lambda_\mu^{\ \nu} U_\nu, \tag{15.78}$$

are called *covariant* vectors.

For every contravariant vector U^μ there is a dual vector

$$U_\mu = g_{\mu\nu} U^\nu. \tag{15.79}$$

We can invert the above equation by multiplying with $g^{\rho\mu}$,

$$g^{\rho\mu} U_\mu = g^{\rho\mu} g_{\mu\nu} U^\nu = \delta^\rho_{\ \nu} U^\nu = U^\rho. \tag{15.80}$$

The dual vector U_μ is a covariant vector. Indeed, under a Lorentz transformation we have

$$U_\mu \to U'_\mu = g_{\mu\nu} U'^\nu = g_{\mu\nu} \Lambda^\nu_{\ \rho} U^\rho = g_{\mu\nu} \Lambda^\nu_{\ \rho} g^{\rho\sigma} U_\sigma = \Lambda_\mu^{\ \sigma} U_\sigma. \tag{15.81}$$

The scalar product of a contravariant and a covariant vector,

$$A \cdot B \equiv A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu, \tag{15.82}$$

is invariant under Lorentz transformations. Indeed,

$$A \cdot B \to A' \cdot B' = A'^\mu B'_\mu = \Lambda^\mu_{\ \rho} A^\rho \Lambda_\mu^{\ \sigma} B_\sigma = \delta^\sigma_{\ \rho} A^\rho B_\sigma = A^\rho B_\rho = A \cdot B. \tag{15.83}$$

Let us define, for short,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{15.84}$$

and the dual contravariant vector

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu. \tag{15.85}$$

The d'Alembert operator is the scalar product

$$\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{15.86}$$

Since it is a scalar product, the d'Alembert operator is invariant under Lorentz transformations.

Finally, we define a tensor with multiple "up" and/or "down" indices to be an object

$$T^{\mu_1 \mu_2 \cdots}_{\nu_1 \nu_2 \ldots} \tag{15.87}$$

which transforms as

$$T^{\mu_1 \mu_2 \cdots}_{\nu_1 \nu_2 \ldots} \to \Lambda^{\mu_1}_{\rho_1} \Lambda^{\mu_2}_{\rho_2} \ldots \Lambda_{\nu_1}^{\ \sigma_1} \Lambda_{\nu_2}^{\ \sigma_2} \ldots T^{\rho_1 \rho_2 \cdots}_{\sigma_1 \sigma_2 \ldots}. \tag{15.88}$$