Quantum Field Theory II

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Contents

Abstract

The subject of the course is modern applications of quantum field theory with emphasis on the quantization of non-Abelian gauge theories. The following topics are discussed:

- Classical gauge transformations.
- Quantization for fermionic and bosonic fields and perturbation theory with path-integrals is developed.
- Quantization of non-Abelian gauge-theories. The Fadeev-Popov method. BRST symmetry.
- The quantum effective action and the effective potential.
- Classical symmetries of the effective action. Slavnov-Taylor identities. The Zinn-Justin equation.

- Physical interpretation of the effective action.
- Spontaneous symmetry breaking. Goldstone theorem. Spontaneous symmetry breaking for theories with local gauge invariance.
- Power-counting and ultraviolet infinities in field theories. Renormalizable Lagrangians.
- Renormalization and symmetries of non-Abelian gauge theories. Renormalization group evolution.
- Infrared divergences. Landau equations. Coleman-Norton physical picture of infrared divergences. Soft and collinear singularities.

1 Path integral quantization in Quantum Mechanics

We have carried out a quantization program for simple field theories in the course of QFT I by means of "canonical quantization" (imposing commutation and anti-commutation relations on fields). Here we will quantise gauge invariant field theories with a different method, using a formalism based on path integrals. The formalism is somewhat imperative to develop. While the canonical formalism can be successfully applied to Quantum Electrodynamics, it is not understood how it can be applied to the gauge field theories which describe the unified strong and electroweak interactions. As a warm-up we revisit quantum mechanics, formulating quantisation using path integrals.

1.1 The propagator

We consider a quantum mechanical state $|\psi\rangle$ which satisfies the Schrödinger equation:

$$i\hbar\partial_t |\psi\rangle = \hat{H} |\psi\rangle . \tag{1}$$

The solution of this equation

$$|\psi(t_2)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} |\psi(t_1)\rangle , \tag{2}$$

determines the evolution of this state from an initial moment t_1 to a later moment t_2. The wave function $\psi(x_2, t_2) \equiv \langle x_2 | \psi(t_2)\rangle$ is then

$$\psi(x_2, t_2) = \langle x_2| e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} |\psi(t_1)\rangle \tag{3}$$

We now insert a unit operator

$$1 = \int d^3x_1 |x_1\rangle \langle x_1| , \tag{4}$$

obtaining

$$\begin{aligned} \psi(x_2, t_2) &= \int d^3x_1 \langle x_2| e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} |x_1\rangle \langle x_1| \psi(t_1)\rangle \\ &= \int d^3x_1 \langle x_2| e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} |x_1\rangle \psi(x_1, t_1) . \end{aligned} \tag{5}$$

In other words, if we know the wave-function at one time, we can determine it fully at a later time by integrating,

$$\psi(x_2, t_2) = \int d^3x' K(x_2, x_1; t_2 - t_1)\psi(x_1, t_1) , \tag{6}$$

with a kernel

$$K(x_2, x_1; t_2 - t_1) = \langle x_2| \, e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} \, |x_1\rangle \tag{7}$$

which depends on the Hamiltonian of the system and the elapsed time $t_2 - t_1$. This integration kernel is called the **"propagator"**. For $t_2 = t_1$, the propagator is a delta function

$$K(x_2, x_1; t_1 - t_1) = \langle x_2| \, x_1\rangle = \delta^3(x_2 - x_1). \tag{8}$$

Exercise: *Prove that the propagator $K(x, x'; t-t')$ satisfies the Schrödinger equation in the variables x, t for times $t > t'$.*
Exercise: *Prove that*

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

Exercise: *Compute the propagator $K(x, x'; t - t')$ for*

- *a free particle,*
- *the simple harmonic oscillator.*

The propagator is the amplitude for a particle measured at position $|x_1\rangle$ at time t_1 to propagate to a new position $|x_2\rangle$ at time $t_2 > t_1$. We can verify this easily. Consider a particle measured at a position x_1. After time $t_2 - t_1$ it will be evolved to a new state

$$e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} \, |x_1\rangle \, . \tag{9}$$

The probability amplitude to be measured at a position x_2 is obtained by taking the inner product of the evolved state and a position ket $|x_2\rangle$,

$$\langle x_2| \, e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} \, |x_1\rangle = K(x_2, x_1; t_2 - t_1). \tag{10}$$

We can attempt to compute the propagator for a transition which takes a very small time $\delta t \to 0$.

$$\begin{aligned} K(x, x'; \delta t) &= \langle x| \, e^{-\frac{i}{\hbar}\hat{H}\delta t} \, |x'\rangle \\ &= \langle x| \, 1 - \frac{i}{\hbar}\hat{H}\delta t \, |x'\rangle + \mathcal{O}\left(\delta t^2\right) \\ &= \langle x| \left(1 - \frac{i}{\hbar}\hat{H}\delta t\right) \left(\int d^3p \, |p\rangle \, \langle p|\right) |x'\rangle + \mathcal{O}\left(\delta t^2\right) \\ &= \int d^3p \left\{ \langle x| \, p\rangle \, \langle p| \, x'\rangle - \frac{i}{\hbar}\delta t \, \langle x| \, \hat{H} \, |p\rangle \, \langle p| \, x'\rangle \right\} + \mathcal{O}\left(\delta t^2\right) \end{aligned} \tag{11}$$

We now specialise to Hamiltonian operators of the form $\hat{H} = f_1(\hat{p}) + f_2(\hat{x})$. Then

$$\langle x| \hat{H} |p\rangle = H \langle x| p\rangle \,, \tag{12}$$

where H is not an operator anymore but the classical Hamiltonian, i.e. a real-valued function of position and momentum. Recall that the position and momentum states are related via a Fourier transform,

$$\langle x| p\rangle = \frac{e^{\frac{i}{\hbar}px}}{\sqrt{2\pi\hbar}}. \tag{13}$$

For simplicity let us consider one only dimension; the three-dimensional case is a faithful repetition of the same steps. Then we find for the propagator at small time intervals:

$$\begin{aligned} K(x,x';\delta t) &= \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}p(x-x')} \left(1 - \frac{i}{\hbar} H\delta t\right) + \mathcal{O}\left(\delta t^2\right) \\ &= \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}\left\{p\left(x-x'\right) - H\delta t\right\}\right) + \mathcal{O}\left(\delta t^2\right) \end{aligned} \tag{14}$$

An interesting form for the propagator for small time intervals arises when the Hamiltonian is of the form

$$H = \frac{p^2}{2m} + V(x). \tag{15}$$

Then,

$$\begin{aligned} K(x,x';\delta t) &= \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}\left\{p\left(x-x'\right) - \frac{p^2}{2m}\delta t - V(x)\delta t\right\}\right) + \mathcal{O}\left(\delta t^2\right) \\ &= \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}\left\{-\frac{\delta t}{2m}\left(p - m\frac{x-x'}{\delta t}\right)^2 + \delta t\frac{1}{2}m\left(\frac{x-x'}{\delta t}\right)^2 - V(x)\delta t\right\}\right) + \mathcal{O}\left(\delta t^2\right) \\ &= \int \frac{d\tilde{p}}{2\pi\hbar} \exp\left(\frac{i\delta t}{\hbar}\left\{-\frac{\tilde{p}^2}{2m} + \frac{1}{2}m\left(\frac{x-x'}{\delta t}\right)^2 - V(x)\right\}\right) + \mathcal{O}\left(\delta t^2\right), \end{aligned} \tag{16}$$

where in the final step we performed a trivial change of integration variable. Our final result reads:

$$K(x,x';\delta t) \simeq \frac{1}{N(\delta t)} \exp\left(\frac{i\delta t}{\hbar}\left\{\frac{1}{2}m\left(\frac{x-x'}{\delta t}\right)^2 - V(x)\right\}\right), \tag{17}$$

with

$$\frac{1}{N(\delta t)} = \int \frac{dp}{2\pi\hbar} \exp\left(\frac{-i\delta t p^2}{2m\hbar}\right) = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2} \tag{18}$$

1.2 The path integral

We consider now the transition from an initial position (x_i, t_i) to a final position (x_f, t_f), which has a propability amplitude given by the propagator:

$$K(x_f, x_i; t_f - t_i).$$

We can take a "snapshot" at an intermediate time t_1 during this transition:

$$t_i < t_1 < t_f.$$

If taking the "snapshot" particle is a measurement of the position x_1 of the particle at the moment t_1, then the amplitude for the full transition will be:

$$K(x_f, x_i; t_f - t_i) = K(x_f, x_1; t_f - t_1) K(x_1, x_i; t_1 - t_i). \tag{19}$$

If taking the "spanshot" is only a thought experiment and we don't actually determine the position x_1 of the particle at t_1 with a real measurement, we should integrate over all probability amplitudes for the particle to have performed this transition via any point. We then have:

$$K(x_f, x_i; t_f - t_i) = \int_{-\infty}^{\infty} dx_1 K(x_f, x_1; t_f - t_1) K(x_1, x_i; t_1 - t_i). \tag{20}$$

We are allowed decide to take several "snapshots" during the transition from x_i to x_f, in times $t_i < t_1 < t_2 < \ldots < t_n < t_f$. Using again the superposition principle we must write:

$$K(x_f, x_i; t_f - t_i) = \int_{-\infty}^{\infty} dx_1 \ldots dx_n K(x_f, x_n; t_f - t_n) K(x_n, x_{n-1}; t_n - t_{n-1}) \times \ldots K(x_2, x_1; t_2 - t_1) K(x_1, x_i; t_1 - t_i). \tag{21}$$

For simplicity, we now consider infinitesimal equally fast $t_{k+1} - t_k = \delta t = \frac{t_f - t_i}{n+1}$ intermediate transitions. Then we obtain

$$K(x_f, x_i; t_f - t_i) = \lim_{n \to \infty} \int_{-\infty}^{\infty} dx_1 \ldots dx_n K(x_f, x_n; \delta t) K(x_n, x_{n-1}; \delta t) \times \ldots K(x_2, x_1; \delta t) K(x_1, x_i; \delta t). \tag{22}$$

We have discretised and taken the infinite limit, which is the defining procedure of an integration. However, this new integral is rather unusual. Think of all the paths which connect the initial and final points x_i and x_f. The points which belong to these paths are accounted for by the limit of infinitesimal $n \to \infty$ transitions which we have taken in Eq. 22. Therefore, the rhs of this equation is an integral over all paths that a particle may take in going from $x_i \to x_f$.

We can insert the expressions for the propagator at small time intervals of Eq. 14 or Eq. 17 into Eq. 22. Notice that in the limit $\delta t \to 0$,

$$\frac{x_n - x_{n-1}}{\delta t} \to \dot{x_n}.$$

Then Eq. 22 becomes:

$$\begin{aligned} K(x_f, x_i; t_f - t_i) = \lim_{n\to\infty} \frac{1}{N(\delta t)^n} \int_{-\infty}^{\infty} & dx_1 \dots dx_n \\ & \exp\left(\frac{i\delta t}{\hbar}\left[\frac{m}{2}\dot{x_1}^2 - V(x_1)\right]\right) \\ & \exp\left(\frac{i\delta t}{\hbar}\left[\frac{m}{2}\dot{x_2}^2 - V(x_2)\right]\right) \\ & \dots \\ & \exp\left(\frac{i\delta t}{\hbar}\left[\frac{m}{2}\dot{x_f}^2 - V(x_f)\right]\right). \end{aligned} \tag{23}$$

Equivalently,

$$\begin{aligned} K(x_f, x_i; t_f - t_i) &= \lim_{n\to\infty} \frac{1}{N(\delta t)^n} \int_{-\infty}^{\infty} dx_1 \dots dx_n \\ & \exp\left(\frac{i}{\hbar}\left\{L(x_1(t_1))\delta t + L(x_2(t_2))\delta t + \dots L(x_f(t_f))\delta t\right\}\right), \end{aligned} \tag{24}$$

where

$$L(x) = \frac{m}{2}\dot{x}^2 - V(x),$$

is the Lagrangian of the system. We now have a more concrete understanding of the above integral. This is an integration over "all paths" connecting the fixed points x_i and x_f, which we write symbolically as

$$K(x_f, x_i; t_f - t_i) = \frac{1}{\mathcal{N}} \int \mathcal{D}x \exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt L[x(t)]\right), \tag{25}$$

and the sum (integral) in the exponential is the classical action as it is evaluated in each of the paths. Even shorter, we can write:

$$K(x_f, x_i; t_f - t_i) = \frac{1}{\mathcal{N}} \int \mathcal{D}x \exp\left(\frac{i}{\hbar}S[x]\right). \tag{26}$$

Exercise: *Consider a Lagrangian of the form*

$$L = \frac{1}{2}f(x)\dot{x}^2 + g(x)\dot{x} - V(x).$$

1. *Compute the Hamiltonian*

2. *Compute the propagator for a small transition*

3. *Write the path-integral expression for the propagator at large time integrals. Notice that the measure of the path integration is modified*

$$\mathcal{D}x \to \mathcal{D}x f(x)^{\frac{1}{2}}$$

1.3 An "adventurous" transition

We look now at a more eventful transition amplitude. We first prepare a particle on an initial position x_i at a time t_i and let it evolve for some time $t - t_i$ according to a Hamiltonian $\hat{H}$:

$$e^{-\frac{i}{\hbar}\hat{H}(t-t_i)} |x_i\rangle$$

At the time t, something abrupt occurs (e.g. an interaction with another particle which was originally far away) and modifies the particle state. We will see later how we can describe interactions of particles using path integrals; now, let us consider an "easy" modification of the state where the state is "mixed up" in a simple way, acting on it with the position operator:

$$\hat{x}e^{-\frac{i}{\hbar}\hat{H}(t-t_i)} |x_i\rangle$$

Then we allow the particle to evolve undistracted for a time $t_f - t$,

$$e^{-\frac{i}{\hbar}\hat{H}\left(t_f-t\right)}\hat{x}e^{-\frac{i}{\hbar}\hat{H}(t-t_i)} |x_i\rangle ,$$

and then we place a detector at x_f:

$$\langle x_f| e^{-\frac{i}{\hbar}\hat{H}\left(t_f-t\right)}\hat{x}e^{-\frac{i}{\hbar}\hat{H}(t-t_i)} |x_i\rangle .$$

We can compute this matrix-element as a path integral. Before we proceed, we should use some language which is more convenient to describe "eventful" transitions. We can write the same transition amplitude as:

$$\begin{aligned} & \left\{\langle x_f| e^{-\frac{i}{\hbar}\hat{H}t_f}\right\} \left\{e^{\frac{i}{\hbar}\hat{H}t}\hat{x}e^{-\frac{i}{\hbar}\hat{H}t}\right\} \left\{e^{\frac{i}{\hbar}\hat{H}t_f} |x_f\rangle\right\} \\ = \; & \langle x_f, t_f| \,\hat{x}(t)\, |x_i, t_i\rangle . \end{aligned} \quad (27)$$

We have defined states

$$|\psi, t\rangle \equiv e^{\frac{i}{\hbar}\hat{H}t} |\psi\rangle , \quad (28)$$

which refer to a fixed moment t only and do not evolve

$$\begin{aligned} & i\hbar\frac{\partial}{\partial t} |\psi, t\rangle = i\hbar\frac{\partial}{\partial t}\left(e^{\frac{i}{\hbar}\hat{H}t} |\psi\rangle\right) \\ = \; & e^{\frac{i}{\hbar}\hat{H}t}\left(i\hbar\frac{\partial}{\partial} |\psi\rangle\right) + i\hbar\left(\frac{\partial}{\partial t}e^{\frac{i}{\hbar}\hat{H}t}\right) |\psi\rangle \\ = \; & e^{\frac{i}{\hbar}\hat{H}t}\left(\hat{H} - \hat{H}\right) |\psi\rangle = 0. \end{aligned} \quad (29)$$

We have defined operators which do change with time

$$\hat{O}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}. \tag{30}$$

As you recognise, this is the "Heisenberg picture" of evolution. For us, it is convenient to assign a time date on a state which denotes a particle or a collection of particles at the beginning of an experiment or at the end of it. However, one could have equally well chosen to work in the probably more familiar "Schrödinger picture".

Lets us now compute this "eventful" transition:

$$\langle x_f, t_f | \hat{x}(t_j) | x_i, t_i \rangle \qquad \text{with } t_i < t < t_f,$$

following the method we used for the simple transition $\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{\frac{i}{\hbar}\hat{H}(t_f - t_i)} | x_i \rangle$, and subdividing the transition in small time intervals. We will find a similar/related path integral for the new case as well.

$$\begin{aligned}
\langle x_f, t_f | \hat{x}(t_j) | x_i, t_i \rangle &= \int dx_1 \ldots dx_{j-1} dx_j \ldots dx_n \\
&\times \langle x_f, t_f | x_n, t_n \rangle \ldots \langle x_{j+1}, t_{j+1} | x_j, t_j \rangle \\
&\times \langle x_j, t_j | \hat{x}(t_j) | x_{j-1}, t_{j-1} \rangle \\
&\times \langle x_{j-1}, t_{j-1} | x_{j-2}, t_{j-2} \rangle \ldots \langle x_1, t_1 | x_i t_i \rangle
\end{aligned} \tag{31}$$

The subdivision of time is carefully chosen. We have

$$\begin{aligned}
\hat{x}(t_j) | x_j, t_j \rangle &= \left(e^{\frac{i}{\hbar}\hat{H}t_j} \hat{x} e^{-\frac{i}{\hbar}\hat{H}t_j} \right) \left(e^{\frac{i}{\hbar}\hat{H}t_j} | x_j \rangle \right) \\
&= e^{\frac{i}{\hbar}\hat{H}t_j} \left(\hat{x} | x_j \rangle \right) \\
&= x_j e^{\frac{i}{\hbar}\hat{H}t_j} | x_j \rangle \\
&= x_j | x_j, t_j \rangle \\
\rightsquigarrow \langle x_j, t_j | \hat{x}(t_j) | x_{j-1}, t_{j-1} \rangle &= x_j \langle x_j, t_j | x_{j-1}, t_{j-1} \rangle .
\end{aligned} \tag{32}$$

We then obtain for the transition amplitude the same succession of propagators as for the simple transition multiplied with an additional factor x_j:

$$\begin{aligned}
&\langle x_f, t_f | \hat{x}(t_j) | x_i, t_i \rangle = \int dx_1 \ldots dx_j x_j \ldots dx_n \\
&\times \langle x_f, t_f | x_n, t_n \rangle \ldots \langle x_{j+1}, t_{j+1} | x_j, t_j \rangle \langle x_j, t_j | x_{j-1}, t_{j-1} \rangle \ldots \langle x_1, t_1 | x_i, t_i \rangle
\end{aligned} \tag{33}$$

Introducing, as before, the explicit form for the propagator during a small time transition

$$\langle x_b, t_{n+1} | x_a, t_n \rangle \approx \left(\frac{m}{2\pi i \hbar \left(t_{n+1} - t_n \right)} \right)^{\frac{1}{2}} \exp \left[\frac{i}{\hbar} \left\{ \frac{m}{2} \left(\frac{x_b - x_a}{t_{n+1} - t_n} \right)^2 - V(x_a) \right\} (t_{m+1} - t_m) \right]$$

and taking equal time intervals, we obtain the path integral

$$\langle x_f, t_f | \, \hat{x}(t_j) \, | x_i, t_i \rangle = \lim_{n\to\infty} \frac{1}{N(\delta t)^n} \int_{-\infty}^{\infty} dx_1 \dots dx_j x_j \dots dx_n$$
$$\times \exp\left(\sum_{r=1}^{n} L(x_r, \dot{x}_r)\delta t \right) \qquad \text{with } \delta t = \frac{t_f - t_i}{n+1}. \tag{34}$$

In compact notation we can write

$$\langle x_f, t_f | \, \hat{x}(\tau) \, | x_i, t_i \rangle = \int \mathcal{D}x x(\tau) e^{\frac{i}{\hbar} S[x]}. \tag{35}$$

Exercise:

1. *Prove that*

$$\langle x_f, t_f | \, \hat{x}(\tau) \, | x_i, t_i \rangle = \int_{-\infty}^{\infty} dx x \, \langle x_f, t_f | \, x, \tau \rangle \, \langle x, \tau | \, x_i, t_i \rangle \, .$$

2. *Evaluate the above integral explicitly for a free particle*
3. *Observe the dependence of the result on the intermediate time* τ
4. *You (could) have considered the simpler matrix element* $\langle x_f, t_f | \, x_i, t_i \rangle$ *and written down an analogous integral. Observe how the intermediate time* τ *drops out from the final expression, when nothing special occurs then!*

We now elaborate further on the form of the path integral that we have just found. Recall the method for computing exponential integrals of the form

$$I_n = \int_{-\infty}^{\infty} dx x^n e^{-ax^2} \qquad \text{with } n = 1, 2, \dots \tag{36}$$

from the result of the integral

$$I_0 \equiv \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}. \tag{37}$$

We can compute I_n with a rather very simple differentiations, after we add a "source" term on the exponent of the integrand

$$\begin{aligned} I_J &= \int_{-\infty}^{\infty} dx e^{-ax^2+Jx} = \int_{-\infty}^{\infty} dx e^{-a\left(x-\frac{J}{2a}\right)^2+\frac{J^2}{4a}} \\ &= \int_{-\infty}^{\infty} d\tilde{x} e^{-a\tilde{x}^2+\frac{J^2}{4a}} = \sqrt{\frac{\pi}{a}} e^{\frac{J^2}{4a}} \end{aligned} \tag{38}$$

To compute the I_1 it is sufficient to differentiate the last expression with respect to the source and substitute $J=0$.

$$\left.\frac{dI_J}{dJ}\right|_{J=0} = \int_{-\infty}^{\infty} dx \left.\frac{de^{-ax^2+Jx}}{dJ}\right|_{J=0} = \int_{-\infty}^{\infty} dx x e^{-ax^2}.$$

Similarly,

$$\left.\frac{d^nI_J}{d^nJ}\right|_{J=0} = \int_{-\infty}^{\infty} dx x^n e^{-ax^2}.$$

Adding a source term to the exponent does not increase the difficulty of the computation and it allows us to calculate all integrals where the integrand is multiplied with a polynomial in the integration variable. This is very suggestive, and we will do the same trick for path integrals such as the one that we found in Eq. 34. We then add a linear term (source) in the Lagrangian; this is only a computational trick and eventually we will compute all interesting physical quantities as in the above example with the source term set to zero. The simple transition from a state $|x_i, t_i\rangle$ to a state $|x_f, t_f\rangle$ in the presence of the source has a probability amplitude:

$$\langle x_f, t_f | x_i, t_i\rangle_J = \lim_{n\to\infty} \frac{1}{N(\delta t)^n} \int dx_1 \dots dx_n e^{\frac{i}{\hbar}\sum_k dt_k (L(x_k,\dot{x}_k)+J_k x_k)} \quad (39)$$

We can then compute

$$\langle x_f, t_f | \hat{x}(t_l) | x_i, t_i\rangle = \frac{\hbar}{i}\frac{\partial}{\partial J_l} \left.\langle x_f, t_f | x_i, t_i\rangle_J\right|_{J_l=0}. \quad (40)$$

Let us now differentiate two times with respect to the source.

$$\left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial J_l \partial J_q} \left.\langle x_f, t_f | x_i, t_i\rangle_J\right|_{J_{l,q}=0} = \lim_{n\to\infty} \frac{1}{N(\delta t)^n} \int dx_1 \dots dx_l x_l \dots dx_q x_q \dots dx_n$$
$$\times e^{\frac{i}{\hbar}\sum_k dt_k (L(x_k,\dot{x}_k)+J_k x_k)}. \quad (41)$$

We can recognise the rhs as the expectation value for the product of two position operators $\hat{x}(t_l)\hat{x}(t_q)$ if t_q is earlier than t_l or $\hat{x}(t_q)\hat{x}(t_l)$ otherwise. We then write

$$\langle x_f, t_f | T\left(\hat{x}(t_l)\hat{x}(t_q)\right) | x_i, t_i\rangle = \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial J_l \partial J_q} \left.\langle x_f, t_f | x_i, t_i\rangle_J\right|_{J=0}, \quad (42)$$

where we have introduced the notation $T(\hat{O}(t_1)\hat{O}(t_3)\hat{O}(t_2)\dots\hat{O}(t_n))$ to remind us that we should put the operators in the correct time order once the sequence of the moments t_i is known. For example, if $t_1 < t_2 < t_3 < \dots < t_n$ we should write

$$T(\hat{O}(t_1)\hat{O}(t_3)\hat{O}(t_2)\dots\hat{O}(t_n)) = \hat{O}(t_n)\dots\hat{O}(t_3)\hat{O}(t_2)\hat{O}(t_1).$$

<u>Exercise:</u> *Compute $\langle x_f, t_f | T\left(\hat{x}(t_l)\hat{x}(t_q)\right) | x_i, t_i\rangle$ for a free particle.*

1.4 Functional differentiation

It is cumbersome to work with path integrals by writing explicitly the infinite limit of discretised paths. We introduced earlier a more compact notation,

$$\langle x_f, t_f | \, x_i, t_i \rangle_J = \frac{1}{\mathcal{N}} \int Dx e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt [L(x(t),\dot{x}(t)) + J(t)x(t)]}. \tag{43}$$

We can write neatly expressions for the expectation values of operators,

$$\langle x_f, t_f | \, \hat{x}(t_1) \, | x_i, t_i \rangle = \frac{1}{\mathcal{N}} \int Dx \; x(t_1) \; e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x(t),\dot{x}(t))}$$

or

$$\langle x_f, t_f | \, T \left(\hat{x}(t_1) \hat{x}(t_2) \right) | x_i, t_i \rangle = \frac{1}{\mathcal{N}} \int Dx \; x(t_1) \; x(t_2) \; e^{\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x(t),\dot{x}(t))}$$

by defining a functional derivative. We consider an integral F[f] over a function f(t) (for example a path-line). We define a functional derivative by changing slightly the function $f(y)$:

$$\frac{\delta F[f(y)]}{\delta f(t)} = \lim_{\epsilon \to 0} \frac{F\left[f(y) + \epsilon \delta(y-t)\right] - F\left[f(y)\right]}{\epsilon}. \tag{44}$$

For example, consider the derivative of the action integral in the presence of a source with respect to the source:

$$\begin{aligned} \frac{\delta \int dy \left\{ L\left(x(y), \dot{x}(y)\right) + J(y)x(y) \right\}}{\delta J(t)} &= \int dy \delta(t-y) x(y) \\ &= x(t). \end{aligned} \tag{45}$$

Practically, we need the chain rule and to remember that

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y).$$

The expectation values of time-ordered operators can then be written as

$$\langle x_f, t_f | \, T \left(\hat{x}(t_l) \ldots \hat{x}(t_n) \right) | x_i, t_i \rangle = \left(\frac{\hbar}{i} \right)^n \frac{\delta^n}{\delta J(t_1) \ldots \delta J(t_n)} \langle x_f, t_f | \, x_i, t_i \rangle_J \Big|_{J=0} . \tag{46}$$

END OF WEEK 1

1.5 Vacuum to vacuum transitions

Field theory allows us to compute transitions between states with different particle content. Interestingly, we can build particle states acting with creation (or field) operators on the ground state which contains no particles (vacuum). All transition amplitudes can be described as expectation values of operators in the vacuum:

$$\langle 0, t_f | T(\ldots) | 0, t_i \rangle .$$

We can compute expectation values of operators in the vacuum with path integrals. First, we try a direct approach

$$\langle 0, t_f | T(\ldots) | 0, t_i \rangle = \int dx dx' \langle 0, t_f | x, t \rangle \langle x, t | T(\ldots) | x', t' \rangle \langle x', t' | 0, t_i \rangle . \quad (47)$$

This formula is complicated. It requires that we know the wave function of the vacuum and that we are able to convolute it with the result for a path integral. There is a rather simpler way with less integrations.

Consider a Hamiltonian, $\hat{H}$ with eigenstates $|n\rangle$,

$$\hat{H} |n\rangle = E_n |n\rangle \qquad \text{(in the Scrhödinger picture)}$$

and a general state

$$|\psi\rangle = \sum_n c_n |n\rangle .$$

Taking a "Heisenberg photograph" of the state in the very past ($-t \to -\infty$), we find:

$$|\psi, -t\rangle = e^{i\bar{H}(-t)} \sum_n c_n |n\rangle . \quad (48)$$

Now, we play a mathematical trick; we give a very small imaginary part to the Hamiltonian

$$\hat{H} \to \hat{H}(1 - i\epsilon), \qquad (\epsilon \to 0^+),$$

which has the same energy eigenstates as the physical Hamiltonian in the limit. The general state in the very past with the modified Hamiltonian is

$$\begin{aligned}
|\psi, -t\rangle &= e^{i\hat{H}(-t)(1-i\epsilon)} \sum_n c_n |n\rangle \\
&= \sum_n c_n e^{-(\epsilon+i)E_n t} |n\rangle \\
&= e^{-(\epsilon+i)E_0 t} \left[c_0 |0\rangle + \sum_{n=1}^{\infty} e^{-(\epsilon+i)(E_n - E_0)t} c_n |n\rangle \right] \\
&= e^{-\epsilon E_0 t} \left[c_0 |0, -t\rangle + \sum_{n=1}^{\infty} e^{-\epsilon(E_n - E_0)t} c_n |n, -t\rangle \right] \quad (49)
\end{aligned}$$

Recall that the states in Eq. (49) are in the Heisenberg picture, and thus independent of time. The time argument in the ket is only a label. For a very long time t in the past the exponential $e^{-\epsilon(E_n-E_0)t}$ vanishes; it vanishes faster for larger energy eigenvalues. We can then approximate,

$$|\psi,-t\rangle \quad \approx \quad e^{-\epsilon E_0 t} c_0 \, |0,-t\rangle \, . \tag{50}$$

This is a very convenient. An arbitrary Heisenberg state in the very past with the slightly complex Hamiltonian is, essentially, the vacuum state in the very past. Higher energy eigenstates do not contribute to the superposition since the small imaginary part forces them to decay as we take the time $-t$ further back in the past.

Similarly, for a general Heisenberg state $\langle\psi',t|$ labeled in the far future ($t\to\infty$), we have

$$\begin{aligned}\left\langle\psi',+t\right| &= \sum_n c_n'^* \langle n| \, e^{-iHt(1-i\epsilon)} \\ &\approx c_0'^* \langle 0| \, e^{-iE_0 t(1-i\epsilon)} \\ &\approx c_0'^* e^{-\epsilon E_0 t} \langle 0,t| \, , \qquad t\to+\infty.\end{aligned} \tag{51}$$

For amplitudes of vacuum-to-vacuum transition over very long times we can therefore write

$$\langle 0,t| \, T(\ldots) \, |0,-t\rangle \quad = \quad \frac{e^{2\epsilon E_0 t}}{c_0 c_0'^*} \left\langle\psi',t\right| T(\ldots) \, |\psi,-t\rangle \; , \quad t\to\infty. \tag{52}$$

as long as we set $H\to H(1-i\epsilon)$ in the right hand side.

We can fix the normalization by dividing with the simplest amplitude of the kind that corresponds to a vacuum-to-vcuum tensition without an “interaction” occuring in an intermediate time. We obtain

$$\frac{\langle 0,t| \, T(\ldots) \, |0,-t\rangle}{\langle 0,t| \, 0,-t\rangle} \quad = \quad \frac{\langle\psi',t| \, T(\ldots) \, |\psi,-t\rangle}{\langle\psi',t| \, \psi,-t\rangle} \quad t\to\infty, \tag{53}$$

We can choose as states in the right-hand side of Eq. (53) two positions Heisenberg states; one labelled to be the eigenvalue of the position operator in the very far past and the second in the very far future.

$$\frac{\langle 0,t| \, T(\ldots) \, |0,-t\rangle}{\langle 0,t| \, 0,-t\rangle} \quad = \quad \frac{\langle x_2,t| \, T(\ldots) \, |x_1,-t\rangle}{\langle x_2,t| \, x_1,-t\rangle}, \quad t\to\infty. \tag{54}$$

Then we can express vacuum to vacuum transitions, from the far past ($t \to -\infty$) to the far future ($t \to -\infty$), as a ratio of path integrals,

$$\frac{\langle 0, \infty |\, T(\hat{x}(t_1) \dots \hat{x}(t_r))\, |0, -\infty\rangle}{\langle 0, \infty |\; 0, -\infty\rangle}$$
$$= \lim_{\epsilon \to 0^+} \frac{\int \mathcal{D}x\, x(t_1) \dots x(t_r)\, e^{\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt L[x(t),\epsilon]}}{\int \mathcal{D}x\, e^{\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt L[x(t),\epsilon]}}. \tag{55}$$

The action integral in the exponent of the path integals is the classical action appearing in Eq. (55), with a small modification (as indicated by the ϵ in the argument of the Lagrangian) which is inherited by the $\hat{H} \to \hat{H}(1 - i\epsilon)$ deformation that we employed in order to "decay" excited states.

1.6 The simple harmonic oscillator

An instructive example is the simple harmonic oscillator, with a classical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{56}$$

The Hamiltonian of Eq (56) is positive definite. For vacuum-to-vacuum transitions over large times, we will need to modify it, $H \to H \cdot (1 - i\epsilon)$, giving to it a small imaginary part. We can achieve this in an easy way, by giving a small negative imaginary part to the square of the frequency,

$$\omega^2 \to \omega^2 - i\epsilon, \quad \epsilon \to 0^+. \tag{57}$$

The slightly deformed Hamiltonian reads,

$$H = \frac{p^2}{2m} + \frac{1}{2}m(\omega^2 - i\epsilon)x^2. \tag{58}$$

The Lagrangian, now, acquires an infinitessimal positive imaginary part,

$$\begin{aligned} L &= p\dot{x} - H \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2(1 - i\epsilon)x^2, \end{aligned} \tag{59}$$

We will now compute a so called "generating functional integral"

$$\begin{aligned} W[J] &= \int \mathcal{D}x\, e^{i\int dt(L + J(t)x(t))}, \\ &= \int \mathcal{D}x\, e^{i\int dt\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2(1-i\epsilon)x^2 + J(t)x(t)\right]}, \end{aligned} \tag{60}$$

which also includes a source term in the action. Functional differentiation with respect to the source J will permit as to compute a large variety of vacuum-to-vacuum transitions including interactions or perturbations.

Things become easier if we perform a Fourier transformation of all quantities which depend on time:

$$x(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{x}(E), \tag{61}$$

$$J(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{J}(E). \tag{62}$$

$$\tag{63}$$

The action integral in the exponent of the generating functional becomes

$$\begin{aligned} S[x] &= \int dt \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m(1-i\epsilon)\omega^2 x^2 + J(t)x(t)\right] \\ &= \frac{1}{4\pi^2} \int dE dE' dt \frac{1}{2} e^{-it(E+E')} \times \left\{\left[-mEE' - m\omega^2(1-i\epsilon)\right] \tilde{x}(E)\tilde{x}(E') \right. \\ &\qquad \left. + \tilde{J}(E)\tilde{x}(E') + \tilde{x}(E)\tilde{J}(E')\right\} \\ &= \int \frac{dE}{4\pi} \left\{m\left[E^2 - \omega^2 + i\epsilon\omega^2\right] \tilde{x}(E)\tilde{x}(-E) \right. \\ &\qquad \left. + \tilde{J}(E)\tilde{x}(-E) + \tilde{x}(E)\tilde{J}(-E)\right\} \\ &= \int \frac{dE}{4\pi} \left\{m\left[E^2 - \omega^2 + i\epsilon\right] \tilde{x}(E)\tilde{x}(-E) + \tilde{J}(E)\tilde{x}(-E) + \tilde{x}(E)\tilde{J}(-E)\right\} \\ &= \int \frac{dE}{4\pi} \left\{\left(\tilde{x}(E) + \frac{\tilde{J}(E)}{m\left[E^2-\omega^2+i\epsilon\right]}\right) m\left[E^2-\omega^2+i\epsilon\right] \left(\tilde{x}(-E) + \frac{\tilde{J}(-E)}{m\left[E^2-\omega^2+i\epsilon\right]}\right) \right. \\ &\qquad \left. - \frac{\tilde{J}(E)\tilde{J}(-E)}{m\left[E^2-\omega^2+i\epsilon\right]}\right\} \end{aligned} \tag{64}$$

simplified $(E^2+\omega^2)\epsilon \to \epsilon$. We can also define

$$\tilde{y}(E) = \tilde{x}(E) + \frac{\tilde{J}(E)}{m\left[E^2-\omega^2+i\epsilon\right]} \tag{65}$$

The second term of the right-hand side corresponds to the Fourier transform of the solution of the Euler-Lagrange equations; i.e. it corresponds to the classical path. We can verify it easily. The classical equation of motion for the harmonic oscillator with a source term and a small negative imaginary part for the frequency is

$$m\left[\frac{d^2 x_{cl}}{dt^2} + \left(\omega^2 - i\epsilon\right) x_{cl}\right] = J(t), \tag{66}$$

and taking the Fourier transform

$$x_{cl} = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{x_{cl}},$$

we obtain

$$\begin{aligned} & m\left[E^2 - \omega^2 + i\epsilon\right] \tilde{x}_{cl}(E) = -\tilde{J}(E) \\ \rightsquigarrow \quad & \tilde{x}_{cl}(E) = -\frac{\tilde{J}(E)}{m\left[E^2 - \omega^2 + i\epsilon\right]}. \end{aligned} \tag{67}$$

Then the path integral over paths $x(t)$, under a shift

$$x(t) = x_{cl}(t) + y(t), \tag{68}$$

becomes

$$W[J] = e^{-\frac{1}{2}\int \frac{dE}{2\pi} \frac{\tilde{J}(E) i \tilde{J}(-E)}{m\left[E^2-\omega^2+i\epsilon\right]}} \int \mathcal{D}y e^{\frac{i}{2}\int \frac{dE}{2\pi} \tilde{y}(E) m\left[E^2-\omega^2+i\epsilon\right]\tilde{y}(-E)} \tag{69}$$

The dependence of the path integral on the source terms becomes a simple pre-factor. It is easy to see that the new path integral corresponds to the harmonic oscillator Hamiltonian with no sources. We can therefore write the above expression as,

$$\langle x_f, +t | \, x_i, -t\rangle_J = e^{\frac{i}{2}\int \frac{dE}{2\pi} \frac{\tilde{J}(E)(-1)\tilde{J}(-E)}{m\left[E^2-\omega^2+i\epsilon\right]}} \langle x_f, +t | \, x_i, -t\rangle , \tag{70}$$

where $t \to \infty$. Performing the inverse Fourier transformations, we have:

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{-\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon} \\ = & \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{-\left(dte^{-iEt}J(t)\right)\left(dt' e^{iEt'} J(t')\right)}{E^2 - \omega^2 + i\epsilon} \\ = & \int dt dt' J(t) G(t-t') J(t'), \end{aligned} \tag{71}$$

with

$$G(t-t') = -\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{i(t-t')E}}{E^2 - \omega^2 + i\epsilon} \tag{72}$$

We then write:

$$\langle x_f, t | \, x_i, -t\rangle_J = e^{\frac{i}{2}\int dt dt' J(t) G(t-t') J(t')} \langle x_f, +t | \, x_i, -t\rangle . \tag{73}$$

We can now differentiate this expression with respect to the sources as many times as we need. We then obtain:

$$\begin{aligned}\langle x_f,+t|\, T\hat{x}(t_n)\ldots\hat{x}(t_1)\,|x_i,-t\rangle &= \left(\frac{1}{i^n}\frac{\delta^n}{\delta J(t_1)\ldots\delta J(t_n)}e^{\frac{i}{2}\int dtdt' J(t)G(t-t')J(t')}\right)\\ &\quad\times\langle x_f,+t|\; x_i,-t\rangle\,. \end{aligned}\tag{74}$$

Taking the limit of $\epsilon \to 0^+$ will select the ground state on both sides of the equation as asymptotic states (the normalization factors are also the same on both sides of the equation). We therefore have for the vacuum expectation value of time ordered position operators:

$$\frac{\langle 0,t|\, T\hat{x}(t_n)\ldots\hat{x}(t_1)\,|0,-t\rangle}{\langle 0,t|\; 0,-t\rangle} = \left(\frac{1}{i^n}\frac{\delta^n}{\delta J(t_1)\ldots\delta J(t_n)}e^{\frac{i}{2}\int dtdt' J(t)G(t-t')J(t')}\right)\tag{75}$$

Exercise:

1. *Show that*
$$\langle 0,t|\, T\hat{x}(t_n)\ldots\hat{x}(t_1)\,|0,-t\rangle = 0,$$
for n odd.

2. *Compute $\langle 0,t|\, T\hat{x}(t_2)\hat{x}(t_1)\,|0,-t\rangle$ and $\langle 0,t|\, T\hat{x}(t_4)\ldots\hat{x}(t_1)\,|0,-t\rangle$ in terms of $G(t-t')$.*

3. *Find a general expression for $\langle 0,t|\, T\hat{x}(t_{2n})\ldots\hat{x}(t_1)\,|0,-t\rangle$ in terms of $G(t-t')$.*

We now compute the propagator $G(t_1,t_2)$ explicitly.

$$\begin{aligned}G(t_1-t_2) &= = -\int\frac{dE}{2\pi}\frac{e^{-i(t_2-t_1)E}}{E^2-\omega^2+i\epsilon}\\ &= -\int\frac{dE}{2\pi}\frac{e^{-i(t_2-t_1)E}}{(E-\omega+i\epsilon)\,(E+\omega-i\epsilon)}\\ &= -\int\frac{dE}{2\pi}\frac{e^{-i(t_2-t_1)E}}{2\omega}\left[\frac{1}{E+\omega-i\epsilon}-\frac{1}{E-\omega+i\epsilon}\right]\end{aligned}\tag{76}$$

We can compute this integral using the residue theorem. If $t_2 > t_1$ we can close the contour of integration on the lower complex half plane where the exponential vanishes at infinity. If $t_1 < t_2$ we can only close the contour on the upper complex half-plane. For both cases, we obtain:

$$G(t_1-t_2) = \frac{i}{2\omega}e^{-i\omega|t_2-t_1|}.\tag{77}$$

END OF WEEK 2

2 Path integrals and scalar fields

In this Chapter, we will write down path integrals for quantum field theory. Earlier, in Quantum Mechanics, we found that the canonical quantization condition and the law of time-evolution,

$$[\hat{x}(t), \hat{p}(t)] = i, \qquad i\partial_t \left|\psi(t)\right\rangle = \hat{H} \left|\psi(t)\right\rangle \tag{78}$$

give rise to a path integral formulation for quantum mechanical correlation functions,

$$\begin{aligned} &\frac{\langle 0, +\infty | \, T\left(\hat{x}(t_1)\ldots\hat{x}(t_n)\right) |0, -\infty\rangle}{\langle 0, +\infty | \, 0, -\infty\rangle} \\ &\qquad = \lim_{\epsilon\to 0^+} \frac{\int \mathcal{D}\, x(t_1)\ldots x(t_n)\, e^{i\int dt L[x(t),\epsilon]}}{\int \mathcal{D}\, e^{i\int dt L[x(t),\epsilon]}}. \end{aligned} \tag{79}$$

In quantum field theory, the canonical quantization formalism postulates commutation relations, which for scalar fields read

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y}). \tag{80}$$

Comparing the commutation relations in QM and QFT, we observe that in passing from the former to the latter, the role of the time coordinate is played by the set of all four space-time coordinates x^μ and the role of the position x(t) in a path should be played by the value $\phi(x^\mu)$ of the field.

To formulate path integrals in QFT, we will heuristically try out a QM to QFT dictionary, which reads

$$\begin{aligned} t &\to x^\mu\,, \\ x(t) &\to \phi(x^\mu)\,, \\ \mathcal{D}x &\to \mathcal{D}\phi\,. \end{aligned} \tag{81}$$

We will not attempt to derive a path integral for correlation functions in QFT directly, starting from the canonical quantization conditions as we did in QM [1]. Instead, we will write down correlation functions in quantum field theory in terms of path integrals applying the dictionary of Eqs. (81) on Eq. (79). We will test the validity of this heuristic approach by verifying that the QFT correlation functions (or, equivalently, the Feynman rules of perturbation theory) which we will derive in this way, are the same as the ones we obtain with the canonical quantization formalism [2].

[1] although, this is possible for scalar fields

[2] We described the canconical formalism in the lecture series of QFT I.

We will start with a field theory of a single scalar field $\phi(x)$. Motivated from our results for path integrals in quantum mechanics, we postulate a generating functional for the field

$$Z[J] \;=\; \mathcal{N} \int \mathcal{D}\phi e^{i \int d^4x \left(\mathcal{L} + J(x)\phi(x) + i\epsilon\phi^2\right)} \tag{82}$$

In Eq. (82), the Lagrangian has already been expressed as the space integral of a Lagrangian density,

$$L = \int d^3\vec{x}\mathcal{L},$$

The role of the $i\epsilon\phi^2$, with $\epsilon \to 0^+$, will be, as in our QM treatment, to dissipate the contributions of excited states in expectation values of operators, yielding expectation values only in the ground state (the "vacuum" state). We will postulate that correlations functions in QFT are given by

$$\begin{aligned} G(x_1, \ldots, x_n) &\equiv \frac{\langle \Omega, +\infty | \, T\left(\hat{\phi}(x_1) \ldots \hat{\phi}(x_n)\right) |\Omega, -\infty\rangle}{\langle \Omega, +\infty | \, \Omega, -\infty\rangle} \\ &= \lim_{\epsilon \to 0^+} \frac{(-i)^n}{Z[0]} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} . \end{aligned} \tag{83}$$

2.1 Real Klein-Gordon field

Let us now try to compute some correlation functions using the path integral of Eq. (82) and Eq. (83) in the simplest possible case of the real Klein-Gordon field. The Lagrangian density for a free real scalar field is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right). \tag{84}$$

The action integral in the exponent of Eq. (82) is

$$\begin{aligned} S &= \int d^4x \left\{ \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) + i\epsilon\phi^2 + J\phi \right\} \\ &= \int d^4x \left\{ \frac{1}{2}\left[\partial_\mu\left(\phi\partial^\mu\phi\right) - \phi\partial^2\phi - m^2\phi^2\right] + i\epsilon\phi^2 + J\phi \right\} \\ &= -\int d^4x \left\{ \frac{1}{2}\phi\left[\partial^2 + m^2 - i\epsilon\right]\phi - J\phi \right\} \end{aligned} \tag{85}$$

In the above, we performed an integration by parts, where we could discard a surface term. The path integral for this action

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi e^{iS},$$

is a straightforward generalization in four dimensions of the path-integral of the simple harmonic oscillator.

We can then compute this path integral repeating the steps of the previous lectures, for the harmonic oscillator in QM. An important step has been to change the path integral integration variable from a generic path $x(t)$ to its difference $y(t) = x(t) - x_{cl}(t)$ from the solution of the classical equations of motion. We will do the same here, for the QFT path integral. We shift

$$\phi(x^\mu) \to \phi(x^\mu) + \phi_{classical}(x^\mu)$$

We then find

$$\begin{aligned} Z[J] &= e^{-\frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)} \times \\ &\quad \mathcal{N}\int \mathcal{D}\phi e^{-\frac{i}{2}\int d^4x \phi\left(\partial^2+m^2-i\epsilon\right)\phi}, \end{aligned} \tag{86}$$

where we reconginse the Feynman propagator of a free scalar field,

$$\Delta_F(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}. \tag{87}$$

Exercise:

a) *Write the Euler-Lagrange equations for the free real scalar field.*

b) *Evaluate the generating path integral for the free real scalar field working in Fourier space and following the analogous derivation of the simple harmonic oscillator.*

c *You can also obtain the result of Eq.* (86) *directly, without resorting to a Fourier transform. Try it out!*

d) *Find the Fourier representation of the propagator.*

e) *Integrate the Fourier representation of the propagator over the energy using the Cauchy theorem. Pay attention to the conditions on the time variable in order to be able to use a closed contour of integration.*

As you can easily compute, Eq. (83) gives for the one, two, three and four-point correlation functions the following results,

$$G(x_1) = 0 \tag{88}$$

$$G(x_1, x_2) = \Delta_F(x_1 - x_2) \tag{89}$$

$$G(x_1, x_2, x_3) = 0 \tag{90}$$

$$\begin{aligned} G(x_1,x_2,x_3,x_4) &= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\ &+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3) \end{aligned} \tag{91}$$

The above results are in agreement with what we obtain in the canonical formalism, by applying Wick's theorem.

For interacting theories, the path integral will serve as a very convenient formalism enabling to develop a method for deriving Feynman rules and performing perturbative expansions. Also for interacting theories, we will find that the path integral formalism for scalar fields reproduces the results of the canonical formalism.

There is a convenient graphical representation which facilitates computations of correlation functions, such as in deriving the correlation functions of Eqs (88)- (91). In a Feynman graphical notation, we represent the propagator of Eq. (87) with a line,

$$\underset{x \qquad\quad y}{\overline{\qquad\qquad}} \equiv \Delta(x-y). \tag{92}$$

The integral over one endpoint of a propagator times a source evaluated at this endpoint, is represented graphically with

$$\times\!\!\underset{\qquad\qquad y}{\overline{\qquad\qquad}} \equiv \int d^4x J(x)\Delta(x-y). \tag{93}$$

Finally, the integral over both endpoints of a propagator multiplied with sources at each endpoint is represented with

$$\times\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\times \equiv \int d^4x d^4y J(x)\Delta(x-y)J(y). \tag{94}$$

With this notation, Eq. (83) takes the form,

$$G(x_1,\ldots,x_n) = (-i)^n \left.\frac{\delta^n\, e^{-\frac{1}{2}\times\!-\!\!\!-\!\!\!-\!\!\times}}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}. \tag{95}$$

The above differentiations are straightforward with our graphical mnemonic. For example, we have that

$$\frac{\delta}{\delta J(x_1)} \times\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\times = \underset{z\qquad\quad}{\overline{\qquad\qquad}}\!\!\times + \times\!\!\underset{\qquad\quad z}{\overline{\qquad\qquad}} = 2\times\!\!\underset{\qquad\quad z}{\overline{\qquad\qquad}}, \tag{96}$$

and

$$\frac{\delta}{\delta J(w)} \times\!\!\underset{\qquad\quad z}{\overline{\qquad\qquad}} = \underset{w \qquad\quad z}{\overline{\qquad\qquad}}. \tag{97}$$

For the one-point correlation function, we find

$$
\begin{aligned}
G(x_1) &= -i \left. \frac{\delta e^{-\frac{1}{2}\times\!\!\!-\!\!\!-\!\!\!-\!\!\!\times}}{\delta J(x_1)} \right|_{J=0} \\
&= i \left. \underset{x_1}{\overline{\quad\quad\quad}}\!\!\times\, e^{-\frac{1}{2}\times\!\!\!-\!\!\!-\!\!\!-\!\!\!\times} \right|_{J=0} \\
&= 0. \qquad (98)
\end{aligned}
$$

For the two-point correlation function we have,

$$
\begin{aligned}
G(x_1, x_2) &= (-i)^2 \left. \frac{\delta^2 e^{-\frac{1}{2}\times\!\!\!-\!\!\!-\!\!\!-\!\!\!\times}}{\delta J(x_1)\delta J(x_2)} \right|_{J=0} \\
&= \left. \frac{\delta}{\delta J(x_2)} \left(\underset{x_1}{\overline{\quad\quad\quad}}\!\!\times\, e^{-\frac{1}{2}\times\!\!\!-\!\!\!-\!\!\!-\!\!\!\times} \right) \right|_{J=0} \\
&= \left. \left(\underset{x_1 \quad\quad x_2}{\overline{\quad\quad\quad\quad}} - \underset{x_1}{\overline{\quad\quad\quad}}\!\!\times \;\; \underset{x_2}{\overline{\quad\quad\quad}}\!\!\times \right) e^{-\frac{1}{2}\times\!\!\!-\!\!\!-\!\!\!-\!\!\!\times} \right|_{J=0} \\
&= \underset{x_1 \quad\quad x_2}{\overline{\quad\quad\quad\quad}}, \qquad (99)
\end{aligned}
$$

and so on.

2.2 Functional Integration and determinants of differential operators

So far, we have avoided to carry out any path integration, as all dependence on the sources $J(x)$ was a "constant" with respect to the integration over paths after a shift of the integration variable by the classical path. The path integral factor was eliminated by the normalization in the denominator of the right-hand side of Eq. (83). Nevertheless, it will be useful to gain an insight on the meaning of the path integral itself. In what follows, we will see that the path integral has a meaning as a "determinant" of the differential operator which appears in the classical Euler-Lagrange equations. For this purpose, we will develop some additional simple mathematical tools.

We start with the familiar integral

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \qquad a > 0. \tag{100}$$

We then obtain

$$\int_{-\infty}^{\infty} dx_1 \dots dx_n e^{-\sum_{i=1}^{n} a_i x_i^2} = \frac{\pi^{\frac{n}{2}}}{(a_1 \dots a_n)^{\frac{1}{2}}}, \qquad a_i > 0. \tag{101}$$

We can define

$$A = diag\,(a_1, a_2, \dots, a_n)\,, \tag{102}$$

$$x^T = (x_1, \dots, x_n). \tag{103}$$

We then rewrite the above integral as

$$\int_{-\infty}^{\infty} \left(\prod_i \frac{dx_i}{\sqrt{\pi}} \right) e^{-x^T A x} = \frac{1}{(detA)^{\frac{1}{2}}}. \tag{104}$$

Let us now perform a transformation

$$x_i = R_{ij} y_j, \qquad \text{or} \qquad x = Ry. \tag{105}$$

The integral can be written as

$$\int_{-\infty}^{\infty} \left(\prod_{i=1}^{n} \frac{dy_i}{\sqrt{\pi}} \right) (detR)\, e^{-y^T (R^T A R) y} = \frac{1}{(detA)^{\frac{1}{2}}}. \tag{106}$$

We define the matrix

$$B = R^T A R. \tag{107}$$

We can easily verify that B is symmetric.

$$B^T = (R^T A R)^T = R^T A^T \left(R^T\right)^T = R^T A R = B.$$

The determinant of B is then

$$detB = det(R^T A R) = (detA)(detR)^2.$$

We then find

$$\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n}\frac{dy_i}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}y^T By} = \frac{1}{(detB)^{\frac{1}{2}}}, \tag{108}$$

where B is any real, positive definite (positive eigenvalues), symmetric matrix.

Let us finally take the limit $n \to \infty$. The exponent in the rhs is originally a double sum over the two indices of the $n \times n$ matrix B. We can then write:

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\int \phi(x)\hat{A}(x)\phi(x)} = \frac{1}{\sqrt{det\hat{A}}}, \tag{109}$$

which we can view as a definition for the determinant of a continuous differential operator.

We can also define the determinant of a Hermitian operator via path integration over complex fields. Consider

$$\begin{aligned}\frac{\pi}{a} &= \int_{-\infty}^{\infty} dxdye^{-a\left(x^2+y^2\right)} \\ &= \int_{-\infty}^{\infty} dxdye^{-a(x-iy)(x+iy)}.\end{aligned} \tag{110}$$

We define

$$x = \frac{z+z^*}{2}, \qquad y = \frac{z-z^*}{2i}$$

We obtain

$$\int \frac{dzdz^*}{2\pi i}e^{-azz^*} = \frac{1}{a}. \tag{111}$$

Following the same steps as before, we can write the determinant of a Hermitian operator as:

$$\int \mathcal{D}\phi\mathcal{D}\phi^* e^{-\int d^4x\phi^*(x)\hat{A}(x)\phi(x)} = \frac{1}{detA}. \tag{112}$$

2.3 Path integrals and interacting fields

For a free scalar field, the computation of correlation functions with the path integral formalism was straightforward. The only integration required was for the evaluation of the Feynman propagator,

$$\Delta_F(x_1 - x_2) \equiv \overline{x_1 \qquad x_2}$$

Using the Feynman propagator as a kernel of a generating functional, we could obtain Green's functions by taking functional derivatives with respect to sources avoiding to carry out explicit functional integrations.

In general, for field theories with interactions, it is not possible to compute exactly correlation functions in four spacetime dimensions. However, the path integral formalism can be very useful, as it enables rather straightforwardly the use of perturbative expansions, where one can compute a few [3] terms in the series with advanced but established computational methods. The terms of the perturbative xpansion will be represented graphically, in terms of Feynman diagrams. In this Section, we will see how they arise with the path integral approach. The results will be the same as with the canonical formalism and we will verify that this is so in some cases.

Let us consider a real scalar field which can interact with itself,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \tag{113}$$

where

$$\mathcal{L}_0 = -\frac{1}{2}\phi(x)\left(\partial^2 + m^2 - i\epsilon\right)\phi(x), \tag{114}$$

and

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi(x)^4. \tag{115}$$

According to our previous conclusions, we can obtain the correlation functions of this theory by taking functional derivatives on the following path integral,

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi e^{i\int d^4x\{\mathcal{L}_0 + \phi(x)J(x) + \mathcal{L}_I\}}. \tag{116}$$

We consider the case of a small coupling constant $\lambda \ll 1$ and assume this condition to suffice for a perturbative expansion around $\lambda \sim 0$. We separate the action into a "free" and an "interacting" part,

$$S = S_0 + S_I, \tag{117}$$

where the "free" part includes the sources as well as the quadratic terms of the Lagrangian in the field ϕ,

$$S_0[J,\phi] = \int d^4x \left\{\mathcal{L}_0 + \phi(x)J(x)\right\}, \tag{118}$$

and the "interacting" part includes,

$$S_I[\phi] = \int d^4x\, \mathcal{L}_I. \tag{119}$$

[3] typically three

We were able to compute correlation functions rcactly for $\lambda = 0$. We will attempt to build solutions for small λ as perturbations of the known solutions at $\lambda = 0$, which we hope that they will furnish the leading contributions to the correlation functions of the interacting theory. A Taylor series expansion of the path integral in λ gives,

$$
\begin{aligned}
Z[J] &= \mathcal{N} \int \mathcal{D}\phi e^{iS_0[J,\phi]+iS_I[\phi]} = \mathcal{N} \int \mathcal{D}\phi \sum_{n=0}^{\infty} \frac{(iS_I[\phi])^n}{n!} e^{iS_0[J,\phi]} \\
&= \mathcal{N} \int \mathcal{D}\phi \sum_{n=0}^{\infty} \frac{\left[i \int d^4y \, \left(\frac{-\lambda}{4!}\phi(y)^4\right)\right]^n}{n!} e^{iS_0[J,\phi]} \\
&= \mathcal{N} \sum_{n=0}^{\infty} \frac{\left[i \int d^4y \left(\frac{-\lambda}{4!}\left(\frac{\delta}{i\delta J(y)}\right)^4\right)\right]^n}{n!} \int \mathcal{D}\phi e^{iS_0[J,\phi]}. \qquad (120)
\end{aligned}
$$

In the last step, we used the fact that we can generate powers of ϕ in the integrand of the path integral by acting with functional derivatives. In a short notation, we "sum back" the series into an exponential of functional derivatives. The result for the generating functional is compactly written in the form,

$$
Z[J] = \mathcal{N} \, e^{i \int d^4y \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)} \int \mathcal{D}\phi e^{iS_0[J,\phi]} \qquad (121)
$$

Let us stress that the exponential of functional derivatives is only meant as a short notation. The precise meaning of it is defined through the Taylor series in the last line of Eq. (120). In the righ-hand side of Eq. (121), we find the path integral of the free field theory,

$$
\begin{aligned}
Z_0[J] &\equiv \mathcal{N}_0 \int \mathcal{D}\phi e^{iS_0[J,\phi]} \\
&= \mathcal{N}_0 \int \mathcal{D}\phi e^{iS_0[0,\phi]} \, e^{-\frac{1}{2} \times\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\times} \qquad (122)
\end{aligned}
$$

Substituting Eq. (122) into Eq. (121) and defining an overall normalization,

$$
\mathcal{N}' = \mathcal{N} \int \mathcal{D}\phi e^{iS_0[0,\phi]} ,
$$

we obtain

$$
Z[J] = \mathcal{N}' \, e^{i \int d^4y \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)} \, e^{-\frac{1}{2} \times\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\times} \qquad (123)
$$

We fix the normalization of the generating functional by requiring that the amplitude for a vacuum to vacuum transition over infinitely long times is the unity,

$$Z[0] = 1.$$

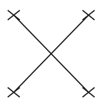
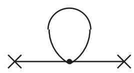
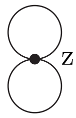
We then obtain,

$$Z[J] \;=\; \frac{e^{i\int d^4y \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)}\, e^{-\frac{1}{2}\, \times\!\!-\!\!\!-\!\!\!-\!\!\times}}{e^{i\int d^4y \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)}\, e^{-\frac{1}{2}\, \times\!\!-\!\!\!-\!\!\!-\!\!\times}\Big|_{J=0}} \tag{124}$$

We will compute explicitly the generating functional $Z[J]$ of Eq. (124) through order $\mathcal{O}(\lambda^2)$. Let us consider the numerator of the right-hand side of Eq. (124),

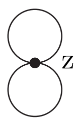
$$\begin{aligned} \mathrm{Num}[J] &= e^{i\int d^4y \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)}\, e^{-\frac{1}{2}\, \times\!\!-\!\!\!-\!\!\!-\!\!\times} \\ &= \left(1 - \frac{i\lambda}{4!}\int d^4z \frac{\delta^4}{\delta J(z)^4}\right) e^{-\frac{1}{2}\, \times\!\!-\!\!\!-\!\!\!-\!\!\times} + \mathcal{O}(\lambda^2). \end{aligned} \tag{125}$$

After carrying out the four functional differentiations, we obtain

$$\mathrm{Num}[J] \;=\; e^{-\frac{1}{2}\, \times\!\!-\!\!\!-\!\!\!-\!\!\times}\left(1 - i\frac{\lambda}{4!}\left[\right.\right.$$

$$-\,6$$

$$+\,3$$


$$\left.\left.\right]\right) + \mathcal{O}(\lambda^2) \tag{126}$$

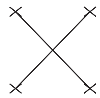
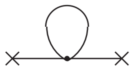
The denominator of $Z[J]$ in Eq. (82) is given by

$$\mathrm{Den} = \mathrm{Num[J]}|_{J=0} = 1 - i\frac{\lambda}{4!}\left[3\right.$$


$$\left.\right] + \mathcal{O}(\lambda^2) \tag{127}$$

We can then compute the generating function, dividing our results for numerator and denominator and expanding consistently in λ. We find,

$$Z[J] = \frac{\mathrm{Num}[J]}{\mathrm{Den}}$$

$$= e^{-\frac{1}{2}}$$  $$\left(1 - i\frac{\lambda}{4!}\left[\right.\right.$$  $$- 6$$  $$\left.\left.\right]\right)$$

$$+\mathcal{O}(\lambda^2) \tag{128}$$

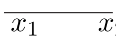
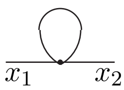
Notice the effect of dividing with the denominator, Den, which guarantees the normalization $Z[0] = 1$, on our result of Eq. (128). After expanding in λ, the denominator cancels all vacuum graphs which also appear in the numerator Num$[J]$. All graphs which remain in the final result are connected to a source.

We can now compute Green's functions perturbatively, through order $\mathcal{O}(\lambda)$, using the master formula

$$G(x_1, \ldots x_n) = (-i)^n \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}, \tag{129}$$

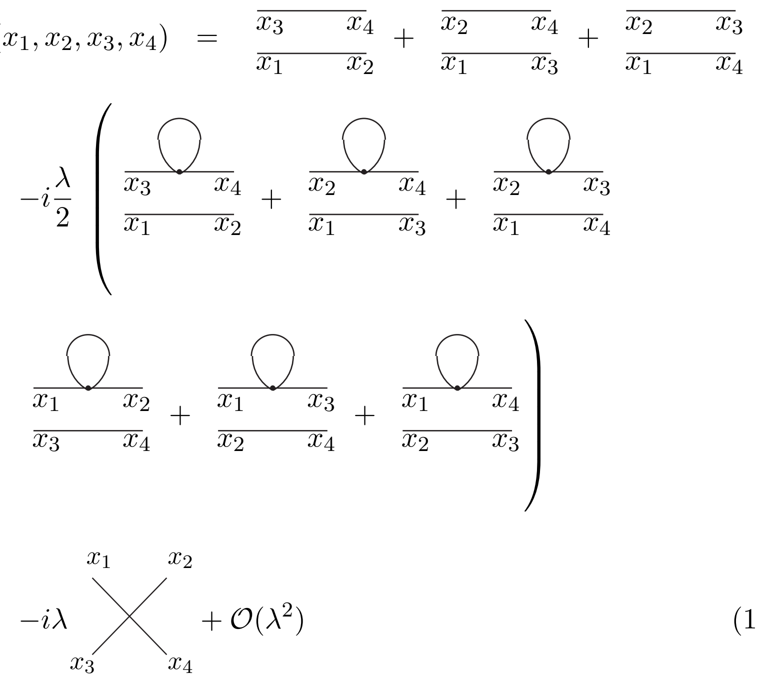
and substituting the perturbative expansion of Eq. (128). We find,

$$G(x_1) = 0\,, \tag{130}$$

$$G(x_1, x_2) =$$  $$- \frac{i\lambda}{2}$$  $$+ \mathcal{O}(\lambda^2)\,, \tag{131}$$

$$G(x_1, x_2, x_3) = 0\,, \tag{132}$$

$$G(x_1, x_2, x_3, x_4) =$$

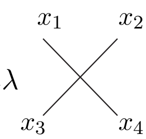


(133)

and so on. We observe that we observe the same Feynman diagram expressions as one obtains with perturbation theory in the canonical quantization formalism.

Notice that Eq. (129) generated *all* Feynman diagrams for the four-point Green's function at the right-hand side of Eq. (133). These comprise diagrams which do not connect all four external points x_1, x_2, x_3, and x_4, along the one diagram (the last graph in the right-hand side of Eq. (133)) which is fully connected. Indeed, we can recast Eq. (129) as

$$G(x_1, x_2, x_3, x_4) = -i\lambda$$



$$+G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3)$$
$$+\mathcal{O}(\lambda^2)\,. \qquad (134)$$

As it is known from scattering theory and the LSZ-formula, the disconnected graphs which belong in products of lower point Green's functions, do not contribute to the scattering probability amplitudes.

2.3.1 Generating functional of connected Feynman diagrams

There is a simple and general method to generate directly fully connected Feynman diagrams, by differentiating the logarithm of the path integral $\log Z[J]$, rather than differentiating $Z[J]$. Let us define a path integral $W[J]$ via,

$$Z[J] = e^{iW[J]}. \tag{135}$$

For the one-point function, we find

$$G(x_1) = (-i)\frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \frac{\delta W[J]}{\delta J(x_1)}\bigg|_{J=0} \tag{136}$$

For the two-point correlation function, we obtain

$$\begin{aligned} G(x_1, x_2) &= (-i)^2\, \frac{1}{i^2} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\ &= -i\,\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} + G(x_1)\, G(x_2)\,. \end{aligned} \tag{137}$$

Rearranging, we find that acting with two derivatives on the logarithm $W[Z]$ of the path integral $Z[J]$ has the effect of removing the product of one-point correlation functions, leaving only the fully connected duagrams,

$$\begin{aligned} -i\,\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} &= G(x_1, x_2) - G(x_1)G(x_2) \\ &\equiv G_{\text{conn.}}(x_1, x_2) \end{aligned} \tag{138}$$

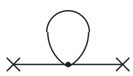
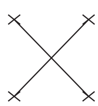
Similarly, for the three point function we find

$$\begin{aligned} &(-i)^2\,\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\bigg|_{J=0} = G_{\text{conn}}(x_1, x_2, x_3) \\ &\equiv G(x_1, x_2, x_3) - G_{\text{conn.}}(x_1, x_2)\, G(x_3) - G_{\text{conn.}}(x_2, x_3)\, G(x_1) \\ &-G_{\text{conn.}}(x_1, x_3)\, G(x_2) - G(x_1)\, G(x_3)\, G(x_3)\,, \end{aligned} \tag{139}$$

and so on. The general result reads,

$$G_{\text{conn.}}(x_1, x_2, \ldots, x_n) = (-i)^{n-1}\,\frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\bigg|_{J=0} \tag{140}$$

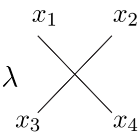
In the case of the ϕ^4 scalar theory, we have.

$W[J] = \frac{i}{2}$  $+ \frac{\lambda}{4}$  $- \frac{\lambda}{4!}$  $+ \mathcal{O}(\lambda^2).$ (141)

Applying Eq. (140), we find indeed only the connected graph for the four point function,

$$G_{\text{conn.}}(x_1, x_2, x_3, x_4) = (-i)^3 \frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0}$$

$$= -i\lambda \qquad (142)$$



2.4 Fermionic path integrals

We have discussed the path integral for scalar fields which are bosonic. To describe fermions, we need to formulate path integration over anti-commuting functions.

We will introduce a new kind of numbers, the *Grassmann numbers*, which are defined to obey

$$\{c_i, c_j\} = c_i c_j + c_j c_i = 0. \tag{143}$$

A Grassmann number c is also defined to commute with a regular number x,

$$x\,c = c\,x\,. \tag{144}$$

From the definition of Eq. (143), we have that

$$c_i^2 = 0. \tag{145}$$

Therefore, any function constructed out of Grassmann variables is at most linear in any of these variables.

Exercise:*Show that the product of two Grassmann variables $a = c_1 c_2$ commutes wth a Grassmann variable c_3*

$$[a, c_3] = 0$$

Exercise: *Show that linear combinations of Grassmann variables are also Grassmann variables.*

We will now develop our calculus for functions of Grassmann variables, defining derivatives and integration. This will lead us to a definition of a path integration over Grassmann fields, which we will postulate to describe fermions.

We start by defining a derivative similarly to derivatives of commuting numbers,

$$\frac{\partial c_i}{\partial c_j} = \delta_{ij}. \tag{146}$$

We will also take that we can apply a chain rule for differentiating functions of Grassmann variables.

Our definition of Grassmann differentiation, as it stands now, is a bit naive. When it is used to differentiate functions of multiple Grassmann variables it leads to ambiguities. To obtain unambiguous results, we must introduce a further rule.

Let us consider a concrete example of a function of two Grassmann variables c_1, c_2. The general form of the function contains up to linear terms, and it is given by

$$f\left(c_1, c_2\right) = a_0 + a_1 c_1 + a_2 c_2 + a_{12} c_1 c_2, \tag{147}$$

where the coefficients a_0, a_1, a_2, a_{12} are regular commuting numbers. Ambiguities arise because a reordering of Grassmann variables in a product may give rise to minus signs. For example, we can decide to commute the c_2 variable to the left of the c_1 Grassmann variable in the last term of the right-hand side of Eq. (147), obtaining an equivalent expression.

$$f(c_1, c_2) = a_0 + a_1 c_1 + a_2 c_2 - a_{12} c_2 c_1, \tag{148}$$

A naive application of Eq. (146) could yield two different results,

$$\frac{\partial}{\partial c_2} = a_2 + a_{12} c_1, \quad \text{or} \quad \frac{\partial}{\partial c_2} = a_2 - a_{12} c_1.$$

We will need to fix this ambiguity before we take derivatives.

We will define a "left" derivative with respect to c_2 which acts in the following way,

$$\begin{aligned} \frac{\partial^L f}{\partial c_2} &= \frac{\partial^L}{\partial c_2}(a_0 + a_1 c_1 + a_2 c_2 + a_{12} c_1 c_2) \\ &= \frac{\partial^L}{\partial c_2}(a_0 + a_1 c_1 + a_2 c_2 - a_{12} c_2 c_1) \\ &= a_2 - a_{12} c_1. \end{aligned} \tag{149}$$

The meaning of "left" (as the superscript denotes) is that we must bring the variable which we differentiate to the left of any other Grassmann variable in a product, anticommuting by means of Eq. (143), before we eliminate the variable by differentiating with the rule of Eq. (146).

Analogously, we could have defined a "right" derivative, where we always anti-commute a Grassmann variable to the right of any other variable before we eliminate it. In our example, we have:

$$\frac{\partial^R f}{\partial c_2} = a_2 + a_{12} c_1. \tag{150}$$

In this lecture series, we will always use left derivatives, unless it is explicilty stated otherwise. From now on, we will drop the superscript L on derivatives of Grassmann variables and we will assume implicitly that the derivative is a left one.

Exercise: *Prove that*

$$\left\{ C_i, \frac{\partial}{\partial C_j} \right\} = \delta_{ij}, \tag{151}$$

and

$$\left\{ \frac{\partial}{\partial C_j}, \frac{\partial}{\partial C_j} \right\} = 0. \tag{152}$$

Our next step is to define an integration over Grassmann variables c_i. As functions of Grassmann variables are linear, we only require to know two integrals

$$\int dc\, 1 =?$$

and

$$\int dc\, c =?$$

We will set

$$\int dc\, 1 = 0. \tag{153}$$

In addition, we will set

$$\int dc\, c = x,$$

where x is a usual commuting number of a fixed value. As the only other possible integral, Eq. (153) vanishes, the actual value of x can be absorbed into a redefinition of the integration measure. It is convenient to take,

$$\int dc\, c = 1. \tag{154}$$

For integration, as well as for differentiation, we will need to fix sign ambiguities when integrating over functions of multiple Grassmann variables. Our rule will be to order products of Grassmann variables so that the Grassmann variable that we integrate over to appear to the left. For example, in integrating a product of two Grassmann variables, we have

$$\int dc_1 c_2 c_1 = -\int dc_1 c_1 c_2 = -c_2.$$

We can now reveal a striking fact about our definitions of differentiation and integration on Grassmann variables. Differentiation and integration are identical operations.

$$\begin{aligned}
\int dc_2 f &= \int dc_2\, (a_0 + a_1 c_1 + a_2 c_2 + a_{12} c_1 c_2) \\
&= \int dc_2\, (a_0 + a_1 c_1 + a_2 c_2 - a_{12} c_2 c_1) \\
&= a_2 - a_{12} c_1 \\
&= \frac{\partial f}{\partial c_2}.
\end{aligned} \tag{155}$$

Multiple integrals over Grassmann variables can be used to express determinants. This property, will turn out to be very useful in defining a path integral for non-abelian gauge theories. Consider a general integral

$$\int dc_1 dc_2 \dots dc_n f(c_1, c_2, \dots, c_n),$$

and perform a linear transformation

$$c_i = M_{ij} b_j. \tag{156}$$

We shall have

$$\int dc_1 dc_2 \dots dc_n f(c_i) = (Jacobian) \int db_1 db_2 \dots db_n f\left(M_{ij} b_j\right). \tag{157}$$

What is the Jacobian? In the case of a double integral, we have:

$$\begin{aligned}
\int dc_1 dc_2 c_1 c_2 &= (Jacobian) \int db_1 db_2 \left(M_{11} b_1 + M_{12} b_2\right)\left(M_{21} b_1 + M_{22} b_2\right) \\
&= (Jacobian) \int db_1 db_2 \left(M_{11} M_{22} b_1 b_2 + M_{12} M_{21} b_2 b_1\right) \\
&= (Jacobian) \int db_1 db_2 b_1 b_2 \left(M_{11} M_{22} - M_{12} M_{21}\right) \\
&= (Jacobian) det(M) \int db_1 db_2 b_1 b_2 \\
&\leadsto Jacobian = \frac{1}{det(M)}.
\end{aligned} \tag{158}$$

Exercise: *Prove that the same results holds for multiple integrals of arbitrary dimensions.*

If we recall that Grassmann integration is in reality differentiation, it is not surprising that the Jacobian of the transformation on Grassmann variables is the inverse of what emerges in integrations of normal commuting variables.

Exercise: *Consider the complex linear combinations*

$$y = \frac{c_1 + ic_2}{\sqrt{2}}, \bar{y} = \frac{c_1 - ic_2}{\sqrt{2}},$$

of two real Grassmann variables c_1, c_2. *Show that:*

$$\begin{aligned}
&\{y, \bar{y}\} = 0 \\
&\int dc_1 dc_2 f(c_1, c_2) = i \int dy d\bar{y} f\left(\frac{y + \bar{y}}{\sqrt{2}}, \frac{y - \bar{y}}{\sqrt{2} i}\right)
\end{aligned}$$

After we have defined integration over Grassmann variables, we can now study multiple integrals over exponentials of Grassmann variables. This will be an intermediate step in formulating a path integral for fermionic fields, as an analogous step emerged in the formulation of a path integral for scalar bosonic fields. Let us start with two independent vectors of Grassmann variables $x^T = (x_1, x_2)$ and $y^T = (y_1, y_2)$. We have

$$x^T y = x_1 y_1 + x_2 y_2, \tag{159}$$

and

$$\begin{aligned}\left(x^T y\right)^2 &= (x_1y_1 + x_2y_2)(x_1y_1 + x_2y_2) \\ &= x_1y_1x_2y_2 + x_2y_2x_1y_1 \\ &= 2x_1y_1x_2y_2. \end{aligned} \tag{160}$$

We can easily see that

$$\left(x^T y\right)^n = 0, \qquad n > 2. \tag{161}$$

Then, for the exponential, we have

$$\begin{aligned} e^{-x^T y} &= 1 - x^T y + \frac{1}{2}(x^T y)^2 - \frac{1}{3!}(x^T y)^3 + \dots \\ &= 1 - x_1y_1 - x_2y_2 + x_1y_1x_2y_2 \end{aligned} \tag{162}$$

Therefore, using a Taylor expansion, we have

$$\begin{aligned} \int dx_1dy_1dx_2dy_2 e^{-x^T y} &= \int dx_1dy_1dx_2dy_2 \left[1 - (x_1y_1 + x_2y_2) + x_1y_1x_2y_2\right] \\ &= 1. \end{aligned} \tag{163}$$

We can now perform linear transformations on both x and y.

$$x \to Mx',$$

$$y \to Ny'.$$

We find,

$$\begin{aligned} 1 = \int dxdye^{-x^T y} = det(M^T)^{-1} det(N)^{-1} \int dxdye^{-x^T M^T N y} \\ = det(M^T N)^{-1} \int dxdye^{x^T M^T N y}. \end{aligned} \tag{164}$$

Defining $A = M^T N$, we obtain that

$$\int dxdye^{-x^T A y} = det(A). \tag{165}$$

Exercise: *Prove the above for* $x^T = (x_1, x_2, x_3)$ *and* $y^T = (y_1, y_2, y_3)$. *Generalize to arbitrary dimensionality*

Recall that for normal commuting variables we have

$$\int dxdx^\dagger e^{-x^\dagger A x} \sim \frac{1}{detA}. \tag{166}$$

Let us now define a path integral over Grassmann variables by considering an infinite number of them and taking the continuous limit. This path integral will give correlation functions for the fermionic field ψ by taking functional derivatives with respect to sources, similarly to what we have seen in the scalar field case. It will differ however from the Green's functions of the bosonic field, in that the order of differentiations will matter. Different orderings will yield minus signs which are expected in Fermi spin-statistics. Let us consider the Lagrangian of a free Dirac fermion.

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{167}$$

We can write a generating functional

$$Z_0\left[a, \bar{a}\right] = \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^4x\left[\mathcal{L}+\bar{a}\psi+\bar{\psi}a\right]}. \tag{168}$$

In this path integral the integration variables $\psi, \bar{\psi}$ and the sources $a, \bar{a}$ are all independent Grassmann functions. The constant $\mathcal{N}$ is fixed as usual by requiring that

$$Z_0\left[0, 0\right] = \langle 0|\ 0\rangle = 1.$$

Everything works in an analogous way as in the case of the path integral for the scalar field. For a theory without interactions, we can "complete the square" and compute the generating functional explicitly if we shift he fields $\psi, \bar{\psi}$ by a constant corresponding to their classical value which minimizes the action. We need the inverse of the Dirac wave operator,

$$(i\partial\!\!\!/ - m\mathbf{1})S(x-y) = i\,\delta^4\left(x-y\right)\mathbf{1}. \tag{169}$$

As for the scalar field and the harmonic oscillator we can write a Fourier representation,

$$S(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{k\!\!\!/ + m}{k^2 - m^2}e^{-ik\cdot(x-y)} \tag{170}$$

Exercise: *Write the Euler-Lagrange equations for the action with sources $a, \bar{a}$. Solve these equations in Fourier space.* The values of $\psi, \bar{\psi}$ which minimize the action are now given by

$$\psi_{cl} = i\int d^4y S(x-y)a(y),$$

$$\bar{\psi_{cl}} = -i\int d^4y \bar{a}(y)S(x-y),$$

and the change of variables

$$\psi = \psi_{cl} + \eta, \qquad \bar{\psi} = \bar{\psi}_{cl} + \bar{\eta},$$

yields

$$Z_0[a, \bar{a}] = e^{-\int d^4x d^4y \bar{a}(x) S(x-y) a(y)}. \tag{171}$$

We obtain expectation values of time-ordered products of field operators from the generating functional via

$$\langle 0, +\infty | \, T \ldots \psi_i(x) \ldots \bar{\psi}_j(y) \ldots |0, -\infty\rangle = \ldots \frac{1}{i} \frac{\delta}{\delta \bar{a}_i(x)} \ldots i \frac{\delta}{\delta a_j(y)} \ldots Z_0[a, \bar{a}], \tag{172}$$

where the indices i, j are spinor indices. **Exercise:** *Calculate the expectation values*

- $\langle 0, +\infty | \, T \psi_i(x_1) \bar{\psi}_j(x_2) \, |0, -\infty\rangle$
- $\langle 0, +\infty | \, T \psi_i(x_1) \bar{\psi}_j(x_2) \psi_k(x_3) \bar{\psi}_l(x_4) \, |0, -\infty\rangle$

and compare them with your results from canonical quantization. We can also develop a perturbative expansion repeating faithfully the steps we performed in the ϕ^4 scalar field theory. It is not hard to convince ourselves that a completely analogous formula should be valid here for the generating functional when an interaction term $\mathcal{L}_I$ is present in the Lagrangian:

$$Z\left[a, \bar{a}\right] = \frac{e^{i \int d^4z \mathcal{L}_I \left(i \frac{\delta}{\delta a}(x), \frac{1}{i} \frac{\delta}{\delta \bar{a}(x)}, \right)} Z_0\left[a, \bar{a}\right]}{e^{i \int d^4z \mathcal{L}_I \left(i \frac{\delta}{\delta a}(x), \frac{1}{i} \frac{\delta}{\delta \bar{a}(x)}, \right)} Z_0\left[a, \bar{a}\right] \Big|_{a=\bar{a}=0}}. \tag{173}$$

Besides the main similarities in the appearance of the formulae there are also very important differences which are encoded in the Grassmann algebra of the functions which we integrate upon. What is very important to remember, is the fact that functional derivatives anticommute, generalizing the result that we found for the derivatives of discrete Grassmann variables:

$$\left\{ \frac{\delta}{\delta a(x)}, \frac{\delta}{\delta a(y)} \right\} = \left\{ \frac{\delta}{\delta \bar{a}(x)}, \frac{\delta}{\delta \bar{a}(y)} \right\} = \left\{ \frac{\delta}{\delta a(x)}, \frac{\delta}{\delta \bar{a}(y)} \right\} = 0. \tag{174}$$

We should also remember that these derivatives are left (by convention) derivatives, and the order in which sources appear in the integrand matters indeed.

Exercise: *Consider a theory with a fermion a real scalar and a Yukawa interaction $\lambda \bar{\psi} \psi \phi$. Calculate the fermion and scalar propagators through $\mathcal{O}(\lambda^2)$. Derive the Feynman diagrams contributing to the scattering of four fermions. Derive the Feynman diagrams contributing to the scattering of two fermions and two scalars.* [4]

[4] This exercise is very important for checking your understanding of the material covered

3 Non-abelian gauge theories

We will now discuss quantum field theories with a gauge symmetry. Realisitc theories such as QED, QCD and the full Standard Model are all symmetric under local gauge transformations.

3.1 Gauge invariance and QED

We shall start with the familiar case of Quantum Electrodynamics. We will see how the QED Lagrangian emerges from extending the Dirac Lagrangian of free electrons, by requiring that the "global" symmetry of $U(1)$ transformations found in the Dirac Lagrangian becomes "local".

The Dirac Lagrangian for a fermion field ψ is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(x)\,(i\not\partial - m)\,\psi(x) \tag{175}$$

This is clearly invariant under a "global" transformation,

$$\psi(x) \to \psi'(x) = \exp(i\,g\,\theta)\,\psi(x). \tag{176}$$

where the parameter θ is constant $\left(\frac{\partial\theta}{\partial x} = 0\right)$. Under a local gauge transformation,

$$U(x) = \exp\left(i\,g\,\theta(x)\right), \tag{177}$$

the free Dirac Lagrangian is not invariant.

$$\bar{\psi}\not\partial\psi \to \bar{\psi}\not\partial\psi + \bar{\psi}e^{-ig\theta}\left[\not\partial e^{ig\theta}\right]\psi. \tag{178}$$

We will now modify the Dirac Lagrangian in order to make it invariant under *local* gauge transformations with $\theta = \theta(x)$. To achieve this goal, we must introduce another field A^{μ}, which we will recognise as the photon field.

The term which causes the Dirac Lagrangian not to be invariant under *local* transformations is the derivative term,

$$\not\partial\psi \to \not\partial\psi' = e^{i\,g\,\theta}\left[\not\partial + ig\,(\not\partial\theta)\right]\psi.$$

We will substitute the simple derivative ∂_{μ} with a covariant derivative D_{μ} which transforms more conveniently. Specifically, we look for a covariant derivative which, under a *local* gauge transformation, transforms as:

$$\not D\psi \to U(x)\not D\psi. \tag{179}$$

so far. Please spend as much time as needed until you get it right.

To obtain a covariant derivative, we add to the usual derivative a new function (field):

$$D_\mu = \partial_\mu - igA_\mu(x). \tag{180}$$

The new field $A_\mu(x)$ must be a vector field, as it must have the same Lorentz transformation as the derivative ∂_μ which is a vector. The field A_μ will also have its own gauge transformation,

$$A_\mu \to A'_\mu$$

which we will define in such a manner so that we obtain the covariant derivative transformation of Eq. (179) We need:

$$\begin{aligned}
& D_\mu\psi(x) \to D'_\mu\psi' = U(x)D_\mu\psi \\
\rightsquigarrow\quad & \left(\partial_\mu - igA'_\mu\right)(U(x)\psi) = U(x)\left(\partial_\mu - igA_\mu\right)\psi \\
\rightsquigarrow\quad & U(x)\partial_\mu\psi + \left[\partial_\mu U(x)\right]\psi - igA'_\mu U(x)\psi = U(x)\partial_\mu\psi - igA_\mu U(x)\psi \\
\rightsquigarrow\quad & A'_\mu = A_\mu - \frac{i}{g}U^{-1}(x)\partial_\mu U(x) \qquad (181)
\end{aligned}$$

The covariant derivative transforms as:

$$\begin{aligned}
D_\mu \to D'_\mu &= \partial_\mu - igA'_\mu \\
&= \partial_\mu - ig\left(A_\mu - \frac{i}{g}U^{-1}\partial\mu U\right) \\
&= \partial_\mu - igA_\mu - U^{-1}\left(\partial_\mu U\right) \\
&= \partial_\mu - igA_\mu + U\left(\partial_\mu U^{-1}\right) \\
&= U(x)\left(\partial_\mu - igA_\mu\right)U^{-1}(x) \qquad (182)
\end{aligned}$$

Therefore:

$$D_\mu \to D'_\mu = U(x)D_\mu U^{-1}(x) \tag{183}$$

We can now replace the free Lagrangian of the spin-1/2 field with a new Lagrangian which is also gauge invariant.

$$\begin{aligned}
&\mathcal{L}' = \bar\psi\left[i\not{D} - m\right]\psi \\
&\to \bar\psi U^{-1}U\left[i\not{D} - m\right]U^{-1}U\psi \\
&\bar\psi\left[i\not{D} - m\right]\psi.
\end{aligned}$$

If A_μ is a physical field, we need to introduce a kinetic term in the Lagrangian for it. We will insist on constructing a fully gauge invariant Lagrangian. To this purpose, we can use the covariant derivative as a building block. Consider the gauge transformation of the product of two covariant derivatives:

$$\begin{aligned}
&D_\mu D_\nu \to D'_\mu D'_\nu = UD_\mu U^{-1}UD_\nu U^{-1} \\
&= UD_\mu D_\nu U^{-1}.
\end{aligned}$$

This is not a gauge invariant object. Now look at the commutator:

$$\begin{aligned} [D_\mu, D_\nu] &\to \left[D'_\mu, D'_\nu\right] \\ &= U\left[D_\mu, D_\nu\right]U^{-1} \end{aligned} \tag{184}$$

This is gauge invariant. To convince ourselves we write the commutator explicitly:

$$\begin{aligned} & [D_\mu, D_\nu] = (\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu) - [\mu \leftrightarrow \nu] \\ = \; & \partial_\mu\partial_\nu - ig(\partial_\mu A_\nu) - igA_\nu\partial_\mu - igA_\mu\partial_\nu + (ig)^2 A_\mu A_\nu - [\mu \leftrightarrow \nu] \\ = \; & -ig\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right]. \end{aligned} \tag{185}$$

Inserting Eq. 185 into Eq. 184, we find that the commutator of covariant derivatives (in the abelian $U(1)$ case) is gauge invariant. We have also found that it is proportional to the field strength tensor of the gauge (photon) field:

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{186}$$

We now have invariant terms for a Lagrangian with an "electron" and a "photon" field. The Lagrangian for QED reads

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{187}$$

Exercise: *Find the Noether current and conserved charge due to the invariance under the $U(1)$ gauge transformation of the QED Lagrangian.*
Exercise: *Think of at least five more operators that one can add to the QED Lagrangian without spoiling gauge invariance. There is an infinite number of them. Which of them do not have mass dimension four? As we shall see, operators of higher dimension spoil renormalizability*

3.2 Non-abelian (global) $SU(N)$ transformations

Lagrangians for theories with more than one fields may be symmetric under more complicated transformations than a $U(1)$ symmetry. In particular, a symmetry group which is relevant for quantum chromodynamics and weak theory is the $SU(N)$ symmetry group. Let's consider, as an example, a collection of N scalar fields and the Lagrangian.

$$\mathcal{L} = (\partial_\mu \phi_i)(\partial^\mu \phi_i^*) - m^2(\phi_i\phi_i^*) - \frac{\lambda}{4}(\phi_i\phi_i^*)^2, \qquad i = 1\ldots N, \tag{188}$$

where we have used Einstein's double index summation. The Lagrangian is symmetric under $SU(N)$ transformations of the ϕ_i fields,

$$\phi_i \to \phi_i' = U_{ij}\,\phi_j\,, \quad U^\dagger = U^{-1}\,,\ detU = 1\,. \tag{189}$$

Indeed,

$$\begin{aligned}\phi_i^*\phi_i \to \phi_i'^*\phi_i' &= U_{ij}^*\phi_j^* U_{ik}\phi_k \\ &= U_{ji}^\dagger U_{ik}\phi_j^*\phi_k = \delta_{jk}\phi_j^*\phi_k = \phi_i^*\phi_i\,.\end{aligned} \tag{190}$$

We can learn a lot by studying small $SU(N)$ transformations. Due to them forming a group, large transformation can be obtained by repeating (infinitely) many small ones. We write:

$$U_{ij} = \delta_{ij} - i\theta^a T_{ij}^a + \mathcal{O}(\theta^2), \tag{191}$$

where we choose θ^a to be real parameters.

The $N \times N$ matrices T^a are generators of $SU(N)$ matrices. They are N^2-1: An arbitrary $N \times N$ complex matrix has $2N^2$ real elements. For a unitary matrix $U^\dagger = U^{-1}$, only N^2 elements are independent. The specialty condition $detU = 1$ adds one more constraint, leaving $N^2 - 1$ independent elements. Remember the dimensionality of the indices defining the generators,

$$T_{i,j}^a : \qquad a = 1\ldots(N^2-1)\ \textbf{and}\ i,j = 1\ldots N\,,$$

as they will be needed in various situations.

Exercise *$SO(N)$ group: $N \times N$ matrices with*

$$R_{ij}R_{kl}\delta_{jl} = \delta_{ik}.$$

Find the number of generators.

Exercise *The symplectic group $Sp(2N)$ can be defined as $2N \times 2N$ matrices S with*

$$S_{ij}S_{kl}\delta_{jl} = \eta_{ik},$$

where,

$$\eta_{ij} = -\eta_{ji} \qquad \textit{and} \qquad \eta^2 = -\mathbf{1}.$$

Find the number of generators.

The $SU(N)$ generators are **hermitian**:

$$\begin{aligned} & U^\dagger U = 1 \\ \rightsquigarrow\ & \left(1 + ig(T^a)^\dagger\theta^a\right)\left(1 - igT^b\theta^b\right) = \mathbf{1} + \mathcal{O}(\theta^2) \\ \rightsquigarrow\ & T^{a\dagger} = T^a \end{aligned} \tag{192}$$

and **traceless**:

$$
\begin{aligned}
& detU = 1 \rightsquigarrow \log\left(detU\right) = 0 \\
\rightsquigarrow \quad & Tr\left(\log U\right) = 0 \\
\rightsquigarrow \quad & Tr \log\left(1 - i\theta^a T^a\right) = 0 \\
\rightsquigarrow \quad & Tr\left(i\theta^a T^a\right) = 0 \\
\rightsquigarrow \quad & Tr\left(T^a\right) = 0
\end{aligned}
\tag{193}
$$

We can choose a normalization condition for the $SU(N)$ generators. By convention, we choose,

$$
Tr\left(T^a T^b\right) \equiv T^a_{ij} T^b_{ji} = T_R\, \delta_{ab}, \quad T_R = \frac{1}{2}\,. \tag{194}
$$

A very basic property of the generators is that they satisfy a Lie algebra

$$
\left[T^a, T^b\right] = i f^{abc} T^c, \tag{195}
$$

where f^{abc} are the structure constants of the algebra.
Exercise: *Prove Eq. 195 by considering a transformation* $U'^{-1}U^{-1}U'U$*, with* U, U' *independent* $SU(N)$ *transformations.*
From Eq. 195, we can derive

$$
\begin{aligned}
& \left[T^a, T^b\right] = i f^{abd} T^d \rightsquigarrow \left[T^a, T^b\right] T^c = i f^{abd} T^d T^c \\
\rightsquigarrow \quad & f^{abc} = -2i Tr\left(\left[T^a, T^b\right] T^c\right).
\end{aligned}
\tag{196}
$$

Exercise: *Prove that the structure constants are fully antisymmetric and real*

3.3 Local non-abelian gauge symmetries

We now consider N fields ϕ_i (scalar or spinor). We are interested in Lagrangians which are invariant under a local $SU(N)$ transformation:

$$
\phi_i(x) \to \phi'_i(x) = U_{ij}(x)\phi_j(x). \tag{197}
$$

As in QED, the building block for the construction of a gauge invariant Lagrangian will be a covariant derivative:

$$
D_\mu = \partial_\mu - igA_\mu, \tag{198}
$$

such that

$$
D_\mu \to D'_\mu = U D^\mu U^\dagger, \qquad \text{with } U^\dagger = U^{-1}. \tag{199}
$$

For a scalar field ϕ we have

$$(D'_\mu \phi'^\dagger)(D'^\mu \phi') \to (D_\mu \phi^\dagger)(D^\mu \phi)$$

Similarly, the kinetic term with the same covariant derivative for a fermion field is invariant. There are many similarities with QED, however there are also many important differences. Let's start by pointing out that the gauge field A_μ is an $N \times N$ matrix.

We can easily find the transformation for the gauge field:

$$\begin{aligned} & D_\mu \to D'_\mu = U(x) D_\mu U^\dagger(x) \\ \rightsquigarrow \quad & \left(\partial_\mu - igA'_\mu\right) = U\left(\partial_\mu - igA_\mu\right) U^\dagger \\ \rightsquigarrow \quad & \partial_\mu - igA'_\mu = \partial_\mu + U\left(\partial_\mu U^\dagger\right) - UigA_\mu U^\dagger \\ \rightsquigarrow \quad & A'_\mu = U(x) A_\mu U^\dagger(x) + \frac{i}{g} U(x)\left(\partial_\mu U^\dagger(x)\right) \end{aligned} \tag{200}$$

This formula is analogous to the gauge transformation of the photon in QED. However, here the gauge field A_μ and the gauge transformation U are complex $N \times N$ matrices rather than complex numbers.

We now compute the corresponding commutator:

$$\begin{aligned} & [D_\mu, D_\nu] = \left(\partial_\mu - igA_\mu\right)\left(\partial_\nu - igA_\nu\right) - [\mu \leftrightarrow \nu] \\ = \quad & \partial_\mu \partial_\nu - ig\left(\partial_\mu A_\nu\right) - igA_\nu \partial_\mu - igA_\mu \partial_\nu + (ig)^2 A_\mu A_\nu - [\mu \leftrightarrow \nu] \\ = \quad & -ig\left\{\partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]\right\}. \end{aligned} \tag{201}$$

The commutator term was absent in the case of QED.

The gauge field strength:

$$G_{\mu\nu} \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right], \tag{202}$$

is no longer gauge invariant:

$$G_{\mu\nu} \to G'_{\mu\nu} = U(x) G_{\mu\nu} U^\dagger(x).$$

However, the trace

$$\begin{aligned} & Tr(G_{\mu\nu} G^{\mu\nu}) \\ \to \quad & Tr(U G_{\mu\nu} U^\dagger U G^{\mu\nu} U^\dagger) \\ \to \quad & Tr(U^\dagger U G_{\mu\nu} U^\dagger U G^{\mu\nu}) \\ \to \quad & Tr(G_{\mu\nu} G^{\mu\nu}) \end{aligned}$$

is gauge invariant.

We can expand the gauge field in the basis of generators:

$$A_\mu = A_\mu^a T^a, \tag{203}$$

Equivalently,

$$A_\mu^a = \frac{Tr(A_\mu T^a)}{T_R} \tag{204}$$

We also have

$$G_{\mu\nu} = G_{\mu\nu}^a T^a, \tag{205}$$

with

$$G_{\mu\nu}^a = \frac{Tr(G_{\mu\nu} T^a)}{T_R}. \tag{206}$$

It is

$$\begin{aligned} G_{\mu\nu}^a &= \frac{1}{T_R} Tr(G_{\mu\nu} T^a) \\ &= \frac{1}{T_R} Tr\left(\partial_\mu A_\nu T^a - \partial_\nu A_\mu T^a - ig\left[A_\mu, A_\nu\right] T^c\right) \\ &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \end{aligned} \tag{207}$$

A prominent example of a theory with a non-Abelian local gauge symmetry is QCD which is invariant under $SU(3)$ gauge transformations.

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left(i\not{D} - m\mathbf{1}\right)\psi - \frac{1}{4} G^{c\mu\nu} G_{\mu\nu}^c. \tag{208}$$

Exercise: *Expand all terms in the QCD Lagrangian using the explicit expressions in terms of the gauge field for the covariant derivative and the gauge field strength. Sketch the interactions (the precise Feynman rules will be derived in forthcoming lectures)*

Exercise: *Write a gauge invariant Lagrangian under $SU(N)$ transformations for a scalar field. This case appears in supersymmetric theories for the scalar partners of quarks. Sketch the interactions.*

3.3.1 Adjoint representation:

By expanding the commutators we can easily prove the Jacobi identity,

$$\left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^b, T^c\right], T^a\right] + \left[\left[T^c, T^a\right], T^b\right] = 0. \tag{209}$$

From this we derive a relation for the structure constants:

$$\begin{aligned} &\rightsquigarrow \left[f^{abd} T^d, T^c\right] + \left[f^{bcd} T^d, T^a\right] + \left[f^{cad} T^d, T^b\right] = 0 \\ &\rightsquigarrow f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \end{aligned} \tag{210}$$

We define the matrices

$$\tilde{T}_{ac}^b = i f^{abc}. \tag{211}$$

Then the above relation can be written as

$$\left[\tilde{T}^b, \tilde{T}^c\right] = if^{bcd}\tilde{T}^d. \tag{212}$$

Therefore, the matrices $\tilde{T}$ furnish a representation of the same Lie algebra. This is called the adjoint representation.

We can now consider fields ψ_a which transform in the adjoint representation. An example emerges in supersymmetric theories of QCD where the gluino, the supersymmetric partner of the gluon, transforms in the adjoint. Specifically, the transformation is

$$\psi_a \to \psi'_a = {U_A}^{ab}\psi_b \quad \text{with} \quad U_A^{ab} = e^{-i\theta^c \tilde{T}^c_{ab}} \quad \text{and} \quad a, b, c = 1 \ldots (N^2-1). \tag{213}$$

Or, for small transformations,

$$\begin{aligned}
&\psi_a \to \psi'_a = \psi_a - i\theta^b \tilde{T}^b_{ac}\psi_c \\
\leadsto\ &\psi'_a = \psi_a - i\theta^b \left(if^{abc}\psi_c\right) \\
\leadsto\ &\psi'_a = \psi_a + f^{abc}\theta^b\psi_c
\end{aligned} \tag{214}$$

We can find a covariant derivative for the transformations of the adjoint representation in a complete analogy as for the "fundamental" representation:

$$\begin{aligned}
D_\mu\psi_a &= \partial_\mu\psi_a - igA^b_\mu \tilde{T}^b_{ac}\psi_c \\
&= \partial_\mu\psi_a + gf^{abc}A^b_\mu\psi_c.
\end{aligned} \tag{215}$$

Now consider a general member of the represenation $\psi = \psi_a T^a$. The covariant derivative acts on it like:

$$\begin{aligned}
D_\mu\psi &= \left(\partial_\mu\psi_a + gf^{abc}A^b_\mu\psi_c\right)T^a \\
&= \partial_\mu\psi - ig\left[T^b, T^c\right]\psi_c A^b_\mu,
\end{aligned} \tag{216}$$

or equivalently

$$D_\mu\psi = \partial_\mu\psi - ig\left[A_\mu, \psi\right], \quad \psi \equiv \psi_a T^a, \quad A_\mu \equiv A^a_\mu T^a. \tag{217}$$

Exercise: *Take an element $\xi = \xi^a T^a$ of the Lie algebra, transforming in the adjoint representation, and a field χ transforming in the fundamental representation. Prove the Leibniz rule for the covariant derivative:*

$$D_\mu\left(\xi\chi\right) = \left(D_\mu\xi\right)\chi + \xi\left(D_\mu\chi\right). \tag{218}$$

3.3.2 Euler-Lagrange equations / conserved currents*

We consider the variation of the gauge field

$$A_\mu \to A_\mu + \delta A_\mu$$

and

$$\partial_\nu A_\mu \to \partial_\nu A_\mu + \delta\left(\partial_\nu A_\mu\right).$$

The corresponding variation of the gauge field strength is

$$\begin{aligned} &\delta G_{\mu\nu} = \delta\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]\right) \\ &= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu - ig\left[\delta A_\mu, A_\nu\right] - ig\left[A_\mu, \delta A_\nu\right] \\ \rightsquigarrow\quad &\delta G_{\mu\nu} = D_\mu\left(\delta A_\nu\right) - D_\nu\left(\delta A_\mu\right). \end{aligned} \tag{219}$$

We can now look at the variation of this term in the QCD action

$$\begin{aligned} &\delta\int d^4x Tr\left(G_{\mu\nu}G^{\mu\nu}\right) \\ =\;& 2\int d^4x Tr\left(G_{\mu\nu}\delta G^{\mu\nu}\right) \\ =\;& 2\int d^4x Tr\left(G^{\mu\nu}\left[D_\mu\left(\delta A_\nu\right) - D_\nu\left(\delta A_\mu\right)\right]\right) \\ =\;& 4\int d^4x Tr\left(G^{\mu\nu}D_\mu\left(\delta A_\nu\right)\right) \qquad \text{antisymmetry} \\ =\;& 4\int d^4x\left\{D_\mu\left[Tr\left(G^{\mu\nu}\left(\delta A_\nu\right)\right)\right] - Tr\left[D_\mu\left(G^{\mu\nu}\right)\delta A_\nu\right]\right\} \end{aligned} \tag{220}$$

The first trace is gauge invariant. We can then find a gauge transformation for the gluon field so that $D_\mu \to U D_\mu U^\dagger = \partial_\mu$, and drop the surface term. So, we have:

$$\delta\int d^4x Tr\left[\frac{-1}{2}G_{\mu\nu}G^{\mu\nu}\right] = 2\int d^4x Tr\left[D_\mu\left(G^{\mu\nu}\right)\delta A_\nu\right]. \tag{221}$$

The fermionic term in the Lagrangian varies as:

$$\begin{aligned} &\delta\int d^4x\bar{\psi}\left(i\not{D} - m\right)\psi = \delta\int d^4x\bar{\psi}\left(i\not{\partial} + g\not{A}\right)\psi \\ =\;& g\int d^4x\bar{\psi}\delta\not{A}\psi = g\int d^4x\bar{\psi}\gamma^\mu T^a\psi\delta A^a_\mu = \\ =\;& 2g\int d^4x\left(\bar{\psi}_i\gamma^\mu T^a_{ij}\psi_j\right)Tr\left[\delta A_\mu T^a\right] \\ =\;& -2gTr\left[J_\mu\delta A^\mu\right], \end{aligned} \tag{222}$$

with

$$J_\mu = J^a_\mu T^a, \tag{223}$$

and

$$J_\mu^a = -g\bar{\psi}\gamma^\mu T^a \psi. \tag{224}$$

Combining the variation of both fermionic and gauge boson terms in the Lagrangian of Eq. 208 we derive the Euler-Lagrange equation:

$$D_\mu G^{\mu\nu} = J^\nu. \tag{225}$$

We can act with a second covariant derivative on the above equation:

$$D_\nu D_\mu G^{\mu\nu} = D_\nu J^\nu \tag{226}$$

The lhs is:

$$\begin{aligned} D_\nu D_\mu G^{\mu\nu} &= D_\nu \left(\partial_\mu G^{\mu\nu} - ig\left[A_\mu, G^{\mu\nu}\right]\right) \\ &= \partial_\nu \partial_\mu G^{\mu\nu} \\ &-ig\left[\partial_\nu A_\mu, G^{\mu\nu}\right] \\ &-ig\left[A_\mu, \partial_\nu G^{\mu\nu}\right] - ig\left[A_\nu, \partial_\mu G^{\mu\nu}\right] \\ &-g^2\left[A_\nu\left[A_\mu, G^{\mu\nu}\right]\right] \end{aligned} \tag{227}$$

Using that $G^{\mu\nu} = -G^{\nu\mu}$ and

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

you can prove that

$$D_\nu D_\mu G^{\mu\nu} = \frac{-ig}{2}\left[G_{\mu\nu}, G^{\mu\nu}\right] = 0 \tag{228}$$

Therefore:

$$D_\mu J^\mu = 0. \tag{229}$$

The fermionic current is thus no longer (as in QED) a conserved current. It is rather covariantly conserved!

Exercise: *In an abelian gauge theory, consider the dual tensor*

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

Show that

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K^\mu, \tag{230}$$

with

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$$

Exercise: *In a non-abelian gauge theory, consider the dual tensor*

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}.$$

Show that

$$G^{\mu\nu}\tilde{G}_{\mu\nu} = \partial_\mu K^\mu, \tag{231}$$

with

$$K^\mu = \epsilon^{\rho\sigma\mu\nu}Tr\left[G_{\sigma\mu}A_\nu + \frac{2}{3}A_\sigma A_\mu A_\nu\right]$$

.

END oF WEEK 4

4 Quantization of non-abelian gauge theories

We now have a sufficient formalism to study correlation fucntions with a path integral formalism in a non-abelian gauge theory. The classical Yang-Mills Lagrangian is,

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu}. \tag{232}$$

with

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{233}$$

Proceeding in analogy with the path integral quantization of the scalar field theory, we may write the following path integral for the Yang-Mills theory,

$$Z[J^\mu_a] = \mathcal{N} \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\rm YM}(A^\mu_a) + J^\mu_a A_{a\mu}\right]} \tag{234}$$

We would like to develop a perturbation theory program. Remember what steps emerged in the scalar $-\lambda\frac{\phi^4}{4!}$ case. We could try to define the analogous steps here. Namely,

- find the propagator of the free-field by inverting the differential operator in the quadratic part of the Lagrangian. This would give us an expression for the path integral when all interactions are switched off ($g = 0$)

$$Z_0[J^\mu_a] = \mathcal{N}' e^{i\int d^4x d^4y J^\mu_a(x) \Delta_{\mu\nu ab}(x-y) J^\nu_b(x)}. \tag{235}$$

- derive the perturbative expansion from

$$Z[J] \sim e^{-\int d^4z \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J^\mu_a(z)}\right)} Z_0[J^\mu_a]. \tag{236}$$

We will see that the first of the two steps is problematic in a naive treatment.

The free-field action ($g = 0$) is

$$\begin{aligned} S_{free} &= -\frac{1}{4}\int d^4x \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{a\mu}\right) \\ &= -\frac{1}{2}\int d^4x \left[\left(\partial_\mu A^a_\nu\right)\left(\partial^\mu A^{\nu a}\right) - \left(\partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu a}\right)\right] \\ &= -\frac{1}{2}\int d^4x \left[\partial_\mu\left(A^a_\nu \partial^\mu A^{\nu a}\right) - A^a_\nu \partial_\mu\partial^\mu A^{\nu a} - \partial_\nu\left(A^a_\mu \partial^\mu A^{\nu a}\right) + A^a_\mu \partial_\nu \partial^\mu A^{\nu a}\right] \\ &= -\frac{1}{2}\int d^4x A^{a\mu}\delta^{ab}\left[-\partial^2 g_{\mu\nu} + \partial_\mu \partial\nu\right] A^{\nu b}. \end{aligned} \tag{237}$$

We now need to find the inverse of the operator

$$\left[-\partial^2 g_{\mu\nu} + \partial_\mu \partial \nu\right] \delta^{ab}.$$

However, it turns out that there is none! This operator has zero eigenvalues, and its determinant is zero. In particular we obtain zero when it acts on any function that can be written as a total derivative:

$$\left[-\partial^2 g_{\mu\nu} + \partial_\mu \partial \nu\right] \partial^\nu \Lambda(x) = -\partial^2 \partial_\mu \Lambda(x) + \partial^2 \partial_\mu \Lambda(x) = 0. \qquad (238)$$

Our naive attempt to establish a perturbation expansion in g using a path integral formalism has failed at the first step. However, there is a property of the theory, gauge invariance, which we have not yet used and we can exploit it to remove the zero-modes of the operator in the free part of the Lagrangian.

Let us start by defining a δ-functional, in analogy to a δ-function, which we will need in a while. The integral over a δ-function is

$$\int df \delta(f) = 1.$$

We can change variables, $f = f(w)$, and we obtain

$$\int dw \frac{\partial f}{\partial w} \delta(f(w)) = 1.$$

The multidimensional generalization of this equation is:

$$1 = \int dw_1 \ldots dw_n det \left(\frac{\partial f_i}{\partial w_j}\right) \delta(f_1(w_1, \ldots, w_n)) \ldots \delta(f_n(w_1, \ldots, w_n)).$$

We can take the limit $n \to \infty$ which yields a functional integral over $w(x)$, where x is the continuous variable corresponding to the index $i = 1 \ldots n$. We define the infinte product of delta functions as a delta functional. We write:

$$\begin{aligned} & \int \mathcal{D}w det \left(\frac{\delta f(x)}{\delta w(y)}\right) \delta\left[f(w)\right] \\ \equiv & \lim_{n\to\infty} \int dw_1 \ldots dw_1 det \left(\frac{\partial f_i}{\partial w_j}\right) \delta(f_1(w_1, \ldots, w_n)) \ldots \delta(f_n(w_1, \ldots, w_n)) \\ = & \ 1. \end{aligned} \qquad (239)$$

Notice the emergence of a functional determinant, due to changing variables in the measure of a functional integral.

We now return to the gauge-theory; the action is of course invariant under gauge transformations

$$A_\mu \to A'_\mu = U(x) A_\mu U^\dagger(x) + \frac{i}{g} U(x) \left(\partial_\mu U^\dagger(x)\right), \qquad (240)$$

with $A_\mu \equiv A^a_\mu T^a$. The transformation matrices U are determined by as many independent parameters as the generators of the Lie group,

$$U(x) = e^{-i\theta^a(x)T^a} = 1 - i\theta^a(x)T^a + \mathcal{O}(\theta^2). \qquad (241)$$

The integration $\int \mathcal{D}A^a_\mu$ in the path integral formalism does not discriminate among the fields which are connected via a gauge transformation.

Consider all the fields $A_\mu^{a(independent)}$ which cannot be connected via a gauge transformation. We never know them explicitly, but we can impose that they satisfy gauge-fixing conditions of the form

$$F(A^a_\mu) = 0,$$

which remove the superfluous degrees of freedom. For example, a very common choice is a Lorentz gauge fixing condition,

$$\partial_\mu A^{a\mu} = 0.$$

Of course, we will need to apply this gauge condition an infinite ammount of times, corresponding to all possible values of gauge transformation $\theta^a_\mu(x)$ values at each x which are sampled in the path integral.

Let us now go back to the path integral for the gauge theory without sources:

$$Z = \mathcal{N} \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\mathrm{YM}}(A^\mu_a)\right]} \qquad (242)$$

This integrates over all fields including the ones related by gauge transformations. In other words, had we thought of all possible gauge fixings, it integrates over all these possibilities. We can write:

$$\begin{aligned} Z &= \mathcal{N} \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\mathrm{YM}}(A^\mu_a)\right]} \times \mathbf{1} \\ &= \mathcal{N} \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\mathrm{YM}}(A^\mu_a)\right]} \times \int \mathcal{D}F^a \delta[F^a(A^{a\mu})], \end{aligned} \qquad (243)$$

where

$$F^a(A^{\mu a}) = 0 \rightsquigarrow A^{a\mu} = A^{a\mu}(\theta^b).$$

Different gauge fixings F^a correspond to different group parameters, so we may change variables $F^a \to \theta^a$ in the second functional integration.

$$\begin{aligned} Z &= \mathcal{N} \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\mathrm{YM}}(A^\mu_a)\right]} \int \mathcal{D}\theta^b \det\left(\frac{\delta F^a(A^{a\mu})}{\delta\theta^b}\right) \delta[F^a(A^{a\mu}(\theta^b))] \\ &= \mathcal{N} \int \mathcal{D}\theta^b \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\mathrm{YM}}(A^\mu_a)\right]} \det\left(\frac{\delta F^a(A^a_\mu)}{\delta\theta^b}\right)\Bigg|_{F^a(A^{a\mu}(\theta^b))=0} \delta[F^a(A^{a\mu}(\theta^b))] \end{aligned} \qquad (244)$$

Now there is a crucial observation to be made. No term in the inner functional integral depends on θ^b. Let us justify this statement. We have,

$$1 = \int \mathcal{D}F^a \delta\left[F^a(A^{a\mu})\right]$$

$$\rightsquigarrow \frac{1}{\det\left(\frac{\delta F^a(A^{a\mu})}{\delta\theta^b}\right)\Big|_{F^a(A^{a\mu}(\theta^b))=0}} = \int \mathcal{D}\theta^b \delta(F^a(A^{a\mu}(\theta^b))) \quad (245)$$

This determinant is therefore just a gauge-invariant number, as it does not depend on gauge parameters which are ommtegrated over in the right-hand side. The exponential of the Yang-Mills action is of course gauge invariant:

$$e^{i\int d^4x \mathcal{L}_{\rm YM}(A_\mu)} = e^{i\int d^4x \mathcal{L}_{\rm YM}(A^a_\mu(\theta^b))}.$$

Finally, the gauge-boson fields are also members of the Lie algebra and only get reshuffled by gauge transformations. The measure $\mathcal{D}A^a_\mu$ can be evaluated at any gauge.

$$\mathcal{D}A^a_\mu = \mathcal{D}A^a_\mu(\theta^b). \quad (246)$$

<u>**Exercise:**</u> *Consider a gauge transformation $A^a_\mu \to A^{a\prime}_\mu$. Prove that $\mathcal{D}A^a_\mu = \mathcal{D}A^{a\prime}_\mu$. You only need to consider an infinitesimal gauge transformation.*

We then compute all terms in the inner path inegral at the specific-gauge chosen by the delta-functional.

$$\begin{aligned} Z &= \mathcal{N}\int \mathcal{D}\theta^b \int \mathcal{D}A^a_\mu e^{i\int d^4x[\mathcal{L}_{\rm YM}(A^\mu_a)]} \det\left(\frac{\delta F^a(A^a_\mu)}{\delta\theta^b}\right)\Big|_{F^a(A^{a\mu}(\theta^b))=0} \delta[F^a(A^{a\mu}(\theta^b))] \\ &= \mathcal{N}\int \mathcal{D}\theta^b \int \mathcal{D}A^a_\mu(\theta^b) e^{i\int d^4x[\mathcal{L}_{\rm YM}(A^\mu_a(\theta^b))]} \det\left(\frac{\delta F^a(A^a_\mu(\theta^b))}{\delta\theta^b}\right) \delta[F^a(A^{a\mu}(\theta^b))] \\ &= \mathcal{N}\left(\int \mathcal{D}\theta^b\right) \int \mathcal{D}A^a_\mu e^{i\int d^4x[\mathcal{L}_{\rm YM}(A^\mu_a)]} \det\left(\frac{\delta F^a(A^a_\mu)}{\delta\theta^b}\right) \delta[F^a(A^{a\mu})] \quad (247) \end{aligned}$$

In the last line we noticed that the gauge-fixed field variable $A^a_\mu(\theta^b)$ is a dummy integration variable. The path integration over all possible gauge transformations (corresponding to all possible gauge fixings) is an overall normalization factor. This is an infinite integration over the measure of infinite Lie algebra parameters. However, this is not a problem if we want to compute physical quantities, since the overall normalization will cancel. We therefore end up with the gauge-fixed path integral

$$Z = \mathcal{N}'\left(\int \mathcal{D}\theta^b\right) \int \mathcal{D}A^a_\mu e^{i\int d^4x[\mathcal{L}_{\rm YM}(A^\mu_a)]} \det\left(\frac{\delta F^a(A^a_\mu)}{\delta\theta^b}\right) \delta[F^a(A^{a\mu})]. \quad (248)$$

It is very important to notice that (up to an irrelevant infinite normalization) this path integral does not depend on the gauge-fixing condition $F(A^{a\mu})$ that we may choose!

This new-path integral in Eq. 248 integrates over fields which cannot be related via gauge transformations; it should therefore be fine to derive Green's functions for fields which are physically distinct. However, the new expression has two features, the gauge-fixing delta-functional and the determinant, which were absent in our formulation of perturbation theory. It is therefore unclear at first sight how to establish a calculable perturbative expansion with the mathematics that we now. Two very clever tricks will come to our rescue.

Without loss of generality we write

$$F^a(A^{a\mu}) = \mathcal{G}^a(A^{a\mu}) - w^a(x). \tag{249}$$

We are allowed to multiply the path-integral with an overall constant without any physical consequences. We then multiply with the factor,

$$C = \int \mathcal{D}w^a e^{-i\int d^4x \frac{w^a(x)^2}{2\xi}}. \tag{250}$$

We can do this because $Z[J_a^\mu]$ in Eq. 248 does not depend on $w(x)$. We have

$$\begin{aligned} Z &\sim \left(\int \mathcal{D}w^a e^{-i\int d^4x \frac{w^a(x)w^a(x)}{2\xi}}\right) \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\rm YM}(A^\mu_a)\right]} \det\left(\frac{\delta \mathcal{G}^a(A^a_\mu)}{\delta\theta^b}\right) \delta[\mathcal{G}^a(A^{a\mu}) - w^a(x)] \\ &= \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\rm YM}(A^\mu_a)\right]} \det\left(\frac{\delta \mathcal{G}^a(A^a_\mu)}{\delta\theta^b}\right) \int \mathcal{D}w^a e^{-i\int d^4x \frac{w(x)^2}{2\xi}} \delta[\mathcal{G}^a(A^{a\mu}) - w(x)] \\ &= \int \mathcal{D}A^a_\mu e^{i\int d^4x \left[\mathcal{L}_{\rm YM}(A^\mu_a) - \frac{1}{2\xi}(\mathcal{G}^a(A^{a\mu}))^2\right]} \det\left(\frac{\delta \mathcal{G}^a(A^a_\mu)}{\delta\theta^b}\right) \end{aligned} \tag{251}$$

With this trick, we remove the delta-function from the integrand and modify the exponent of the path-integral. This also yields a well-defined propagator for the gauge boson field. If we choose for example the gauge-fixing condition

$$\mathcal{G}^a(A^{a\mu}) = \partial_\mu A^{a\mu},$$

the free-part ($g = 0$) of the action in the exponent becomes:

$$S_{free} = -\frac{1}{2}\int d^4x A^{a\mu}\delta^{ab}\left[-\partial^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial\nu\right] A^{b\nu}. \tag{252}$$

Now, the new differential operator has an inverse:

$$\Delta^{ab}_{\mu\nu} = \delta^{ab}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right]. \tag{253}$$

Exercise: *Find the inverse of the operator:*

$$\left[\left(-\partial^2 + M^2\right) g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial \nu.\right]$$

This will be the case of a massive gauge-boson such as W, Z.
Exercise: *Find the gauge boson propagator in an axial gauge*

$$G(A) = n_\mu A^{a\mu},$$

where n is a light-like vector $n^2 = 0$.

We now need to deal with the determinant in the integrand of the path integral. Here we will use a result that we found from infinite integration over Grassmann variables. We proved earlier that:

$$\int dx_1 \dots dx_n dy_1 \dots dy_n e^{-x^T A y} = det(A). \tag{254}$$

where A is an $n \times n$ matrix and x, y are *Grassmann* variables. We can take the limit of $n \to \infty$. We then obtain express a functional determinant as a fermionic path integrals over two independent Grassmann functions x and y:

$$ig\, det(A) = \int \mathcal{D}y \mathcal{D}x e^{i \int d^4x_1 d^4x_2\, y(x_1)(gA(x_1 - x_2))x(x_2)}. \tag{255}$$

The Fadeev-Popov idea was to introduce two new fields with odd spin-statistics (Grassmann variables in the path-integral), a ghost and an anti-ghost, and write the determinant as functional integral over an exponential. We write,

$$\begin{aligned} Z &\sim \int \mathcal{D}A_\mu^a e^{i\int d^4x \left[\mathcal{L}_{\text{YM}}(A_a^\mu) - \frac{1}{2\xi}(\mathcal{G}^a(A^{a\mu}))^2\right]} \det\left(ig \frac{\delta \mathcal{G}^a(A_\mu^a)}{\delta\theta^b}\right) \\ &\sim \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\eta}^a \mathcal{D}\eta^a e^{i\int d^4x \left[\mathcal{L}_{\text{YM}}(A_a^\mu) - \frac{1}{2\xi}(\mathcal{G}^a(A^{a\mu}))^2\right]} \times \\ &\qquad \times e^{i\int d^4x_1 d^4x_2 \bar{\eta}^a(x_1)\left(g\frac{\delta\mathcal{G}^a(A_\mu^a(\theta^a))}{\delta\theta^b}\right)\eta^b(x_2)}. \end{aligned} \tag{256}$$

We need not to worry about computing precisely the overall normalization of the path-integral. This will drop out when we require that vacuum to vacuum transitions have a unit amplitude. The factor $-ig$ is convenient, in order to later combine easierly all terms under a common exponential.

Let us consider a local gauge transformation

$$U(x) = e^{-i\theta^a T^a},$$

under which a gauge-field transforms as

$$A_\mu(\theta) = U(x) A_\mu U^\dagger(x) + \frac{i}{g} U(x) \left(\partial_\mu U^\dagger(x) \right), \tag{257}$$

where, as usual, $A_\mu = A_\mu^a T^a$. For a small transformation, we obtain

$$A^{\mu,a}(\theta) = A^{\mu,a} - \frac{1}{g} \left[\partial^\mu \delta^{ab} - g f^{abc} A^{\mu,c} \right] \theta^b. \tag{258}$$

We recognize in the above expression the covariant derivative in the adjoint representation:

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c, \tag{259}$$

so we can write:

$$A^{\mu,a}(\theta) = A^{\mu,a} - \frac{1}{g} D^{\mu,ab} \theta^b. \tag{260}$$

We then have,

$$\frac{\delta A^{\mu,a}(\theta(x))}{\delta \theta^b(y)} = -\frac{1}{g} \frac{\delta \left(D^{\mu,ac}(x) \theta^c(x) \right)}{\delta \theta^b(y)} = -\frac{1}{g} D^{\mu,ab}(y) \delta(x-y). \tag{261}$$

We can now compute the functional derivative

$$\begin{aligned} g \frac{\delta G(A^{\mu,a}(\theta(x)))}{\delta \theta^b(y)} &= g \int d^4 z \frac{\delta G(A^{\mu,a}(\theta(x)))}{\delta (A^{\nu,c}(\theta(z)))} \frac{\delta (A^{\nu,c}(\theta(z)))}{\delta \theta^b(y)} \\ &= -\int d^4 z \frac{\delta G(A^{\mu,a}(x))}{\delta (A^{\nu,c}(z))} D^{\nu,cb}(z) \delta(z-y) \\ &= -\frac{\delta G(A^{\mu,a}(x))}{\delta (A^{\nu,c}(y))} D^{\nu,cb}(y) \end{aligned} \tag{262}$$

We can then write the following expression for the path integral:

$$\begin{aligned} Z \sim & \int \mathcal{D} A_\mu^a \mathcal{D} \bar{\eta}^a \mathcal{D} \eta^a e^{i \int d^4 x \left[\mathcal{L}_{\mathrm{YM}}(A_a^\mu) - \frac{1}{2\xi} (\mathcal{G}^a(A^{a\mu}))^2 \right]} \times \\ & \times e^{i \int d^4 x_1 d^4 x_2 \bar{\eta}^a(x_1) \left(-\frac{\delta \mathcal{G}^a(A_\mu^a(x_1))}{\delta A_\nu^c(x_2)} D_\nu^{cb}(x_2) \right) \eta^b(x_2)}. \end{aligned} \tag{263}$$

This is valid for any arbitrary gauge fixing condition $\mathcal{G}^a(A_\mu^a)$.

It will be instructive and practical (this is what we need to do when we compute elements of the S-matrix) to choose a gauge. A common choice is the Lorentz gauge:

$$\mathcal{G}^a(A_\mu^a) = \partial_\mu A^{\mu,a}, \tag{264}$$

where

$$\frac{\delta \partial_\mu A^{\mu,a}(x_1)}{\delta A^{\nu,c}(x_2)} = \partial_\mu g_\nu^\mu \delta^{ac} \delta(x_1 - x_2) = \partial_\nu \delta^{ac} \delta(x_1 - x_2). \tag{265}$$

Therefore, in the Lorentz gauge, the path integral is (up to a normalization):

$$Z = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\eta}^a \mathcal{D}\eta^a e^{i\int d^4x\left[-\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{\eta}^a(x)\partial_\mu D^{\mu;ab}\eta^b(x)\right]}. \tag{266}$$

After a partial integration, we obtain:

$$Z = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\eta}^a \mathcal{D}\eta^a e^{i\int d^4x\left[-\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu\bar{\eta}^a)D^{\mu;ab}\eta^b\right]}. \tag{267}$$

In this last expression, we have exponentiated all terms which arose from gauge-fixing. The argument of the exponential is a new action, modified by a gauge-fixing term and contributions from the ghost fields.

Notice that the form of the quadratic term in the ghost fields is the same as for a complex *scalar* field. However, the variables $\eta, \bar{\eta}$ in the path-integral are anti-commuting Grassmann variables. Therefore, the ghost field is a scalar field with the "wrong" spin-statistics.

We also observe that for an abelian gauge theory $f^{abc} = 0$ we have $D_\mu^{ab} = \delta^{ab}\partial_\mu$ and there is no interaction term for the ghost field and the gauge-boson. In this case, the ghost field is a free field and we can integrate out its contribution to the path integral (changing the irrelevant overall normalization). This is why in QED you never needed to introduce it.

Exercise: *Find the expression of the path-integral for an $SU(N)$ Yang-Mills theory in an axial gauge*

$$G(A) = n_\mu A^{a\mu},$$

where n is a light-like vector $n^2 = 0$.

4.1 Perturbative QCD

After gauge-fixing and the Fadeev-Popov method, we can formulate a path integral for QCD, where the path-integral action is:

$$S = \int d^4x \mathcal{L},$$

with

$$\mathcal{L} = \mathcal{L}_{\mathrm{YANG-MILLS}} + \mathcal{L}_{\mathrm{FERMION}} + \mathcal{L}_{\mathrm{GAUGE-FIXING}} + \mathcal{L}_{\mathrm{FADEEV-POPOV}}.$$

The classical Yang-Mills Lagrangian is:

$$\mathcal{L}_{\mathrm{YANG-MILLS}} = -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu},$$

and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

The gauge fixing and the Fadeev-Popov terms, in the Lorentz gauge, are:

$$\mathcal{L}_{\text{GAUGE-FIXING}} = -\frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2.$$

$$\mathcal{L}_{\text{FADEEV-POPOV}} = \left(\partial^\mu \bar{\eta}^a\right) D^{ab}_\mu \eta^b,$$

and the fermion term:

$$\mathcal{L}_{\text{FERMION}} = \bar{\psi}^i \left(i\gamma^\mu D^{ij}_\mu - m\delta^{ij}\right)\psi^j.$$

The covariant derivtives in the adjoint and fundamental representation are

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - g f^{abc} A^c_\mu$$

and

$$D^{ij}_\mu = \partial_\mu \delta^{ij} - i g T^c_{ij} A^c_\mu$$

accordingly.

We would like to compute Green's function using perturbation theory. If we switch off the coupling, $g = 0$, then we are left with terms which are quadratic in the fields, and we can compute the corresponding path integral for the free action. We define,

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{interacting}, \tag{268}$$

with

$$\mathcal{L}_{free} = \mathcal{L}|_{g=0}.$$

Explicitly,

$$\begin{aligned}\mathcal{L}_{free} \quad = \quad & -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu}\right) - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)\left(\partial^\nu A^a_\nu\right) \\ & + \left(\partial_\mu \bar{\eta}^a\right)\left(\partial^\mu \bar{\eta}^a\right) \\ & + \bar{\psi}^i \left(i\gamma^\mu \partial_\mu - m\right)\psi^i \end{aligned} \tag{269}$$

It is convenient to use integration by parts and cast the free Lagrangian in a "standard form":

$$-\frac{1}{2}\left(\text{Field}_A\right)\hat{O}\left(\text{Field}_A\right) + \partial_\mu\left(\ldots\right)$$

or, for terms with independent fields (appearing separately in the measure of the path-integral),

$$-\left(\text{Field}_A\right)\hat{O}\left(\text{Field}_B\right) + \partial_\mu\left(\ldots\right).$$

We have

$$\begin{aligned}
\mathcal{L}_{free} \quad = \quad & -\frac{1}{2} A_\mu^a K^{ab,\mu\nu} A_\nu^b \\
& -\bar{\eta}^a K^{ab} \eta^b \\
& -\bar{\psi}^i \Lambda^{ij} \psi^j \\
& +\partial_\mu (\ldots) , \qquad (270)
\end{aligned}$$

with ($\partial^2 \equiv \partial_\mu \partial^\mu$)

$$K^{ab,\mu\nu} = \delta^{ab} \left[-g^{\mu\nu} \partial^2 + \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right], \qquad (271)$$

$$K^{ab} = \delta^{ab} \partial^2, \qquad (272)$$

$$\Lambda^{ij} = \delta^{ij} \left(m - i\not{\partial} \right). \qquad (273)$$

An essential step in order to compute the generating functional for the free path-integral is to find the inverse of the above operators, i.e. objects which diagonilize them in all indices. We define:

$$\begin{aligned}
& K_{\mu\rho}^{ac}(x) D^{cb,\rho\nu}(x-y) = \delta^{ab} g_\mu^\nu \delta^{(4)}(x-y), \\
& K^{ac}(x) D^{cb}(x-y) = \delta^{ab} \delta^{(4)}(x-y), \\
& \Lambda^{ik}(x) S^{kj}(x-y) = \delta^{ij} \delta^{(4)}(x-y). \qquad (274)
\end{aligned}$$

Exercise: *Solve these equations. The solutions are given next in the text and it is easy to verify whether they are correct or not by insertibg them above. However, it will be instructive to think how to find them if nobody told you the answer!*

We find that

$$D_{\mu\nu}^{ab}(x) = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 + i\epsilon} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right] \qquad (275)$$

$$D^{ab}(x) = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 + i\epsilon} (-1) \qquad (276)$$

$$S^{ij}(x) = \delta^{ij} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} (-1) \left(\not{k} + m \right). \qquad (277)$$

The generating functional for the free-Lagrangian is

$$\begin{aligned}
Z_0 \left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}} J_A \right] = & \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\eta \mathcal{D}\bar{\eta} \mathcal{D}A_\mu \exp \Bigg(i \int d^4x \\
& \Bigg[-\frac{1}{2} A_\mu^a K^{ab,\mu\nu} A_\nu^b - \bar{\eta}^a K^{ab} \eta^b - \bar{\psi}^i \Lambda^{ij} \psi^j \\
& + J_A^{a,\mu} A_\mu^a + J_{\bar{\psi}}^i \psi^i + \bar{\psi}^i J_\psi^i + J_{\bar{\eta}}^a \eta^a + \bar{\eta}^a J_\eta^a \Bigg] \Bigg)
\end{aligned} \qquad (278)$$

where we have included independent sources for all fermion, anti-fermion, ghost, anti-ghost, and gauge fields. We should keep in mind that only the source $J_A^{a,\mu}$ for the gauge field is a bosonic (commuting) variable; all other sources are Grassmann variables. We can now "complete squares" by shifting the fields as,

$$A^{a,\mu}(x) \to A^{a,\mu}(x) + \int d^4y D^{ab}_{\mu\nu}(x-y) J_A^{b,\nu}(y) \tag{279}$$

$$\eta^a(x) \to \eta^a(x) + \int d^4y D^{ab}(x-y) J_\eta^b(y) \tag{280}$$

$$\bar{\eta}^a(x) \to \bar{\eta}^a(x) + \int d^4y J_{\bar{\eta}}^b(y) D^{ba}(x-y) \tag{281}$$

$$\psi^i(x) \to \psi^i(x) + \int d^4y S^{ij}(x-y) J_\psi^j(y) \tag{282}$$

$$\bar{\psi}^i(x) \to \bar{\psi}^i(x) + \int d^4y J_{\bar{\psi}}^j(y) S^{ji}(x-y). \tag{283}$$

We then obtain (with an undetermiend overall constant factor),

$$Z_0\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right] = \mathcal{N} \exp\left(i\int d^4x d^4y \left[\frac{1}{2} J_A^{a,\mu}(x) D^{ab}_{\mu\nu}(x-y) J_A^{b,\nu}(y) + J_{\bar{\eta}}^a(x) D^{ab}(x-y) J_\eta^b(y) + J_{\bar{\psi}}^i(x) S^{ij}(x-y) J_\psi^j(y)\right]\right) \tag{284}$$

We will now deal with the interaction Lagrangian; this is defined as

$$\mathcal{L}_{interaction} = \mathcal{L} - \mathcal{L}_{free}. \tag{285}$$

We find

$$\begin{aligned}\mathcal{L}_{interaction} &= g\bar{\psi}^i T^a_{ij} \not{A}^a \psi^j \\ &\quad - g\left(\partial_\mu \bar{\eta}^a\right) f^{abc} A^{a,\mu} \eta^b \\ &\quad - 2g f^{abc} A^{b,\mu} A^{c,\nu}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \\ &\quad - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d,\mu} A^{e,\nu}.\end{aligned} \tag{286}$$

The generating functional for the full theory can be obtained perturbatively, expanding in g,

$$\begin{aligned} & Z\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right] \\ = {} & \mathcal{N}\exp\left\{i\int d^4z \mathcal{L}_{interaction}\left(-i\frac{\delta}{\delta J_A}, i\frac{\delta}{\delta J_\psi}, -i\frac{\delta}{\delta J_{\bar{\psi}}}, i\frac{\delta}{\delta J_\eta}, -i\frac{\delta}{\delta J_{\bar{\eta}}}\right)\right\} \\ & Z_0\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right], \end{aligned} \tag{287}$$

where we must replace in the interaction Lagrangian the field variables with functional derivatives with respect to their corresponding sources.

Exercise:

- *Compute* $Z\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right]$ *through* $\mathcal{O}(g^2)$ *in the expansion around* $g = 0$.
- *Compute the generating functional of connected diagrams*

$$W\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right] = -i \log Z\left[J_\psi, J_{\bar{\psi}}, J_\eta, J_{\bar{\eta}}, J_A\right]$$

through the same order.

- *Compute the gauge-boson propagator through the same order*

$$\langle 0, +\infty | \, T A^{a,\mu}(x) A^{b,\nu}(0) \, | 0, -\infty \rangle$$

- *Transform this expression in momentum space*

END OF WEEK 5

5 BRST symmetry

We recall here the path-integral for a non-abelian gauge theory with a fermion field,

$$Z \sim \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\psi}^i \mathcal{D}\psi^i \mathcal{D}\bar{\eta}^a \mathcal{D}\eta^a e^{i\int d^4x \left[\mathcal{L}_{\rm YM}+\mathcal{L}_{fermion}-\frac{1}{2\xi}(\mathcal{G}^a(A^{\mu a}))^2+\bar{\eta}^a\Delta^a\right]}, \tag{288}$$

where we have defined,

$$\Delta^a(x) = -\int d^4y \left(\frac{\delta\mathcal{G}^a(A_\mu^a(x))}{\delta A_\nu^c(y)} D_\nu^{cb}(y)\right)\eta^b(y). \tag{289}$$

It is convenient to linearize the action in the gauge-fixing term. If we introduce (yet) another (bosonic) field, w^a, we can re-write,

$$\begin{aligned}\int \mathcal{D}w^a e^{i\int d^4x\left(\frac{\xi}{2}w^a w^a + w^a\mathcal{G}^a\right)} &= \int \mathcal{D}w^a e^{i\frac{\xi}{2}\int d^4x\left[\left(w^a+\frac{\mathcal{G}^a}{2}\right)^2-\frac{\mathcal{G}^a\mathcal{G}^a}{\xi^2}\right]} \\ &= \mathcal{N}_x e^{i\int d^4x \frac{-\mathcal{G}^a\mathcal{G}^a}{2\xi}}\end{aligned} \tag{290}$$

We will denote,

$$\mathcal{L}_{cl} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm fermion}, \tag{291}$$

the classical Lagrangian which satisfies local gauge-invariance. The path-integral, up to a normalization, is equal to

$$Z = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{\psi}^i \mathcal{D}\psi^i \mathcal{D}\bar{\eta}^a \mathcal{D}\eta^a \mathcal{D}w^a e^{i\int d^4x\left[\mathcal{L}_{cl}+\bar{\eta}^a\Delta^a+w^a\mathcal{G}^a+\frac{\xi}{2}w^a w^a\right]}. \tag{292}$$

If we want to go back to the original form of Z we simply need to integrate out the field w^a. The exponent is not gauge invariant, except the term with the classical Lagrangian $\mathcal{L}_{cl}$.

We find a closely related symmetry which is called Becchi-Rouet-Stora; Tyutin or, shorter,BRST symmetry which leaves invariant not only the classical Lagrangian but also the sum gauge-fixing and Fadeev-Popov terms. The BRST symmetry transformations are gauge transformations of a special form for the gauge boson A_μ^a and fermion ψ fields which enter the classical Lagrangian $\mathcal{L}_{cl}$. the gauge boson A_μ^a and fermion ψ. The gauge parameter is made out of the ghost field and a Grassmann variable. Specifically, the fermion transforms as:

$$\delta_\theta\psi = -iT^a\left(\theta\eta^a\right)\psi. \tag{293}$$

This is equivalent to classical gauge transformation with the replacement

$$\theta^a(x) \to \theta\eta^a.$$

where both η and the ghost field η^a are Grassmann variables. Of course, their product is a bosonic variable as expected by a classical gauge transformation. We take the parameter θ to be global

$$\partial_\mu \theta = 0.$$

For the gauge boson, we require that the BRST transformations is also a gauge transformation with gauge parameter $\theta\eta^a$:

$$\delta_\theta A_\mu^a = -\frac{\theta}{g} D_\mu^{ab} \eta^b. \tag{294}$$

Similarly, the anti-fermion field transforms as:

$$\delta_\theta \bar{\psi} = \bar{\psi} i T^a \left(\theta \eta^a\right).$$

Notice that the Grassmann variable entering here is η, so that we perform the same gauge-transformation on all classical fields $\psi, \bar{\psi}, A_\mu^a$.

Before we present the trasnformations for the remaining fields, we state a characteristic property of the transformation: **Two successive BRST transformations on an arbitrary function of fields leave the function invariant (nilpotent transformation)**.

$$\delta_{\theta_2} \delta_{\theta_1} F(A, \psi, \bar{\psi}, \eta, \bar{\eta}) = 0. \tag{295}$$

If we insist on this property, we obtain:

$$\begin{aligned} 0 &= \delta_{\theta_2}\left(\delta_{\theta_1}\psi\right) \\ &= \delta_{\theta_2}\left(-iT^a\theta_1\eta^a\psi\right) \\ &= -iT^a\theta_1\left[(\delta_{\theta_2}\eta^a)\psi + \eta^a(\delta_{\theta_2}\psi)\right] \\ &= -iT^a\theta_1\left[(\delta_{\theta_2}\eta^a)\psi + \eta^a\left(-iT^b\theta_2\eta^b\psi\right)\right] \\ &= -iT^a\theta_1\left[(\delta_{\theta_2}\eta^a) - i\eta^a\theta_2\eta^b T^b\right]\psi \end{aligned} \tag{296}$$

Equivalently,

$$\begin{aligned} & T^c\delta_{\theta_2}\eta^c = iT^bT^c\eta^b\theta_2\eta^c \\ \rightsquigarrow\ & tr(T^aT^c)\delta_{\theta_2}\eta^c = -i\theta_2 tr(T^aT^bT^c)\eta^b\eta^c \qquad \text{beware of Grassmann!} \\ \rightsquigarrow\ & \frac{\delta^{ac}}{2}\delta_{\theta_2}\eta^c = -i\theta_2 tr\left(T^aT^bT^c\right)\frac{\eta^b\eta^c - \eta^c\eta^b}{2} \\ \rightsquigarrow\ & \delta_{\theta_2}\eta^c = -i\theta_2 tr\left(T^a\left[T^b, T^c\right]\right)\eta^b\eta^c \\ \rightsquigarrow\ & \delta_{\theta_2}\eta^c = f^{bcd} tr(T^aT^d)\eta^b\eta^c, \end{aligned} \tag{297}$$

Therefore we require the ghost field to transform as:

$$\delta_\theta \eta^a = \frac{\theta}{2} f^{abc}\eta^b\eta^c. \tag{298}$$

Two successive transformations on the gauge field produce,

$$\begin{aligned}
\delta_{\theta_2}\delta_{\theta_1} A^a_\mu &= -\frac{\theta_1}{g}\delta_{\theta_2}\left[\partial_\mu\eta^a - gf^{abc}\eta^b A^c_\mu\right] \\
&= -\frac{\theta_1}{g}\left[\partial_\mu\left(\delta_{\theta_2}\eta^a\right) - gf^{abc}\left(\delta_{\theta_2}\eta^b\right)A^c_\mu - gf^{abc}\eta^b\left(\delta_{\theta_2}A^c_\mu\right)\right] \\
&= -\frac{\theta_1}{g}\left[D^{ab}_\mu\left(\delta_{\theta_2}\eta^a\right) + \frac{1}{2}f^{abc}\eta^b\theta_2 D^{cd}_\mu\eta^d\right] = \ldots = 0. \quad (299)
\end{aligned}$$

Exercise: *Fill the dots... Prove the above statement using the anti-commutation of Grassmann variables and the Jacobi identity for the structure constants*

For the remaining two independent fields in the action of the path integral, we have no unabiguous guidance in order to construct their BRST transformations. We will make two very simple choices. We perform no transformation on the auxiliary bosonic field,

$$\delta_\theta w^a = 0.$$

For the anti-ghost we require that under a a BRST transformation it gets shifted by a a constant.

$$\delta_\theta \bar{\eta}^a = \frac{1}{g}\theta w^a.$$

This choice as we will see guarantees BRST invariance of the quantum action. Notice that

$$\delta_{\theta_1}\delta_{\theta_2}\bar{\eta}^a = \delta_{\theta_1}\delta_{\theta_2} w^a = 0.$$

Let us now compactify the notation. We consider any field F from $\{A_\mu, \psi, \bar{\psi}, \eta, \bar{\eta}, w\}$. We will introduce the short-hand notation:

$$\delta_\theta F \equiv \theta(sF)$$

We have pulled an explicit prefactor of the BRST transformation parameter, and the notation sF denotes the remainder of the expression after we transformed the field F. For example, we write

$$\delta_\theta\psi = -igT^a\theta\eta^a\psi \rightsquigarrow s\psi = -igT^a\eta^a\psi.$$

Notice that if the field F is a bosonic field then sF is Grassmannian and vice versa. If we perform two consecutive BRST transformations, we have:

$$\delta_{\theta_1}\delta_{\theta_2} F = \theta_1\theta_2 s^2 F.$$

Nilpotency means that:

$$s^2 F = 0. \quad (300)$$

Nilpotency is a property valid for a product of two variables as well. Performing a BRST transformation on the product of two fields we have:

$$\begin{aligned}\delta_{\theta_1}(F_1F_2) &= (\delta_{\theta_1}F_1)F_2 + F_1(\delta_{\theta_1}F_2)\\ &\quad \theta_1(sF_1)F_2 + F_1\theta_1(sF_2)\\ &\quad \theta_1\left[(sF_1)F_2 \pm F_1(sF_2)\right],\end{aligned} \tag{301}$$

where the minus sign arises if F_1 is a Grassmann variable. We have used here that the field F and sF have always opposite spin-statistics. If we perform a double BRST transformation on the product F_1F_2 we then find,

$$\begin{aligned}\delta_{\theta_2}\delta_{\theta_1}(F_1F_2) &= \theta_1\delta_{\theta_2}\left[(sF_1)F_2 \pm F_1(sF_2)\right]\\ &= \theta_1\left[(sF_1)\theta_2(sF_2) \pm \theta_2(sF_1)(sF_2)\right]\\ &= \theta_1\theta_2\left[\mp(sF_1)(sF_2) \pm (sF_1)(sF_2)\right]\\ &= 0.\end{aligned} \tag{302}$$

We can continue in the same spirit. We find that:

$$\delta_{\theta_2}\delta_{\theta_1}(F_1F_2\ldots F_n) = 0. \tag{303}$$

In fact every functional of these fields satisfies,

$$\delta_{\theta_2}\delta_{\theta_1}G[(F_1, F_2, \ldots, F_n)] = 0.$$

We will return to this shortly.

We should investigate the effect of a BRST transforamtion on the gauge fixing term $\mathcal{G}^a$ in the Lagrangian. $\mathcal{G}^a$ is a function of the gauge field and we should use the chain-rule.

$$\begin{aligned}\delta_\theta\mathcal{G}^a(A^a_\mu(x)) &= \int d^4y \frac{\delta\mathcal{G}^a(x)}{\delta A^b_\mu(y)}\delta_\theta A^b_\mu(y)\\ &= -\frac{1}{g}\theta\int d^4y \frac{\delta\mathcal{G}^a(x)}{\delta A^b_\mu(y)}D^{ab}_\mu\eta^b(y)\\ &= \frac{\theta}{g}\Delta^a\end{aligned} \tag{304}$$

Let us now consider the variation:

$$\begin{aligned}\delta_\theta\left[\bar{\eta}^a\left(\mathcal{G}^a + \frac{\xi}{2}w^a\right)\right] &= (\delta_\theta\bar{\eta}^a)\left(\mathcal{G}^a + \frac{\xi}{2}w^a\right) + \bar{\eta}^a(\delta_\theta\mathcal{G}^a)\\ &= \frac{\theta}{g}\left(\bar{\eta}^a\Delta^a + w^a\mathcal{G}^a + \frac{\xi}{2}w^a w^a\right).\end{aligned} \tag{305}$$

In other words, the non-classical part of the Lagrangian is already a total variation under a BRST-transformation. Due to the property of nilpotency, such terms remain invariant under a BRST-transformation.

$$\delta_\theta\left[\mathcal{L}_{cl}+\bar{\eta}^a\Delta^a+w^a\mathcal{G}^a+\frac{\xi}{2}w^aw^a\right]=(\delta_\theta\mathcal{L}_{cl})+g\delta_\theta\left(s\left[\bar{\eta}^a\left(\mathcal{G}^a+\frac{\xi}{2}w^a\right)\right]\right)=0 \tag{306}$$

We remind here that the classical-part of the Lagrangian is BRST-invariant due to its gauge-invariance.

Exercise:*Prove that the Jacobian of a BRST transformation is unit. This completes a proof that the full path-integral is BRST-invariant.*

The BRST symmetry provides us with the asymptotic states of the S-matrix. Let us compute the S-matrix element

$$\langle\alpha|\,\beta\rangle\,,$$

in a two different gauges, $\mathcal{G}_1$ and $\mathcal{G}_2$, where the two gauge fixing conditions differ by little:

$$\mathcal{G}_2=\mathcal{G}_1+\tilde{\delta}\mathcal{G}.$$

We require that the initial $|\beta\rangle$ and final $\langle\alpha|$ states are physical and thus the same in both gauges:

$$\langle\alpha|_{\mathcal{G}_1}=\langle\alpha|_{\mathcal{G}_2}\,,\qquad|\beta\rangle_{\mathcal{G}_1}=|\beta\rangle_{\mathcal{G}_2}\,.$$

The matrix-element is computed through the path integral Z, which has an apparent but fake dependence on the gauge. A difference due to the different gauge-fixing conditions shows up due to the change of the action in the exponent of the path-integral. We will have:

$$\tilde{\delta}Z=Z_{\mathcal{G}_2}-Z_{\mathcal{G}_1}=\int DA\ldots e^{i\int d^4\mathcal{L}cl+gs\,K|_{\mathcal{G}_1}}-\int DA\ldots e^{i\int d^4\mathcal{L}cl+gs\,K|_{\mathcal{G}_2}}$$

Since we have considered infinitesimally different gauge-fixing conditions, we have that:

$$\tilde{\delta}Z=Z_{\mathcal{G}_1}igs\tilde{\delta}K.$$

Demanding that this difference has no physical consequences leads to a selection criterion for physical states. Promoting $\tilde{\delta}K$ into an operator, we would like that it yields a zero expectation value when inserted in a matrix-element for the transition in between physical states:

$$0=\langle\alpha|\,s\tilde{\delta}K\,|\beta\rangle\,.$$

We can construct an operator Q which is the generator of the BRST transformations for canonical fields.

$$\delta_\theta(Field)=i\left[\theta Q,Field\right]_\pm$$

where a commutator for a field with even spin and an anti-commutator for a field with odd spin are understood with the $\pm$ subscript: $[A, B]_{\pm} = AB \pm BA$. We can write:

$$0 = s(s\,Field) = \left[Q, [Q, Field]_{\mp}\right]_{\pm} = \left[Q^2, Field\right]_{-}.$$

For the above to be satisfied for every field we need:

$$Q^2 = 0.$$

Using the BRST generator we write:

$$\langle\alpha|\, s\tilde{\delta}K\,|\beta\rangle = \langle\alpha|\left[Q, \tilde{\delta}K\right]_{\pm}|\beta\rangle = \langle\alpha|\, QK\,|\beta\rangle \pm \langle\alpha|\, KQ\,|\beta\rangle.$$

The matrix-element is the same in all gauges if the above vanishes; this provides us with a condition for physical fields. Computations of matrix-elements in different gauges yield the same result if the physical external states annihilate the generator of BRST transformations:

$$\langle\alpha|\, Q = Q\,|\beta\rangle = 0. \tag{307}$$

We have discovered a new symmetry transformation which leaves the gauge-fixed path-integral of a non-abelian gauge theory invariant. The BRST transformation is very important for the renormalization of the theory since it constrains the UV divergencesof S-matrix elements.

5.1 Application to the free electromagnetic field

If we trun off the coupling constant g non-abelian gauge theories reduce to the free QED Lagrangian. Also, the in and out states of the S-matrix are constructed from the requirement that asymptotic states at times very far in the future and the past are states of the interaction-free Lagrangian. It is therefore very useful to study how BRST symmetry can be used to select physical states in the case of the free electromagnetic field. The generating functional for QED in the Lorentz gauge takes the form:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\eta}\mathcal{D}\eta\mathcal{D}w \exp\left\{i\int d^4x \left[\mathcal{L}_{cl} + \frac{\xi}{2}w^2 + w\partial_\mu A^\mu + (\partial_\mu\bar{\eta})(\partial^\mu\eta)\right]\right\}, \tag{308}$$

We recover a form which is non-linear in the gauge-fixing condition if we integrate the bosonic field w:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\eta}\mathcal{D}\eta \exp\left\{i\int d^4x \left[\mathcal{L}_{cl} - \frac{(\partial_\mu A^\mu)^2}{2\xi} + (\partial_\mu\bar{\eta})(\partial^\mu\eta)\right]\right\} \tag{309}$$

The classical Euler-Lagrange equation of motion for the auxiliary field w is

$$w = -\frac{1}{\xi}\partial_\mu A^\mu \tag{310}$$

Notice that we can obtain the part-integral with the field integrated out in Eq. 309 by substituting in the path-integral of Eq. 308 the equation of motion of Eq. 310.

The BRST transformations with the w field set to its classical value (or integrated out) are:

$$\delta_\theta \eta = 0, \quad \delta_\theta \bar{\eta} = -\frac{\theta}{g\xi}\partial_\mu A^\mu, \quad \delta_\theta A^\mu = -\frac{\theta}{g}\partial^\mu \eta. \tag{311}$$

We now write the quantum fields as a superposition of plane waves:

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \epsilon_\lambda^\mu \left[a_\lambda e^{ik\cdot x} + a_\lambda^* e^{-ik\cdot x}\right] \tag{312}$$

$$\bar{\eta}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[b e^{ik\cdot x} + b^* e^{-ik\cdot x}\right] \tag{313}$$

$$\eta(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[c e^{ik\cdot x} + c^* e^{-ik\cdot x}\right], \tag{314}$$

where the operators b, b^* and c, c^* are not necessarily Hermitian conjugates and ϵ_λ^μ are polarization vectors. We take them to satisfy the normalization

$$\epsilon_\lambda \cdot \epsilon_{\lambda'} = g^{\lambda\lambda'}. \tag{315}$$

Substituting these equations into the BRST transformations of Eq. 311, we obtain (**exercise**):

$$[Q, a_\lambda] = -(k\cdot\epsilon_\lambda)\, c \tag{316}$$

$$[Q, a_\lambda^*] = (k\cdot\epsilon_\lambda)\, c^* \tag{317}$$

$$\{Q, b\} = \frac{1}{\xi}\sum_\lambda (\epsilon_\lambda\cdot k)\, a_\lambda, \tag{318}$$

$$\{Q, b^*\} = \frac{1}{\xi}\sum_\lambda (\epsilon_\lambda\cdot k)\, a_\lambda^*, \tag{319}$$

$$\{Q, c\} = 0. \tag{320}$$

$$\{Q, c^*\} = 0. \tag{321}$$

$$\tag{322}$$

A state is physical if it annihilates the BRST operator:

$$Q\,|\text{phys}\rangle = 0.$$

Let us consider a state

$$|\lambda, \text{phys}\rangle = \alpha^*_\lambda \, |\text{phys}\rangle \,,$$

which has in adddition a photon with polarization ϵ_λ. Acting with the BRST generator on it, we obtain:

$$\begin{aligned} Q \, |\lambda, \text{phys}\rangle &= Q\alpha^*_\lambda \, |\text{phys}\rangle \\ &= [Q, \alpha^*_\lambda] \, |\text{phys}\rangle \\ &= (k \cdot \epsilon_\lambda) \, c^* \, |\text{phys}\rangle \,, \end{aligned} \tag{323}$$

which is zero if $k \cdot \epsilon_\lambda = 0$. We conclude that a photon state is physical if the polarization vector is transverse to the momentum, in agreement with our expectations.

Now, we consider a state

$$b^* \, |\text{phys}\rangle$$

which also contains an anti-ghost field. Acting on it with the BRST generator we obtain

$$\begin{aligned} Qb^* \, |\text{phys}\rangle &= \{Q, b^*\} \, |\text{phys}\rangle \\ &= \frac{1}{\xi} \sum_\lambda (k \cdot \epsilon_\lambda) \, a^*_\lambda \, |\text{phys}\rangle \,, \end{aligned} \tag{324}$$

The polarizations which are transverse to the momentum yield a zero contribution to the sum. However, a polarization vector ϵ^μ_{II} will survive. We thus have that the BRST generator transforms an anti-ghost state into a photon-state with a longitutidal polarization.

$$Qb^* \, |\text{phys}\rangle = \epsilon_\parallel \cdot k \, |\parallel, \text{phys}\rangle \,. \tag{325}$$

Reading this equation in the opposite direction,

$$|\parallel, \text{phys}\rangle \sim Qb^* \, |\text{phys}\rangle \tag{326}$$

a state with a longitutidal photon is a "BRST exact" state, meaning that it can be written as the BRST generator acting on a different state. While such a state satisfies the physical condition

$$Q \, |\parallel, \text{phys}\rangle \sim Q^2 b^* \, |\text{phys}\rangle = 0,$$

it does not contribute to the S-matrix. Indeed,

$$\left\langle \text{phys}' \right| \, \parallel, \text{phys}\rangle \sim \left\langle \text{phys}' \right| Qb^* \, |\text{phys}\rangle = 0. \tag{327}$$

Similarly, we find that a state with a ghost is unphysical

$$.... \qquad (328)$$

Suggested further reading:
"A Brst Primer"
D. Nemeschansky, C. R. Preitschopf and M. Weinstein.
10.1016/0003-4916(88)90233-3
Annals Phys. **183**, 226 (1988).

END OF WEEK 6

6 Quantum effective action and the effective potential

We have started to collect essential tools for the renormalization of gauge field theories by proving the existence of the BRST symmetry. In the following lectures we will convince ourselves that gauge theories, such as QCD, are renormalizable. It turns out that for renormalization we need to worry only about one-particle-irreducible (1PI) Feynman diagrams. If these are rendered finite, then the full S-matrix elements will possess no other ultraviolet singularities. In this Section, we wll introduce a new functional, the quantum effective action, which generates only 1PI graphs. The quantum effective action is also a very important tool in order to define the ground state of the quantum field theory, and to study symmetry breaking via quantum effects.

We have worked with the generating functional

$$Z[J] = \int \mathcal{D}\phi e^{i\left(S[\phi]+\int d^4x J(x)\phi(x)\right)};$$

Green's functions are obtained via:

$$\langle 0| T\phi(x_1)\dots\phi(x_n) |0\rangle = \frac{1}{i^n}\frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\bigg|_{J=0}.$$

We found that if we require only connected graphs to be generated, which are relevant for computing S-matrix elements, we should use the generating functional $W[Z]$, with

$$Z[J] = e^{iW[J]} \rightsquigarrow W[J] = -i\log Z[J].$$

Then,

$$\langle 0| T\phi(x_1)\dots\phi(x_n) |0\rangle_{\text{CONECTED}} = \frac{1}{i^{n-1}}\frac{\delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)}\bigg|_{J=0}.$$

An important Green's function is the one-point function

$$\langle 0| \phi |0\rangle (x) = \frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0},$$

which we typically find it to vanish for physical fields that can create asymptotic states. However, there are situations that a field (external field) fills the vacuum $\langle 0| \phi |0\rangle (x) \neq 0$. Such a field cannot be in the final or initial state of a scattering process, but it can be a background where scattering of other fields takes place. For example, the Higgs field may have such a role. The field vacuum expectation value does not have to vanish either when the corresponding source term in the functional is not set to zero. In the presence of sources, we have

$$\langle\phi\rangle_J \equiv \langle 0|\,\phi\,|0\rangle_J(x) = \frac{\delta W[J]}{\delta J(x)}.$$

The above equation defines a relation between the source $J(x)$ and the vacuum expectation of the corresponding field. We can then trade source functions in the path-integral and in functional derivatives with the corresponding vev's (vacuum expectation values).

$$\langle\phi\rangle_J = \text{function}(J), \qquad J = \text{function}^{-1}(\langle\phi\rangle_J).$$

Considering it as differential equation, we solve

$$W[J] = \int d^4x \,\langle\phi\rangle_j(x) J(x) + \Gamma\left[\langle\phi\rangle_J\right] \tag{329}$$

where the last term is a constant of integration and does not depend on the source $J(x)$. This constant of integration is the quantum effective action

$$\Gamma\left[\langle\phi\rangle\right] = W[J] - \int d^4x J(x)\,\langle\phi\rangle\, J(x). \tag{330}$$

and it is a functional of field vacuum expecation values with very interesting properties. From the above we find the simple equation,

$$\frac{\delta\Gamma\left[\langle\phi\rangle\right]}{\delta\,\langle\phi\rangle\,(x)} = -J(x). \tag{331}$$

Recall the role of the classical action $S[\phi]$. The equations of motion for the classical field are found by requiring that the action takes a minimal value

$$\left.\frac{\delta S[\phi]}{\delta\phi}\right|_{\phi=\phi_{classical}} = 0.$$

At the quantum level, fields are promoted into operators. The analogue of the classical fields in quantum field theory is the expectation value of the field operator in the state, usually ground state, of the system. In the absence of sources, the quantum effective action yields the equations of motion for the average values of quantum fields:

$$\frac{\delta\Gamma\left[\langle\phi\rangle\right]}{\delta\,\langle\phi\rangle_0\,(x)} = 0. \tag{332}$$

6.1 The quantum effective action as a generating functional

We may take a next step and promote the quantum effective action to generate new Green's functions. We use it in the exponent of a path integral:

$$e^{iW_\Gamma[g,J]} = \int \mathcal{D}\langle\phi\rangle\, e^{\frac{i}{g}\left\{\Gamma[\langle\phi\rangle]+\int d^4x J(x)\langle\phi\rangle(x)\right\}}. \tag{333}$$

We can establish perturbation theory using this path integral. It is possible to derive propagators by identifying the quadratic terms in the action and inverting the corresponding operator. We will not do this explicitly here; we are rather interested in counting powers of the arbitrary constant g.

Every propagator in a graph (since it is produced by inversion), will contribute a single power of g. Vertices are derived from the non-quadratic terms in the Lagrangian without any inversion. Thus, each vertex will contribute a power $1/g$ to a Feynman graph. For a graph with N_I propagators and N_V vertices the overall power of the coupling is:

$$g^{N_I-N_V}.$$

All connected graphs generated by W_Γ [5] *with* N_I *propagators and* N_V *vertices have* $L = 1 + N_I - N_V$ *loops.* It only takes trying a few examples out in order to convince ourselves about the above statement. Otherwise, assign N_I unconstrained momenta for each propagator. Each vertex will provide one constraint, of which one combination is an overall delta function stating that the sum of momenta of incoming an outgoing particles is zero. $N_I - N_V + 1$ momenta are left unconstrained and they are thus loop momenta. The power of g for a graph is therefore determined exclusively from the number of loops that it posseses:

$$g^{L-1}.$$

We can then perform an expansion:

$$W_\Gamma[g,J] = \sum_{L=0}^{\infty} g^{L-1} W_\Gamma^{(L)}[J]. \tag{334}$$

Of course, we can still be interested in the case with $g = 1$. What the above expression tells us,

$$W_\Gamma[1,J] = \sum_{L=0}^{\infty} W_\Gamma^{(L)}[J], \tag{335}$$

[5] they are connected because W_Γ is the logarithm of a path integral

is that the generating functional $W_\Gamma[1,J]$ can be decomposed as a sum of independent generating functionals $W_\Gamma^{(L)}[J]$ corresponding to different loop orders. The functionals $W_\Gamma^{(L)}$ are independent in the sense that shifts in the measure do not mix them; symmetries of the full action should therefore be symmetries of each one of the loop contributions sepearately.

Let us explore the possibility of a very small parameter g. We can expand the exponent in the path integral around the value:

$$\langle\phi\rangle = \langle\phi\rangle_J + \eta, \qquad \text{with} \qquad \frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle_J(x)} = -J(x).$$

We have for the exponent of the path integral:

$$\begin{aligned}\Gamma[\langle\phi\rangle] + \int d^4xJ\langle\phi\rangle &= \Gamma[\langle\phi\rangle_J] + \int d^4xJ\langle\phi\rangle_J + \\ &+ \int d^4x\eta(x)\left[\frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle_J} + J\right] \\ &+ \int d^4xd^4y\eta(x)\frac{\delta^2\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle_J(x)\delta\langle\phi\rangle_J(y)}\eta(y) + \dots\end{aligned} \tag{336}$$

The linear term in η vanishes. We therefore have,

$$\Gamma[\langle\phi\rangle]+\int d^4xJ\langle\phi\rangle = \left\{\Gamma[\langle\phi\rangle_J] + \int d^4xJ\langle\phi\rangle_J\right\}+\mathcal{O}(\eta^2) = W[J]+\mathcal{O}(\eta^2). \tag{337}$$

We then have,

$$e^{\frac{i}{g}\sum_{L=0}^{\infty}g^L W_\Gamma^{(L)}[J]} = e^{\frac{i}{g}W[J]}\int \mathcal{D}\eta e^{\frac{i}{g}\int\mathcal{O}(\eta^2)}. \tag{338}$$

The path integral over η can be computed perturbatively. The factor $\frac{1}{g}$ can be eliminated by redefining $\eta = \eta' g^{1/2}$. Then we are left with a "canonically" normalized quadratic part. The important observation to make is that this perturbative expansion will start at order $\mathcal{O}(g^0)$ the earliest. After taking the logarithm of both sides of the above equation, and by comparing the $\frac{1}{g}$ coefficients, we find that:

$$W[J] = W_\Gamma^{(0)}[J]. \tag{339}$$

In other words, the full generating functional of connected graphs $W[J]$ can be obtained by a generating function where we have replaced the classical action with the quantum effective action:

$$-i\log\int\mathcal{D}\phi e^{i\left[S[\phi]+\int d^4xJ(x)\phi(x)\right]} = \int_{\mathbf{TREE}}\mathcal{D}\phi e^{i\left[\Gamma[\phi]+\int d^4xJ(x)\phi(x)\right]}, \tag{340}$$

and **keeping only the tree-contributions** (denoted with the subscript in the integral symbol).

This is a remarkable result; it states that it is possible to reorganize the perturbative expansion, which gives rise to both tree and loop graphs, into a new expansion where only tree-graphs appear. Of course, $W[J]$ and $W_\Gamma^{(0)}[J]$ are equal. The corresponding perturbative expansions are therefore equivalent; the apparent lack of loops in the expansion from the path integral wth the effective action $W_\Gamma^{(0)}[J]$ should be explained by a re-writing of the usual expansion from $W[J]$ with modified vertices and propagators. These new exotic vertices and propagators should be "dressed" to account for all loop effects that we have encountered in the path integral of the classical action.

The above result is of great importance for renormalization. "Trees" do not have any ultraviolet divergences. Therefore, we only need to establish a renormalization procedure which renders finite the "dressed" propagators and vertices of the quantum effective action.

Let us take the "tree-only" statement seriously, and write down all possible graphs that we might have for two-, three-, and four-point functions. This will be sufficient to establish a pattern for the Green's functions derived from the effective action. Actually, we can only have a very small number of tree-graphs for small number of external legs.

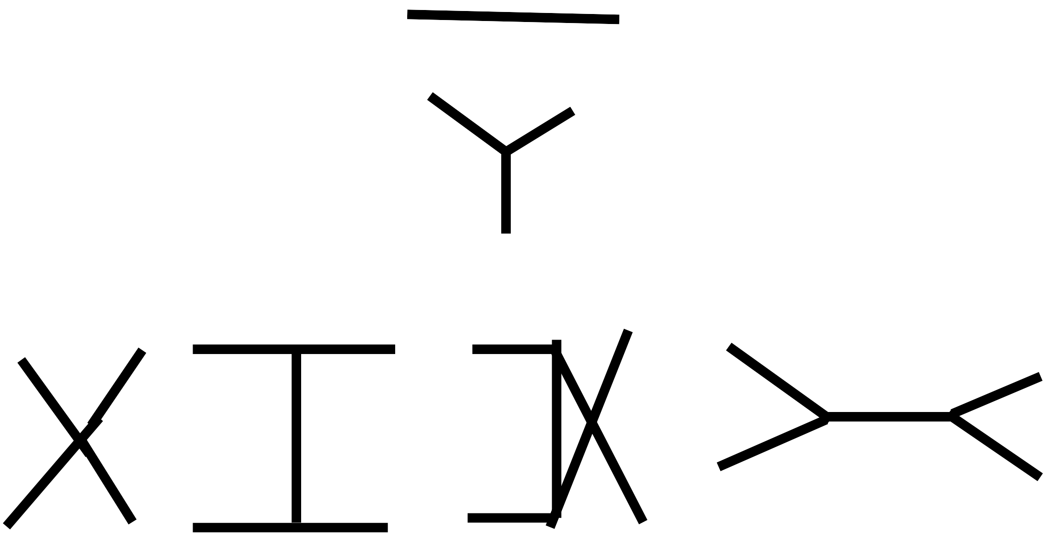
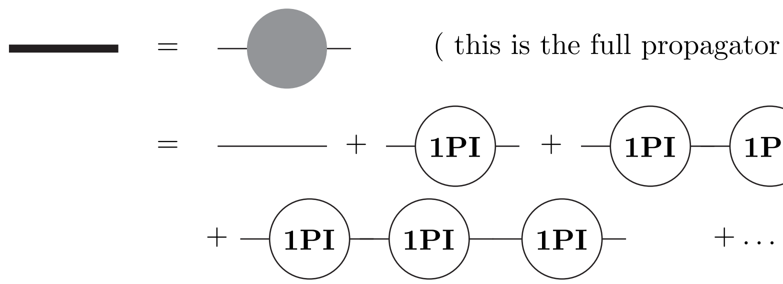


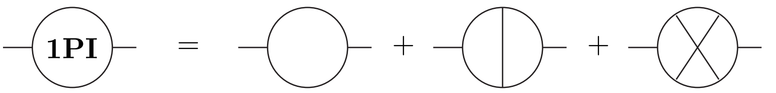
Figure 1: Possible tree-graphs for two-, three-, and four-point functions. This very small number of connected graphs, which arises from the perturbative expansion of the path integral $W_\Gamma[J]$ with the effective action $\Gamma[\langle\phi\rangle]$, should contain in the propagators and vertices all loop effects found in the usual path integral $W[J]$.

We can figure out the propagators and vertices of the tree diagrams of Fig 1 by comparing with the usual Feynman diagrams which we obtain by expanding $W[J]$. The two-point function in Fig 1 must be equivalent to the full propagator, computed at all orders in perturbation theory from $W[J]$:



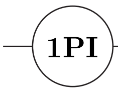
(341)

where we sum all possible Feynman diagrams with two external legs. We can write the sum of all graphs contributing to the full propagator as a geometric series of one-particle-irreducible two-point loop Feynman diagrams.



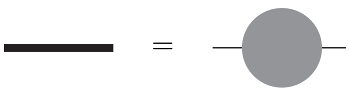
(342)

One-particle irreducible diagrams are these which cannot be separated into two diagrams after we cut one of the propagators. Knowledge of the 1PI propagator graphs is sufficient to determine the full propagator. Let us work, as an example, with a scalar field theory. We denote, in Fourier-space,



$$= \frac{i}{p^2 - m^2} \left(-i\Sigma(p^2) \right) \frac{i}{p^2 - m^2}.$$

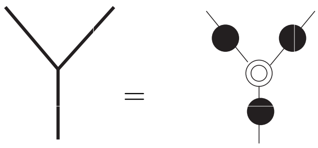
From Eq. 341 we find,



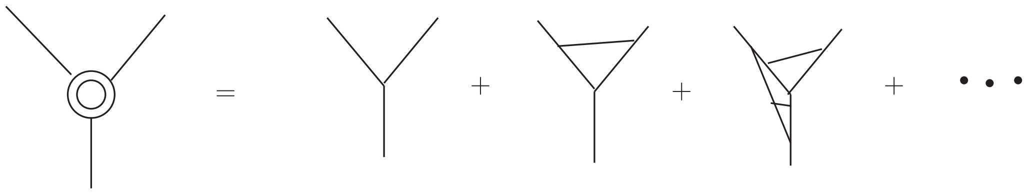
$$= \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left[\frac{\Sigma(p^2)}{p^2 - m^2} \right]^n = \frac{i}{p^2 - m^2 - \Sigma(p^2)}.$$

Indeed, the full propagator is then computed using only 1PI graphs.

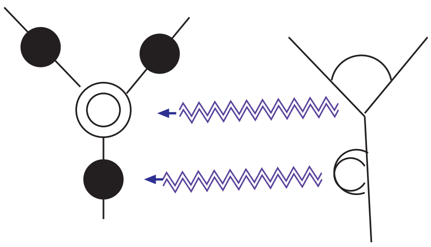
We proceed to compare the three-point Green's functions of Fig. 1 with the result of the full perturbative expansion from $W[J]$. We now that the propagators connected to the triple-vertex are full propagators.



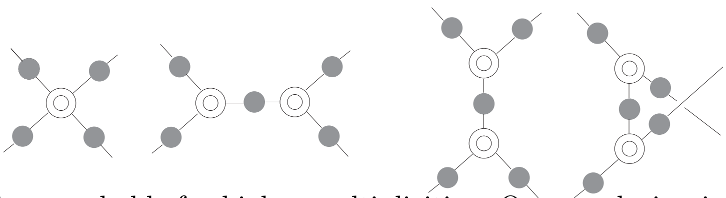
The triple vertex must then be only the sum of all 1PI three-point functions.



Three-point graphs which are one-particle reducible, always contribute a two-point subgraph to the full propagators of the external legs and an 1PI 3-point subgraph to the vertex.



We can now convince ourselves easily that the four point vertex in Fig. 1 contains all one-particle irreducible four point functions.



The same of course holds for higher multiplicities. Our conclusion is that we can always rearrange the sum of graphs in the perturbative exansion, derived via $W[J]$ and containing both loops and trees, to an equivalent "tree-level graphs only" expansion where the propagator is the full "two-point" function and the vertices are all one-particle irreducible graphs with the same number of external legs as in the vertex.

The important statement is that $W[J] = W_\Gamma^{(0)}[J]$ and that all Green's functions can be obtained automatically from the tree-level expansion of a generating functional with the effective action replacing the classical action. Let us verify that the two, three, and four point functions are derivable from the effective action $\Gamma[J]$.

We first introduce the short notation

$$\langle\phi\rangle_J(x) \equiv \phi_x, \quad J(x) \equiv J_x.$$

We start from the equation,

$$\frac{\delta\Gamma}{\delta\phi_x} = -J_x.$$

Differentiating with a source, we obtain:

$$\begin{aligned}\delta(x-y) = -\frac{\delta}{\delta J_y}\frac{\delta\Gamma}{\delta\phi_x} & \\ &= -\int d^4z \frac{\delta\phi_z}{\delta J_y}\frac{\delta^2\Gamma}{\delta\phi_z\delta\phi_x} \\ &= \int d^4z \left[\frac{\delta^2\Gamma}{i\delta\phi_x\delta\phi_z}\right]\left[\frac{\delta^2 W}{i\delta J_z\delta J_y}\right]. \end{aligned} \tag{343}$$

From the above we see that

$$\frac{1}{i}\frac{\delta^2\Gamma}{\delta\phi_x\delta\phi_y}$$

is the inverse of the full two-point function

$$\Delta(x_1-x_2) \equiv \langle 0|\, T\phi(x_1)\phi(x_2)\,|0\rangle = \frac{1}{i}\frac{\delta^2 W}{\delta J_x\delta J_y}.$$

Before we compute the three-point function we need two tricks.

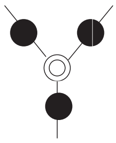
- Chain rule:

$$\begin{aligned}\frac{\delta}{\delta J_x} &= \int d^4z \frac{\delta\phi_z}{\delta J_x}\frac{\delta}{\delta\phi_z} \\ &= \int d^4z \frac{\delta^2 W[J]}{\delta J_x\delta J_z}\frac{\delta}{\delta\phi_z} = i\int d^4z\Delta(x-z)\frac{\delta}{\delta\phi_z}. \end{aligned} \tag{344}$$

- Differentiation of an inverse matrix

$$\begin{aligned} & 1 = MM^{-1} \\ \rightsquigarrow\ & 0 = \frac{\partial\left(MM^{-1}\right)}{\partial\lambda} = \frac{\partial M}{\partial\lambda}M^{-1} + M\frac{\partial M^{-1}}{\partial\lambda} \\ \rightsquigarrow\ & \frac{\partial M^{-1}}{\partial\lambda} = -M^{-1}\frac{\partial M}{\partial\lambda}M^{-1}. \end{aligned} \tag{345}$$

We have:



$$
\begin{aligned}
&= \frac{1}{i^2}\frac{\delta^3 W[J]}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}} \\
&= \int d^4y_1 \Delta(x_1-y_1)\frac{\delta}{\delta\phi_{y_1}}\left[\frac{\delta^2 W}{i\delta J_{x_2}\delta J_{x_3}}\right] \\
&= \int d^4y_1 \Delta(x_1-y_1)\frac{\delta}{\delta\phi_{y_1}}\left[\frac{\delta^2 \Gamma}{i\delta \phi_{x_2}\delta \phi_{x_3}}\right]^{-1} \\
&= \int d^4y_1 d^4y_2 d^4y_3 \Delta(x_1-y_1)\left[\frac{\delta^2 \Gamma}{i\delta \phi_{x_2}\delta \phi_{y_2}}\right]^{-1}\frac{\delta^3\Gamma}{i\delta\phi_{y_1}\phi_{y_2}\delta\phi_{y_3}}\left[\frac{\delta^2 \Gamma}{i\delta \phi_{y_3}\delta \phi_{x_3}}\right]^{-1} \\
&= \int d^4y_1 d^4y_2 d^4y_3 \Delta(x_1-y_1)\Delta(x_2-y_2)\Delta(x_3-y_3)\frac{\delta^3\Gamma}{i\delta\phi_{y_1}\delta\phi_{y_2}\delta\phi_{y_3}} \qquad (346)
\end{aligned}
$$

Now we may compare the graph on the lhs and the rhs of this equation. We have explicitly found that the full three-point function is the convolution of propagators, one for each external leg, and the third derivative of the effective action. From our earlier discussion we now that after we factor out full propagators for the external legs, the remainder is the sum of one-particle irreducible three-point diagrams.

Exercise: *Prove that*

$$\frac{\delta^4\Gamma}{i\delta\phi_{y_1}\delta\phi_{y_2}\delta\phi_{y_3}\delta\phi_{y_4}}$$

is the sum of 1PI 4-point functions.

In summary, we can deduce from the Quantum Effective Action all physical predictions in a quantum field theory.

- The second derivative of $\Gamma[\langle\phi\rangle]$ is the inverse propagator. The zeros of the inverse propagator yield the mass values of the physical particles in the theory.
- Higher derivatives of the effective action are 1PI Green's function. Connecting them with full propagators to from trees we can derive akk connected amplitudes which are required for S-matrix element computations.

Additionally, solving the equation

$$\frac{\delta\Gamma}{\delta\langle\phi\rangle} = 0$$

yields the values of vevs where the effective action is minimal. This will serve to define the true ground-state of the theory. The location of the minimum will also reveal whether any symmetries of the Lagrangian are broken spontaneously.

6.2 The effective potential

We have just observed that by differentiating the effective action functional with respect to the field vevs, we generate one-particle-irreducible Feynman diagrams. All functional derivatives of $\Gamma[\langle\phi\rangle]$ are therefore represnted in terms of Feynman diagrams; if we could compute all these diagrams we could compute the full effective action by summing up all the terms of a Taylor series expansion.

Specifically, we can expand

$$\begin{aligned}\Gamma\left[\langle\phi\rangle\right] &= \sum_{n=1}^{\infty}\frac{1}{n!}\frac{\delta\Gamma\left[\langle\phi\rangle\right]}{\delta\left\langle\phi(x_1)\right\rangle\ldots\delta\left\langle\phi(x_n)\right\rangle}\left\langle\phi(x_1)\right\rangle\ldots\left\langle\phi(x_n)\right\rangle \\ &= \sum_{n=1}^{\infty}\frac{i}{n!}\Gamma^{(n)}(x_1,\ldots,x_n)\left\langle\phi(x_1)\right\rangle\ldots\left\langle\phi(x_n)\right\rangle, \qquad (347)\end{aligned}$$

where

$$\Gamma^{(n)}(x_1,\ldots,x_n) \equiv \frac{1}{i}\frac{\delta\Gamma\left[\langle\phi\rangle\right]}{\delta\left\langle\phi(x_1)\right\rangle\ldots\delta\left\langle\phi(x_n)\right\rangle}. \qquad (348)$$

are one-particle-irreducible Green's functions (in coordinate space).

We consider the case where the ground state (vacuum) is translation invariant; it does not distinguish among different points in space-time. There are situations where this is not true (instantons), however the space-time blind vacuum case is interesting and common. We then have:

$$\left\langle\phi(x)\right\rangle = \text{constant} \equiv \phi. \qquad (349)$$

In this case, the Green's functions simplify enormously if we use a Fourier transformation (momentum space):

$$\begin{aligned}&\int d^4x_1\ldots d^4x_n\Gamma^{(n)}(x_1,\ldots,x_n)\left\langle\phi(x_1)\right\rangle\ldots\left\langle\phi(x_n)\right\rangle \\ =\;& \phi^n\int d^4x_1\ldots d^4x_n\Gamma^{(n)}(x_1,\ldots,x_n) \\ =\;& \phi^n\int d^4x_1\ldots d^4x_n \\ &\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}e^{-ik_1x_1}\ldots e^{-ik_nx_n}(2\pi)^4\delta^{(4)}(k_1+k_2+\ldots+k_n) \\ &\times\tilde{\Gamma}^{(n)}(k_1,\ldots,k_n) \\ =\;& \phi^n\int d^4k_1\ldots d^4k_n\delta^{(4)}(k_1)\ldots\delta^{(4)}(k_n)(2\pi)^4\delta^{(4)}(k_1+k_2+\ldots+k_n) \\ &\times\tilde{\Gamma}^{(n)}(k_1,\ldots,k_n) \\ =\;& \phi^n(2\pi)^4\tilde{\Gamma}^{(n)}(0,0,\ldots,0)\delta^{(4)}(0). \qquad (350)\end{aligned}$$

Notice that we have explicitly shown the delta-function which imposes momentum conversation. The multiple integrations over space-time x_i are simple because of the assumption of x-independent vev ϕ. The factor

$$(2\pi)^4\delta(0) = \int d^4x e^{-i0\cdot x} = \left(\int d^4x\right).$$

We then have for the effective action,

$$\Gamma[\phi] = \left(\int d^4x\right) \sum_{n=1}^{\infty} \frac{\phi^n}{n!} \tilde{\Gamma}^{(n)}(0, 0, \ldots, 0). \tag{351}$$

The effective potential is defined from the effective action, factoring out the space-time volume:

$$V_{eff}(\phi)] \equiv -\frac{\Gamma[\phi]}{\left(\int d^4x\right)} \tag{352}$$

We obtain:

$$V_{eff}(\phi) = -\sum_{n=1} \frac{\phi^n}{n!} \tilde{\Gamma}^{(n)}(0, 0, \ldots, 0). \tag{353}$$

Therefore, the recipe to compute the effective potential is:

- Compute all 1PI Green's fucntions with increasing number of external legs in momentum-space and setting all external momenta to zero.
- For each external leg include a power of the classical vev of the corresponding field.
- Sum the series up without forgetting to include the $i/n!$ from the Taylor series expansion.

Let us consider the Lagrangian of a real scalar field with a quartic interaction,

$$\mathcal{L} = \frac{1}{2}\left(\partial_m u\phi\right)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{354}$$

A computation of the effective potential including all orders in perturbation theory is impossible. We can compute the effective potential easily in the tree and one-loop approximation.

We observe that the only two **1PI** Green's functions that we can write in the tree approximation are the 2-point (amputated propagator) and 4-point (vertex). From Eq. 353 we find,

$$V_{eff}^{tree} = +\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{355}$$

It turns out that the effective potential at tree-level is the same as the potential of the classical Lagrangian.

The 1-loop computation of the effective potential will be discussed in the exercise class.

END OF WEEK 7

7 Symmetries of the path integral and the effective action

Our guiding principle in constructing realistic theories for particle interactions is invariance of the classical action under certain symmetries (e.g. BRST symmetry for Yang-Mills theories). Symmetries of the classical action S may not be automatically symmetries of the effective action Γ. However, the effective action Γ satisfies very general equations (Slavnov-Taylor identities) due to these classical symmetry constraints.

7.1 Slavnov-Taylor identities

Consider a theory of ϕ_i interacting fields with arbitrary (bosonic or fermionic) spin-statistics. We assume that this theory is symmetric under an infinitesimal symmetry transformation:

$$\phi_i \to {\phi_i}' = \phi_i + \epsilon F_i(x, \phi_i),$$

where ϵ is a small parameter and F^i is usually an ordinary function of the fields ϕ_i and their derivatives. Then, we require that both the action and the path-integral measure of the fields are invariant under this transformation:

$$\begin{aligned} S[\phi_i + \epsilon F_i(x, \phi_i)] &= S[\phi_i] \\ \mathcal{D}\left(\phi_i + \epsilon F_i(x, \phi_i)\right) &= \mathcal{D}\phi_i. \end{aligned}$$

After transforming the fields, the generating path-integral is

$$\begin{aligned} Z[J_i] &= \int \mathcal{D}\phi_i' e^{iS[\phi_i'] + i\int d^4x \phi_i' J_i} \\ &= \int \mathcal{D}\left(\phi_i + \epsilon F_i(x, \phi_i)\right) e^{iS[\phi_i + \epsilon F_i(x,\phi_i)] + i\int d^4x (\phi_i + \epsilon F_i(x,\phi_i)) J_i} \\ &= \int \mathcal{D}\phi_i e^{iS[\phi_i] + i\int d^4x (\phi_i + \epsilon F_i(x,\phi_i)) J_i}. \end{aligned}$$

We can now expand in the small parameter ϵ,

$$\begin{aligned} Z[J_i] &= \int \mathcal{D}\phi_i e^{iS[\phi_i] + i\int d^4x \phi_i J_i} \left(1 + i\epsilon F_i(x, \phi_i) + \mathcal{O}(\epsilon^2)\right) \\ &= Z[J_i] + i\epsilon \int \mathcal{D}\phi_i \left(\int d^4y F_i(y, \phi_i) J_i(y)\right) e^{iS[\phi_i] + i\int d^4x \phi_i J_i} \\ &\rightsquigarrow \int \mathcal{D}\phi_i \left(\int d^4y F_i(y, \phi_i) J_i(y)\right) e^{iS[\phi_i] + i\int d^4x \phi_i J_i} = 0, \quad (356) \end{aligned}$$

or, dividing by the path inegral,

$$\int d^4y \left[\frac{\int \mathcal{D}\phi_i F_i(y,\phi_i) e^{iS[\phi_i]+i\int d^4x \phi_i J_i}}{Z[J_i]} \right] J_i(y) = 0 \tag{357}$$

In the square brackets we recognize the average of the transformation over all field configurations,

$$\langle F_i(y,\phi_i) \rangle_J \equiv \frac{\int \mathcal{D}\phi_i F_i(y,\phi_i) e^{iS[\phi_i]+i\int d^4x \phi_i J_i}}{Z[J_i]}. \tag{358}$$

We then find the identity,

$$\int d^4y \langle F_i(y,\phi_i) \rangle_J J_i(y) = 0, \tag{359}$$

concluding that if there exists an infnitesimal symmetry transformation of the classical action, then there is a constraint on the "**average**" value of the transformation. Eq. 359 depends on abritrary sources, and by differentiating multiple times with the sources, we can obtain an infinite number of identities. These are called Slavnov-Taylor identities; we shall consider an example soon.

7.2 Symmetry constraints on the effective action

The generating Slavnov-Taylor identity of Eq. 359 identity tells us that there exists a symmetry for the effective action. Substituting

$$J_i(y) = -\frac{\delta\Gamma}{\delta \langle \phi_i(y) \rangle_J},$$

we obtain:

$$\int d^4y \langle F_i(y,\phi_i) \rangle_J \frac{\delta\Gamma}{\delta \langle \phi_i(y) \rangle_J} = 0. \tag{360}$$

Equivalently,

$$\begin{aligned} & \frac{\Gamma[\langle\phi_i\rangle_J]}{\epsilon} + \int d^4y \langle F_i(y,\phi_i) \rangle_J \frac{\delta\Gamma}{\delta \langle \phi_i(y) \rangle_J} = \frac{\Gamma[\langle\phi_i\rangle_J]}{\epsilon} \\ \rightsquigarrow \quad & \Gamma[\langle\phi_i\rangle_J] + \epsilon \int d^4y \langle F_i(y,\phi_i) \rangle_J \frac{\delta\Gamma}{\delta \langle \phi_i(y) \rangle_J} = \Gamma[\langle\phi_i\rangle_J] \\ \rightsquigarrow \quad & \Gamma[\langle\phi_i\rangle_J + \epsilon \langle F_i(y,\phi_i) \rangle_J] = \Gamma[\langle\phi_i\rangle_J] + \mathcal{O}\left(\epsilon^2\right). \end{aligned} \tag{361}$$

Therefore, the effective action is symmetric under the transformation

$$\langle\phi_i\rangle \to \langle\phi_i\rangle' = \langle\phi_i\rangle + \epsilon \langle F_i(\phi_i) \rangle. \tag{362}$$

We recall that the classical action is symmetric under the transformation

$$\phi_i \to \phi_i' = \phi_i + \epsilon F_i(\phi_i)\,. \tag{363}$$

Are these two transformations the same? Otherwise, is it $F_i = \langle F_i \rangle$? In general they are not! The symmetries of the classical action are usually no symmetries of the quantum effective action. Consider an example of a classical action symmetric under a field transformation

$$\phi(x) \to \phi(x) + \epsilon\phi^2(x)$$

The quantum action should be symmetric under the transformation

$$\langle\phi(x)\rangle \to \langle\phi(x)\rangle + \epsilon\left\langle\phi^2(x)\right\rangle.$$

Given that

$$\langle\phi(x)\rangle^2 \neq \left\langle\phi^2(x)\right\rangle,$$

the two transformations are different.

Nevertheless, we can identify many symmetries in classical actions for realistic field theories which are linear:

$$F_i[\phi_i, x] = c_i(x) + \int d^4y T^{ij}(x,y)\phi_j(y). \tag{364}$$

The equivalent symmetry transformation for the effective action is

$$\begin{aligned} \langle F_i[\phi_i, x]\rangle &= \left\langle c_i(x) + \int d^4y T^{ij}(x,y)\phi_j(y)\right\rangle \\ &= c_i(x) + \int d^4y T^{ij}(x,y)\langle\phi_j(y)\rangle \\ &= F_i[\langle\phi_i\rangle, x], \end{aligned} \tag{365}$$

and it is identical to the classical transformation. It is usefull to remember that **linear symmetry transformations of the classical action, are automatically symmetry transformations of the effective action.**

7.3 Contraints on the effective action from BRST symmetry transformations of the classical action

The BRST transformations are not linear; therefore they are only a symmetry of the classical action [6] and not of the effective action. Nevertheless, the effective action is constrained by the BRST symmetry

[6] From now on, by "classical action" for a gauge theory we mean the action obtained after gauge-fixing using the Fadeev-Popov method.

of the classical Lagrangian (Eq. 359). For these nillpotent transformations Eq. 359 takes a very special form, the so called Zinn-Justin equation.

We start with a classical action $S[\phi_i]$ of fields ϕ_i which is symmetric under the BRST transformation

$$\delta_\theta \phi_i = \theta B_i. \tag{366}$$

Since B_i is nilpotent we also have

$$\begin{aligned} & \delta_{\theta'} \delta_\theta \phi_i = 0 \\ \rightsquigarrow \quad & \delta_{\theta'} B_i = 0 \end{aligned} \tag{367}$$

We realize that, because of Eq. 367, there is a more general action which has the same symmetry as the original $S[\phi_i]$. It is easy to verify that the action,

$$S[\phi_i, K_i] = S[\phi_i] + \int d^4x B_i(x) K_i(x), \tag{368}$$

is indeed symmetric under the same transformation.

The functions K_i are arbitrary (sources). We recall, however, that the functions B_i have the opposite spin-statistics of the corresponding field ϕ_i. Since the product $B_i K_i$ must have even spin-statistics (the same as the action S), we deduce that the source K_i and the field ϕ_i have also opposite spin-statistics.

We can write the generating functional W for connected graphs,

$$e^{iW[J_i, K_i]} = \int \mathcal{D}\phi_i e^{iS[\phi_i] + i\int d^4x B_i K_i + i \int d^4x \phi_i J_i}. \tag{369}$$

The fields ϕ_i may be bosonic or fermionic, therefore the order that we have chosen in writing the integrands in the exponential is important. Conventionally, we have placed source terms J_i, K_i to the right.

The vacuum expecation value $\langle \phi_i \rangle$ is given by

$$\begin{aligned} \langle \phi_i(y) \rangle &= \frac{\int \mathcal{D}\phi_i(y) e^{iS[\phi_i] + i\int d^4x B_i K_i + i \int d^4x \phi_i J_i}}{\int \mathcal{D}x_i e^{iS[\phi_i] + i\int d^4x B_i K_i + i \int d^4x \phi_i J_i}} \\ &= \frac{\delta_R W[J_i, K_i]}{\delta J_i(y)}. \end{aligned} \tag{370}$$

This is an implicit relationship among $J_i, K_i, \langle \phi_i \rangle$, and we will consider $K_i, \langle \phi_i \rangle$ as independent variables, and the sources J_i as being expressed in terms of these two variables:

$$J_i = J_i(\langle \phi_i \rangle, K_i).$$

The exact form of $J_i(\langle\phi_i\rangle, K_i)$ can be found if we evaluate explicitly the effective action,

$$\Gamma[\langle\phi_i\rangle, K_i] = W[J_i(\langle\phi_i\rangle, K_i), K_i] - \int d^4x \langle\phi_i\rangle J_i(\langle\phi_i\rangle, K_i), \quad (371)$$

and take a left derivative,

$$\frac{\delta_L \Gamma[\langle\phi_i\rangle, K_i]}{\delta\langle\phi_i(x)\rangle} = -J_i(\langle\phi_i\rangle, K_i)(x). \quad (372)$$

We now compute the derivative of the effective action with respect to the sources K_i.

$$\begin{aligned}
\frac{\delta_R \Gamma[\langle\phi_i\rangle, K_i]}{\delta K_i(x)} &= \frac{\delta_R}{\delta K_i(x)}\left(W[J_i(\langle\phi_i\rangle, K_i), K_i] - \int d^4x \langle\phi_i\rangle J_i(\langle\phi_i\rangle, K_i)\right) \\
&= \left.\frac{\delta_R W[J_i, K_i]}{\delta K_i(x)}\right|_{J_i = J_i(\langle\phi_i\rangle, K_i)} \\
&\quad + \int d^4y \left(\left.\frac{\delta_R W[J_i, K_i]}{\delta J_m(y)}\right|_{J_m = J_m(\langle\phi_i\rangle, K_i)}\right)\left(\frac{\delta_R J_m(\langle\phi_i\rangle, K_i)(y)}{\delta K_i(x)}\right) \\
&\quad - \int d^4y \langle\phi_m\rangle(y) \frac{\delta_R J_m(\langle\phi_i\rangle, K_i)(y)}{\delta K_i(x)} \\
&= \left.\frac{\delta_R W[J_i, K_i]}{\delta K_i(x)}\right|_{J_i = J_i(\langle\phi_i\rangle, K_i)} \\
&\quad + \int d^4y \langle\phi_m\rangle(y) \frac{\delta_R J_m(\langle\phi_i\rangle, K_i)(y)}{\delta K_i(x)} \\
&\quad - \int d^4y \langle\phi_m\rangle(y) \frac{\delta_R J_m(\langle\phi_i\rangle, K_i)(y)}{\delta K_i(x)} \\
&= \left.\frac{\delta_R W[J_i, K_i]}{\delta K_i(x)}\right|_{J_i = J_i(\langle\phi_i\rangle, K_i)} \\
&= -i \frac{\delta_R}{\delta K_i(x)} \ln\left.\left(\int \mathcal{D}\phi_i e^{iS[\phi_i] + i\int d^4x \phi_i J_i + \int d^4x B_i K_i}\right)\right|_{J_i = \text{FIXED}} \\
&= \langle B_i\rangle. \qquad (373)
\end{aligned}$$

From the general Slavnov-Taylor identity we have,

$$\begin{aligned}
&\int d^4x \langle B_i\rangle J_i = 0 \\
\rightsquigarrow &\int d^4x \langle B_i\rangle \frac{\delta_L \Gamma}{\delta\langle\phi_i\rangle} = 0 \\
\rightsquigarrow &\int d^4x \frac{\delta_R \Gamma}{\delta K_i} \frac{\delta_L \Gamma}{\delta\langle\phi_i\rangle} = 0. \qquad (374)
\end{aligned}$$

This is a constraint which depends only on the effective action $\Gamma[\langle\phi_i\rangle, K_i]$ (Zinn-Justin equation). It is a very useful form in order to study the consequences of symmetry for the effective action, especially in connection with renormalization proofs and studying anomalies.

For later use, we define the product

$$(F,G) \equiv \int d^4x \left(\frac{\delta_R F}{\delta K_i} \frac{\delta_L G}{\delta \langle\phi_i\rangle} - \frac{\delta_R F}{\delta \langle\phi_i\rangle} \frac{\delta_L G}{\delta K_i} \right). \tag{375}$$

for functionals $F[\langle\phi_i\rangle, K_i], F[\langle\phi_i\rangle, K_i]$ of the functions $\langle\phi_i\rangle, K_i$. Recall that $\langle\phi_i\rangle$ and K_i have opposite spin-statistics. Then, the Zinn-Justin equation can be written as

$$(\Gamma, \Gamma) = 0. \tag{376}$$

7.4 Slavnov-Taylor identities in QED

We now consider an example of Slavnov-Taylor identities in QED. The classical Lagrangian is, in the Lorentz gauge,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\left(i \not{D} - m\right)\psi - \frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2. \tag{377}$$

The corresponding path integral is,

$$Z\left[J^\mu, \bar{\rho}, \rho\right] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^4x\left[\mathcal{L} + A^\mu J_\mu + \bar{\psi}\rho + \bar{\rho}\psi\right]}. \tag{378}$$

An infiniterimal local gauge transformation is:

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \frac{1}{q}\partial_\mu \Theta(x), \\ \psi &\rightarrow (1 - iq\Theta(x))\psi, \\ \bar{\psi} &\rightarrow (1 + iq\Theta(x))\bar{\psi}. \end{aligned}$$

The path integral measure is invariant under the gauge transformation. In the integrand of the path-integral exponent, we can identify a part which is invariant under these transformations, while the remaining, which includes tha gauge-fixing term and the source term, is not invariant.

$$\mathcal{L}_{non-invariant} = -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 + A^\mu J_\mu + \bar{\psi}\rho + \bar{\rho}\psi. \tag{379}$$

By performing a gauge-transformation on the path-integral we can derive, as before, the Slavnov-Taylor identity,

$$\int d^4x \left\langle \delta\mathcal{L}_{non-invariant} \right\rangle = 0. \tag{380}$$

We can work out what is the change in the non-invariant part of the Lagrangian. The gauge-fixing transforms as:

$$\begin{aligned} -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 &\to -\frac{1}{2\lambda}\left[\partial_\mu\left(A^\mu - \frac{1}{q}\partial^\mu\Theta\right)\right]^2 \\ &= -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2 + \frac{1}{q\lambda}\partial_\mu A^\mu \partial^2\Theta(x) + \mathcal{O}\left(\Theta^2\right) \\ \rightsquigarrow \delta\left(-\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2\right) &= \frac{1}{q\lambda}\partial_\mu A^\mu \partial^2\Theta(x). \end{aligned} \tag{381}$$

Adding the variation of the source terms, we obtain:

$$\delta\mathcal{L}_{non-invariant} = \frac{1}{q\lambda}\partial_\mu A^\mu\partial^2\Theta(x) - iq\Theta(x)\bar{\rho}\psi + iq\Theta(x)\bar{\psi}\rho - \frac{1}{q}J^\mu\partial_\mu\Theta(x). \tag{382}$$

The Slavov-Taylor identity is:

$$\begin{aligned} &\int d^4x \left\langle \frac{1}{q\lambda}\partial_\mu A^\mu\partial^2\Theta(x) - iq\Theta(x)\bar{\rho}\psi + iq\Theta(x)\bar{\psi}\rho - \frac{1}{q}J^\mu\partial_\mu\Theta(x)\right\rangle = 0 \\ \rightsquigarrow\ &\int d^4x \left[\frac{1}{q\lambda}\partial_\mu\left\langle A^\mu\right\rangle\partial^2\Theta(x) - iq\Theta(x)\bar{\rho}\left\langle\psi\right\rangle + iq\Theta(x)\left\langle\bar{\psi}\right\rangle\rho - \frac{1}{q}J^\mu\partial_\mu\Theta(x)\right] = 0 \\ \rightsquigarrow\ &\int d^4x\Theta(x)\left[\frac{1}{\lambda}\partial_\mu\partial^2\left\langle A^\mu\right\rangle - iq^2\bar{\rho}\left\langle\psi\right\rangle + iq^2\left\langle\bar{\psi}\right\rangle\rho + \partial_\mu J^\mu\right] = 0, \end{aligned} \tag{383}$$

where we have used integration by parts. The above should be valid for arbitrary $\Theta(x)$, therefore the kernel of the integration in the square brackets should be identically zero.

$$\frac{1}{\lambda}\partial_\mu\partial^2\left\langle A^\mu\right\rangle - iq^2\bar{\rho}\left\langle\psi\right\rangle + iq^2\left\langle\bar{\psi}\right\rangle\rho + \partial_\mu J^\mu = 0. \tag{384}$$

Substituting vacuum expectation values with functional derivatives of the path-integral for connected graphs, $W = -ilnZ$, we write:

$$\frac{1}{\lambda}\partial_\mu\partial^2\frac{\delta W}{\delta J_\mu} - iq^2\bar{\rho}\frac{\delta W}{\delta\bar{\rho}} - iq^2\frac{\delta W}{\delta\rho}\rho + \partial_\mu J^\mu = 0, \tag{385}$$

where the functional derivatives are left derivatives for the fermionic sources.

Eq. 385, provides constraints for Green's functions in QED at all orders in perturbation theory. We find the simplest example, by differentiating this equation with a photon source and then set all sources to zero,

$$\begin{aligned} &\frac{\delta}{\delta J^\nu(y)}\left(\frac{1}{\lambda}\partial_\mu\partial^2\frac{\delta W}{\delta J_\mu} - iq^2\bar{\rho}\frac{\delta W}{\delta\bar{\rho}} - iq^2\frac{\delta W}{\delta\rho}\rho + \partial_\mu J^\mu\right)\Bigg|_{J,\rho,\bar{\rho}=0} = 0, \\ \rightsquigarrow\ &\frac{1}{\lambda}\partial_\mu\partial^2\ \frac{\delta^2 W}{\delta J_\mu(x)\delta J_\nu(y)}\Bigg|_{J,\rho,\bar{\rho}=0} + \partial_\mu\delta(x-y) = 0. \\ \rightsquigarrow\ &\frac{1}{\lambda}\partial_\mu\partial^2\left\langle 0\right| TA^\mu(x)A^\nu(y)\left|0\right\rangle = -\partial_\mu\delta(x-y). \end{aligned} \tag{386}$$

We now write the Fourier representations,

$$\delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)},$$

and

$$\langle 0|\, T A^\mu(x) A^\nu(y)\, |0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} D^{\mu\nu}(k).$$

Substituting into Eq. 386 we find that the photon propagator in momentum space should satisfy,

$$k_\mu D^{\mu\nu}(k) = \lambda \frac{k^\nu}{k^2} \tag{387}$$

We can write, in complete generality,

$$D_{\mu\nu} = A(k^2) g_{\mu\nu} + B(k^2) \frac{k_\mu k_\nu}{k^2}. \tag{388}$$

Substituting in Eq. 387, we find

$$A(k^2) + B(k^2) = \frac{\lambda}{k^2}. \tag{389}$$

Thus, the photon propagator in momentum space has the form,

$$D^{\mu\nu}(k) = A(k^2) \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{\lambda}{k^2} \frac{k^\mu k^\nu}{k^2} \tag{390}$$

This is a result valid at all orders in perturbation theory.

Notice that the term which depends on the gauge-fixing parameter is fully known. We can compare this with the result at leading order in perturbation theory,

$$D^{\mu\nu}(k) = \frac{-1}{k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \frac{\lambda}{k^2} \frac{k^\mu k^\nu}{k^2} + \mathcal{O}(g^2). \tag{391}$$

We can see that the gauge-fixing contribution is accounted fully in the leading order result, and therefore it is not modified at higher orders in perturbation theory. Higher order corrections modify only the function $A(k^2)$. For this reason, the gauge-fixing parameter λ does not receive any renormalization.

A second important observation to make is that the part of the propagator which does not depend on λ,

$$D_T^{\mu\nu} = A(k^2) \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right),$$

is transverse to the photon-momentum. Indeed, we easily find that

$$D_T^{\mu\nu}(k) k_\mu = 0.$$

END OF WEEK 8

8 Spontaneous symmetry breaking

Symmetry transformations that leave the effective action invariant may not be symmetries of the physical states and the vacuum state. These symmetries are "spontaneously broken". Spontateous symmetry breaking is associated with a degeneracy of the ground state (vacuum). Let's assume that an effective action is symmetric under

$$\langle\phi\rangle \to -\langle\phi\rangle \rightsquigarrow \Gamma[\langle\phi\rangle] = \Gamma[-\langle\phi\rangle]$$

a symmetry which is inherited unchanged from the classical action (as we have seen earlier). Assume also that the physical vacuum expectation value of the field,

$$\langle\phi\rangle_a \quad : \quad \frac{\delta\Gamma}{\delta\langle\phi\rangle_a} = 0, \tag{392}$$

with $\langle\phi\rangle_a \neq 0$. In other words, there exists a state $|v_a\rangle$ with

$$\langle\phi\rangle_a \equiv \frac{\langle v_a|\,\hat{\phi}(x)\,|v_a\rangle}{\langle v_a|\,v_a\rangle} \neq 0, \tag{393}$$

for which $\Gamma[\langle\phi\rangle_a] \equiv \Gamma_a$ is a minimum. Then, there should be a second value of the field vev in a different state which also gives the same value for the effective action:

$$\exists\,|b\rangle \quad : \quad \langle\phi\rangle_b = -\langle\phi\rangle_a \quad \text{with} \quad \Gamma[\langle\phi\rangle_a] = \Gamma[\langle\phi\rangle_b] = \text{ minimum } .$$

So, while the transformation $\phi \to -\phi$ preserves the action and the effective action, it does not preserve the states and transforms one state into another:

$$|v_a\rangle \to |v_b\rangle\,.$$

The symmetry is broken as long as the system is in one of the degenerate states.

8.1 Goldstone theorem

Consider a symmetry transformation

$$\phi_n(x) \to \phi_n'(x) = \phi_n(x) + i\epsilon \sum_m t_{nm}\phi_m(x) \tag{394}$$

with ϵ a small parameter and t_{nm} generators of the transformation. The transformation leaves the effective action intact:

$$\Gamma\left[\langle\phi_n(x)\rangle\right] = \Gamma\left[\langle\phi_n'(x)\rangle\right]. \tag{395}$$

The symmetry of the effective action gives rise to Slavnov-Taylor identities:

$$\int d^4x \frac{\delta\Gamma}{\delta\langle\phi_n(x)\rangle} t_{nm}\langle\phi_m(x)\rangle = 0. \tag{396}$$

Taking a second derivative, we have:

$$0 = \int d^4x \frac{\delta^2\Gamma}{\delta\langle\phi_l(y)\rangle\,\delta\langle\phi_n(x)\rangle} t_{nm}\langle\phi_m(x)\rangle + \frac{\delta\Gamma}{\delta\langle\phi_n(y)\rangle} t_{nl}. \tag{397}$$

For physical systems with zero external sources, $J_n(x) = 0$, we have that:

$$\frac{\delta\Gamma}{\delta\langle\phi_l(y)\rangle} = 0 \tag{398}$$

and we arrive to the equation

$$0 = \int d^4x \frac{\delta^2\Gamma}{\delta\langle\phi_l(y)\rangle\,\delta\langle\phi_n(x)\rangle} t_{nm}\langle\phi_m(x)\rangle. \tag{399}$$

We now make an important assumption that the vacuum state $|\Omega\rangle$ is translation invariant. We then find that the vacuum expectation value of the field is the same in all space-time. Since we can find the value of the field operator at a space-time point x from the value of the field at the origin with a translation using the momentum operator as a generator, we can write:

$$\begin{aligned}\langle\phi(x)\rangle &= \langle\Omega|\,\hat{\phi}(x)\,|\Omega\rangle \\ &= \langle\Omega|\,e^{i\hat{P}x}\hat{\phi}(0)e^{-i\hat{P}x}\,|\Omega\rangle \\ &= \langle\Omega|\,\hat{\phi}(0)\,|\Omega\rangle = \text{ constant } \equiv \langle\phi\rangle.\end{aligned} \tag{400}$$

Then, the effective action can be written as:

$$\Gamma[\langle\phi_n\rangle] = -\int d^4x V_{eff}(\langle\phi_n\rangle) = -\left(\int d^4x\right) V_{eff}(\langle\phi_n\rangle) \tag{401}$$

Eq. 399 then yields for the effective potential V_{eff} the following constraint:

$$\sum_{nm} t_{nm}\langle\phi_m\rangle \frac{\partial^2 V_{eff}}{\partial\langle\phi_l\rangle\,\partial\langle\phi_n\rangle} = 0. \tag{402}$$

This is a constraint on the mass spectrum of the theory. To see that, we recall that the second derivative of the effective action is the inverse of a two-point Green's function:

$$\begin{aligned}
&\frac{\delta W[J]}{\delta J_m(x)} = \langle \phi_m(x) \rangle \\
\rightsquigarrow \quad &\frac{\delta^2 W[J]}{\delta \langle \phi_n(y) \rangle \delta J_m(x)} = \delta(x-y)\delta_{nm} \\
&\ldots \\
\rightsquigarrow \quad &\int d^4z \, \langle \Omega | T\phi_n(y)\phi_k(z) |\Omega\rangle \frac{\delta^2 \Gamma}{\delta \langle \phi_k(z) \rangle \delta \langle \phi_m(x) \rangle} = \delta(x-y)\delta_{nm} \\
\rightsquigarrow \quad &\int d^4z d^4y \, \langle \Omega | T\phi_n(y)\phi_k(z) |\Omega\rangle \frac{\delta^2 \Gamma}{\delta \langle \phi_k(z) \rangle \delta \langle \phi_m(x) \rangle} = \delta_{nm} \\
\rightsquigarrow \quad &\int d^4z d^4y \, \langle \Omega | T\phi_n(y)\phi_k(z) |\Omega\rangle \frac{\partial^2 V_{eff}}{\partial \langle \phi_k \rangle \partial \langle \phi_m \rangle} \delta(z-x) = -\delta_{nm} \\
\rightsquigarrow \quad &\frac{\partial^2 V_{eff}}{\partial \langle \phi_k \rangle \partial \langle \phi_m \rangle} \int d^4y \, \langle \Omega | T\phi_n(y)\phi_k(x) |\Omega\rangle = -\delta_{nm} \qquad (403)
\end{aligned}$$

Substituting the Fourier transformation of the 2-point function:

$$\langle \Omega | T\phi_n(y)\phi_k(x) |\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} D_{nk}(p^2) e^{-ip\cdot(x-y)} \qquad (404)$$

the integration over the y variable yields a delta function setting the momentum $p^\mu = 0$. We therefore have:

$$D_{nk}(0) \frac{\partial^2 V_{eff}}{\partial \langle \phi_k \rangle \partial \langle \phi_m \rangle} = -\delta_{nm}. \qquad (405)$$

Or, equivalently,

$$\frac{\partial^2 V_{eff}}{\partial \langle \phi_n \rangle \partial \langle \phi_m \rangle} = -D^{-1}_{nm}(0) \qquad (406)$$

Eq. 402 yields that

$$\sum_{nm} D^{-1}_{ln}(0) t_{nm} \langle \phi_m \rangle = 0. \qquad (407)$$

When is this equation satisfied? Let's write the combination $\sum_m t_{nm} \langle \phi_m \rangle = \delta \langle \phi_n \rangle$ as the variation of the vev under the symmetry transformation. Then Eq. 407 becomes:

$$\sum_n D^{-1}_{ln}(0) \delta \langle \phi_n \rangle = 0. \qquad (408)$$

If the transformation leaves the vacuum state and, thus, the vacuum expectation value of the fields invariant, $\delta \langle \phi_n \rangle = 0$, then Eq. 408 is fulfilled. What if the symmetry is broken and the symmetry transformation of the effective action changes the vacuum, so that there are some $\delta \langle \phi_i \rangle \neq 0$? Let us rewrite Eq. 408 in a matrix notation:

$$\left(\quad D^{-1}_{ln}(0) \quad \right) \left(\langle \phi_n \rangle \right) = \mathbf{0} \left(\langle \phi_n \rangle \right) \tag{409}$$

We observe that the matrix $D^{-1}_{nl}(0)$ has zero eigenvalues, as many as the independent vectors $\delta \langle \phi_n \rangle$ which are non-vanishing. In the simplest case of only one field, the inverse propagator of the field at zero momentum is proportional to the mass of the particle excitation of the field:

$$D(p) = \frac{iZ}{p^2 - m^2} + \text{continuum} \rightsquigarrow D^{-1}(0) \propto m^2.$$

In general, $D^{-1}_{nl}(0)$ is the mass-matrix of the theory. Redefining appropriately the fields, $\phi_n = R_{nm}\tilde{\phi}_m$ eliminates non-diagonal terms and the diagonal terms, the eigenvalues of the matrix, are the masses of the physical particle excitations of the fields $\tilde{\phi}_i$.

We have just proven Goldstone's theorem. Namely, for each independent $\delta \langle \phi_n \rangle = \sum_m T_{nm} \langle \phi_m \rangle \neq 0$ there exists a massless particle in the spectrum of the theory. The symmetry generators T_{nm} which change the vev of the fields are called "broken" generators. There is an alternative proof [7] of Goldstone's theorem due to Weinberg. This proof also demonstrates that

- The massless states are one-particle states.
- They are also invariant under rotations and correspond to spin-0 particles, the so called Goldstone bosons.
- The Goldstone bosons have the same "quantum numbers" as the conserved currents corresponding to the broken generators.

Goldstone's theorem seems very powerful and its proof appears to leave no room for exceptions. Nevertheless, we will be able to find a loophole soon: it is possible to have spontaneous symmetry breaking without giving rise to massless particles. We note that our proof requires translation invariance of the vacuum states as well as positive norms. These requirements cannot be satisfied simultaneously for quantum theories with local gauge invariance.

[7] to be taught in the course of *The physics of Electroweak Symmetry Breaking*

8.2 General broken global symmetries

Let's assume a pattern of spontaneous symmetry breaking:

$$G \to H,$$

where G is the symmetry group that leaves invariant the effective action and H a subgroup of G which leaves invariant the vacuum. We will also assume that the symmetry group is global. In other words, the effective action Γ remains invariant $\Gamma[\psi_n] = \Gamma[\psi'_n]$ for

$$\psi'_n = \sum_m g_{nm}\psi_m, \quad \frac{\partial g_{nm}}{\partial x^\mu} = 0, \quad g_{nm} \in G, \tag{410}$$

and the vacuum remains invariant

$$\sum_m h_{nm} \langle \psi_m \rangle = \langle \psi_n \rangle, \quad \forall\, h_{nm} \in H. \tag{411}$$

According to Goldstone's theorem, the mass matrix of the theory has zero eigenvalues for the eigenvectors:

$$\sum_m T^a_{nm} \langle \psi_m \rangle \equiv \delta \langle \psi_n \rangle, \tag{412}$$

where T^a_{nm} is a broken generator.

Which independent linear combinations of the fiels in the Lagrangian of the theory correspond to Goldstone fields and which are not? We shall prove that all fields ψ_n (including non-Goldstones) can be obtained from Goldstone-free fields $\tilde{\psi}_n$ by performing a local group transformation:

$$\psi_n(x) = \sum_m \gamma_{nm}(x)\tilde{\psi}_m(x). \tag{413}$$

We start by observing that Goldstone-free field combinations $\tilde{\psi}_n$ (the "heavy" fields of the theory) must be orthogonal to the vectors of Eq. 412, that is:

$$\sum_{nm} \tilde{\psi}_n(x) T^a_{nm} \langle \psi_m \rangle = 0. \tag{414}$$

Without loss of generality, we will assume that the elemebts $g \in G$ belong to a real and orthogonal representation of the group which is compact. Then, the quantity:

$$V_\psi(g) = \psi_n g_{nm} \langle \psi_m \rangle \tag{415}$$

is a bounded, continuous, real-valued function.

Exercise: ... Let us now find an appropriate $g = \gamma$ for which $V_{\psi(x)}(g)$

is a maximum at every space-time point x. Then, under a small variation of the group parameter

$$\delta\gamma_{nm} = i\sum_a \epsilon^a \gamma_{nl} T^a_{lm}, \tag{416}$$

$V_\psi(g)$ is stationary:

$$0 = \delta V_\psi(g) = i\sum_a \epsilon^a \sum_{nlm} \psi_n(x)\gamma_{nl}(x) T^a_{lm} \langle\psi_m\rangle \tag{417}$$

Recalling that we have chosen an orthogonal and real representation of the group, we have:

$$[\gamma_{nl}] = [\gamma_{ln}]^{-1}. \tag{418}$$

Thus,

$$0 = i\sum_a \epsilon^a \sum_{lm} \left[\sum_n \gamma^{-1}_{ln}\psi_n\right] (T^a_{lm}\langle\psi_m\rangle) \tag{419}$$

Therefore, the field combinations:

$$\tilde{\psi}_l = \sum_n \gamma^{-1}_{ln}\psi_n \tag{420}$$

are orthogonal to the vectors

$$\sum_m T^a_{lm}\langle\psi_m\rangle = \delta\langle\psi_l\rangle \tag{421}$$

and they are **not** Goldstone bosons.

Let's rewrite the Lagrangian of the theory by making the substitution which we have just found:

$$\psi(x) = \gamma(x)\tilde{\psi}(x), \tag{422}$$

rewriting the fields of the theory as explicit non-Goldstones $\tilde{\psi}()$ and the remaining Goldstone fields contained in $\gamma(x)$. We remind that the Lagrangian is only invariant under a global gauge transformation, while the above transformation is a local gauge transformation which does **not** leave the Lagrangian invariant. We have:

$$\begin{aligned}\mathcal{L}\left[\gamma(x)\tilde{\psi}(x)\right] &= \mathcal{L}\left[\gamma(x_0)\tilde{\psi}(x)\right] \\ &\quad + \text{derivatives of } \gamma(x), \tilde{\psi}(x)\end{aligned} \tag{423}$$

Due to the global gauge invariance of the theory, $\mathcal{L}\left[\gamma(x_0)\tilde{\psi}(x)\right] = \mathcal{L}\left[\tilde{\psi}(x)\right]$, we have that:

$$\begin{aligned}\mathcal{L}\left[\gamma(x)\tilde{\psi}(x)\right] &= \mathcal{L}\left[\tilde{\psi}(x)\right] \\ &\quad + \text{derivatives of } \gamma(x), \tilde{\psi}(x)\end{aligned} \tag{424}$$

where the first term does not have any Goldstone bosons. Goldstone bosons appear only as derivatives. This forbids mass terms:

$$m_B^2 B(x)B(x)$$

for them. Also, ar low energies, Goldstone interactions vanish. Indeed, the Feynman rules for fields that appear as derivatives will be proportional to the momenta of the particles:

$$\partial_\mu \gamma(x) \to \partial_\mu B(x) \to p_\mu \qquad \text{(in Ferynman rules)}$$

and vanish for zero momenta $p^\mu\ to 0$.

8.3 Spontaneous symmetry breaking of local gauge symmetries

Let us now assume that our Lagrangian is invariant under a local gauge symmetry. Repeating the reasoning of the previous section and rewriting

$$\psi = \gamma\tilde{\psi},$$

we have that

$$\mathcal{L}\left[\gamma(x)\tilde{\psi}(x)\right] = \mathcal{L}\left[\tilde{\psi}(x)\right]. \tag{425}$$

In other words, our carefully selected gauge transformation eliminates all Goldstone boson fields from the Lagrangian. We have just found an exception of Goldstone's theorem in theories with local gauge invariance, where the symmetry is spontaneously broken but there are no physical massless Goldstone fields due to the breaking of the symmetry. The rewriting $\psi = \gamma\tilde{\psi}$ is equivalent to choosing a gauge fixing condition:

$$\tilde{\psi} \cdot (T^a \langle\psi\rangle) = 0. \tag{426}$$

Lagrangians which are locally invariant under a continuous symmetry transformation require gauge bosons in order to form covariant derivatives. Let us look at the quadratic terms in the covariant derivatives:

9 Renormalization: counting the degree of ultraviolet divergences

Consider a Lagrangian with fisdimensionalityelds f and generic interaction operators O_i.

$$\mathcal{L} = \text{kinetic terms} + g_1 O_1 + g_2 O_2 + \ldots + g_N O_N. \tag{427}$$

Each operator O_i is a product of fields and/or their derivatives. In QED for example, we have one such interaction term:

$$\bar{\psi} \not{A} \psi;$$

In QCD more operators emerge, e.g.

$$f^{abc} \partial_\mu A_\nu^a A^{\mu,b} A^{\nu,c}, \qquad f^{abe} f^{cde} A_\mu^a A_\nu^b A^{\mu,c} A^{\nu,d}, \ldots$$

We would like to keep this discussion as general as possible; At the end, we will be able to make statements on whether we can remove via renormalization ultraviolet infinities from arbitrary Lagrangians. Most of our arguments will be derived using simple dimensional analysis.

We consider a generic one-particle irreducible Feynman diagram in perturbation theory. We will first find a simple formula to test whether it has the most obvious of all possible divergences, the so called superficial ultra-violet divergence. If the diagram has L loops, a superficial divergence corresponds to an infinity of the diagram in the limit

$$|k_1| = |k_2| = \ldots = |k_L| = \kappa \to \infty,$$

where k_i are the loop-momenta. A Feynman diagram might have divergences in other limits, where only some momenta or linear combinations of them are taken to infinity while the remaining independent momenta remain fixed. A Feynman diagram in the superficial ultraviolet limit behaves as

$$\int^\infty d\kappa \kappa^{D-1}, \tag{428}$$

where D is an integer, called the *superficial degree of divergence*.

- If $D > 0$ the Feynman diagram has a powerlike divergence,
- if $D = 0$ it diverges logarithmically,
- if $D < 0$ it is convergent *(only in the superficial limit, since it might have other divergences)*.

We can compute D (or an upper bound of it) for any Feynman diagram on general grounds. We assume that our 1PI Feynman diagram has

- I_f internal propagators for each of the fields f,
- E_f external legs for each of the fields f and
- N_i vertices corresponding to the term $g_i O^i$ in the Lagrangian.

Recall, as examples, the Feynman rules for propagators in gauge theories, and how they behave at the limit of infinite momentum.

- a photon propagator,

$$\sim \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}}{k^2} \sim \kappa^{-2};$$

- a fermion propagator,

$$\sim \frac{\not{k} + m}{k^2 - m^2} \sim \kappa^{-1};$$

- for a scalar,

$$\sim \frac{1}{k^2 - m^2} \sim \kappa^{-2}.$$

For each of the internal propagators of the field f in the Feynman diagram there is a contribution to the superficial divergence,

$$\Delta_f \sim k^{-2+2s_f},$$

where, $s_f = 0$ for a boson and $s_f = \frac{1}{2}$ for a fermion. The total contribution to the asymptotic limit from propagators is then

$$\kappa^{\sum_f 2I_f(s_f-1)}. \tag{429}$$

The contribution from vertices is easy to find if we know how many loop momenta appear in the corresponding Feynman rules. For a vertex due to an operator O_i this number is equal to the number of space-time derivatives d_i which can be found in the expression for O_i. Recall that a Feynman rule for a vertex is esentially the Fourier transform of the expression of its operator and therefore momenta arise only from derivatives. The total contribution from vertices to the superficial ultraviolet limit is

$$\kappa^{\sum_i N_i d_i}. \tag{430}$$

Finally, due to the integration measure $d^4 k_i$ for each loop, the total contribution from the loop-momenta to the superficial UV limit is

$$\kappa^{4L}, \tag{431}$$

where L is the number of loops in the graph. L is known if we are given the number of internal propagators I_f and the number of vertices

N_i in the graph. The number of loop-momenta carried from internal propagators is $\sum_f I_f$. The vertices provide $\sum_i N_i$ constraints of which one is not for loop-momenta but for the external momenta. Therefore, the number of loop-momenta is

$$L = \sum_f I_f - \sum_i N_i + 1.$$

Putting together the contributions from the loop integration measures, vertices, and internal propagators, we find that the asymptotic behavior at infinity has a superficial degree of divergence

$$D = \sum_f 2I_f(1 + s_f) - \sum_i N_i(4 - d_i) + 4. \qquad (432)$$

We can express the number of internal propagators in terms of the number of external legs. Let us assume that we have N_{if} particles f in the vertex corresponding to the operator O_i in the Lagrangian. The total number of (internal) legs of the particle f connected in all the vertices of the graph are

$$\sum_i N_i N_{if}.$$

From these E_f are external and the remaining are internal. Every propagator of f has two edges, so the number of internal legs is

$$2I_f.$$

We then have the identity

$$2I_f + E_f = \sum_i N_i N_{if}. \qquad (433)$$

We can therefore write the degree of divergece as

$$D = 4 - \sum_f E_f(1 + s_f) - \sum_i N_i \left[4 - d_i - \sum_f N_{if}(1 + s_f). \right] \qquad (434)$$

Notice that the square bracket in the last expression depends only on the functional form of the operator O_i. If this operator is multiplied with a coupling constant g_i in the Lagrangian, i.e. $\mathcal{L} = g_i O_i + \dots$ we can prove that this square bracket is exactly the mass dimensionality of the coupling g_i:

$$[g_i] = 4 - d_i - \sum_f N_{if}(1 + s_f). \qquad (435)$$

Indeed. The mass dimensionality of each term in the action should be zero. We then have that

$$\left[d^4x\right] + [g_i] + [O_i] = 0,$$

where the operator O_i has d_i derivatives and N_{if} fields f. Thus,

$$-4 + [g_i] + d_i - \sum_f N_{if}[f] = 0,$$

and $[f] = 1 + s_f$ is the mass dimensionality of the field. Indeed,

$$\begin{aligned} & \langle 0| Tf(x_1)f(x_2) |0\rangle \sim \int_{k\to\infty} d^4k k^{-2+2s_f} e^{-ikx} \\ \rightsquigarrow \quad & 2[f] = 4 - 2 + 2s_f \rightsquigarrow [f] = 1 + s_f. \end{aligned} \tag{436}$$

In conclusion, we can write a very suggestive expression for the superficial degree of divergence:

$$D = 4 - \sum_f E_f(1 + s_f) - \sum_i N_i[g_i]. \tag{437}$$

If the Lagrangian does not contain any couplings with negative mass dimensions, $[g_i] \geq 0$, we find a superficial ultraviolet divergence, $D \geq 0$, only in Feynman diagrams with a small number of external legs.

$$D \geq 0 \rightsquigarrow \sum_f E_f(1 + s_f) \leq 4.$$

In particular, superficial divergences do not appear in *(one-particle-irreducible)* Feynman diagrams with five external legs or more.

Examples of theories where superficial divergences may appear in only a limited number of Green's fucntions are QED and QCD. All interaction operators have dimension four and their coefficients are dimensionless. Superficial ultraviolet divergences are limited in 1PI Green's functions, such as $\langle 0| T\bar{\psi}(x_1)\psi(x_2) |0\rangle$, $\langle 0| TA^a_\mu(x_1)A^b_\nu(x_2) |0\rangle$, $\langle 0| T\bar{\psi}(x_1)A^b_\nu(x_2)\psi(x_3) |0\rangle$, . On the contrary $\langle 0| T\bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4) |0\rangle_{1PI}$ is (superficially) finite.

Theories with $[g_i] \geq 0$ are called **renormalizable**. As we shall see, these superficial divergences in a finite number of Green's functions can be removed by adding a finite number of extra terms in the original Lagrangian (counterterms).

If the Lagrangian contains a coupling with negative mass dimension $[g_j] < 0$, then from Eq. 437 we see that **all** Green's functions, at some loop-order, will develop a superficial divergence. It is therefore impossible to cancel the infinities by adding a finite number of counterterms. Such theories are called non-renormalizable.

9.1 Subdivergences

We should stress that the counting of the superficial degree of divergence is not sufficient to prove that a Feynman is finite. Consider the two example graphs of Fig. 2. Both two-loop graphs have the same

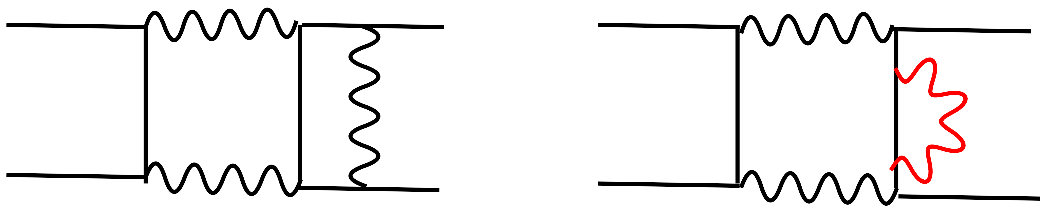


Figure 2: Both two-loop graphs have the same superficial degree of divergence $D = -2$. However the Feynman diagram on the right has a self-energy one-particle-irreducible subgraph which has a superficial degree of divergence $D = 1$. A necessary condition for a graph to be UV finite is that the graph and all its subgraphs have $D < 0$

superficial degree of divergence $D = -2$. One could naively conclude that both Feynman two-loop graphs are likely to be finite. We know, though, that this is not the case. Let us compute the superficial degree of divergence for all one-loop subgraphs that we can spot in the two diagrams. For the left diagram, we find that all subgraphs have a negative superficial degree of divergence. It also turns out with an explicit calculation (beyond the scope of this lecture) that the diagram is indeed UV finite. However the two-loop Feynman diagram on the right has an one-loop self-energy subgraph; this has a superficial degree of divergence $D = 1$. The self-energy is $1/(d-4)$ divergent, where d is the space-time dimensionality. Such a divergence remains even after we embed the one-loop self-energy as a subgraph inside a two-loop graph (there is no mechanism to cancel it). Against our naive counting for the global superficial degree of divergence, the two-loop diagram on the right is divergent. The lesson from the above examples is that for a diagram to be UV finite it is necessary that the superficial degrees of divergence for the full graph and all of its sub-graphs must be negative.

9.2 Cancelation of superficial divergences with counterterms

We derived a criterion to decide whether a Green's function will develop the most "obvious" type of divergence (superficial) in the limit where the magnitudes of all loop momenta tend simultaneously to infinity. We also found that for renormalizable theories this type of divergence appears in only a finite number of Green's functions with a small number of external legs.

Infinities are not acceptable for physical theories. A way out of this problem is to recognize that the Lagrangian that we started with has a certain degree of arbitrariness. The guiding principle for constructing a Lagrangian is to respect a set of symmetries (e.g. BRST symmetry). However, this is not a tight enough constraint to fix, for example, the actual values of independent mass and coupling parameters. It may be possible to redefine the parameters and fields of the Lagrangian or even add more operators to it without destroying the symmetries of the Lagrangian. How can we fix the fields and parameters of the Lagrangian, choosing among their various possible redefinitions? In renormalizable theories, we fix (partially) this arbitrariness so that all Green's functions calculated with the redefined ("renormalized") fields, couplings and masses are finite.

We have seen that for "renormalizable theories" the "disease" of infinities is only spread to a few Green's functions. Redefinining the fields and parameters of the Lagrangian ($\psi = Z\psi_R = \psi_R + \delta Z \psi_R, \ldots$) gives rise to a few new terms (counterterms) with coefficients engineered to cancel exactly the UV infinities which emerge order by order in perturbation theory. But, is it possible mathematically that we can cancel the infinities from loop diagrams with counterterms? For this method to work, it is essential that diagrams with counterterms at a loop order have the same kinematic dependence as the UV infinities of loop diagrams without counterterms at higher orders. At the first two orders in perturbation theory, this statement means that tree-diagrams with counterterms must have the same kinematic depenence as the infinities of one-loop diagrams without counterterms.

If, for example, the $1/\epsilon$ terms of a Green's function at the one-loop order (where $d = 4 - 2\epsilon$ in dimensional regularisation) are logarithms of external momenta,

$$\frac{\ln(p^2)}{\epsilon}$$

such a contribution cannot be cancelled by the tree-level contribution of a countertem. Recalling the Feynman rules for vertices which enter tree-level calculations in all theories that we have examined so far, we find no such logarithms in their expressions. Feynman rules always yield simple polynomial expressions for the vertices of tree-diagrams. For such tree-level expressions mde out of counterterms to cancel the infinities of one-loop diagrams the latter have also to be constants or simple polynomials of momenta. A necessary condition for the counterterm program to be succesfull is that one-loop infinities are “local”, i.e. they appear as simple polynomials in the external momenta as the usual Feynman rules do.

Let us look at the fuctional form of the superficial infinities at-one loop order in perturbation theory, and convince ourselves that indeed this is exactly what happens in practice. Take, as an example, the one-loop correction to the four-point function in the $-\frac{\lambda}{4!}\phi^4$ theory. The superficial degree of divergence is $D = 4 - 4 \times 1 = 0$ and the graph is indeed divergent. In dimensional regularization, the corresponding Feynman parameter integral yields,

$$\begin{aligned} \langle 0| T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle|_{1-loop} &\sim \lambda^2 \int d^dk \frac{1}{(k^2-m^2)\left[(k+p)^2-m^2\right]} \\ &\sim \lambda^2\Gamma(\epsilon)\int_0^1 dx_1 dx_2 \frac{\delta(1-x_1-x_2)}{(m^2-x_1x_2 s)^\epsilon} \\ &\sim \frac{\lambda^2}{\epsilon} - \lambda^2\gamma - \lambda^2 \int_0^1 dx \ln\left(m^2 - x(1-x)s\right) \end{aligned} \tag{438}$$

with $s = (p_1+p_2)^2 = (p_3+p_4)^2$.

Loop integrals, in general, contain logarithms or integrals of logarithms (polylogaritms) with arguments kinematic invariants formed from external momenta. Our example result is not an exception and we indeed find logarithmic contributions in the finite part. We cannot escape to observe however, that the divergent part is very simple; it is just a constant. We can then modify the interaction terms in the Lagrangian,

$$\frac{-\lambda}{4!}\phi^4 \to \frac{-\lambda}{4!}\phi^4 + \frac{\sharp\lambda^2}{\epsilon}\phi^4$$

and adjust the coefficient $\sharp$ so that it cancels exactly the divergent part of this one-loop integral.

The divergent parts of one-loop integrals are simple polynomials of the external momenta, as in the above example. If we use Feynman parameters, any one-loop integral may be written as,

$$I_{1-loop} \sim \Gamma\left(N - \frac{d}{2}\right)\int_0^1 \frac{dx_1\ldots dx_n \delta(1-x_1-\ldots x_n)}{\left(m_1^2 x_1 + \ldots + m_n^2 x_n - \sum s_{i\ldots j}x_i x_j - i\delta\right)^{N-\frac{d}{2}}} \tag{439}$$

where N is an integer (equal to the number of propagators) and $d = 4-2\epsilon$ the dimension. The denominator contains a sum over masses and kinematic invariants of the external momenta. Divergences may arise from two terms; the Gamma function $\Gamma(N-2+\epsilon)$ and the denominator of the integrand. The argument of the Gamma function $N - d/2 = D/2$ is proportional to the superficial degree of divergence

D of the integral. The denominator of the Feynman integral does not have any ultraviolet divergences. It could become divergent when masses or invariants become zero, but it is finite when all the propagators are massive. If this is not the case, and there are massless particles propagating in a loop, singularities from the denominator are of infrared nature connected to the small or zero values of $|k|$ rather than the UV $|k| \to \infty$ limit. Infrared singularities can also be regulated by attributing a small mass to massless particles and/or considering them to be slighlty off-shell. We shall not worry here about infrared one-loop singularities and focus on the ultraviolet divergences which can be found, at one-loop order, in the Gamma function pre-factor of the Feynman parameters integral:

$$\Gamma(N - d/2) = \Gamma(-1 + \epsilon), \Gamma(\epsilon)$$

for $N = 1, 2$. Using the identity,

$$\Gamma(x) = \frac{\Gamma(1 + x)}{x},$$

and

$$\Gamma(1 + \epsilon) = 1 - \gamma\epsilon + \mathcal{O}(\epsilon^2),$$

and the fact that for $N = 1, 2$ (responsible for the UV divergences) the denominator of Eq. 439 turns into a numerator $N - d/2 < 0$ in four dimensions, we can see that the coefficient of the $1/\epsilon$ pole can only be a polynomial in the external momenta. A loop diagram with superficial degree of divergence D

$$\int_{|k| \to \infty} |k|^{D-1}, \tag{440}$$

has a mass dimensionality D. Therefore, the polynomial can only be of rank D in the external momenta. Each term in this polynomial multiplying $1/\epsilon$ must be cancelled by a separate counterterm operator with a different number of derivatives. Naturally, the number of derivatives must be:

$$d_i \leq D. \tag{441}$$

Exercise: *Prove that for the cancelation of UV divergences we need at most as many counterterms in the Lagrangian as the divergent Green's functions.*

9.3 Nested and overlapping divergences

It can be proven that we only to worry about removing superficial divergences from loop integrals. Nested and overlapping singularities are "automatically" removed with this procedure as well. We refer to original literature for this topic:

- Hepp:1966eg K. Hepp, "Proof of the Bogolyubov-Parasiuk theorem on renormalization," Commun. Math. Phys. **2** (1966) 301.

END OF WEEK 11

10 Proof of renormalizability for non-abelian gauge theories

Consider a theory with action $S[\phi]$ which is invariant under BRST trnsformations of the fields ϕ_i: $\delta_\theta \phi_i = \theta B_i$. We can add to the classical action source terms which preserve the invariance under BRST transformations due to their nilpotency.

$$S[\phi_i, K_i] = S[\phi_i] + \int d^4x B_i K_i. \tag{442}$$

We now split the action into two terms,

$$S[\phi_i, K_i] = S_R[\phi_i, K_i] + S_\infty[\phi_i, K_i]. \tag{443}$$

The first term is the action with the fields, masses and coupling constants set to their renormalized values. The second term contains the counterterms. S and S_R have the same functional form. Therefore, they possess the same set of symmetries. It also follows that S_∞ must also possess the same set of symmetries.

The effective action can be cast as an expansion in loops:

$$\Gamma[\phi_i, K_i] = \sum_{L=0}^{\infty} \Gamma_L[\phi_i, K_i]. \tag{444}$$

We recall that all terms in the expansion are seperately symmetric and that we can perform independent shifts to the measure of the path integral for each one of them.

The Slavnov Taylor identities for the BRST symmetry transformations result to the Zinn-Justin equation (Eq. 374) which is written in a short notation as

$$(\Gamma, \Gamma) = 0. \tag{445}$$

Inserting the loop expansion of the effective action, we obtain:

$$\begin{aligned} 0 \;=\; & (\Gamma_0, \Gamma_0) \\ & +(\Gamma_0, \Gamma_1) + (\Gamma_1, \Gamma_0) \\ & +(\Gamma_0, \Gamma_2) + (\Gamma_1, \Gamma_1) + (\Gamma_2, \Gamma_0) \\ & + \ldots \end{aligned} \tag{446}$$

Every line in the above expression must be separetely zero, since it corresponds to a different order in the loop expansion (equivalently, the $\hbar$ expansion). For the $N-$th term of the expansion we have

$$\sum_{L=0}^{N} (\Gamma_L, \Gamma_{N-L}) = 0. \tag{447}$$

At each loop order we find UV infinities. We decompose the L-loop effective action into a finite and a divergent part:

$$\Gamma_L = \Gamma_{L,fin} + \Gamma_{L,\infty}. \tag{448}$$

At zeroth order we only find tree-graphs and there are no infinities. In addition, the tree-level effective action is equal to the classical action. We therefore have

$$\Gamma_{0,fin} = S_R, \quad \Gamma_{0,\infty} = 0. \tag{449}$$

We will prove using induction that we can removing all infinities from the effective action, rendering all $\Gamma_{L,\infty} = 0$, with the counterterms in S_R. Let's assume that we have achieved this for all loops up to $N-1$,

$$\Gamma_{L,\infty} = 0, \quad L = 1 \ldots N-1. \tag{450}$$

Then, taking the infinite part of Eq. 447 we obtain that

$$(\Gamma_{0,fin}, \Gamma_{N,\infty}) + (\Gamma_{N,\infty}, \Gamma_{0,fin}) = 0. \tag{451}$$

Or, equivalently,

$$(S_R, \Gamma_{N,\infty}) = 0. \tag{452}$$

As we have discussed in a previous section, we expect the infinties of momentum space Green's functons in $\Gamma_{N,\infty}$ to have a simple polynomial dependence in the momenta, given that all divergences at the previous loop orders are cancelled. We now make two observations:

- As we have shown earlier, the infinities of $\Gamma_{N,\infty}$ arise in Green's functions with a small number of external legs. As we have assumed that the infinities of all loop previous orders have been cancelled, at the $N-$th loop order we cannot have any subdivergences. Thus, the $N-$th loop order divergences correspond to the superficial limit where all loop momenta are taken to infinity. For the superficial divergences we have derived that they should originate from local field operators (products of fields and their derivatives as well as sources K_i) in $\Gamma_{N,\infty}$ whose dimensionality is less than or equal to four.
- $\Gamma_{N,\infty}$ has all the linear symmetries of S_R. These are:
  - Lorentz transformations
  - Global gauge transformations
  - Anti-ghost translations
  - Ghost phase-transformations ($\rightsquigarrow$ ghost number conservation)

The last two symmetries are apparent by inspecting the ghost-terms of the Lagramgian:

$$\mathcal{L}_{\mathrm{FDEEV-POPOV}} = (\partial^\mu \bar{\eta}^a)\, D^{ab}_\mu \eta^b. \tag{453}$$

The anti-ghost field enter the Lagrangian only with its derivative, and thus the Lagrangian is invariant if we shift globally the field by a constant. In addition, the Lagrangian is invariant under a phase-transformation of the ghost and anti-ghost fields:

$$\eta^a(x) \to e^{i(+1)\rho}\eta^a(x), \quad \eta^a(x) \to e^{i(-1)\rho}\eta^a,$$
$$A^{a\mu} \to e^{i(+0)\rho}A^{a\mu}, \quad \psi \to e^{i(+0)\rho}\psi.$$

The conserved charge of this symmetry is called the ghost number. The ghost numbers of the $\phi_i = \{A^{a\mu}, \psi, \eta^a, \bar{\eta}^a\}$ fields are $\gamma_i = \{0, 0, +1, -1\}$ respectively. The above phase-transformations leave the action $S[\phi_i]$ invariant. For the extended action $S[\phi_i, K_i]$ to be invariant, we need to assign ghost numbers to the sources K_i as well. From the BRST transformations we see that if a field ϕ_i has a ghost number γ_i, the variation under the transformation B_i of the field has a ghost number $\gamma_i + 1$. The term $\int d^4x B_i K_i$ ought to remain invariant under the pghost-phase transformation. We must therefore assign ghost numbers $-\gamma_i - 1$ for the sources K_i. Specifically, the ghost numbers for $K_A, K_\psi, K_{\bar{\eta}}, K_\eta$ are $-1, -1, 0, -2$ respectively.

Lemma: $\Gamma_{N,\infty}$ is linear in the sources K_i.
Proof: To prove this we shall use dimensional analysis and symmetries. First we determine the mass dimension of the sources K_i. For a field ϕ_i with dimensionality d_i the operators B_i have dimensionality $d_i + 1$, as can be seen from the expressions of the BRST transformations. The term $\int d^4x B_i K_i$ ought to have zero dimensionality. Therefore we conclude that the sources K_i have $3 - d_i$ dimensionality. Therefore the dimensionality of BRST sources corresponding to scalar and vector fields, $K_A, K_\eta, K_{\bar{\eta}}$ is 2 while the dimensionality of the BRST source corresponfing to fermion fields K_ψ is 3/2.

Since the operators of $\Gamma_{N,\infty}$ are of dimensionality four at most, we can have operators with at most two sources K_i:

- $K_{\mathrm{scalar/vector}} K_{\mathrm{scalar/vector}}$,
- $K_{\mathrm{fermion}} K_{\mathrm{fermion}}$,
- $K_{\mathrm{fermion}} K_{\mathrm{fermion}}$field, with a dimensionality [field] ≤ 1.

All quadratic terms in the sources K_i have a non-zero ghost-number and they are therefore excluded, with the exception of

$$K_{\bar{\eta}^a} K_{\bar{\eta}^a}$$

operator which has a zero ghost-number. We can exclude this operator for a different reason. The BRST symmetry transformation of the classical action for an anti-ghost is linear and not quadratic,

$$\delta_\theta \bar{\eta}^a = -\theta \omega^a. \tag{454}$$

Therefore,

$$\frac{\delta_L \Gamma_N[\langle \phi_i \rangle, K_i]}{\delta K_{\bar{\eta}^a}} = -\langle \omega^a \rangle \rightsquigarrow \frac{\delta_L \Gamma_N[\phi_i, K_i]}{\delta K_{\bar{\eta}^a}} = -\omega^a \tag{455}$$

where in the last step we used that the transformation is linear so that, in that case, the transformation of the "average" is equal to the "average" of the transformation. The above differential equation tells us that Γ_N is at most linear in the source $K_{\bar{\eta}^a}$. We have just shown that $\Gamma_{N,\infty}$ is at most linear in all sources K_i. We write

$$\Gamma_{N,\infty}[\phi_i, K_i] = \Gamma_{N,\infty}[\phi_i, 0] + \int d^4x \tilde{B}_i K_i. \tag{456}$$

Recall that the classical action is also linear:

$$S_R[\phi_i, K_i] = S_R[\phi_i] + \int d^4x B_i K_i. \tag{457}$$

Substituting in $(S_R, \Gamma_{N,\infty}) = 0$ we obtain two equations for the zeroth and the first order term in K_i. Namely,

$$\int d^4x \left[B_i \frac{\delta_L \Gamma_{N,\infty}}{\delta \phi_i} + \tilde{B}_i \frac{\delta_L S_R}{\delta \phi_i} \right] = 0, \tag{458}$$

$$\int d^4x \left[B_i \frac{\delta_L \tilde{B}_j}{\delta \phi_i} + \tilde{B}_i \frac{\delta_L B_j}{\delta \phi_i} \right] = 0. \tag{459}$$

We now define:

$$\Gamma^{(\epsilon)} \equiv S_R + \epsilon \Gamma_{N,\infty}, \tag{460}$$

$$B_i^\epsilon \equiv B_i + \epsilon \tilde{B}_i, \tag{461}$$

where ϵ is a very small parameter. With the Eqs 458, we can prove that under a field transformation:

$$\phi_i \to \phi_i + \theta B_i^\epsilon, \tag{462}$$

- $\Gamma^{(\epsilon)}$ is invariant
- The tranformation is nilpotent (up to $\mathcal{O}(\epsilon)$).

We leave the proof of the above statements to the reader as an **exercise**.

From the above equations we infer that the dimensionality of $\tilde{B}_i$ is at most the dimensionality of B_i. From Eq 459, we also infer that B_i, $\tilde{B}_i$ and thus B_i^ϵ have all the same ghost number (**exercise**). With these constraints, the allowed form of the transformations Eq. 462 is

$$\psi \rightarrow \psi + i\,(\theta\eta^a)\,T_a\psi \tag{463}$$

$$A^{\mu,a} \rightarrow A^{\mu a} + \theta\left[B^{ab}\partial^\mu\eta^b + D^{abc}A^{\mu,b}\eta^c\right] \tag{464}$$

$$\eta^a \rightarrow \eta^a - \frac{1}{2}\theta E^{abc}\eta^b\eta^c \tag{465}$$

with $E^{abc} = -E^{acb}$ due to the ghost field η^b being a Grassmann variable.

We can place more constraints on the coefficients of Eqs 463 by exploiting that the transformations are nilpotent:

- From $\delta_{\theta_1}\delta_{\theta_2}\eta^a = 0$, we find that

$$E^{abc}E^{bde} + E^{abe}E^{bcd} + E^{abd}E^{bec} = 0, \tag{466}$$

which reveals that E^{abc} must be a structure constant of some Lie algebra. It would not be a surprise if this Lie algebra is the same as the one of the non-Abelian gauge group of the classical action S_R. Indeed, if we set the small parameter ϵ exactly to its zero value then $\Gamma^{(\epsilon)}\Big|_{\epsilon=0} = S_R$. The structure constants E^{abc} must therefore be proportional to the structure constants of the non-abelian gauge group of the classical action:

$$E^{abc} = \lambda f^{abc}. \tag{467}$$

- The nilpotency of the transformation of the gauge field $A^{\mu,a}$ yields two constraints. Namely

$$D^{abc}D^{bde} - D^{abe}D^{bdc} = E^{bec}D^{adb} = \lambda f^{bec}D^{adb} \tag{468}$$

$$B^{ab}E^{bcd} = D^{abd}B^{bc} \tag{469}$$

Eq. 468 tells us that the the matrices $\tilde{t}^a_{bc} = iD^{bca}$ satisfy the commutation relation of generators in some representation of the non-abelian gauge group:

$$\left[\tilde{t}^c, \tilde{t}^e\right] = i f^{ceb}\tilde{t}^b.$$

The only representation of the Lie group with the dimensionality of D^{abc} is the adjoint representation. Therefore, the solution of Eq. 468 is:

$$D^{abc} = \lambda f^{abc}. \tag{470}$$

Eq 469 reveals that the matrix B_{ab} commutes with the structure constants which can be chosen to be totally antisymmetric. The only possible solution is therefore a diagonal matrix (**exercise**). You can verify this easily in the special case of an $SU(2)$ group where the structure constants are the totally antisymmetric Levi-Civita symbol. Eq 469 takes the form:

$$B^{ab}\epsilon_{bcd} = B^{bc}\epsilon_{abd}$$

which, as examples, for $(a,c,d) = (1,2,3)$ yields $B^{11} = B_{22}$ and for $(a,c,d) = (1,2,2)$ yields $B^{23} = 0$. Similarly, one finds all diagonal terms to be equal and the non-diagonal terms to vanish. We write

$$B^{ab} = N\lambda\delta_{ab}. \tag{471}$$

- Nilpotency of the fermion field transformation yields for the matrices T^a that the also satisfy the Lie algebra of the non-abelian group of the classical action,

  $$\left[T^b, T^c\right] = iE^{abc}T^a = i\lambda f^{abc}T^a. \tag{472}$$

  Therefore, as suggested from the $\epsilon = 0$ limit, we have

  $$T^a = \lambda t^a, \tag{473}$$

  where t^a are the generators of the representation for the fermions in S_R.

We have therefore found that the $\Gamma^{(\epsilon)}$ is symmetric under the same BRST symmetry transformation as S_R up to some re-scalings. Explicitly, the BRST symmetry transformations of $\Gamma^{(\epsilon)}$ take the form:

$$\psi \rightarrow \psi + i\left(\lambda\theta\eta^a\right)t^a\psi \tag{474}$$

$$A^{\mu,a} \rightarrow A^{\mu a} + \lambda\theta\left[N\partial^\mu\eta^a + f^{abc}A^{\mu,b}\eta^c\right] \tag{475}$$

$$\eta^a \rightarrow \eta^a - \frac{1}{2}\lambda\theta f^{abc}\eta^b\eta^c \tag{476}$$

$$\bar{\eta}^a \rightarrow \bar{\eta}^a - \theta\omega^a \tag{477}$$

$$\omega_a \rightarrow \omega_a. \tag{478}$$

The last two transformations are linear symmetry transformations of the classical action and they are automatically symmetry transformations of the effective action and Γ^ϵ as well.

Recall that we expect $\Gamma^{(\epsilon)}$ to be made out of local operators. We write

$$\Gamma^{(\epsilon)} = \int d^4x\mathcal{L}^{(\epsilon)}. \tag{479}$$

The dimensionality of the operators is bounded by the power-counting arguments of the previous chapter. In addition, $\mathcal{L}^{(\epsilon)}$ should consist of a combination of operators that they respect the BRST symmetry which we have just discovered (Eqs 474). Finally, $\mathcal{L}^{\epsilon}$ is constrained further to respect all the linear symmetries of the classical Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{fermion}} - \frac{1}{4} G^{a\mu\nu} G^{a}_{\mu\nu} - (\partial_\mu \bar{\eta}^a)(\partial^\mu \eta^a) + f^{abc} (\partial_\mu \bar{\eta}^a) A^{b,\mu} \eta^c + \omega^a \partial_\mu A^{a,\mu} + \frac{\xi}{2} \omega^a \omega^a.$$

These linear symmetries are:

- Lorentz invariance
- Global gauge invariance. Explicitly, the global symmetry transformations are:

$$\delta\psi = i\epsilon^a t^a \psi,\ \delta A^b_\mu = f_{bca} \epsilon^a A^c_\mu,\ \delta\eta^b = f_{bca}\epsilon^a \eta^c,\ \delta\bar{\eta}^b = f_{bca}\epsilon^a \bar{\eta}^c,\ \delta\omega^b = f_{bca}\epsilon^a \omega^c.$$

- Anti-ghost translation invariance: $\bar{\eta}^a \to \bar{\eta}^a + c$,
- Ghost-number conservation.

Can we write a Lagrangian density $\mathcal{L}^{(\epsilon)}$ with additional operators than the ones that we find in the classical $\mathcal{L}$ and still satisfying the list of constraints that we have found above? If such operators exist, then we can establish some simple rules for them. To preserve ghost-number, the ghost and anti-ghost fields must appear in pairs or not appear at all in such novel operators. Because of anti-ghost translation invariance, the anti-ghost must always be differentiated. We therefore conclude that the ghost fields should appear in the form

$$(\partial_\mu \bar{\eta}^a) \tag{480}$$

Let us recall the dimensionalities of the fields

$$[A^{a,\mu}] = [\eta^\mu] = [\bar{\eta}^a] = 1, \quad [\omega^a] = 2.$$

The combination of fields in Eq. 480 has a dimensionality three. Operators must have a dimensionality less than four, thus they can include at most one such combination of ghost fields. Altogether, we can have the following operators:

- ghost-operators

$$(\partial_\mu \bar{\eta}^a)\left(\partial^\mu \eta^b\right),\ (\partial_\mu \bar{\eta}^a) A^{c,\mu} \eta^b, \tag{481}$$

- auxiliary field operators

$$\omega^a \left(\partial_\mu A^{b,\mu}\right),\ \omega^a A^c_\mu A^{b,\mu}, \tag{482}$$

- and operators which contain only fermion and gauge boson fields. We denote the sum of them as

$$\mathcal{L}_{\psi A}. \tag{483}$$

Therefore, the most general Lagrangian density $\mathcal{L}^{(\epsilon)}$ is

$$\begin{aligned}\mathcal{L}^{(\epsilon)} &= \mathcal{L}_{\psi A} + \frac{\xi'}{2}\omega^a\omega^a + C\omega^a\left(\partial_\mu A^{a,\mu}\right) \\ &\quad - e_{abc}\omega^a A^b_\mu A^{c,\mu} - Z_\eta\left(\partial_\mu\bar{\eta}^a\right)\left(\partial^\mu\eta^a\right) - d_{abc}\left(\partial_\mu\bar{\eta}^a\right)A^{c,\mu}\eta^b,\end{aligned} \tag{484}$$

where $\xi', C, d_{abc}, e_{abc}$ are unknown constants with e_{abc} being symmetric in the last two indices: $e_{abc} = e_{acb}$, which are however constrained by global gauge invariance.

We now use that $\mathcal{L}^{(\epsilon)}$ is invariant under the BRST transformations of Eqs 474. We recall that for fermion and gauge boson fields, the BRST transformation has the same functional form as a classical local gauge transformation with a local gauge parameter made out of a Grassmann constant and the ghost field. Thus, the $\mathcal{L}_{\psi A}$ part of the Lagrangian has to be not only globally gauge invariant but also locally gauge invariant with a gauge parameter:

$$\epsilon^a \to \lambda N \theta \eta^a.$$

and with a gauge coupling $g_s \to g_s/N$ (equivalently, replacing the generators and structure constants by $\left(t^a, f^{abc}\right) \to \left(\tilde{t}^a = t^a/N, \tilde{f}^{abc} = f^{abc}/N\right)$). BRST invariance of the ghost and auxiliary field part of $\mathcal{L}^{(\epsilon)}$ leads to a determination of the constants. Specifically, we find (**exercise**):

$$C = \frac{Z_\eta}{\lambda N} \tag{485}$$

$$d_{abc} = -\frac{Z_\eta}{N} \tag{486}$$

$$e_{abc} = 0. \tag{487}$$

Summarising the effect of all constraints, we can cast $\mathcal{L}^{(\epsilon)}$ in the form:

$$\begin{aligned}\mathcal{L}^{(\epsilon)} &= -Z_A \tilde{G}^{a\mu\nu}\tilde{G}^a_{\mu\nu} - Z_\psi\bar{\psi}\gamma^\mu\left[\partial_\mu - i\tilde{t}^a A^a_\mu\right]\psi \\ &\quad + \frac{\xi'}{2}\omega^a\omega^a + \left(\frac{Z_\eta}{N\lambda}\right)\omega_a\partial_\mu A^{a\mu} - Z_\eta\left(\partial_\mu\bar{\eta}^a\right)\left(\partial^\mu\eta^a\right) \\ &\quad + Z_\eta\tilde{f}^{abc}\left(\partial_\mu\bar{\eta}^a\right)A^{c,\mu}\eta^b,\end{aligned} \tag{488}$$

where the field strength tensor $\tilde{G}^{a\mu\nu}$ is evaluated as in $G^{a\mu\nu}$ with the replacement $f^{abc} \to \tilde{f}^{abc}$.

This is a Lagrangian which is vert similar to the classical Lagrangian $\mathcal{L}$, differing only in multiplicative constants. This tells us that the two Lagrangians describe the same physics, since we are allowed to rescale at wish the definitions of fields, couplings and masses. We can exploit this freedom to remove all ultraviolet divergences. With explicit calculations of a few Green's functions at the $N-$th loop order, we can find how the constants that emerged in $\mathcal{L}^{(\epsilon)}$ (which contain necessarily the infinities of all matrix-elements) are related to the original parameters and field definitions of the classical Lagrangian. With this information at hand and reverse-engineering we can redefine the fields, fermion masses, and coupling constant so that at the $N-$th loop order we have $\Gamma^{(\epsilon)} = S_R$, which renders $\Gamma_{N,\infty} = 0$.

We have proven that non-abelian gauge theories are renormalizable, in the sense that multiplicative redefinitions of fields and parameters order by order in the loop expansion can remove all ultraviolet infinities from Green's functions. This is one of the biggest successes in Quantum Field Theory since we have realistic theories with predictive power for physical (i.e. finite) observables.

END OF WEEK 12

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