Quantum Field Theory I

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Conventions for Special Relativity

Our metric convention is

$$g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{1}$$

A contravariant position four-vector is

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) = (ct, \vec{x}). \tag{2}$$

A covariant position four-vector is

$$x_\mu = g_{\mu\nu} x^\nu, \tag{3}$$

which gives

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z) = (ct, -\vec{x}). \tag{4}$$

Space-time derivatives form four-vectors,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{5}$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) \tag{6}$$

The D' Alambert scalar second order differential operator is

$$\partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2. \tag{7}$$

Chapter 1

Quantum Field Theory. Why?

The goal of this lecture series is to introduce a synthesis of quantum mechanics and special relativity into a unified theory, the theory of quantised fields. Rendering the two theories consistent with each other was a challenge for the physicists of the last century. Naive generalisations of the Schrödinger equation to incorporate relativity were giving non-physical results, such as particles with negative kinetic energies. Quantum field theory provided the solution to this and other problems.

Quantum field theory allows us to tackle deep questions. What is a particle? Why particles have a spin? Why particles carry electric charge? What types of charge may exist beyond the electric charge? Why do particles have mass?

Novel phenomena emerge at high energies in collider experiments. As an example, which has been exhaustively studied at the Large-Electron-Positron (LEP) collider in Geneva, think of the production of a muon and its anti-particle out of the annihilation of an electron and a positron:

$$e^- + e^+ \to \mu^- + \mu^+. \tag{1.1}$$

Such a common reaction cannot be explained with quantum mechanics as we have known it so far. While we could assign a wave-function for the electron/positron system before the reaction takes place and similarly a different wave-function for the muon/anti-muon system, the Schrodinger equation does not predict that the latter is the evolution of the former.

Quantised fields, on the other hand, allow for the annihilation of particles and the creation of others, as long as this is consistent with symmetries and the corresponding conservation laws. Quantum field theory is a predictive framework. Together with symmetries, it tells us precisely how particles may interact at the shortest distances and higher energies that we have explored so far in nature.

Chapter 2

Review of principles of classical and quantum mechanics

Before we introduce Quantum Field Theory, it will be useful to recall how we described the dynamics of simple mechanical systems in classical and quantum physics. In QFT, we will postulate principles that we have already seen there, such as the principle of least action and canonical quantization. Conservation theorems derived in classical and quantum mechanics will also apply to QFT.

2.1 Time evolution in classical mechanics

Consider, for simplicity, a one-dimensional mechanical system whose dynamical behavior is encoded in a Lagrangian

$$L\left(x(t), \dot{x}(t)\right).$$

The Lagrangian depends on the position and the velocity, which are functions of time. We will focus on energy conserving systems, in which the Lagrangian acquires all of its time dependence through these functions and has no other explicit time dependence,

$$\frac{\partial L}{\partial t} = 0. \tag{2.1}$$

The time evolution of the position and the velocity obeys the principle of least action. This states that the action is stable,

$$\delta S = 0\,, \tag{2.2}$$

under small variations of the physical trajectory

$$x(t) \to x(t) + \delta x(t). \tag{2.3}$$

Eq. 2.2, leads to the Euler-Lagrange differential equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0. \tag{2.4}$$

For example, a particle moving in one-dimension under the influence of a force potential $V(x)$ has a Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{2.5}$$

and the corresponding Euler-Lagrange equation yields Newtons law

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0. \tag{2.6}$$

An alternative description of the same dynamics can be obtained through the Hamiltonian formalism. We define first the canonical momentum,

$$p = \frac{\partial L}{\partial \dot{x}}, \tag{2.7}$$

which, in general, is a function of both the position and velocity,

$$p = p\left(x(t), \dot{x}(t)\right). \tag{2.8}$$

We now define the Hamiltonian

$$H(x, p) = p\dot{x} - L, \tag{2.9}$$

and pick the pair of (x, p) as our independent variables, "inverting" Eq. (2.8), and considering the velocity as a function of position and momentum,

$$\dot{x} = \dot{x}\left(x, p\right). \tag{2.10}$$

Differentiating the Hamiltonian with the position, we have

$$\frac{\partial H}{\partial x} = \ldots = -\dot{p}, \tag{2.11}$$

while differentiating with respect to the momentum we arrive at

$$\frac{\partial H}{\partial p} = \ldots = \dot{x}. \tag{2.12}$$

Therefore, alternatively to the second-order Euler-Lagrangian differential equation, we can describe the dynamical system with the pair of first-order differential equations of Hamilton

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}. \tag{2.13}$$

Let us now consider a generic physical quantity,

$$A = A(x, p, t). \tag{2.14}$$

Differentiating with respect to time, we have

$$\frac{dA}{dt} = \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial t}. \tag{2.15}$$

Substituting above the Hamilton equations (2.13), we obtain

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}. \tag{2.16}$$

where the Poisson bracket for two functions $f(x, p)$ and $g(x, p)$ is given by

$$\{f, g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{2.17}$$

For example, the canonical momentum for a particle in a potential, corresponding to the Lagrangian of Eq. (2.5), is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{2.18}$$

which we invert to read

$$\dot{x} = \frac{p}{m}. \tag{2.19}$$

The Hamiltonian is then computed to be

$$H = p\dot{x} - L = \frac{p^2}{2m} + V(x), \tag{2.20}$$

and Hamilton's equations are

$$\dot{x} = \frac{p}{m}\,, \dot{p} = -\frac{\partial V}{\partial x}. \tag{2.21}$$

and the general evolution equation reads

$$\frac{dA}{dt} = \left\{ A, \frac{p^2}{2m} + V(x) \right\} + \frac{\partial A}{\partial t} \tag{2.22}$$

We will study the theoretical implications of the time-evolution equation for generic physical quantities A above and the Poisson brackets, soon.

2.1.1 Properties of Poisson brackets

We can derive the following properties of Poisson brackets from their definition in Eq. (2.17).

- Anti-commutation,

$$\{A, B\} = -\{B, A\} \tag{2.23}$$

- Linearity

$$\{A, B + C\} = \{A, B\} + \{A, C\}\,, \tag{2.24}$$

- Product rule,

$$\{A, B\,C\} = B\ \{A, C\} + \{A, B\}\ C \tag{2.25}$$

- Action on position and momentum

$$\{x, p\} = 1\,, \quad \{x, x\} = 0\,, \quad \{p, p\} = 0\,. \tag{2.26}$$

In addition, Poisson brackets satisfy the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{2.27}$$

We can now also show the inverse. For the class of functions of position and momentum which have a series representation as in

$$f(x, p) = \sum_{n,m=0}^{\infty} c_{nm} x^n p^m, \tag{2.28}$$

the properties of Eqs. (2.23) - (2.26), lead back to the definition of Eq. (2.17). The proof can be somewhat lengthy but relies on simple induction.

We first prove inductively that

$$\{x, x^m\} = 0, \quad \{p, p^m\} = 0, \tag{2.29}$$

and

$$\{x, p^m\} = mp^{m-1} = \frac{\partial}{\partial p} p^m, \quad \{x^m, p\} = mx^{m-1} = \frac{\partial}{\partial x} x^m. \tag{2.30}$$

Then, it follows that for a generic function $f(x, p)$, as of Eq. 2.17, the Poisson bracket with respect to position (momentum) is equivalent to the derivative with respect to momentum (position),

$$\{x, f\} = \frac{\partial f}{\partial p}, \quad \{f, p\} = \frac{\partial f}{\partial x}. \tag{2.31}$$

Now, we can show inductively that

$$\{x^n, f(x,p)\} = nx^{n-1}\frac{\partial f}{\partial p} = \frac{\partial x^n}{\partial x}\frac{\partial f}{\partial p}, \quad \{f(x,p), p^n\} = np^{n-1}\frac{\partial f}{\partial x} = \frac{\partial p^n}{\partial p}\frac{\partial f}{\partial x}. \tag{2.32}$$

Now, in Eq. (2.32) and in the anticommutation, Eq. (2.23), linearity, Eq. (2.24), and product rule, Eq. (2.25), of the Poisson brackets we have all the ingredients needed to return back to Eq. (2.17).

$$\begin{aligned} \{f(x,p), g(x,p)\} &= \left\{\sum_{nm} c_{nm} x^n p^m, g(x,p)\right\} = \sum_{nm} c_{nm} \{x^n p^m, g\} \\ &= \sum_{nm} c_{nm}\, x^n \{p^m, g\} + \sum_{nm} c_{nm}\, \{x^n, g\} p^m = \sum_{nm} c_{nm} \left[-x^n \{g, p^m\} + \{x^n, g\} p^m\right] \\ &= \sum_{nm} c_{nm} \left[-x^n \frac{\partial p^m}{\partial p}\frac{\partial g}{\partial x} + \frac{\partial x^n}{\partial x} p^m \frac{\partial g}{\partial p}\right] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \end{aligned} \tag{2.33}$$

2.1.2 A way to think of classical time evolution

We have just then shown, that the definition of Poisson brackets in Eq. (2.17) is equivalent to the properties of Eqs (2.23)-Eqs (2.26). We can then choose to formulate time evolution as follows.

The dynamics of a physical system is encoded in its Hamiltonian,

$$H(x_i, p_i)$$

which depends on space coordinates x_i and canonical momenta p_i. A physical quantity $A(x_i, p_i, t)$ evolves in time according to

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}, \tag{2.34}$$

and the Poisson brackets are defined to satisfy

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \tag{2.35}$$

as well as, anticommutation, linearity, and the product rule of Eqs. (2.23)-(2.25). The above statements are all what one needs to describe the dynamic evolution of a classical system.

2.2 Time evolution in quantum mechanics

Physical systems are in quantum states $|\psi\rangle$ of a Hilbert space. Physical quantities, such as the position and momentum, are hermitian operators acting on this space,

$$\hat{x}_i\,,\hat{p}_i\,,\ldots$$

The mean value of experimental measurements of an observable quantity $\mathcal{O}(x_i, p_i)$, which is represented by an operator $\hat{\mathcal{O}}$ in Hilbert space, is given by

$$\langle \mathcal{O} \rangle = \langle \psi | \, \hat{\mathcal{O}}(\hat{x}_i, \hat{p}_i) \, | \psi \rangle \,. \tag{2.36}$$

Quantum time evolution has a very similar form as classical evolution, in the form presented in the subsection 2.1.2. A central object in classical evolution has been the Poisson brackets of position and momentum. What is the analogous object for the corresponding quantum mechanical operators? Dirac observed that the commutator of two operators,

$$[A, B] \equiv AB - BA$$

possesses algebraic properties analogous to the classical Poisson brackets, i.e.

$$[A, B] \;=\; -[B, A]\;, \tag{2.37}$$
$$[A, B+C] \;=\; [A, B] + [A, C]\;, \tag{2.38}$$
$$[A, BC] \;=\; B\,[A, C] + [A, B]\,C\,, \tag{2.39}$$
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\,. \tag{2.40}$$

In classical mechanics, the Poisson bracket of a coordinate and its canonical momentum is a constant (unit). We will postulate the same for the commutator of their quantum mechanical operators,

$$[\hat{x}, \hat{p}] = \text{constant}$$

The constant must then be imaginary, given that the positions and momenta are hermitian,

$$\text{constant}^* = [\hat{x}, \hat{p}]^\dagger = \left[\hat{p}^\dagger, \hat{x}^\dagger\right] = -\,[\hat{x}, \hat{p}] = -\text{constant}. \tag{2.41}$$

In analogy to classical mechanics, it comes natural to postulate the following commutation relations for positions and momenta

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0\,, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\,. \tag{2.42}$$

Comparing Eqs. (2.42) and the Poisson bracket "postulates" for classical mechanics in Eq. (2.35) it is motivated to identify the quantum mechanical analogue of the Poisson bracket as the commutator,

$$\{A, B\} \to \frac{1}{i\hbar}\left[\hat{A}, \hat{B}\right]. \tag{2.43}$$

Inspired by this analogy and Eq. (2.34), it seems natural to postulate the following time evolution for operators of physical operators $\hat{\mathcal{O}}\left(\hat{x}_i, \hat{p}_i\right)$ in quantum mechanics,

$$\frac{d\hat{\mathcal{O}}}{dt} = \frac{1}{i\hbar}\left[\hat{\mathcal{O}}, \hat{H}\right], \tag{2.44}$$

where $\hat{H}$ is the Hamiltonian operator.

As we know, this is the correct evolution for operators (in the Heisenberg picture) in quantum mechanics. Eq. (2.44) has the solution,

$$\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}Ht}\hat{\mathcal{O}}_0 e^{-\frac{i}{\hbar}Ht}. \tag{2.45}$$

The Heisenberg picture will be more convenient for studying time evolution in QFT. Eq. (2.45) is of course equivalent to the more commonly used Schrödinger picture. Indeed, we can change picture by shifting time evolution from the operator to the quantum states

$$\langle\psi_0|\,\hat{O}(t)\,|\psi_0\rangle = \langle\psi(t)|\,\hat{O}_0\,|\psi(t)\rangle\,, \tag{2.46}$$

where

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|\psi_0\rangle\,. \tag{2.47}$$

It is easy to see by differentiating the above equation with respect to time, that the time-dependent states (Schrödinger picture) satisfy Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle\,. \tag{2.48}$$

2.3 Conservation and symmetries in classical mechanics

Symmetries will play a very important role in our introduction to Quantum Field Theory and we will discuss them throughout this course. Let us recall here a few selected results from classical mechanics first. We will limit ourselves, for now, to continuous symmetry transformations of coordinates,

$$x_i \to x'_i = G_{ij}(\theta^a)x_j, \tag{2.49}$$

which leave the Hamiltonian invariant. In the above, θ^a are parameters which define the transformation (for example, for three dimensional rotations these parameters can be chosen as $\theta^a = \{\omega_{xy}, \omega_{yz}, \omega_{zx}$ rotation angles on the $x-y$, $y-z$ and $z-x$ planes). We can assume that there exist a set of values of the θ^a parameters for which the transformation leaves the coordinates unchanged (unit transformation). Conventionally, we calibrate the θ^a's so that

$$G_{ij}(\theta^a)|_{\theta^a=0} = \delta_{ij}. \tag{2.50}$$

For small values of the parameters θ^a we can approximate

$$G_{ij}(\theta^a) \approx \delta_{ij} + i\theta^a J^a_{ij}, \tag{2.51}$$

where the matrices J^a_{ij} are the generators of the symmetry on coordinates. Actually, the generators are the only quantities needed in order to produce any symmetry transformation since "large" symmetry transformations can always be broken down to infinitely small symmetry transformations for which Eq. (2.51) holds. The generators satisfy the Lie algebra

$$\left[J^a, J^b\right] = if^{abc}J^c, \tag{2.52}$$

where f^{abc} are the structure constants of the group.

The symmetry transformation should preserve Poisson brackets for positions and their corresponding momenta,

$$\{x'_i, p'_j\} = \delta_{ij}. \tag{2.53}$$

If the momenta transform as

$$p_i \to p'_i = \tilde{G}_{ij} p_j, \tag{2.54}$$

then Eq. (2.53) yields

$$G_{ik} \tilde{G}_{jk} = \delta_{ij}. \tag{2.55}$$

For small transformations, the above equation gives

$$\tilde{G}(\theta^a)_{ij} = \delta_{ij} - i\theta^a J^{aT}_{ij}. \tag{2.56}$$

Let us now consider a general quantity $A(x_i, p_i)$. Under an infinitesimal symmetry transformation as described above,

$$x_i \to x'_i = x_i + \delta x_i = x_i + i\theta^a J^a_{ik} x_k, \quad p_i \to p'_i = p_i + \delta p_i = p_i - i\theta^a J^{aT}_{ik} p_k, \tag{2.57}$$

A changes by

$$A \to A' = A + \delta A, \tag{2.58}$$

with

$$\begin{aligned} \delta A &= \frac{\partial A}{\partial x_i}\delta x_i + \delta p_i \frac{\partial A}{\partial p_i} = \{A, p_i\}\delta x_i + \delta p_i \{x_i, A\} \\ &= \{A, p_i\}\delta x_i - \delta p_i \{A, x_i\} = \{A, p_i\}\delta x_i - \left(-i\theta^a J^{aT}_{ik} p_k\right)\{A, x_i\} \\ &= \{A, p_i\}\delta x_i + p_k \left\{A, \left(i\theta^a J^{aT}_{ik}\right) x_i\right\} = \{A, p_i\}\delta x_i + p_k \{A, (i\theta^a J^a{}_{ki}) x_i\} \\ &= \{A, p_i\}\delta x_i + p_k \{A, \delta x_k\} = \{A, p_i \delta x_i\}. \end{aligned} \tag{2.59}$$

We have arrived to an important result, for the effect of a symmetry transformation on a generic quantity

$$\delta A = \{A, p_i \delta x_i\}. \tag{2.60}$$

For the Hamiltonian, which we have assumed to stay invariant under our symmetry transformations, we have that

$$\delta H = 0 \rightsquigarrow \{H, p_i \delta x_i\} = 0. \tag{2.61}$$

This leads to the well known conclusion (Noether's theorem) that there exist conserved quantities as a consequence of the symmetry, namely,

$$\frac{d}{dt}(p_i \delta x_i) = i\frac{d}{dt}(\theta^a J^a_{ik} p_i x_k) = 0. \tag{2.62}$$

Since the θ^a's are independent, we obtain that for every symmetry generator there is a conserved quantity

$$J^a_{ik} p_i x_k = constant, \quad \forall\, a. \tag{2.63}$$

2.3.1 A classical example

As an example, consider a system with N complex coordinates $q_i,\ i = 1 \dots N$ which is dictated by the Lagrangian

$$L = m \sum_{i=1}^{3} \dot{q}_i^* \dot{q}_i - \lambda \left(\sum_{i=1}^{3} q_i^* q_i - v \right)^2 \tag{2.64}$$

One can easily check that the above Lagrangian is invariant under the transformation

$$q_i \to q_i' = U_{ij} q_j, \quad q_i^* \to q_i^{*\prime} = U_{ij}^* q_j^*, \tag{2.65}$$

where the transformation matrix U is unitary,

$$U_{ki}^* U_{kj} = \delta_{ij} \qquad (U^\dagger U = \mathbf{1}). \tag{2.66}$$

Differentiating with respect to time, we obtain the symmetry transformations of velocities,

$$\dot{q}_i \to \dot{q}_i' = U_{ij} \dot{q}_j, \quad \dot{q}_i^* \to \dot{q}_i^{*\prime} = U_{ij}^* \dot{q}_j^*. \tag{2.67}$$

Therefore, the velocities transform exactly as their corresponding coordinates.

We now compute the canonical momenta,

$$p_i = \frac{\partial L}{\partial q_i} = m \dot{q}_i^*, \quad p_i^* = \frac{\partial L}{\partial q_i^*} = m \dot{q}_i. \tag{2.68}$$

Notice that the canonical momenta are

$$p_i = m \dot{q}_i^* \neq m \dot{q}_i,$$

and they do not transfrom like their velocities,

$$p_i = m \dot{q}_i^* \to p_i' = m \dot{q}_i^{*\prime} = m U_{ij}^* q_j^* = U_{ij}^* p_j = \left[U^\dagger \right]_{ij}^T p_j = \left[U^{-1} \right]_{ij}^T p_j. \tag{2.69}$$

Exercise: *What are the conserved quantities predicted by Noether's theorem, which originate from the $U(N)$ symmetry of this Lagrangian?*

2.4 Symmetries in Quantum mechanics

A continuous symmetry transformation acts on quantum states, transforming them as

$$|\psi\rangle \to |\psi'\rangle = U(\theta^a) |\psi\rangle, \tag{2.70}$$

where θ^a are parameters of the transformation. From Wigner's theorem we know that the transformation matrix U is linear and unitary (rather than antilinear and antiunitary, which is the second allowed option of the theorem). Indeed, continuous symmetry transformations, such as rotations or translations, contain the unit transformation (no transformation) as an element of the group, which is linear and unitary. For small values of the parameters we have that

$$U(\theta^a) \approx \mathbf{1} + i \theta^a t^a, \tag{2.71}$$

where the generators of the symmetry (as represented on Hilbert space) satisfy the Lie algebra

$$\left[t^a, t^b\right] = i f^{abc} t^c. \tag{2.72}$$

We remind that the generators J^a in the "classical mechanics"representation of a symmetry transformation on coordinates, as in Eq. (2.52), are different than the generators of the representation of the symmetry on quantum states, as in Eq. (2.72). Of course, the structure constants f^{abc} are the same, as they are common to the generators of all representations.

From unitarity, we can easily derive that the generators are hermitian,

$$U^\dagger U = \mathbf{1} \rightsquigarrow t^{a\dagger} = t^a, \tag{2.73}$$

which renders them good candidates for operators of physical (measurable) quantities. If we assume that the expectation value of the energy of a system in an arbitrary quantum state is invariant under the symmetry transformation,

$$\langle\psi| H |\psi\rangle = \langle\psi'| H |\psi'\rangle, \tag{2.74}$$

then we obtain that

$$U^\dagger(\theta^a) H U(\theta^a) = H, \tag{2.75}$$

or, equivalently, that the commutator of the symmetry generators and the Hamiltonian vanishes

$$[t^a, H] = 0. \tag{2.76}$$

On one hand, the above result implies that the generators t^a are quantum mechanical operators of conserved physical quantities,

$$i\hbar \frac{d}{dt} t^a = [t^a, H] = 0. \tag{2.77}$$

On the other hand, it implies that the energy spectrum is degenerate. Consider an energy eigenstate $|E\rangle$,

$$H\,|E\rangle = E\,|E\rangle. \tag{2.78}$$

Then, the states $(t^a\,|E\rangle)$ are also energy eigenstates,

$$[t^a, H]\,|E\rangle = 0 \rightsquigarrow H\,(t^a\,|E\rangle) = E\,(t^a\,|E\rangle). \tag{2.79}$$

Under a symmetry transformation, an expectaiton value of an operator changes to

$$\langle\psi| A |\psi\rangle \to \langle\psi'| A |\psi'\rangle = \langle\psi| U^\dagger A U |\psi\rangle. \tag{2.80}$$

It is then convenient to calculate the effect of a symmetry operator on expectation values or matrix-elements by leaving the states unchanged and performing a similarity transformation on operators

$$A \to A' = U^\dagger A U, \tag{2.81}$$

which, for small symmetry transformations gives

$$\delta A = A' - A = i\theta^a\,[A, t^a]. \tag{2.82}$$

We would now like to see how continuous symmetry transformations change a generic operator and compare with our classical mechanics intuition. To study the variation, we first need to introduce the notion of a "derivative" for quantum mechanical operators. We can do that by means of commutators with position and momenta. For a generic operator $A(x_i, p_i)$ we define the following derivatives

$$i\hbar \frac{\partial A}{\partial x_i} = [A, p_i] \,, \tag{2.83}$$

and

$$i\hbar \frac{\partial A}{\partial p_i} = [x_i, A] \,. \tag{2.84}$$

You can check with explicit calculations that the right-hand sides behave, algebraically, like derivatives.

Now let us introduce a convention for writing down an operator which is a function of position and momenta. We can agree to order products with positions to the right and momenta to the left. For example, we will use the commutation relation of positino and momenta to write $A = \hat{p}\hat{x} + i\hbar$ rather than $A = \hat{x}\hat{p}$. With this convention for casting operators, a generic operator $A(x_i, p_i)$, under a symmetry transformation, varies by

$$\begin{aligned} \delta A &= \frac{\partial A}{\partial x_i}\delta x_i + \delta p_i \frac{\partial A}{\partial p_i} \\ &= \frac{1}{i\hbar}\, [A, p_i]\, \delta x_i + \frac{1}{i\hbar}\delta p_i\, [x_i, A] \,. \end{aligned} \tag{2.85}$$

It is now not difficult to repeat the steps in the classical mechanics derivation and to find the quantum mechanics analogue of Eq. (2.60),

$$i\hbar \delta A = [A, p_i \delta x_i] \,. \tag{2.86}$$

Analogously to classical mechanics, for $A = H$, we find that

$$[H, p_i \delta x_i] = 0 \rightsquigarrow \frac{d}{dt}\left(p_i \delta x_i\right) = 0. \tag{2.87}$$

This is (a case of) Noether's theorem in quantum mechanics. Given a symmetry, it informs us which quantum mechanical operators are conserved.

In conclusion, we have seen that by quantizing a classical system in a canonical quantization formalism, we can get guidance about what quantities are conserved from our classical studies of the system. The conserved operators commute with the Hamiltonian. Once we have identified them from symmetries, we can find their common eigenstates with the Hamiltonian.

We will apply exactly the same strategy for systems of an infinite number of generalised coordinates (fields). For free (non-interacting) fields, the common eigenstates of the Hamiltonian and the conserved quantities will be the ground state (vacuum), particle and multi-particle states.

Chapter 3

Theory of Classical Fields

We start by introducing a Lagrangian formalism for fields at the classical level. A classical field is a continuous function defined at every point in space-time. For example, the amplitude of a mechanical wave or an electromagnetic wave are fields. Fields contain an infinite number of degrees of freedom, as necessitated by their existence at every point in spacetime. However, the Lagrangian formalism for them is completely analogous to discrete systems by a simple extension.

3.1 Fields from a discretised space (lattice)

A generalisation of the Lagrangian formalism to fields is made, if we first assume that a field function takes non-zero values only at discrete points of a lattice which spans all space, and then take the "continuous limit" of a zero distance between the lattice points.

As a simple example, we shall study the vibration motion in an one-dimensional elastic rod. The rod has a mass density μ. For an elastic rod, the force applied on it is proportional to the elongation or the compression per unit length (ξ) of the rod,

$$F = -Y\xi, \tag{3.1}$$

where Y is a constant called Young's modulus.

A non-vibrational rod can be seen as an infinite number of equally spaced particles at rest, all having the same mass m, with a relative distance a. For a small spacing a, we have

$$\mu = \frac{dm}{dx} = \lim_{a\to 0} \frac{m}{a}. \tag{3.2}$$

A vibration is created when these particles are displaced from their positions at rest. We will assume that each particle in the lattice can interact with its immediate neighbours only. When two neighbouring particles are displaced to a larger relative distance, an attractive force tends to bring them back to their resting position, while a repulsive force is developed when they are found at a shorter relative distance. Such a force can be approximated (elastic rod) by considering an elastic spring between the particles, with a Hooke's constant κ. The force required to elongate one spring is,

$$\begin{aligned} F &= -\kappa\left(y_{i+1} - y_i\right) = -\left(\kappa a\right)\frac{y_{i+1} - y_i}{a} \\ &= -\left(\kappa a\right)\xi. \end{aligned} \tag{3.3}$$

We can then relate the microscopic Hooke's constant to the macroscopic Young's modulus:

$$Y = \lim_{a\to 0} (\kappa a) . \tag{3.4}$$

The potential energy in the rod is equal to the sum of the potential energies in all springs,

$$V = \sum_i \frac{1}{2}\kappa \Delta y_i^2 = \sum_i \frac{1}{2}\kappa (y_{i+1} - y_i)^2, \tag{3.5}$$

where $\Delta y_i =$ the expansion or contraction of the spring in between particle i and particle $i+1$, and y_i is the displacement of particle i from its position at rest. The kinetic energy of all particles is,

$$T = \sum_i \frac{1}{2} m \dot{y}_i^2. \tag{3.6}$$

The Lagrangian of the system is simply given by,

$$\begin{aligned} & L = T - V \\ \rightsquigarrow\quad & L = \sum_i \left[\frac{1}{2} m \dot{y}_i^2 - \frac{1}{2}\kappa \left(y_{i+1} - y_i\right)^2\right] . \end{aligned} \tag{3.7}$$

We can easily find the equations of motion for each particle j in the discretised rod, from the Euler-Lagrange equations **??**,

$$\begin{aligned} & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_k}\right) - \frac{\partial L}{\partial y_k} = 0 \\ \rightsquigarrow\quad & \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_k}\right) + \frac{\partial V}{\partial y_k} = 0 \\ \rightsquigarrow\quad & \frac{d}{dt}\left(\frac{\partial \sum_i \frac{1}{2} m \dot{y}_i^2}{\partial \dot{y}_k}\right) + \frac{\partial \sum_i \frac{1}{2}\kappa (y_{i+1} - y_i)^2}{\partial y_k} = 0 \\ \rightsquigarrow\quad & \frac{d}{dt}\left(\frac{\partial \frac{1}{2} m \dot{y}_k^2}{\partial \dot{y}_k}\right) + \frac{\partial \left[\frac{1}{2}\kappa (y_{k+1} - y_k)^2 + \frac{1}{2}\kappa (y_k - y_{k-1})^2\right]}{\partial y_k} = 0 \\ \rightsquigarrow\quad & m\ddot{y}_k + \kappa (y_k - y_{k-1}) - \kappa (y_{k+1} - y_k) = 0. \end{aligned} \tag{3.8}$$

Let us now consider the limit that the spacing in the discretised rod tends to zero. The vibration amplitude $y_k(t)$, defines a continuous function at each position x in the one-dimensional rod.

$$\begin{aligned} & y_k(t) \to y(x,t) \\ & y_{k+1}(t) \to y(x+a,t) \\ & y_{k-1}(t) \to y(x-a,t) \\ & y_{k+2}(t) \to y(x+2a,t) \\ & y_{k-2}(t) \to y(x-2a,t) \\ & \dots \end{aligned}$$

The equations of motion are written as,

$$\begin{aligned} & m\ddot{y}(x,t) - \kappa \left[(y(x+a,t) - y(x,t)) - (y(x,t) - y(x-a,t))\right] = 0 \\ \rightsquigarrow\quad & \frac{m}{a}\ddot{y}(x,t) - \kappa \left[\frac{y(x+a,t) - y(x,t)}{a} - \frac{y(x,t) - y(x-a,t)}{a}\right] = 0 \\ \rightsquigarrow\quad & \frac{m}{a}\ddot{y}(x,t) - (\kappa a) \frac{\frac{y(x+a,t)-y(x,t)}{a} - \frac{y(x,t)-y(x-a,t)}{a}}{a} = 0. \end{aligned} \tag{3.9}$$

Taking the limit $a \to 0$, we obtain the equation of motion for the vibration field $y(x,t)$,

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} - Y \frac{\partial^2 y(x,t)}{\partial x^2} = 0. \tag{3.10}$$

It is interesting to take the continuum limit in the expression of Eq. 3.7 for the Lagrangian as well. We have,

$$L = \sum_i a \left[\frac{1}{2} \frac{m}{a} \dot{y}_i^2 - \frac{1}{2} (\kappa a) \left(\frac{y_{i+1} - y_i}{a} \right)^2 \right]. \tag{3.11}$$

In the continuous limit, the summation turns into an integration over the position variable x, leading to

$$L = \int dx \left[\frac{1}{2} \mu \left(\frac{\partial y(x,t)}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial y(x,t)}{\partial x} \right)^2 \right]. \tag{3.12}$$

The action is then written as an integral in time and space dimensions,

$$S = \int dt dx \mathcal{L}, \tag{3.13}$$

over a *Lagrangian density*,

$$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial y(x,t)}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial y(x,t)}{\partial x} \right)^2. \tag{3.14}$$

Can we obtain the equations of motion (Eq. 3.10) from the action integral over the Lagrangian density of Eq. 3.13 using the principle of least action? If so, we can avoid the cumbersome discretisation derivation and work using a direct formalism for continuum systems. We require that

$$\delta \int dt dx \mathcal{L} \left[y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x} \right] = 0, \tag{3.15}$$

if we vary the field $y(x,t)$ from its physical solution,

$$y(x,t) \to y(x,t) + \delta y(x,t). \tag{3.16}$$

We also require that the variation of the field vanishes at the boundaries of the integrations,

$$\delta y(x, t_1) = \delta y(x, t_2) = 0, \delta y(x_1, t) = \delta y(x_2, t) = 0. \tag{3.17}$$

We then have,

$$\begin{aligned} & \delta \int dt dx \mathcal{L} \left[y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x} \right] = 0 \\ \rightsquigarrow & \int dt dx \left[\frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial (\partial_t y)} \delta(\partial_t y) + \frac{\partial \mathcal{L}}{\partial (\partial_x y)} \delta(\partial_x y) \right] = 0 \end{aligned} \tag{3.18}$$

The field variation commutes with space and time derivatives, and we can rewrite the above equation as,

$$\begin{aligned} & \int dt dx \left[\frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial (\partial_t y)} \partial_t (\delta y) + \frac{\partial \mathcal{L}}{\partial (\partial_x y)} \partial_x (\delta y) \right] = 0 \\ \rightsquigarrow & \int dt dx \left\{ \left[\frac{\partial \mathcal{L}}{\partial y} - \partial_t \frac{\partial \mathcal{L}}{\partial (\partial_t y)} - \partial_x \frac{\partial \mathcal{L}}{\partial (\partial_x y)} \right] \delta y \right. \\ & \left. + \partial_t \left(\frac{\partial \mathcal{L}}{\partial (\partial_t y)} \delta y \right) + \partial_x \left(\frac{\partial \mathcal{L}}{\partial (\partial_x y)} \delta y \right) \right\} = 0. \end{aligned} \tag{3.19}$$

Since the field variation δy is generic and it vanishes at the boundaries, we obtain an Euler-Lagrange differential equation for $y(x,t)$,

$$\frac{\partial \mathcal{L}}{\partial y} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t y)} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x y)} = 0. \tag{3.20}$$

Substituting into this equation the Lagrangian density for the elastic rod, we find the following equation of motion:

$$\mu \frac{\partial^2 y(x,t)}{\partial t^2} - Y \frac{\partial^2 y(x,t)}{\partial x^2} = 0, \tag{3.21}$$

which is the same as Eq. 3.8

3.2 Euler-Lagrange equations for a classical field from a Lagrangian density

We generalize readily the Lagrangian formalism developed for the one-dimensional elastic rod example, to fields which are defined in four space-time dimensions $x^\mu = (x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$. In this lecture series, we use natural units, setting $\hbar = c = 1$, and the space-time metric is defined as $g^{\mu\nu} = diag(1, -1, -1, -1)$. We consider a system of N fields $\phi^{(i)}, i = 1, \ldots, N$ with a Lagrangian density which depends only on the fields and their first (for simplicity) space-time derivatives

$$\mathcal{L} = \mathcal{L}\left[\phi^{(i)}, \partial_\mu \phi^{(i)}\right]. \tag{3.22}$$

The action is given by the integral,

$$S \equiv \int_{t_1}^{t_2} dx_0 \int_V d^3\vec{x} \mathcal{L}\left[\phi^{(i)}, \partial_\mu \phi^{(i)}\right]. \tag{3.23}$$

and we consider small variations $\phi^{(i)}(\vec{x},t) \to \phi^{(i)}(\vec{x},t) + \delta\phi^{(i)}(\vec{x},t)$ of the fields from their physical values which are zero at the times t_1, t_2 or at the boundary surface $S(V)$ of the volume V,

$$\delta\phi^{(i)}(\vec{x},t)\big|_{t=t_1,t_2} = \delta\phi^{(i)}(\vec{x},t)\big|_{\vec{x}\in S(V)} = 0, \tag{3.24}$$

but arbitrary otherwise. Applying the variational principle, we obtain

$$\begin{aligned}
& \delta S = 0 \\
\rightsquigarrow\ & \delta \int d^4x \mathcal{L} = 0 \\
\rightsquigarrow\ & \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^{(i)}} \delta\phi^{(i)} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{(i)}\right)} \delta\left(\partial_\mu \phi^{(i)}\right)\right] = 0 \\
\rightsquigarrow\ & \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^{(i)}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{(i)}\right)}\right] \delta\phi^{(i)} + \int d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{(i)}\right)} \delta\phi^{(i)}\right] = 0 \\
\rightsquigarrow\ & \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi^{(i)}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{(i)}\right)}\right] \delta\phi^{(i)} = 0,
\end{aligned} \tag{3.25}$$

where we have used the conditions for vanishing field variations at the boundaries. Given that the functional form of the field variation $\delta\phi^{(i)}$ is arbitrary, we obtain the Euler-Lagrange differential equations,

$$\frac{\partial \mathcal{L}}{\partial \phi^{(i)}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^{(i)}\right)} = 0. \tag{3.26}$$

From the above derivation we can easily see that two Lagrangian densities, $\mathcal{L}$ and $\mathcal{L} + \partial_\nu F^\nu(\phi^{(i)}, \partial\phi^{(i)})$, which differ by a total divergence yield identical Euler-Lagrange identities. Indeed the variation of the total divergence term in the action integral is zero,

$$\delta \int_{V,T} d^4x \partial_\nu F^\nu = \delta \int_{S(V,T)} d\sigma_\nu F^\nu = \int_{S(V,T)} d\sigma_\nu \delta F^\nu = 0. \tag{3.27}$$

In the above we have used Gauss' theorem and that the variation of a smooth function F of the fields and their derivatives vanishes at the boundaries of the action integral.

3.3 Noether's theorem

We can prove a very powerful theorem for Lagrangian systems wich exhibit symmetries, which is known as the *Noether theorem*. The theorem states that for every symmetry of the system, there exists a conserved current and a corresponding conserved charge.

3.3.1 Internal field symmetry transformations

Consider a Lagrangian density

$$\mathcal{L}\left[\phi^{(i)}, \partial_\nu \phi^{(i)}\right], \tag{3.28}$$

and an infinitesimal symmetry transformation of the fields

$$\phi^{(i)}(x) \to \phi^{(i)}(x) + \delta\phi^{(i)}(x), \quad \partial_\nu\phi^{(i)}(x) \to \partial_\nu\phi^{(i)}(x) + \partial_\nu\delta\phi^{(i)}(x). \tag{3.29}$$

Under a symmetry transformation the equations of motion remain the same, therefore the Lagrangian density can only change up to a total derivative:

$$\delta\mathcal{L} = \partial_\mu X^\mu. \tag{3.30}$$

The change in the Lagrangian density can be computed as,

$$\begin{aligned}
\delta\mathcal{L} &= \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\delta\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\partial_\nu\delta\phi^{(i)}\right] \\
&= \sum_i \left\{\delta\phi^{(i)}\left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\right] + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\delta\phi^{(i)}\right)\right\} \\
\rightsquigarrow \delta\mathcal{L} &= \sum_i \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\delta\phi^{(i)}\right) = \partial_\nu X^\nu.
\end{aligned} \tag{3.31}$$

Therefore the currents

$$J^\nu = \sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\delta\phi^{(i)} - X^\nu, \tag{3.32}$$

are conserved:

$$\partial_\nu J^\nu = 0. \tag{3.33}$$

The conserved currents J^ν are as many as the number of independent generators for the transformations $\delta\phi^{(i)}$.

To each conserved current, corresponds a conserved charge, i.e. a physical quantity which maintains the same value at all times. Indeed,

$$\begin{aligned}
& \partial_\nu J^\nu = 0 \\
\rightsquigarrow \quad & \frac{\partial J^0}{\partial t} + \frac{\partial J^i}{\partial x^i} = 0 \\
\rightsquigarrow \quad & \frac{\partial J^0}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \\
\rightsquigarrow \quad & \int d^3\vec{x} \frac{\partial J^0}{\partial t} + \int d^3\vec{x} \vec{\nabla} \cdot \vec{J} = 0 \\
\rightsquigarrow \quad & \frac{\partial Q}{\partial t} = 0 \text{ with } Q = \int d^3\vec{x} J^0
\end{aligned} \tag{3.34}$$

3.3.2 Space-Time symmetry transformations

We now consider a more complicated version of the theorem, which deals with space-time symmetries. We consider a classical system of $i = 1 \dots N$ physical fields $\phi^{(i)}$, and we assume that the Lagrangian density does not depend explicitly on space-time coordinates:

$$\mathcal{L} = \mathcal{L}\left[\phi^{(i)}, \frac{\partial}{\partial x^\nu}\phi^{(i)}\right]. \tag{3.35}$$

However, the Lagrangian depends on the space-time coordinates implicitly, through the fields $\phi^{(i)}(x^\mu)$. This is a reasonable assumption if we require that physical laws (the equations of motion) derived from this Lagrangian are valid everywhere and always.

We now require that if we perform a space-time symmetry transformation, such as a translation or a rotation or a boost

$$x^\mu \to x'^\mu, \tag{3.36}$$

the action remains the same:

$$S = \int_V d^4x \mathcal{L}\left[\phi^{(i)}(x), \partial_\nu \phi^{(i)}(x)\right] = \int_{V'} d^4x' \mathcal{L}\left[\phi'^{(i)}(x'), \frac{\partial}{\partial x^{\nu\prime}}\phi'^{(i)}(x')\right]. \tag{3.37}$$

In this way, since the action is invariant under the transformation, the variational principle $\delta S = 0$ is automatically satisfied after the transformation if it was satisfied before the transformation. We consider continuous transformations for which there exists a choice of the transformation parameters resulting to a unit transformation, i.e. no transformation. An example is a Lorentz boost with some velocity $\vec{v}$, where for $v = 0$ we have $x^\mu \to x'^\mu = x^\mu$. There are examples of symmetry transformations where this does not occur. For example a parity transformation does not have this property, and the Noether theorem is not applicable then. For continuous symmetry transformations, we can consider that they are infinitesimally different than the unit transformation,

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu \tag{3.38}$$

with δx_μ very small.

In general, the fields $\phi^{(i)}$ also transform under general space-time tranformations, since they may be, for example, vectors, tensors, etc. We write, to a linear approximation,

$$\phi'^{(i)}(x') = \phi^{(i)}(x) + \Delta\phi^{(i)}(x) \quad (3.39)$$

$$\frac{\partial}{\partial x^{\nu\prime}}\phi'^{(i)}(x') = \frac{\partial}{\partial x^{\nu}}\phi^{(i)}(x) + \Delta\left(\frac{\partial}{\partial x^{\nu}}\phi^{(i)}(x)\right) \quad (3.40)$$

$$(3.41)$$

We decompose the variation of a generic function $f(x)$, which can stand for either a field or its derivative $f(x) = \phi^{(i)}(x), \partial_\nu\phi^{(i)}(x)$, into two parts

$$\Delta f(x) \equiv f'(x') - f(x) = (f'(x') - f(x')) + (f(x') - f(x))\,. \quad (3.42)$$

The difference in the second bracket of the right-hand side describes the change of the field (of its derivative) due to transporting its argument from $x \to x'$ without considering the change that the function itself should undergo in its form due to the transformation. To the linear approximation,

$$\delta f(x) = f(x') - f(x) = (f(x) + \partial_\mu f\delta x^\mu + \ldots) - f(x) = \partial_\mu f(x)\,\delta x^\mu + \mathcal{O}(\delta^2). \quad (3.43)$$

The remaining change originates from changing the form of the function f. We call this difference *variation at a point*,

$$\delta_* f(x') \equiv f'(x') - f(x'). \quad (3.44)$$

We note that, to the linear approximation, we are allowed to evaluate the variation at a point of a field function at either the original or the transformed points as the difference of these two differnces is of quadratic order. Indeed,

$$\delta_* f(x') = \delta_* f(x) + (\partial_\mu\delta_* f(x))\,\delta x^\mu + \ldots = \delta_* f(x) + \mathcal{O}(\delta^2). \quad (3.45)$$

Explicitly, for the transformation of the fields we have

$$\Delta\phi^{(i)} = \delta_*\phi^{(i)}(x) + \delta x^\mu\partial_\mu\phi^{(i)}, \quad (3.46)$$

and, for the the transformation of the field derivatives,

$$\Delta\partial_\nu\phi^{(i)} = \partial_\nu\delta_*\phi^{(i)}(x) + \delta x^\mu\partial_\mu\partial_\nu\phi^{(i)}(x). \quad (3.47)$$

The action, after the transformation, becomes

$$\begin{aligned} S &= \int_{V'} d^4x'\mathcal{L}\left[\phi'^{(i)}(x'), \partial_\nu\phi'^{(i)}(x')\right] \\ &= \int_V d^4x \det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right)\mathcal{L}\left[\phi^{(i)}(x) + \Delta\phi^{(i)}(x), \partial'_\nu\phi^{(i)}(x) + \Delta\left(\partial_\nu\phi^{(i)}(x)\right)\right]. \end{aligned} \quad (3.48)$$

The variation of the Lagrangian density due to the symmetry transformation can be computed as,

$$\begin{aligned} \Delta\mathcal{L} &= \mathcal{L}\left[\phi^{(i)}(x) + \Delta\phi^{(i)}(x), \partial_\nu\phi^{(i)}(x) + \Delta\left(\partial_\nu\phi^{(i)}(x)\right)\right] - \mathcal{L}\left[\phi^{(i)}(x), \partial_\nu\phi^{(i)}(x)\right] \\ \rightsquigarrow \Delta\mathcal{L} &= \sum_i\left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\Delta\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\Delta\left(\partial_\nu\phi^{(i)}\right)\right] + \mathcal{O}\left(\left(\Delta\phi^{(i)}\right)^2\right) \end{aligned} \quad (3.49)$$

where the term inside the square brackets vanishes by requiring that $\phi^{(i)}$ are physical fields satisfying the Euler-Lagrange equations.

The Jacobian of our space-time transformation is,

$$\det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right) = \det\left(\frac{\partial\left[x_\mu + \delta x_\mu\right]}{\partial x_\nu}\right) = \det\left(\delta^\nu_\mu + \partial^\nu \delta x_\mu\right). \tag{3.50}$$

We now need the determinant of a four by four matrix ($\mu, \nu = 0, 1, 2, 3$). We can verify with an explicit calculation that through order $\mathcal{O}(\delta x)$:

$$\det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right) = 1 + \partial^\mu\left(\delta x_\mu\right) + \mathcal{O}((\delta x)^2). \tag{3.51}$$

[**exercise: Verify the above explicitly**]

Substituting the expansions of Eq. 3.49 and Eq. 3.51 in the action integral of Eq. 3.48, and keeping terms through $\mathcal{O}\left(\Delta\phi^{(i)}\right)$, we find

$$\begin{aligned} S &= S + \int d^4x \left[\Delta\mathcal{L} + \mathcal{L}\partial^\mu \delta x_\mu\right] \\ \rightsquigarrow 0 &= \int d^4x \left\{\sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\Delta\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\Delta\left(\partial_\nu\phi^{(i)}\right)\right] + \mathcal{L}\partial^\mu\delta x_\mu\right\} \end{aligned} \tag{3.52}$$

Substituting Eq 3.46 and Eq 3.47 into Eq. 3.52, we obtain:

$$\begin{aligned} 0 = &\int d^4x \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\delta_*\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\partial_\nu\left(\delta_*\phi^{(i)}\right)\right] \\ &+ \int d^4x \left\{\mathcal{L}\partial^\mu\delta x_\mu + \delta x^\mu \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\partial_\mu\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\partial_\mu\partial_\nu\phi^{(i)}\right]\right\} \end{aligned} \tag{3.53}$$

Applying the chain rule, we see that the term in the curly brackets is a total divergence,

$$\partial_\mu\left(\mathcal{L}\delta x^\mu\right) = \mathcal{L}\partial_\mu\delta x^\mu + \delta x^\mu \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\partial_\mu\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\partial_\mu\partial_\nu\phi^{(i)}\right]. \tag{3.54}$$

The term in the square brackets of Eq. 3.53 can also be written as a total divergence, applying integration by parts and using the Euler-Lagrange equations,

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\phi^{(i)}}\delta_*\phi^{(i)} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\partial_\nu\left(\delta_*\phi^{(i)}\right) \\ = &\left[\frac{\partial\mathcal{L}}{\partial\phi^{(i)}} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\right]\delta_*\phi^{(i)} + \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\delta_*\phi^{(i)}\right) \\ = &\ \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\delta_*\phi^{(i)}\right) \end{aligned} \tag{3.55}$$

Eq. 3.53, with Eq. 3.54 and Eq. 3.55, becomes

$$
\begin{aligned}
0 &= \int d^4x \left[\sum_i \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^{(i)} \right)} \delta_* \phi^{(i)} \right) + \partial_\mu \left(\mathcal{L} \delta x^\mu \right) \right] \\
\rightsquigarrow \quad 0 &= \int d^4x \partial_\nu \left[\sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^{(i)} \right)} \delta_* \phi^{(i)} + \mathcal{L} \delta x^\nu \right]. \qquad (3.56)
\end{aligned}
$$

This is essentially the proof of Noether's theorem. The volume of integration in equation Eq. 3.56 is considered to have boundaries which are arbitrary. Then, in order for Eq. 3.56 to hold, the integrand must vanish. This gives:

$$
\partial_\nu J^\nu = 0, \qquad (3.57)
$$

where the conserved currents are:

$$
J^\nu \equiv \sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^{(i)} \right)} \delta_* \phi^{(i)} + \mathcal{L} \delta x^\nu \qquad (3.58)
$$

We have therefore proven, that if the system possesses a symmetry under a space-time transformation, then there exist conserved currents.

3.3.3 Energy-momentum tensor

For simplicity, we restrict ourselves to fields $\phi^{(i)}$ which transform as scalars under a space-time transformation, i.e. they do not transform:

$$
\phi'^{(i)}(x') = \phi^{(i)}(x) \rightsquigarrow \Delta \phi^{(i)} = 0. \qquad (3.59)
$$

For this to happen, the variation of a field at a point $\delta_* \phi^{(i)}$ must compensate the variation $\delta \phi^{(i)}$ due to changing the space-time position:

$$
\begin{aligned}
& \Delta \phi^{(i)} = 0 \rightsquigarrow \delta_* \phi^{(i)} + \delta \phi^{(i)} = 0 \\
\rightsquigarrow \quad & \delta_* \phi^{(i)} = -\delta \phi^{(i)} \rightsquigarrow \delta_* \phi^{(i)} = -\delta x^\mu \partial_\mu \phi^{(i)}. \qquad (3.60)
\end{aligned}
$$

Substituting Eq. 3.59 into Eq 3.58 we obtain

$$
\partial_\nu T^{\mu\nu} \delta x_\mu = 0. \qquad (3.61)
$$

where we have defined:

$$
T^{\mu\nu} \equiv \sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi^{(i)} \right)} \left(\partial^\mu \phi^{(i)} \right) - \mathcal{L} g^{\mu\nu} \qquad (3.62)
$$

The tensor $T^{\mu\nu}$ is known as the *energy-momentum tensor* for reasons that will become obvious in a while. The name *stress-energy tensor* is also encountered very often.

$$
\partial_\nu J^\nu = 0, \qquad J^\nu = \delta x_\mu T^{\mu\nu}. \qquad (3.63)
$$

We note that we can write as many currents J^ν as the number of the independent δx_μ space-time symmetry transformations which are obeyed by the physical system.

Translation symmetry transformations

As a first application of the Noether theorem, we find the conserved currents and the corresponding charges for physical systems which are symmetric under space-time translations. An infinitesimal translation transformation is

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \rightsquigarrow \delta x^\mu = \epsilon^\mu, \tag{3.64}$$

where ϵ^μ is a small constant four-vector.

The corresponding conserved current is

$$J^\nu = T^{\mu\nu}\delta\epsilon_\nu, \tag{3.65}$$

satisfying the continuity equation

$$\begin{aligned} & \partial_\nu J^\nu = 0 \\ \rightsquigarrow \quad & \partial_\nu T^{\mu\nu}\epsilon_\mu = 0. \end{aligned} \tag{3.66}$$

The vector ϵ^μ is small but otherwise arbitrary. Then, we must have that the above equation is satisfied in general if

$$\partial_\nu T^{\mu\nu} = 0, \tag{3.67}$$

for every value of the index $\mu = 0, 1, 2, 3$ separately.

To the four conserved currents correspond four "conserved charges" (Eq. 3.34),

$$P^\mu = \int d^3\vec{x}\, T^{\mu 0}. \tag{3.68}$$

Specifically, the "time-component" of P^μ is

$$\begin{aligned} P^0 &= \int d^3\vec{x}\, T^{00} \\ &= \int d^3\vec{x} \left[\sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi^{(i)}\right)} \left(\partial_0 \phi^{(i)}\right) - g^{00}\mathcal{L} \right]. \end{aligned} \tag{3.69}$$

We identify in the above a conjugate momentum for the field $\phi^{(i)}$ is

$$\pi^{(i)} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi^{(i)}\right)}, \tag{3.70}$$

and thus

$$P^0 = \int d^3\vec{x} \left[\sum_i \pi^{(i)} \partial_0 \phi^{(i)} - \mathcal{L} \right]. \tag{3.71}$$

In the square bracket, we recognize a Hamiltonian density

$$\mathcal{H} = \sum_i \pi^{(i)} \partial_0 \phi^{(i)} - \mathcal{L} \tag{3.72}$$

The conserved charge

$$P^0 = \int d^3\vec{x}\, \mathcal{H}, \tag{3.73}$$

concluding that the charge P^0 is the Hamiltonian of the system and it is conserved. Energy conservation is thus a pure consequence of time translation symmetry. Similarly, we can identify the charges P^i, $i = 1, 2, 3$ as the momentum components of the system in the three space directions, and they are of course also conserved.

3.3.4 Lorentz symmetry transformations and conserved currents

Lorentz transformations,

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \tag{3.74}$$

preserve the distance

$$ds^2 = g^{\mu\nu} dx_{\mu} dx_{\nu} = g^{\mu\nu} dx'_{\nu} dx'_{\nu}. \tag{3.75}$$

This leads to

$$\begin{aligned} g^{\mu\nu} dx_{\mu} dx_{\nu} &= g^{\mu\nu} \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} dx_{\rho} dx_{\sigma} \\ &= g^{\rho\sigma} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} dx_{\mu} dx_{\nu} \\ \rightsquigarrow g^{\mu\nu} &= g^{\rho\sigma} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma}. \end{aligned} \tag{3.76}$$

Considering an infinitesimal transformation,

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}, + \mathcal{O}(\omega^2) \tag{3.77}$$

we obtain

$$\begin{aligned} g^{\mu\nu} &= g^{\rho\sigma} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} \\ \rightsquigarrow g^{\mu\nu} &= g^{\rho\sigma} \left(\delta^{\mu}_{\rho} + \omega^{\mu}_{\rho} + \ldots\right) \left(\delta^{\nu}_{\sigma} + \omega^{\nu}_{\sigma} + \ldots\right) \\ \rightsquigarrow g^{\mu\nu} &= g^{\mu\nu} + \omega^{\nu\mu} + \omega^{\mu\nu} + \mathcal{O}(\omega^2) \\ \rightsquigarrow \omega^{\mu\nu} &= -\omega^{\nu\mu}. \end{aligned} \tag{3.78}$$

The parameters $\omega^{\mu\nu}$ for generating Lorentz transformations are therefore antisymmetric.

A transformed space-time coordinate is given by

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu\rho} x_{\rho} \\ &= \left(g^{\mu\rho} + \omega^{\mu\rho} + \ldots\right) x_{\rho} \\ &= x^{\mu} + \omega^{\mu\rho} x_{\rho} + \mathcal{O}(\omega^2) \\ \rightsquigarrow \delta x^{\mu} &\equiv x'^{\mu} - x^{\mu} = \omega^{\mu\rho} x_{\rho}. \end{aligned} \tag{3.79}$$

If Lorentz transformations are a symmetry of a physical system consisting of scalar fields, then the corresponding conserved currents are

$$\begin{aligned} J^{\nu} &= T^{\mu\nu} \delta x_{\mu} = T^{\mu\nu} x^{\rho} \omega_{\mu\rho} = \frac{1}{2} T^{\mu\nu} x^{\rho} \left[\omega_{\mu\rho} - \omega_{\rho\mu}\right] \\ &= \frac{1}{2} \left[T^{\mu\nu} x^{\rho} \omega_{\mu\rho} - T^{\mu\nu} x^{\rho} \omega_{\rho\mu}\right] = \frac{1}{2} \left[T^{\mu\nu} x^{\rho} \omega_{\mu\rho} - T^{\rho\nu} x^{\mu} \omega_{\mu\rho}\right] \\ \rightsquigarrow J^{\nu} &= \frac{1}{2} \omega_{\mu\rho} \mathcal{J}^{\mu\nu\rho} \text{ with } \mathcal{J}^{\mu\nu\rho} \equiv T^{\mu\nu} x^{\rho} - T^{\rho\nu} x^{\mu}. \end{aligned} \tag{3.80}$$

The continuity equations for the conserved currents J^{ν} are,

$$\partial_{\nu} J^{\nu} = 0 \rightsquigarrow \partial_{\nu} J^{\mu\nu\rho} = 0, \tag{3.81}$$

and the corresponding conserved charges are:

$$M^{\mu\rho} = \int d^3\vec{x} J^{\mu 0 \rho} = \int d^3\vec{x} \left[T^{\mu 0} x^{\rho} - T^{\rho 0} x^{\mu}\right]. \tag{3.82}$$

If we associate with $T^{\mu 0}$ the "four-momentum density" $p^\mu = T^{\mu 0}$ of the fields, then we can interpret the above conserved currents as the field "four-angular-momentum". For example, the space components

$$M^{12} = \int d^3\vec{x}\left[T^{10}x^j - T^{20}x^i\right] = -\int d^3\vec{x}\left[\vec{r}\times\vec{p}\right]_3 = -L_3 \tag{3.83}$$

Exercise: *Find the conserved currents for a Lagrangian system of*

1. *a vector field* A^μ
2. *a tensor field* $X^{\mu\nu}$

if the action is invariant under a Lorentz symmetry transformation.

Poincaré symmetry transformations

We now require that a physical system of scalar fields possesses both space-time translation and Lorentz transformation symmetry,

$$x^\mu \to x'^\mu = \epsilon^\mu + \Lambda^\mu_\rho x^\rho. \tag{3.84}$$

The symmetry group of both translations and Lorentz transformations is called the Poincaré group. The charge conservation equations which we have found before should now hold simultaneously:

$$\partial_\nu T^{\mu\nu} = 0, \text{ due to translation symmetry,} \tag{3.85}$$

$$\partial_\nu \mathcal{J}^{\mu\nu\rho} = 0, \text{ due to Lorentz symmetry.} \tag{3.86}$$

This is only possible if the *energy-momentum tensor* $T^{\mu\nu}$ is symmetric. Indeed,

$$\begin{aligned} 0 &= \partial_\nu \mathcal{J}^{\mu\nu\rho} = \partial_\nu\left[T^{\mu\nu}x^\rho - T^{\rho\nu}x^\mu\right] \\ &= x^\rho\partial_\nu T^{\mu\nu} + T^{\mu\nu}\partial_\nu x^\rho - x^\mu\partial_\nu T^{\rho\nu} - T^{\rho\nu}\partial_\nu x^\mu \\ &= x^\rho 0 + T^{\nu\mu}\delta^\rho_\nu - x^\mu 0 - T^{\rho\nu}\delta^\mu_\nu \\ &= T^{\mu\rho} - T^{\rho\mu} \\ &\rightsquigarrow T^{\mu\rho} = T^{\rho\mu}. \end{aligned} \tag{3.87}$$

It is not obvious that the energy-momentum tensor is symmetric from its definition:

$$T^{\mu\nu} \equiv \sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^{(i)}\right)}\left(\partial^\mu\phi^{(i)}\right) - \mathcal{L}g^{\mu\nu} \tag{3.88}$$

does not exhibit an explicit symmetry in the indices μ and ν. However, the energy momentum tensor is not uniquely defined. Let us consider a general tensor of the form

$$\partial_\rho f^{\rho\nu\mu},$$

which is antisymmetric in the indices ν, ρ:

$$f^{\rho\nu\mu} = -f^{\nu\rho\mu}.$$

We can show that such a tensor is a conserved current,

$$\begin{aligned}
\partial_\nu \left(\partial_\rho f^{\rho\nu\mu}\right) &= \frac{1}{2}\left[\partial_\nu \partial_\rho f^{\rho\nu\mu} - \partial_\nu \partial_\rho f^{\nu\rho\mu}\right] \quad \text{(antisymmetry)} \\
&= \frac{1}{2}\left[\partial_\nu \partial_\rho f^{\rho\nu\mu} - \partial_\rho \partial_\nu f^{\rho\nu\mu}\right] \quad \text{(relabelling)} \\
&= \frac{1}{2}\left[\partial_\nu \partial_\rho f^{\rho\nu\mu} - \partial_\nu \partial_\rho f^{\rho\nu\mu}\right] = 0. \quad \text{(commuting derivatives)}
\end{aligned} \tag{3.89}$$

However it produces a null conserved charge, as we can verify easily:

$$\begin{aligned}
Q &= \int d^3\vec{x}\, \partial_\rho f^{\rho 0\mu} \\
&= \int d^3\vec{x}\, \left(\partial_0 f^{00\mu} + \partial_i f^{i0\mu}\right) \\
&= \int d^3\vec{x}\, \left(\partial_0 0 + \nabla_i f^{i0\mu}\right) \\
&= \int d^3\vec{x}\, \nabla_i f^{i0\mu} \quad \text{(using Gauss' theorem)} \\
&= 0.
\end{aligned} \tag{3.90}$$

We can therefore add such a tensor to the energy-momentum tensor,

$$T^{\mu\nu} \to \bar{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho f^{\rho\nu\mu} \tag{3.91}$$

without modifying the corresponding conserved charges (energy and momenta). The function $f^{\rho\nu\mu}$ can be chosen so that $\bar{T}^{\mu\nu}$ is manifestly symmetric in the indices μ, ν.
Exercise: *Consider the Lagrangian for the electromagnetic field:*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.92}$$

Calculate the energy-momentum tensor using Eq. 3.88. Is this symmetric? Is this gauge invariant? Find an appropriate way to symmetrize it. Is the result gauge invariant?

3.4 Field Hamiltonian Density from discretization

Noether's theorem for systems with time-translation invariance gave rise to the Hamiltonian as a conserved quantity. We can confirm here that indeed the expression for the Hamiltonian density of Eq. (3.72) can be derived from the known definitions in classical mechanics of discrete systems by taking the continuous limit.

For a system with a finite number of degrees of freedom, we find the conjugate momentum of a generalised coordinate by differentiating the Lagrangian with respect to the time derivative of the coordinate:

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{3.93}$$

The Hamiltonian of the system is then given by,

$$H = \sum_j p_j \dot{q}_j - L. \tag{3.94}$$

Let us now start from the Lagrangian of a field (in one dimension for simplicity),

$$L = \int dx \mathcal{L}, \tag{3.95}$$

which after discretisation becomes

$$L = \sum_i a L_i, \tag{3.96}$$

where L_i is the value of the Lagrangian density at the point i of the lattice (discretised line in our case). The field $q(x,t)$ has a value $q_i(t)$, at the same point. The corresponding conjugate momentum is,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_i a \frac{\partial L_i}{\partial \dot{q}_j}. \tag{3.97}$$

The Hamiltonian of discretised system is then,

$$H = \sum_i a \left(\sum_j \dot{q}_j \frac{\partial L_i}{\partial \dot{q}_j} - L_i \right). \tag{3.98}$$

We now assume that the discretised Lagrangian at any point i, contains the time derivative of only one degree of freedom $\dot{q}_i$, and no other (e.g. $\dot{q}_{i+1}, \dot{q}_{i-1}, \ldots$):

$$\frac{\partial L_i}{\partial \dot{q}_j} = \delta_{ij} \frac{\partial L_i}{\partial \dot{q}_i}. \tag{3.99}$$

This was also the assumption that we made for the example of the elastic rod in deriving Euler-Lagrange equations and for all physical systems that we will study. It essentially means that each point in the lattice has its own self-determined kinetic energy. On the other hand, the potential term in L_i can depend on the coordinates of its neighbours but we will assume that this dependence is in the form of differences will turn into derivatives at the point i in the continuum limit. The Hamiltonian now becomes,

$$H = \sum_i a \left(\dot{q}_i \frac{\partial L_i}{\partial \dot{q}_i} - L_i \right). \tag{3.100}$$

Finally, by taking the continuum limit $a \to 0$, we obtain a volume integral over a it Hamiltonian density,

$$H = \int dx \mathcal{H} \tag{3.101}$$

with

$$\mathcal{H} = \dot{q}(x,t)\pi(x,t) - \mathcal{L}, \tag{3.102}$$

where the field conjugate momentum defined as,

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{q}(x,t)}. \tag{3.103}$$

3.4.1 Hamilton equations for fields

Let us consider a Lagrangian density $\mathcal{L}[\phi, \partial_\mu \phi]$ for a set of fields $\phi_i(t, \vec{x})$ which take values in three space dimensions. The conjugate momenta of the fields are defined as:

$$\pi_i(t, \vec{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(t, \vec{x})} \tag{3.104}$$

For the physical Lagrangian densities that we will consider, the canonical field momenta depend on the fields and their time derivatives (but not their space derivatives):

$$\pi_i(t, \vec{x}) = \pi_i \left[\phi(t, \vec{x}), \dot{\phi}(t, \vec{x})\right]. \tag{3.105}$$

We can assume that we are able to invert the above equation and cast it in the form:

$$\dot{\phi}_i(t, \vec{x}) = \dot{\phi}_i \left[\phi(t, \vec{x}), \pi(t, \vec{x})\right]. \tag{3.106}$$

Generalising the result of Eq. 3.102, the Hamiltonian density is:

$$\mathcal{H}\left[\phi, \partial_j \phi, \pi\right] = \pi_i \dot{\phi}_i[\phi, \pi] - \mathcal{L}\left[\phi, \partial_j \phi, \dot{\phi}\left[\phi, \pi\right]\right], \tag{3.107}$$

with $\partial_j,\ j = 1 \ldots 3$ denoting spatial derivatives. It is now straightforward (**exercise**) to show that

$$\frac{\partial \mathcal{H}}{\partial \phi_i(t, \vec{x})} = -\frac{\partial \mathcal{L}}{\partial \phi_i(t, \vec{x})} \tag{3.108}$$

and

$$\frac{\partial \mathcal{H}}{\partial \pi_i(t, \vec{x})} = \dot{\phi}_i(t, \vec{x}). \tag{3.109}$$

We can now separate the time and space derivatives in the Euler-Lagrange equations:

$$\partial_0 \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_j \frac{\partial \mathcal{L}}{\partial \left(\partial_j \phi_i\right)} \tag{3.110}$$

Recalling the definition of the canonica momentum, we arrive to Hamilton equations for fields:

$$\dot{\phi}_i(t, \vec{x}) = \frac{\partial \mathcal{H}}{\partial \pi_i(t, \vec{x})}, \quad \dot{\pi}_i(t, \vec{x}) = -\frac{\partial \mathcal{H}}{\partial \phi_i(t, \vec{x})} + \partial_j \frac{\partial \mathcal{H}}{\partial \left(\partial_j \phi_i\right)} \tag{3.111}$$

3.5 An example: acoustic waves

As an example, we use an elastic medium (e.g. air) for acoustic waves. The Lagrangian density is a generalisation in three space dimensions of the elastic rod Lagrangian that we have already studied.

$$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial y(\vec{r},t)}{\partial t}\right)^2 - \frac{\rho v_{sound}^2}{2}\left(\vec{\nabla} y(\vec{r},t)\right)^2, \tag{3.112}$$

with v_{sound} the speed of sound. Setting $v_{sound} = 1$, we write

$$\begin{aligned} \mathcal{L} &= \frac{\rho}{2}\left(\partial_\mu y\right)\left(\partial^\mu y\right) \\ &\equiv \frac{\rho}{2}\left[\left(\partial_{x_0} y\right)^2 - \left(\partial_{x_1} y\right)^2 - \left(\partial_{x_2} y\right)^2 - \left(\partial_{x_3} y\right)^2\right]. \end{aligned} \tag{3.113}$$

We first find the corresponding Euler-Lagrange equations, where we need the derivatives

$$\frac{\partial \mathcal{L}}{\partial y} = 0, \tag{3.114}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu y\right)} = \rho\, \partial^\mu y, \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu y\right)} = \rho\, \partial_\mu \partial^\mu y = \rho \partial^2 y. \tag{3.115}$$

The equations of motion are

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu y\right)} &= \frac{\partial \mathcal{L}}{\partial y} \\ \rightsquigarrow \qquad \partial^2 y &= 0. \end{aligned} \tag{3.116}$$

A solution of the equation of motion is a plane-wave

$$y = e^{ik_\mu x^\mu} \equiv e^{ik\cdot x}, \qquad \text{with} \qquad k_\mu k^\mu = k^2 = 0. \tag{3.117}$$

The conjugate momentum of the field y is,

$$\begin{aligned} \pi \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_0 y\right)} &= \rho \partial_0 y \\ \rightsquigarrow \quad \partial_0 y &= \frac{\pi}{\rho}. \end{aligned} \tag{3.118}$$

The Hamiltonian density is

$$\begin{aligned} &\mathcal{H} \equiv \pi \partial_0 y - \mathcal{L} \\ \rightsquigarrow \quad &\mathcal{H} = \frac{\pi^2}{2\rho} + \frac{1}{2}\rho\left(\vec{\nabla} y\right)^2. \end{aligned} \tag{3.119}$$

Chapter 4

Quantisation of the Schrödinger field

We start our study of field quantisation, by first reproducing the known results of Quantum Mechanics, for a system of many identical non-interacting particles.

4.1 The Schrödinger equation from a Lagrangian density

As a first step, we introduce a Lagrangian that yields the Schrödinger equation for a free particle with the Euler-Lagrange formalism. We can verify that the following density,

$$\mathcal{L} = i\psi^* \frac{\partial \psi}{\partial t} - \frac{\left(\vec{\nabla}\psi^*\right)\left(\vec{\nabla}\psi\right)}{2m} \tag{4.1}$$

does exactly this. ψ and ψ^* turn out to be complex conjugate when they take their physical values, but this is not an assumption that we have to do now ourselves. It will be a natural consequence of solving the Euler-Lagrange equations, which are:

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} - \frac{\partial \mathcal{L}}{\partial \psi} \\ &= \partial_0 \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)} + \partial_i \frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi\right)} - \frac{\partial \mathcal{L}}{\partial \psi} = i\frac{\partial \psi^*}{\partial t} + \frac{\partial_i \partial^i \psi^*}{2m} + 0 \\ \rightsquigarrow 0 &= \left[i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m}\right]\psi^*, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} 0 &= \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^*\right)} - \frac{\partial \mathcal{L}}{\partial \psi^*} \\ &= \partial_0 \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi^*\right)} + \partial_i \frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi^*\right)} - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 + \frac{\partial_i \partial^i \psi}{2m} - i\frac{\partial \psi}{\partial t} \\ \rightsquigarrow 0 &= \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right]\psi. \end{aligned} \tag{4.3}$$

Indeed, we obtain the Schrödinger equation for ψ and its complex conjugate for ψ^*. A general solution of the above equations is a superposition of plane-wave solutions,

$$\psi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, a(\vec{k}) e^{-i\left(\omega_k t - \vec{k}\vec{x}\right)}, \text{ with } \omega_k = \frac{\vec{k}^2}{2m}. \tag{4.4}$$

It is also useful to compute the conjugate momenta for ψ and ψ^* as well as the Hamiltonian density corresponding to the Schrödinger Lagrangian. We find

$$\pi = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi\right)} = i\psi^*, \tag{4.5}$$

and

$$\pi^* = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi^*\right)} = 0. \tag{4.6}$$

The Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \pi\partial_0\psi + \pi^*\partial_0\psi^* - \mathcal{L} \\ \rightsquigarrow \mathcal{H} &= -\frac{\left(\partial_i\psi^*\right)\left(\partial^i\psi\right)}{2m} = \frac{\left(\vec{\nabla}\psi^*\right)\left(\vec{\nabla}\psi\right)}{2m}. \end{aligned} \tag{4.7}$$

Notice that this Hamiltonian density differs from

$$-\frac{\psi^*\nabla^2\psi}{2m} = \frac{\left(\vec{\nabla}\psi^*\right)\left(\vec{\nabla}\psi\right)}{2m} - \frac{\vec{\nabla}\left(\psi^*\vec{\nabla}\psi\right)}{2m}, \tag{4.8}$$

by a total divergence which does not contribute to the Hamiltonian integral $H = \int d^3\vec{x}\mathcal{H}$.

Substituting the general solution of Eq. 4.4 in the expression for the Hamiltonian density, Eq. 4.7, we obtain

$$\mathcal{H} = \int \frac{d^3\vec{k}_1}{(2\pi)^3}\frac{d^3\vec{k}_2}{(2\pi)^3}\frac{\vec{k}_1\vec{k}_2}{2m}\, e^{-i\left[\left(\omega_{k_2}-\omega_{k_1}\right)t-\left(\vec{k}_2-\vec{k}_1\right)\vec{x}\right]}a^*(k_2)a(k_1) \tag{4.9}$$

The Hamiltonian of the Schrödinger field is,

$$H = \int d^3\vec{x}\,\mathcal{H}, \tag{4.10}$$

and we can perform the volume integration easily using $\int d^3\vec{x}e^{i\vec{k}\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k})$. We find the following expression for the Hamiltonian,

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega_k\, a^*(k)a(k). \tag{4.11}$$

4.2 Symmetries of the Schroedinger field

Under the transformation:

$$\psi(\vec{x},t) \to e^{i\theta}\,\psi(\vec{x},t) \quad \psi^*(\vec{x},t) \to e^{-i\theta}\,\psi^*(\vec{x},t). \tag{4.12}$$

the Schrödinger Lagrangian is invariant:

$$\mathcal{L} \to \mathcal{L}. \tag{4.13}$$

In an infinitesimal form we have

$$\delta\psi = i\theta\psi \qquad \delta\psi^* = -i\theta\psi^*. \tag{4.14}$$

Noether's theorem tells us that the following current:

$$J^\mu = (J^0, \vec{J}), \tag{4.15}$$

with

$$J^0 = \frac{\partial \mathcal{L}}{\partial(\partial\partial_0\psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial\partial_0\psi^*)}\delta\psi^* = -i\theta\psi^*\psi \tag{4.16}$$

and

$$J^i = \frac{\partial \mathcal{L}}{\partial(\partial\partial_i\psi)}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial\partial_i\psi^*)}\delta\psi^* = -i\theta\frac{i\left[\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right]}{2m} \tag{4.17}$$

is conserved:

$$\frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0. \tag{4.18}$$

The conserved charge corresponding to this "phase" symmetry transformation is:

$$N = \int d^3\vec{x} J^0 = \int d^3\vec{x}\psi^*(\vec{x},t)\psi(\vec{x},t). \tag{4.19}$$

where we have dropped an unnecessary $-i\theta$ normalization factor. Substituting the solution of Eq. 4.4 of the equations of motion and performing the integrations over the volume we find:

$$N = \int \frac{d^3\vec{k}}{(2\pi)^3} a^*(\vec{k})a(\vec{k}). \tag{4.20}$$

It satisfies that:

$$\frac{dN}{dt} = 0. \tag{4.21}$$

The Lagrangian is also invariant under space-time translations and space rotations. Symmetry under time translations dictates that the Hamiltonian of Eq. 4.11 is also conserved:

$$\frac{dH}{dt} = 0. \tag{4.22}$$

Symmetry under space translations predicts the existence of a conserved field momentum

$$P^i = \int d^3x T^{0i}. \tag{4.23}$$

Explicitly, for the Schrödinger field we find,

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3}\vec{k}\, a^*(\vec{k})a(\vec{k}), \tag{4.24}$$

and it is, of course, time independent:

$$\frac{d\vec{P}}{dt} = 0. \tag{4.25}$$

Exercise:

- *Calculate the components of the energy-momentum tensor for the Scrödinger field*
- *Prove Eq. 4.24 for the total momentum of the Schrödinger field*

- *Show that the Lagrangian for the Schrödinger field is invariant under space rotations. What are the corresponding conserved charges (angular momentum)? Can you find an expression for the field angular momentum which is manifestly time independent?*

- *Calculate the change of the Lagrangian of the Schrödinger field under an infinitesimal Lorentz boost transformation. Prove that also the action is not invariant under boost transformations.*

4.3 Quantisation of Fields

For a system with a finite number of degrees of freedom $L[q_i \dot{q}_i]$, the quantisation procedure is the following. For each generalised coordinate we find first the corresponding conjugate momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{4.26}$$

Then we promote all q_i, p_i to operators, which satisfy commutation relations,

$$[q_i, p_j] = i\delta_{ij}, \tag{4.27}$$

and

$$[p_i, p_j] = [q_i, q_j] = 0. \tag{4.28}$$

We now need to generalize this quantisation procedure to systems of fields, where the number of degrees of freedom is infinite. Viewing fields as the continuum limit of a discrete spectrum of degrees of freedom is particularly useful. In this way, we can make the "discrete to continuum" correspondence:

$$\begin{aligned} j &\rightarrow \vec{x}, \\ q_j(t) &\rightarrow \phi(\vec{x}, t) \\ p_j(t) &\rightarrow \pi(\vec{x}, t). \end{aligned} \tag{4.29}$$

We are now at place to impose quantisation conditions for fields. The canonical prescription for quantising bosonic fields is:

$$[\phi(\vec{x}_1, t), \pi(\vec{x}_2, t)] = i\delta^{(3)}(\vec{x}_1 - \vec{x}_2), \tag{4.30}$$

and

$$[\phi(\vec{x}_1, t), \phi(\vec{x}_2, t)] = [\pi(\vec{x}_1 t), \pi(\vec{x}_2, t)] = 0. \tag{4.31}$$

where the Kronecker symbol of the discrete equations has been replaced by a Dirac delta function.

We remark that our quantisation procedure should not be viewed as a result of a mathematical derivation but rather as an axiom that defines our quantum theory. We shall need to revise this procedure a few times, for example when we quantise fermionic fields and gauge fields with self-interactions (like gluons).

4.4 Quantised Schrödinger field

We can apply the field quantisation conditions of the previous section (Eqs 4.30 - 4.31) to the Schrödinger field $\psi(\vec{x},t)$ and its conjugate momentum $\pi(\vec{x},t) = i\psi^*(\vec{x},t)$. We obtain

$$\begin{aligned} [\psi(\vec{x}_1,t),\pi(\vec{x}_2,t)] &= i\delta^{(3)}(x_1-x_2) \\ \rightsquigarrow [\psi(\vec{x}_1,t),\psi^*(\vec{x}_2,t)] &= \delta^{(3)}(x_1-x_2), \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} [\psi(\vec{x}_1,t),\psi(\vec{x}_2,t)] &= [\pi(\vec{x}_1,t),\pi(\vec{x}_2,t)] = 0 \\ \rightsquigarrow [\psi(\vec{x}_1,t),\psi(\vec{x}_2,t)] &= [\psi^*(\vec{x}_1,t),\psi^*(\vec{x}_2,t)] = 0. \end{aligned} \tag{4.33}$$

We now run into a problem if we impose a quantisation condition for ψ^* and π^*, since we have found $\pi^* = 0$ in Eq. 4.6. The requirement,

$$\delta^{(3)}(\vec{x}_1,\vec{x}_2) = [\psi^*(\vec{x}_1,t),\pi^*\vec{x}_2,t] = [\psi^*(\vec{x}_1,t),0] = 0, \tag{4.34}$$

is clearly inconsistent.

To circumvent the problem we exploit the freedom we have in writing down the Lagrangian of the Schrödinger field in Eq. 4.1. A different Lagrangian which is classically equivalent, yielding identical equations of motion,

$$\mathcal{L}_{new} = \frac{i}{2}\psi^*\frac{\partial\psi}{\partial t} - \frac{i}{2}\psi\frac{\partial\psi^*}{\partial t} - \frac{\left(\vec{\nabla}\psi^*\right)\left(\vec{\nabla}\psi\right)}{2m}. \tag{4.35}$$

This Lagrangian differs from the one of Eq. 4.1 by a time derivative $\frac{i}{2}\partial_t\left(\psi^*\psi\right)$.
Exercise: *Verify that this Lagrangian yields the Scrödinger equation as Euler-Lagrange equations and the conserved quantities of section 4.2*

With $\mathcal{L}_{new}$ both π and π^* are non-vanishing, and all commutation relations that we can write lead to one set of three quantisation conditions which are equivalent [1] with Eqs 4.32 - 4.33. This is an illustrative example showing that not all Lagrangian densities which are equivalent at the classical level, i.e. yield the same equations of motion, can be subjected to a self-consistent quantisation. In our case, $\mathcal{L}_{new}$ can be quantised while $\mathcal{L}$ cannot.

To summarise, the quantisation conditions for the Schrödinger field and its conjugate are:

$$\begin{aligned} [\psi(\vec{x}_1,t),\psi(\vec{x}_2,t)] = [\psi^*(\vec{x}_1,t),\psi^*(\vec{x}_2,t)] = 0, \\ [\psi(\vec{x}_1,t),\psi^*(\vec{x}_2,t)] = \delta^{(3)}\left(\vec{x}_2-\vec{x}_1\right). \end{aligned} \tag{4.36}$$

From these quantisation conditions for the fields, it is easy to derive the corresponding quantisation conditions for the operators $a(\vec{k})$ and $a^*(\vec{k})$. First we observe, from Eq. 4.4 that the operators $a(\vec{k})$ and $a^*(\vec{k})$ are Fourier transforms of ψ and ψ^* respectively,

$$a(\vec{k}) = \int d^3\vec{x}\phi(\vec{x},t)e^{i\left[\omega_k t - \vec{k}\cdot\vec{x}\right]}. \tag{4.37}$$

[1] to obtain the same quantisation conditions we need to impose $[\psi(\vec{x},t),\pi(\vec{y},t)] = \frac{i}{2}\delta^{(3)}(\vec{x}-\vec{y})$ and $[\psi^*(\vec{x},t),\pi^*(\vec{y},t)] = \frac{i}{2}\delta^{(3)}(\vec{x}-\vec{y})$.

Exercise: *Verify explicitly the above by substituting in the expression of Eq. 4.4 for the field ϕ.*
Then, we have

$$
\begin{aligned}
&\left[a(\vec{k}_1), a^*(\vec{k}_2)\right] \\
&= \int d^3\vec{x}_1 d^3\vec{x}_2 e^{i\left[\omega_{k_1}t - \vec{k}_1\cdot\vec{x}_1\right]} e^{-i\left[\omega_{k_2}t - \vec{k}_2\cdot\vec{x}_2\right]} \left[\phi(\vec{x}_1, t), \phi^*(\vec{x}_2, t)\right] \\
&= \int d^3\vec{x}_1 d^3\vec{x}_2 e^{i\left[\omega_{k_1}t - \vec{k}_1\cdot\vec{x}_1\right]} e^{-i\left[\omega_{k_2}t - \vec{k}_2\cdot\vec{x}_2\right]} \delta^{(3)}\left(\vec{x}_2 - \vec{x}_1\right) \\
&= \int d^3\vec{x} e^{i\left[\left(\omega_{k_1} - \omega_{k_2}\right)t - \left(\vec{k}_1 - \vec{k}_2\right)\cdot\vec{x}\right]} \\
\rightsquigarrow \quad & \left[a(\vec{k}_1), a^*(\vec{k}_2)\right] = (2\pi)^3 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right).
\end{aligned}
\tag{4.38}
$$

Similarly (**exercise**), we can prove that

$$
\left[a(\vec{k}_1), a(\vec{k}_2)\right] = \left[a^*(\vec{k}_1), a^*(\vec{k}_2)\right] = 0. \tag{4.39}
$$

To emphasise that a^* and ψ^* are now operators, we write

$$
a^* \to a^\dagger, \psi^* \to \psi^\dagger.
$$

4.5 Particle states from quantised fields

Let us now look at the so called "number density" operator

$$
\mathcal{N}(k) \equiv a^\dagger(k)a(k). \tag{4.40}
$$

We can prove that

$$
\begin{aligned}
&[\mathcal{N}(k), a(p)] = \left[a^\dagger(k)a(k), a(p)\right] \\
&= a^\dagger(k)\left[a(k), a(p)\right] + \left[a^\dagger(k), a(p)\right] a(k) \\
\rightsquigarrow \quad & [\mathcal{N}(k), a(p)] = -(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})a(p).
\end{aligned}
\tag{4.41}
$$

Also, in the same manner we find that

$$
\left[\mathcal{N}(k), a^\dagger(p)\right] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})a^\dagger(p). \tag{4.42}
$$

It is also easy to prove ((**exercise**)) that two number density operators commute with each other,

$$
\begin{aligned}
&[\mathcal{N}(k), \mathcal{N}(p)] = \left[N(k), a^\dagger(p)a(p)\right] \\
&= \left[\mathcal{N}(k), a^\dagger(p)\right] a(p) + a^\dagger(p)\left[\mathcal{N}(k), a(p)\right] \\
&= (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})a^\dagger(p)a(p) - (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})a^\dagger(p)a(p) \\
\rightsquigarrow \quad & [\mathcal{N}(k), \mathcal{N}(p)] = 0.
\end{aligned}
\tag{4.43}
$$

We recall that the Hamiltonian is (Eq. 4.11),

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega_k\, a^\dagger(k)a(k) \rightsquigarrow H = \int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega_k\, \mathcal{N}(k). \tag{4.44}$$

Obviously, it commutes with the number density operator,

$$[\mathcal{N}(p), H] = \left[\mathcal{N}(p), \int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega_k\, \mathcal{N}(k)\right] = \int \frac{d^3\vec{k}}{(2\pi)^3}\,\omega_k\, [\mathcal{N}(p), \mathcal{N}(k)] = 0. \tag{4.45}$$

As a consequence, the eigenstates of the Hamiltonian can be found as eigenstates of the "number density" operator, or the "particle number operator", defined as

$$\hat{N} \equiv \int \frac{d^3\vec{k}}{(2\pi)^3}\, N(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, a^\dagger(k)a(k), \tag{4.46}$$

and which also obviously commutes with the Hamiltonian. This is not a surprise, since we have found earlier that at the classical level this quantity is conserved due to the phase symmetry of the Lagrangian.

From Eqs 4.41-4.42, we derive that

$$\left[\hat{N}, a^\dagger(p)\right] = +a^\dagger(p), \tag{4.47}$$

and

$$\left[\hat{N}, a(p)\right] = -a(p). \tag{4.48}$$

If a state $|n\rangle$ is eigenstate of the number operator $\hat{N}$ with eigenvalue n,

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \tag{4.49}$$

then the states $a^\dagger(p)\,|n\rangle$ and $a(p)\,|n\rangle$ are also eigenstates with eigenvalues $n+1$ and $n-1$ correspondingly. Indeed,

$$\begin{aligned} \hat{N}\left(a^\dagger(p)\,|n\rangle\right) &= \left\{\left[\hat{N}, a^\dagger(p)\right] + a^\dagger(p)\hat{N}\right\}|n\rangle \\ \rightsquigarrow\quad \hat{N}\left(a^\dagger(p)\,|n\rangle\right) &= \left\{+a^\dagger(p) + a^\dagger(p)n\right\}|n\rangle \\ \rightsquigarrow\quad \hat{N}\left(a^\dagger(p)\,|n\rangle\right) &= (n+1)\left(a^\dagger(p)\,|n\rangle\right), \end{aligned} \tag{4.50}$$

and

$$\begin{aligned} \hat{N}\left(a(p)\,|n\rangle\right) &= \left\{\left[\hat{N}, a(p)\right] + a(p)\hat{N}\right\}|n\rangle \\ \rightsquigarrow\quad \hat{N}\left(a(p)\,|n\rangle\right) &= \left\{-a(p) + a(p)n\right\}|n\rangle \\ \rightsquigarrow\quad \hat{N}\left(a(p)\,|n\rangle\right) &= (n-1)\left(a(p)\,|n\rangle\right). \end{aligned} \tag{4.51}$$

We can prove that the eigenvalues of the number operator must always be positive, if the corresponding eigenstates are states of a Hilbert space with a positive norm. Consider the state $|\omega\rangle = a(p)\,|n\rangle$,

$$\begin{aligned} & 0 \le ||\,|\omega\rangle\,||^2 = \langle\omega|\,\omega\rangle = \langle n|\, a^\dagger(p)a(p)\,|n\rangle = \langle n|\,\mathcal{N}(p)\,|n\rangle \\ \rightsquigarrow\quad & 0 \le \langle n|\left(\int \frac{d^3\vec{p}}{(2\pi)^3}\mathcal{N}(p)\right)|n\rangle = \langle n|\,\hat{N}\,|n\rangle = n\,\langle n|\,n\rangle = n||\,|n\rangle\,||^2 \\ \rightsquigarrow\quad & \qquad n \ge 0. \end{aligned} \tag{4.52}$$

This condition is in an apparent contradiction with Eq. 4.48, since a repeated application of the operator $a(p)$ on a positive eigenvalue will eventually yield an eigenstate with a negative eigenvalue. This is true , if all eigenstates are not annihilated by the $a(p)$ operator. Assume, however, that there is a state $|0\rangle$ which is annihilated by $a(p)$,

$$a(p)|0\rangle = 0. \tag{4.53}$$

This state is an eigenstate of the number operator with a zero eigenvalue:

$$\begin{aligned} a(p)|0\rangle &= 0 \\ \rightsquigarrow a^\dagger(p)a(p)|0\rangle &= 0|0\rangle \rightsquigarrow \left(\int \frac{d^3\vec{p}}{(2\pi)^3} a^\dagger(p)a(p)\right)|0\rangle = 0|0\rangle \\ \rightsquigarrow \hat{N}|0\rangle &= 0|0\rangle \end{aligned} \tag{4.54}$$

Then, Eq. 4.53 only permits to act on the vacuum state with $a^\dagger$ operators, producing eigenstates of the number operator with positive integer eigenvalues. For example, for a general state

$$|1_{particle}\rangle \equiv \int \frac{d^3\vec{p_1}}{(2\pi)^3} f(p_1)a^\dagger(p_1)|0\rangle, \tag{4.55}$$

we have (**exercise**):

$$\hat{N}|1_{particle}\rangle = (+1)|1_{particle}\rangle. \tag{4.56}$$

Similarly, for a general state:

$$|2_{particles}\rangle \equiv \int \frac{d^3\vec{p_1}}{(2\pi)^3}\frac{d^3\vec{p_2}}{(2\pi)^3} f(p_1,p_2)a^\dagger(p_1)a^\dagger(p_2)|0\rangle, \tag{4.57}$$

we have (**exercise**):

$$\hat{N}|2_{particles}\rangle = (+2)|2_{particles}\rangle, \tag{4.58}$$

and so on:

$$\hat{N}|m_{particles}\rangle = (+m)|m_{particles}\rangle, \tag{4.59}$$

with

$$|m_{particles}\rangle \equiv \int \prod_{i=1}^{m}\left(\frac{d^3\vec{p_i}}{(2\pi)^3}a^\dagger(p_i)\right)|0\rangle f(p_1,\ldots,p_m). \tag{4.60}$$

Suggestively, we have labeled the eigenstates of the number operator $\hat{N}$, as states of a discrete number of particles. This becomes better justified when we find common eigenstates with the Hamiltonian H and the field momentum operators. The states of Eq. 4.60 with $f(p_1,\ldots,p_m) = \prod_i (2\pi)^3\delta^{(3)}(p_i - k_i), i = 1\ldots m$:

$$|\vec{p}_1,\ldots,\vec{p}_m\rangle \equiv \left(\prod_{i=1}^{m} a^\dagger(k_i)\right)|0\rangle \tag{4.61}$$

diagonalize the Hamiltonian,

$$H|\vec{p}_1,\ldots,\vec{p}_m\rangle = E|\vec{p}_1,\ldots,\vec{p}_m\rangle, \quad \text{with } E = \sum_{i=1}^{m}\omega_{k_i} \tag{4.62}$$

Exercise:*Show the above for m=1 and m=2. Get convinced that it is valid for arbitrary m.*

It is easy to see that a state $|\vec{p}_1, \ldots, \vec{p}_m\rangle$ of Eq. 4.61 is also an eigenstate of the field momentum operator:

$$\vec{P}\,|\vec{p}_1, \ldots, \vec{p}_m\rangle = \vec{p}\,|\vec{p}_1, \ldots, \vec{p}_m\rangle\,, \quad \text{with } \vec{p} = \sum_{i=1}^{m} \vec{k}_i \tag{4.63}$$

Exercise:*Show the above for m=1 and m=2. Get convinced that it is valid for arbitrary m.*

To summarize, we have proven that the states of Eq. 4.61 are simultaneous eigenstates of the "number-operator", the field Hamiltonian operator, and the field momentum operator. Their eigenvalues are the number of particles, the sum of their energies, and the number of their momenta respectively. We therefore see that the quantisation of the Schrödinger field is a very elegant formalism to describe multi-particle states.

Exercise:*Show that an n-particle eigenstate of the Hamiltonian for a free Schrödinger field is also an eigenstate of an angular momentum operator.*

Exercise: *We can write the Schrödinger field as*

$$\psi(\vec{x}, t) = \rho(\vec{x}, t) e^{-i\phi(\vec{x},t)}, \psi(\vec{x}, t)^* = \rho(\vec{x}, t) e^{i\phi(\vec{x},t)}, \tag{4.64}$$

where ρ, ϕ are real fields.

- *Cast the Lagrangian in terms of the ρ, ϕ fields.*
- *Write down their equations of motion.*
- *What is the Hamiltonian in terms of the ϕ, ρ fields?*
- *Attempt to carry out the quantisation of the theory imposing commutation relations on the ρ, ϕ fields and their conjugate momenta.*

Exercise: *Find the energy, momentum and particle number operator for the following Lagrangian:*

$$\mathcal{L}_{new} = \frac{i}{2}\psi^* \frac{\partial \psi}{\partial t} - \frac{i}{2}\psi \frac{\partial \psi^*}{\partial t} - \frac{\left(\vec{\nabla}\psi^*\right)\left(\vec{\nabla}\psi\right)}{2m} + \lambda \psi^* \psi. \tag{4.65}$$

Show that they commute and find their common eigenstates.

4.6 What is the wave-function in the field quantisation formalism?

In quantum mechanics, the wave function $y(\vec{x}, t)$[2] corresponding to a single particle state is the probability amplitude to find a particle at a position $\vec{x}$,

$$y(\vec{x}, t) = \langle 1_{particle} |\, \vec{x}\rangle\,. \tag{4.66}$$

[2]In this section, we use the letter y to denote the wave-function, and the letter ψ for the Schrödinger field operator.

For a system of N particles the wave-function is the probability amplitude to find the N particles at positions $x_1, x_2, \ldots x_N$.

$$y(\vec{x}_1, \ldots \vec{x}_N, t) = \langle \vec{x}_1, \ldots, \vec{x}_N | \, N_{particles} \rangle \, . \tag{4.67}$$

The wave-function satisfies the Schrödinger equation, which for free particles is:

$$\left[i \frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{\vec{\nabla}^2_{x_i}}{2m} \right] y(\vec{x}_1, \ldots \vec{x}_N, t) = 0. \tag{4.68}$$

Let us now look at a generic N-particle state of the previous section, Eq. 4.60,

$$|N_{particles}\rangle = \int \left(\prod_{i=1}^{N} \frac{d^3\vec{p}_i}{(2\pi)^3} a^\dagger(p_i) \right) |0\rangle \, f(p_1, \ldots, p_N), \tag{4.69}$$

where we can express the creation operators $a^\dagger(p_i)$ as Fourier transforms of the field operators $\phi^\dagger(\vec{x}, t)$. Substituting Eq. 4.37 into Eq. 4.69, we obtain

$$|N_{particles}\rangle = \int \left(\prod_{i=1}^{N} d^3\vec{x}_i \right) |S(x_1, \ldots, x_N)\rangle \, \tilde{f}(x_1, \ldots, x_N, t), \tag{4.70}$$

with,

$$|S(x_1, \ldots, x_N)\rangle = \left(\prod_{i=1}^{N} \psi^\dagger(x_i) \right) |0\rangle \, , \tag{4.71}$$

and

$$\tilde{f}(x_1, \ldots, x_N, t) = \int \left(\frac{d^3\vec{p}_i}{(2\pi)^3} e^{-i\left(\omega_{p_i} t - \vec{p}_i \vec{x}_i\right)} \right) f(p_1, \ldots, p_N). \tag{4.72}$$

By an explicit calculation (**exercise**), we can verify that

$$\left[i \frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{\vec{\nabla}^2_{x_i}}{2m} \right] \tilde{f}(\vec{x}_1, \ldots \vec{x}_N, t) = 0, \tag{4.73}$$

i.e. it satisfies the Schrödinger equation. Therefore the function $\tilde{f}$, is identified as the wave-function $y(x_1, \ldots x_N, t)$,

$$\tilde{f}(x_1, \ldots x_N, t) = y(x_1, \ldots x_N, t) = \langle \vec{x}_1, \ldots, \vec{x}_N | \, N_{particles} \rangle \, . \tag{4.74}$$

Using this observation, we write Eq.4.70 as

$$\begin{aligned} |N_{particles}\rangle &= \int \left(\prod_{i=1}^{N} d^3\vec{x}_i \right) |S(x_1, \ldots, x_N)\rangle \, \langle \vec{x}_1, \ldots, \vec{x}_N | \, N_{particles} \rangle \, , \\ \rightsquigarrow |N_{particles}\rangle &= \left[\int \left(\prod_{i=1}^{N} d^3\vec{x}_i \right) |S(x_1, \ldots, x_N)\rangle \, \langle \vec{x}_1, \ldots, \vec{x}_N | \right] |N_{particles}\rangle \end{aligned} \tag{4.75}$$

Inside the square bracket we have a unit operator, which leads us to the conclusion that

$$|S(x_1, \ldots, x_N)\rangle = |\vec{x}_1, \ldots, \vec{x}_N\rangle \, . \tag{4.76}$$

We summarize the main result of this section, which connect the field operator formalism with the traditional wave-function description of non-relativistic Quantum Mechanics.

- A quantum mechanical state of N particles at positions $\vec{x}_1, \ldots \vec{x}_N$ is given by the action of field operators on the vacuum state:

$$|\vec{x}_1, \ldots \vec{x}_N\rangle = \left(\prod_{i=1}^{N} \psi^\dagger(x_i) \right) |0\rangle \, . \tag{4.77}$$

- A general $N-$particle state is a superposition,

$$|N_{particles}\rangle = \int d^3\vec{x}_1 \ldots d^3\vec{x}_1 \tilde{f}(x_1, \ldots, x_N) \, |\vec{x}_1, \ldots \vec{x}_N\rangle \, . \tag{4.78}$$

- The kernel of the superposition integral is the wave-function, satisfying the Schrödinger equation

$$\left[i\frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{\vec{\nabla}^2_{x_i}}{2m} \right] \tilde{f}(\vec{x}_1, \ldots \vec{x}_N, t) = 0. \tag{4.79}$$

Chapter 5

The Klein-Gordon Field

In the study of the Schrödinger field, we demonstrated how Quantum Field theory can be made to construct quantum states of particles, with certain momentum and an energy momentum relation:

$$E = \frac{\vec{p}^2}{2m}. \tag{5.1}$$

This is the correct energy-momentum relation in the non-relativistic limit. Would it be possible to modify this procedure in such a way so that we obtain the relativistic energy-momentum relation

$$E^2 = \vec{p}^2 + m^2? \tag{5.2}$$

5.1 Real Klein-Gordon field

Let us consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2. \tag{5.3}$$

where ϕ is a real scalar field. This Lagrangian is invariant under Lorentz transformations and therefore consistent with special relativity. We will attempt to write a quantum relativistic theory of free scalar particles (spin-0), by quantising the field ϕ.

5.1.1 Real solution of the Klein-Gordon equation

The Euler-Lagrangian equation for the field ϕ of Eq. 5.3 is,

$$\partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\rightsquigarrow \qquad \left[\partial^2 + m^2\right] \phi(x) = 0. \tag{5.4}$$

This is known as the Klein-Gordon equation.

We shall find a general solution of the Klein-Gordon equation for a real field ϕ. We start with the ansatz:

$$\phi(x) = \int \frac{d^4 k}{(2\pi)^4} \left[f(k) e^{-ik\cdot x} + f^*(k) e^{+ik\cdot x} \right], \tag{5.5}$$

which is manifestly real. In addition, it is invariant under Lorentz transformations, as it is anticipated for a scalar field. Substituting into the Klein-Gordon equation we find,

$$\int \frac{d^4k}{(2\pi)^4} \left(m^2 - k^2\right) \left[f(k)e^{-ik\cdot x} + f^*(k)e^{+ik\cdot x}\right] = 0. \tag{5.6}$$

The above is satisfied if we have,

$$f(k) = (2\pi)\delta(k^2 - m^2)c(k), \qquad f^*(k) = (2\pi)\delta(k^2 - m^2)c^*(k). \tag{5.7}$$

A general real solution of the Klein-Gordon equation is:

$$\phi(x^\mu) = \int \frac{d^4k}{(2\pi)^3} \delta\left(m^2 - k^2\right) \left[c(k)e^{-ik\cdot x} + c^*(k)e^{+ik\cdot x}\right]. \tag{5.8}$$

The delta function requires that

$$k^2 = k_\mu k^\mu = m^2 \rightsquigarrow k_0 = \pm\omega_k, \quad \text{with } \omega_k = \sqrt{\vec{k}^2 + m^2}. \tag{5.9}$$

The delta function can be written as,

$$\delta(k^2 - m^2) = \delta(k_0^2 - \omega_k^2) = \frac{\delta(k_0 - \omega_k) + \delta(k_0 + \omega_k)}{2\omega_k}. \tag{5.10}$$

We can now perform the k_0 integration in Eq. 5.8, obtaining

$$\begin{aligned} \phi(x^\mu) \;=\; \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \Big[& c(\omega_k, \vec{k})e^{-i\left(\omega_k t - \vec{k}\vec{x}\right)} + c(-\omega_k, \vec{k})e^{+i\left(\omega_k t + \vec{k}\vec{x}\right)} \\ & + c^*(\omega_k, \vec{k})e^{+i\left(\omega_k t - \vec{k}\vec{x}\right)} + c^*(-\omega_k, \vec{k})e^{-i\left(\omega_k t + \vec{k}\vec{x}\right)} \Big] \end{aligned} \tag{5.11}$$

We perform a change of integration $\vec{k} \to -\vec{k}$ to the integrals corresponding to the second and fourth term of the sum inside the square bracket. Then, the four terms can be grouped together into a sum of only two exponentials, and we can write the general solution as:

$$\phi(x^\mu) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \left[a(\vec{k})e^{-i\left(\omega_k t - \vec{k}\vec{x}\right)} + a^*(\vec{k})e^{i\left(\omega_k t - \vec{k}\vec{x}\right)}\right], \tag{5.12}$$

where $a(\vec{k}) = c(\omega_k, \vec{k}) + c^*(-\omega_k, -\vec{k})$.

We remark that we often write the integration measure in an explicitly Lorentz invariant form,

$$\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\Theta(k_0 > 0) = \int \frac{d^4k}{(2\pi)^3}\delta^{(+)}(k^2 - m^2), \tag{5.13}$$

with

$$\delta^{(+)}(k^2 - m^2) \equiv \delta(k^2 - m^2)\Theta(k_0 > 0). \tag{5.14}$$

The measure is invariant under orthochronous transformations which do not change the sign of the time-like component of the integration variable.

5.1.2 Quantitation of the real Klein-Gordon field

The conjugate momentum of the Klein-Gordon field, $\phi(\vec{x}, t)$, is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi. \tag{5.15}$$

The Hamiltonian of the Klein-Gordon field is,

$$H = \int d^3\vec{x} \frac{(\partial_0 \phi)^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2}{2}. \tag{5.16}$$

Substituting into the Hamiltonian integral the solution of Eq. 5.12, and performing the integration over the space volume, we find:

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right], \tag{5.17}$$

where we have promoted the field into an operator. We now impose commutation relations for the field and its conjugate momentum,

$$[\phi(\vec{x}_1, t), \phi(\vec{x}_2, t)] = [\pi(\vec{x}_1, t), \pi(\vec{x}_2, t)] = 0, \tag{5.18}$$

and

$$[\phi(\vec{x}_1, t), \pi(\vec{x}_2, t)] = i\delta^{(3)}\left(\vec{x}_2 - \vec{x}_1\right). \tag{5.19}$$

With a straightforward calculation, we can verify (**exercise**) that the operators a and $a^\dagger$ are written in terms of the ϕ and π operators as,

$$a(k) = i \int d^3\vec{x} e^{i\left(\omega_k t - \vec{k}\vec{x}\right)} \left[\pi(\vec{x}, t) - i\omega_k \phi(\vec{x}, t)\right], \tag{5.20}$$

and

$$a^\dagger(k) = -i \int d^3\vec{x} e^{-i\left(\omega_k t - \vec{k}\vec{x}\right)} \left[\pi(\vec{x}, t) + i\omega_k \phi(\vec{x}, t)\right]. \tag{5.21}$$

Then we can compute the commutators that can be formed with a and $a^\dagger$. We find (**exercise**),

$$[a(k_1), a(k_2)] = \left[a^\dagger(k_1), a^\dagger(k_2)\right] = 0, \tag{5.22}$$

and

$$\left[a(k_1), a^\dagger(k_2)\right] = (2\pi)^3(2\omega_{k_1})\delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right). \tag{5.23}$$

5.1.3 Particle states for the real Klein-Gordon field

We can now construct the particle number operator,

$$\hat{N} = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} a^\dagger(k)a(k), \tag{5.24}$$

for which we can prove that

$$\left[\hat{N}, a(p)\right] = -a(p), \tag{5.25}$$

and

$$\left[\hat{N}, a^\dagger(p)\right] = +a^\dagger(p). \tag{5.26}$$

Therefore, the operators $a(p)$ and $a^\dagger(p)$ are ladder operators. In particular, if a state $|\mathbf{S}\rangle$ is an eigenstate of the number operator $\hat{N}$ with eigenvalue n, then the states $a(p)\,|\mathbf{S}\rangle$ and $a^\dagger(p)\,|\mathbf{S}\rangle$ are also eigenstates with eigenvalues $n-1$ and $n+1$ respectively. The construction of particle states proceeds similarly to the quantisation for the Schrödinger field, after we observe that the field Hamiltonian and the number operator commute (**exercise**):

$$\left[\hat{N}, H\right] = 0. \tag{5.27}$$

The above equations lead to the existence of a vacuum state $|0\rangle$, which is annihilated by the operator $a(k)$,

$$a(k)\,|0\rangle = 0. \tag{5.28}$$

The vacuum state is an eigenstate of the number operator with a zero eigenvalue,

$$\hat{N}\,|0\rangle = 0\,|0\rangle\,. \tag{5.29}$$

All other states can be produced from the vacuum state, by acting on it with *creation* operators $a^\dagger(p_i)$. A generic state

$$|\Psi_m\rangle \equiv \left(\prod_{i=1}^{m} a^\dagger(p_i)\right)|0\rangle\,, \tag{5.30}$$

contains m particles,

$$\hat{N}\,|\Psi_m\rangle = m\,|\Psi_m\rangle\,. \tag{5.31}$$

5.1.4 Energy of particles and "normal ordering"

We consider the simplest case for a $|\Psi_m\rangle$ state,

$$|p\rangle \equiv a^\dagger(p)\,|0\rangle\,, \tag{5.32}$$

which is an one-particle state, satisfying

$$\hat{N}\,|p\rangle = 1\,|p\rangle\,. \tag{5.33}$$

We can compute the energy level of the state by acting on it with the field Hamiltonian operator,

$$\begin{aligned}
H\,|p\rangle &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\,\frac{\omega_k}{2}\left\{a^\dagger(k)a(k) + a(k)a^\dagger(k)\right\}|p\rangle \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\,\frac{\omega_k}{2}\left\{2a^\dagger(k)a(k) + \left[a(k), a^\dagger(k)\right]\right\}|p\rangle \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\,\frac{\omega_k}{2}\left\{2a^\dagger(k)a(k) + (2\pi)^3\,2\omega_k\delta^{(3)}(0)\right\}|p\rangle \\
\rightsquigarrow H\,|p\rangle &= \left(\omega_p + \delta^{(3)}(0)\left[\int d^3\vec{k}\frac{\omega_k}{2}\right]\right)|p\rangle\,.
\end{aligned} \tag{5.34}$$

To our surprise, we find an infinite eigenvalue! This is at first sight very embarrassing.

We can make sense of this infinity, if we look at the energy of the vacuum state, which contains no particles.

$$
\begin{aligned}
H|0\rangle &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left\{a^\dagger(k)a(k) + a(k)a^\dagger(k)\right\}|0\rangle \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left\{2a^\dagger(k)a(k) + \left[a(k), a^\dagger(k)\right]\right\}|0\rangle \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left\{2a^\dagger(k)a(k) + (2\pi)^3\, 2\omega_k \delta^{(3)}(0)\right\}|0\rangle \\
&\rightsquigarrow \; H|0\rangle = \delta^{(3)}(0)\left[\int d^3\vec{k}\frac{\omega_k}{2}\right]|0\rangle \\
&\rightsquigarrow \; \delta^{(3)}(0)\left[\int d^3\vec{k}\frac{\omega_k}{2}\right] = \langle 0|H|0\rangle = E_{\text{vacuum}}.
\end{aligned} \tag{5.35}
$$

We find that the field Hamiltonian gives an energy eigenvalue for the vacuum state in Eq. 5.35 which is equal to the same infinite constant that appeared in the energy eigenvalue (Eq. 5.34) of the one-particle state. We can then re-write Eq. 5.34 in an apparently innocent manner,

$$H|p\rangle = \left(\sqrt{\vec{p}^2 + m^2} + E_{\text{vacuum}}\right)|p\rangle . \tag{5.36}$$

Using the property (**exercise**)

$$\left[H, a^\dagger(p_i)\right] = +\omega(p_i)a^\dagger(p_i), \tag{5.37}$$

we can prove that states with more particles, have an energy,

$$H|\vec{p}_1, \dots, \vec{p}_m\rangle = \left[E_{\text{vacuum}} + \sum_i \sqrt{\vec{p}_i^2 + m^2}\right]|\vec{p}_1, \dots, \vec{p}_m\rangle . \tag{5.38}$$

All Hamiltonian eigenvalues contain an identical infinite constant which is equal to the vacuum energy. However, measurements of absolute energy levels are not possible. The vacuum energy is harmless if we want to measure the energy difference of the states $|\Psi_m\rangle$ from the vacuum. We can then "recalibrate" our energy levels (by an infinite constant) removing from the Hamiltonian operator the energy of the vacuum:

$$H \rightarrow : H :\equiv H - E_{\text{vacuum}} = H - \langle 0|H|0\rangle . \tag{5.39}$$

Explicitly, the Hamiltonian operator after we subtract the vacuum constant can be written as

$$: \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] := \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\omega_k\left[a^\dagger(k)a(k)\right]. \tag{5.40}$$

A practical trick to remove from a conserved operator, such as the Hamiltonian, its vacuum expectation value:

$$: \mathcal{O} :\equiv \mathcal{O} - \langle 0|\mathcal{O}|0\rangle \tag{5.41}$$

is to perform what is known as "normal ordering". As you can see in Eq. 5.40, this is simply achieved by putting creation operators to the left of annihilation operators when these refer to the same momentum $\vec{k}$. With this practical rule,

$$: \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] := a^\dagger(k)a(k) + a^\dagger(k)a(k) = 2a^\dagger(k)a(k). \tag{5.42}$$

This is equivalent to using the commutation relation $[a(k), a^\dagger(k)] = (2\pi)^3 2\omega_k \delta^{(3)}(0)$ in order to put the creation operators to the right and dropping infinities $\delta^{(3)}(0)$. Normal ordering leaves annihilation operators to the right, and guarantees that the vacuum is automatically annihilated by the modified operator:

$$: \mathcal{O} : |0\rangle = 0. \tag{5.43}$$

5.1.5 Field momentum conservation

Noether's theorem predicts that the field momentum is a conserved operator, due to the symmetry of the Klein-Gordon action under space translations. The momentum operator is given by,

$$\begin{aligned}
\vec{P} &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \frac{\vec{k}}{2} \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \vec{k} \left[a^\dagger(k)a(k)\right] + \frac{\delta^{(3)}(0)}{2} \int d^3\vec{k}\, \vec{k} \\
\rightsquigarrow \vec{P} &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \vec{k} \left[a^\dagger(k)a(k)\right].
\end{aligned} \tag{5.44}$$

Here, the infinity $\delta^{(3)}(0)$ cancels because the momentum integral multiplying the delta-function term vanishes due to the antisymmetry of the integrand under the transformation $\vec{k} \to -\vec{k}$. $\vec{P}$ commutes with both the Hamiltonian and the number operator, with which it has common eigenstates.

The momentum of the vacuum state is zero,

$$\vec{P}\,|0\rangle = 0. \tag{5.45}$$

The $\vec{P}$ eigenvalues of multi-particle states can be easily found by using,

$$\left[\vec{P}, a^\dagger(p)\right] = +\vec{p}\,a^\dagger(p). \tag{5.46}$$

For example, its eigenvalue for the one-particle state $|p\rangle = a^\dagger(p)\,|0\rangle$ is

$$\vec{P}\,|p\rangle = \left[\vec{P}, a^\dagger(p)\right]|0\rangle = \vec{p}\,a^\dagger(p)\,|0\rangle = \vec{p}\,|p\rangle\,. \tag{5.47}$$

For a multi-particle state

$$|\vec{p}_1, \ldots, \vec{p}_m\rangle \equiv \prod_{i=1}^{m} a_i^\dagger(\vec{p}_i)\,|0\rangle\,, \tag{5.48}$$

we have

$$\vec{P}\,|\vec{p}_1, \ldots, \vec{p}_m\rangle = \left(\sum_{i=1}^{m} \vec{p}_i\right)|\vec{p}_1, \ldots, \vec{p}_m\rangle\,. \tag{5.49}$$

5.1.6 Labels of particle states?

The main outcome of quantising the real Klein-Gordon field, is a spectrum of states which can be identified as states of many particles with definite number of particles, energy and momentum:

$$|\vec{p}_1, \ldots, \vec{p}_m\rangle \equiv \left(\prod_{i=1}^{n} a^\dagger(p_i) \right) |0\rangle \,. \tag{5.50}$$

Such a state describes m identical particles,

$$\hat{N} |\vec{p}_1, \ldots, \vec{p}_m\rangle = m \, |\vec{p}_1, \ldots, \vec{p}_m\rangle \,, \tag{5.51}$$

each of them carrying momentum $\vec{p}_i$,

$$\vec{P} |\vec{p}_1, \ldots, \vec{p}_m\rangle = \left(\sum_{i=1}^{n} \vec{p}_i \right) |\vec{p}_1, \ldots, \vec{p}_m\rangle \,, \tag{5.52}$$

and having a relativistic energy $+\sqrt{\vec{p}_i^2 + m^2}$,

$$H |\vec{p}_1, \ldots, \vec{p}_m\rangle = \left(\sum_{i=1}^{n} \sqrt{\vec{p}_i^2 + m^2} \right) |\vec{p}_1, \ldots, \vec{p}_m\rangle \,. \tag{5.53}$$

We have therefore arrived to a consistent combination of quantum mechanics and special relativity, where field quanta give rise to particles with the correct properties as anticipated from relativity.

5.2 Casimir effect: the energy of the vacuum

A remarkable consequence of the quantisation of the Klein-Gordon field is that a state with no particles, the vacuum $|0\rangle$, has energy. For the real Klein-Gordon field, for example, we find that the vacuum energy is:

$$E_{\text{vacuum}} = \delta^{(3)}(0) \int d^3\vec{k} \frac{\omega_k}{2}. \tag{5.54}$$

We can associate the infinity $\delta^3(0)$, with the volume of our physical system. We have considered that the classical solutions for the fields extend to an infinite volume, and we have performed all space integrations from $-\infty$ to ∞. If we were to consider a finite volume, this delta function would be replaced by the volume of integration. Recall that,

$$\int d^3\vec{x} e^{-i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}). \tag{5.55}$$

For $\vec{k} = 0$, we have

$$V = \int d^3\vec{x} = (2\pi)^3 \delta^{(3)}(\vec{0}), \tag{5.56}$$

and the density of the vacuum energy can be cast in the physically more appealing form,

$$\epsilon = \frac{E_{\text{vacuum}}}{V} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\sqrt{m^2 + \vec{k}^2}}{2}. \tag{5.57}$$

A second source of infinity in the expression for the vacuum energy is the integration over all frequency modes $\omega_k = \sqrt{m^2 + \vec{k}^2}$.

The value of the energy of the vacuum does not enter physical predictions (if one neglects gravitational effects) and we may not worry that it comes out to be infinite. But we should be able to see that the vacuum energy is different if, for a reason, the fields vanish in some region of the space-volume and/or some frequencies ω_k do not contribute to the vacuum. This can be achieved, if the field is forced to satisfy some boundary conditions.

Let us assume, that the Klein-Gordon field is forced to vanish on the planes $x = 0$ and $x = L$,

$$\phi(t, x = 0, y, z) = \phi(t, x = L, , y, z) = 0. \tag{5.58}$$

The general field solution of the Klein-Gordon equation which in addition satisfies the boundary conditions is:

$$\phi(x^\mu) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \int \frac{d^2\vec{k}_\perp}{(2\pi)^2 2\omega_k} \left[a(\vec{k})e^{-i\left(k^0 t - \vec{k}_\perp \cdot \vec{x}_\perp\right)} + a^\dagger(\vec{k})e^{i\left(k^0 t - \vec{k}_\perp \cdot \vec{x}_\perp\right)}\right], \tag{5.59}$$

with

$$\omega_k = \sqrt{m^2 + \vec{k}_\perp^2 + \frac{n^2\pi^2}{L^2}}. \tag{5.60}$$

and

$$\vec{x}_\perp = (y, z), \quad \vec{k} = \left(\frac{n\pi}{L}, \vec{k}_\perp\right), \quad \vec{k}_\perp = (k_y k_z) \tag{5.61}$$

Notice that we only integrate over the directions k_y, k_z of the momenta parallel to the plates, and we sum over the values of k_x which is is discretized.

As in the absence of boundary conditions, the quantization conditions for the real Klein-Gordon field are given by Eqs. (5.18)- (5.19). For the creation and annihilation operators, this yields (**exercise**) the commutation relations

$$\left[a\left(\frac{n\pi}{L}, \vec{k}_\perp\right), a\left(\frac{r\pi}{L}, \vec{l}_\perp\right)\right] = 0 = \left[a^\dagger\left(\frac{n\pi}{L}, \vec{k}_\perp\right), a^\dagger\left(\frac{r\pi}{L}, \vec{l}_\perp\right)\right] \tag{5.62}$$

$$\left[a\left(\frac{n\pi}{L}, \vec{k}_\perp\right), a^\dagger\left(\frac{r\pi}{L}, \vec{l}_\perp\right)\right] = \frac{2}{L}\delta_{nr}(2\pi)^2(2\omega_k)\delta^{(2)}\left(\vec{k}_\perp - \vec{l}_\perp\right) \tag{5.63}$$

Substituting the solution for the field of Eq. (5.59) and the for the corresponding field conjugate momentum into the Hamiltonian integral of Eq. (5.16) we find (**exercise**)

$$H = \frac{L}{2}\sum_{n=1}^{\infty} \frac{d^2\vec{k}_\perp}{(2\pi)^2\, 2\omega_k} \frac{\sqrt{m^2 + \vec{k}_\perp^2 + \frac{n^2\pi^2}{L^2}}}{2} \left\{a\left(\frac{n\pi}{L}, \vec{k}_\perp\right) a^\dagger\left(\frac{n\pi}{L}, \vec{k}_\perp\right) + a^\dagger\left(\frac{n\pi}{L}, \vec{k}_\perp\right) a\left(\frac{n\pi}{L}, \vec{k}_\perp\right)\right\} \tag{5.64}$$

The vacuum energy per unit surface of the planes is (**exercise**),

$$\frac{E_{\text{vacuum}}}{S} = \sum_{n=1}^{\infty} \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \frac{\sqrt{\frac{\pi^2 n^2}{L^2} + \vec{k}_\perp^2 + m^2}}{2}, \tag{5.65}$$

In polar coordinates,

$$\frac{E_{\text{vacuum}}}{S} = \sum_{n=1}^{\infty} \int \frac{dk_\perp k_\perp}{(2\pi)} \frac{\sqrt{\frac{\pi^2 n^2}{L^2} + k_\perp^2 + m^2}}{2}. \tag{5.66}$$

This integral appears to be divergent in the limit $k_\perp \to \infty$. For simplicity, we will study it in the mass to zero $m = 0$ limit.

Let us now work with a slightly modified integrand,

$$\frac{E_{\text{vacuum}}}{S} = \sum_{n=1}^{\infty} \int \frac{dk_\perp k_\perp^{1-\delta}}{(2\pi)} \frac{\sqrt{\frac{\pi^2 n^2}{L^2} + k_\perp^2}}{2}. \tag{5.67}$$

There exists a value of δ for which the integral is well defined. We shall perform our calculation for such a value, and then we shall try to analytically continue the result to $\delta = 0$. We change variables once again,

$$k_\perp = l_\perp \frac{n\pi}{L}, \tag{5.68}$$

and we obtain,

$$\begin{aligned} \frac{E_{\text{vacuum}}}{S} &= \frac{1}{4\pi}\left(\frac{\pi}{L}\right)^{3-\delta}\left(\sum_{n=1}^{\infty} n^{3-\delta}\right)\int_0^\infty dl_\perp l_\perp^{1-\delta}\sqrt{1+l_\perp^2} \\ \rightsquigarrow \epsilon &= \frac{1}{8\pi}\left(\frac{\pi}{L}\right)^{3-\delta}\left(\sum_{n=1}^{\infty}\frac{1}{n^{-3+\delta}}\right)\int_0^\infty dl_\perp^2 (l_\perp^2)^{-\frac{\delta}{2}}\sqrt{1+l_\perp^2} \end{aligned} \tag{5.69}$$

We recognize the sum as the ζ-function,

$$\sum_{n=1}^{\infty} \frac{1}{n^{-3+\delta}} = \zeta(\delta - 3). \tag{5.70}$$

Performing the change of variables,

$$l_\perp^2 \to \frac{x}{1-x}, \tag{5.71}$$

we find that the integral is,

$$\int_0^\infty dl_\perp^2 (l_\perp^2)^{-\frac{\delta}{2}}\sqrt{1+l_\perp^2} = \int_0^1 dx x^{-\frac{\delta}{2}}(1-x)^{\frac{\delta-5}{2}} = B(1-\delta/2, -3/2+\delta/2). \tag{5.72}$$

where we have recognized the beta function,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{5.73}$$

The final result reads,

$$\frac{E_{\text{vacuum}}}{S} = \frac{1}{8\pi}\left(\frac{\pi}{L}\right)^{3-\delta} B(1-\delta/2, -3/2+\delta/2)\zeta(\delta-3). \tag{5.74}$$

Amazingly the limit $\delta \to 0$ exists and it is finite. We find,

$$\frac{E_{\text{vacuum}}}{S} = -\frac{\pi^2}{1440 L^3}. \tag{5.75}$$

The vacuum energy depends on the distance between the two planes, on which the field vanishes. Can we realize this in an experiment?

The electromagnetic field is zero inside a conductor. If we place two "infinite" conducting sheets parallel to each other at a distance L, then we can reproduce the boundary conditions of the setup that we have just studied. The quantisation of the electromagnetic field is rather more complicated than the Klein-Gordon field, but as far as the vacuum energy is concerned, we shall see that they are very similar.

Our analysis, leads to an amazing prediction. Two electrically neutral conductors attract each other. This is known as the Casimir effect. Notice, that the energy of the vacuum E_{vacuum} gets smaller when the conducting plates are closer. Therefore, there is an attractive force between them. The force per unit area (pressure or rather anti-pressure) between the two sheets which set a zero boundary condition for a real Klein-Gordon field is

$$\mathcal{F} = -\frac{\partial \frac{E_{\text{vacuum}}}{S}}{\partial L} = -\frac{\pi^2}{480 L^4}. \tag{5.76}$$

For the experimentally relevant case of the electromagnetic field the pressure is twice the answer we have just found of the real Klein-Gordon field (due to the two physical polarizations of the electromagnetic field). The Casimir effect has been tested experimentally. You can find a review *here*.

Some physics subtleties have been "swept under the carpet" in the above calculation by using a too convenient regularization method for the integrals which may be rather misleading physically. For example, our regularization of the energy integral subtracts, secretely, an infinity which corresponds to the vacuum energy in the absence of boundary conditions. Indeed, computing the integral of Eq. (5.57) with the regularization method introduced here and setting $m = 0$ gives a zero, rather than an infinite, answer. Issues of this type will become clearer when we introduce properly the method of dimensional regularization. For a more rigorous derivation of the Casimir effect with various boundary conditions we refer to this publication *here*.

5.3 Two real Klein-Gordon fields

Our task is to describe all known particles and their interactions. It is then interesting to study the quantisation of a system with more than one field. We can start simply, describing a system of two Klein-Gordon fields which they differ only in their mass parameter,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi_1\right)\left(\partial^\mu \phi_1\right) - \frac{m_1^2}{2}\phi_1^2 \\ &+ \frac{1}{2}\left(\partial_\mu \phi_2\right)\left(\partial^\mu \phi_2\right) - \frac{m_2^2}{2}\phi_2^2. \end{aligned} \tag{5.77}$$

This Lagrangian yields two Klein-Gordon equations as equations of motion for the fields ϕ_1 and ϕ_2:

$$\left(\partial^2 + m_1^2\right)\phi_1 = 0, \qquad \left(\partial^2 + m_2^2\right)\phi_2 = 0, \tag{5.78}$$

with solutions

$$\phi_1(x^\mu) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_1}\left[a_1(\vec{k})e^{-ik\cdot x} + a_1^\dagger(\vec{k})e^{ik\cdot x}\right], \tag{5.79}$$

$$\phi_2(x^\mu) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_2}\left[a_2(\vec{k})e^{-ik\cdot x} + a_2^\dagger(\vec{k})e^{ik\cdot x}\right], \tag{5.80}$$

and

$$\omega_1 = \sqrt{\vec{k}^2 + m_1^2}, \qquad \omega_2 = \sqrt{\vec{k}^2 + m_2^2}. \tag{5.81}$$

The Hamiltonian H, the field momentum $\vec{P}$ and the number operator $\hat{N}$ can be written as

$$\hat{N} = \hat{N}_1 + \hat{N}_2 \qquad \hat{N}_i = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_i} a_i^\dagger(\vec{k}) a_i(\vec{k}), \tag{5.82}$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2 \qquad \vec{P}_i = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_i} \frac{\vec{k}}{2} \left\{ a_i^\dagger, a_i \right\}, \tag{5.83}$$

$$H = H_1 + H_2 \qquad H_i = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_i} \frac{\omega_i}{2} \left\{ a_i^\dagger, a_i \right\}. \tag{5.84}$$

where the anticommutator of two operators is defined as:

$$\{A, B\} = AB + BA. \tag{5.85}$$

We can construct particle states in the same fashion as with the Lagrangian of just a single Klein-Gordon field. Products of $a_1^\dagger$ operators acting on the vacuum state create relativistic particles with mass m_1, while $a_2^\dagger$ operators create particles with mass m_2. For example, the states

$$|S_1\rangle \equiv a_1^\dagger(p) |0\rangle, \qquad |S_2\rangle \equiv a_2^\dagger(p) |0\rangle \tag{5.86}$$

have

$$\hat{N} |S_i\rangle = +1 |S_i\rangle, \tag{5.87}$$

$$\vec{P} |S_i\rangle = +\vec{p} |S_i\rangle, \tag{5.88}$$

$$H |S_1\rangle = \sqrt{\vec{p}^2 + m_1^2} |S_1\rangle \quad \text{and} \quad H |S_2\rangle = \sqrt{\vec{p}^2 + m_2^2} |S_2\rangle. \tag{5.89}$$

They are degenerate in that they are single particle states with the same momentum $\vec{p}$; however, as long as the masses are different $m_1 \neq m_2$, these two states can be distinguished by measuring the energy of the particles.

The possibility of $m_1 = m_2$ is interesting; if in multi-particle state we replace an $a_1^\dagger$ by an $a_2^\dagger$ operators or vice versa the new state is degenerate in all three conserved quantities: the number of particles, the energy and the momentum.

5.3.1 Two equal-mass real Klein-Gordon fields

The special case of $m_1 = m_2 = m$ merits a detailed study, because a new rotation symmetry emerges in the space of fields ϕ_1 and ϕ_2. This leads, according to Noether's theorem, to a conserved quantity which can be interpreted as the charge of particles and antiparticles.

We rewrite the Lagrangian in a form where the symmetry is manifest:

$$\mathcal{L} = \frac{1}{2} \left(\partial^\mu \vec{\phi} \right)^T \left(\partial_\mu \vec{\phi} \right) - \frac{m}{2} \vec{\phi}^T \vec{\phi}, \tag{5.90}$$

where we have defined the field "vector"

$$\vec{\phi} \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ with its transpose } \vec{\phi}^T = \begin{pmatrix} \phi_1 , & \phi_2 \end{pmatrix}. \tag{5.91}$$

Obviously, the Lagrangian is invariant (**exercise**) if we perform an orthogonal transformation,

$$\vec{\phi} \to \vec{\phi}' = R\vec{\phi}, \text{ with } R^T = R^{-1}. \tag{5.92}$$

To compute the conserved current we need to find the change $\delta\phi_i$ of the fields under an infinitesimal symmetry transformation. We then expand,

$$R_{ij} = \delta_{ij} + \theta_{ij} + \mathcal{O}(\theta^2). \tag{5.93}$$

Orthogonality of R implies that the matrix θ_{ij} is antisymmetric,

$$R^T_{ij} = R^{-1}_{ij} \rightsquigarrow \delta_{ji} + \theta_{ji} = \delta_{ij} - \theta_{ij} \rightsquigarrow \theta_{ij} = -\theta_{ji}. \tag{5.94}$$

The variation of the fields is:

$$\begin{aligned} \phi_1 \quad &\to \quad \phi_1' = R_{1j}\phi_j \\ &\qquad = (\delta_{1j} + \theta_{1j})\,\phi_j = \phi_1 + \theta_{11}\phi_1 + \theta_{12}\phi_2 \\ &\qquad = \phi_1 + \theta_{12}\phi_2 \\ \rightsquigarrow \delta\phi_1 \quad &= \quad \theta_{12}\phi_2, \end{aligned} \tag{5.95}$$

and, similarly,

$$\delta\phi_2 = \theta_{21}\phi_1 = -\theta_{12}\phi_1. \tag{5.96}$$

The Noether current is,

$$\begin{aligned} J^\mu \quad &= \quad \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_1\right)}\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_2\right)}\delta\phi_2 \\ \rightsquigarrow J^\mu \quad &= \quad \theta_{12}\left[\left(\partial^\mu\phi_1\right)\phi_2 - \left(\partial^\mu\phi_2\right)\phi_1\right] \end{aligned} \tag{5.97}$$

and the "charge"

$$Q = \int d^3\vec{x}\left[\left(\partial^\mu\phi_1\right)\phi_2 - \left(\partial^\mu\phi_2\right)\phi_1\right], \tag{5.98}$$

is conserved. Substituting in the above expressions the physical solutions for the fields ϕ_i, and performing the $\vec{x}$ integration we obtain (**exercise**):

$$Q = -i\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left[a_1(k)a_2^\dagger(k) - a_2(k)a_1^\dagger(k)\right]. \tag{5.99}$$

A Noether charge is not defined uniquely. For example, we can multiply it with a constant multiplicative factor. Obviously, if Q is a time-independent (conserved) quantity, then also λQ, where λ is a number constant, is also conserved. The normalization in Eq. 5.99 is convenient yielding a Hermitian Q (**exercise**):

$$Q^\dagger = Q. \tag{5.100}$$

We now proceed to find the eigenstates of the Q charge operator. This becomes easy, if we construct ladder operators for Q. We compute first the commutators of the charge with a_1 and a_2 (**exercise**). We find

$$[Q, a_1(p)] = -ia_2(p), \tag{5.101}$$

and

$$[Q, a_2(p)] = +ia_1(p). \tag{5.102}$$

Define the linear combinations,

$$a(p) \equiv \frac{a_1(p) + ia_2(p)}{\sqrt{2}}, \tag{5.103}$$

and

$$b(p) \equiv \frac{a_1(p) - ia_2(p)}{\sqrt{2}}. \tag{5.104}$$

These are indeed ladder operators (**exercise**), satisfying the following commutation relations with the charge Q:

$$[Q, a(p)] = -a(p), \qquad [Q, b(p)] = +b(p). \tag{5.105}$$

By taking the hermitian conjugate of the above equations, and using $Q^\dagger = Q$, we find that

$$\left[Q, a^\dagger(p)\right] = +a^\dagger(p), \qquad \left[Q, b^\dagger(p)\right] = -b^\dagger(p). \tag{5.106}$$

From a state $|S\rangle$ with charge q,

$$Q\,|S\rangle = q\,|S\rangle\,, \tag{5.107}$$

we can obtain states with charges $q \pm 1$ by applying the ladder operators $a^\dagger(p)$ and $b^\dagger(p)$ (**exercise**),

$$Q\left(a^\dagger(p)\,|S\rangle\right) = (q+1)\left(a^\dagger(p)\,|S\rangle\right), \tag{5.108}$$

$$Q\left(b^\dagger(p)\,|S\rangle\right) = (q-1)\left(b^\dagger(p)\,|S\rangle\right). \tag{5.109}$$

We remark that the operators $a^\dagger, b^\dagger$ are also ladder operators for the Hamiltonian H and the field momentum operator $\vec{P}$. This is a consequence of the fact that $a^\dagger$ and $b^\dagger$ are linear combinations of $a_1^\dagger$ and $a_2^\dagger$.

We now construct the common eigenstates of the charge operator Q, the Hamiltonian H, the momentum operator $\vec{P}$ and the number operator $\hat{N}$. We first observe that the vacuum state is also an eigenstate of the charge operator (**exercise**):

$$Q\,|0\rangle = 0. \tag{5.110}$$

Repeated application of the $a^\dagger$ operator on the vacuum, builds states of positive charge:

$$\left|\Psi_n^{(+)}\right\rangle \equiv \left(\prod_{i=1}^{n} a^\dagger(p_i)\right), \tag{5.111}$$

with

$$\hat{N}\left|\Psi_n^{(+)}\right\rangle = n\left|\Psi_n^{(+)}\right\rangle \tag{5.112}$$

$$\vec{P}\left|\Psi_n^{(+)}\right\rangle = \left(\sum_{i=1}^{n}\vec{p_i}\right)\left|\Psi_n^{(+)}\right\rangle \tag{5.113}$$

$$H\left|\Psi_n^{(+)}\right\rangle = \left(\sum_{i=1}^{n}\sqrt{\vec{p_i}^{\,2} + m^2}\right)\left|\Psi_n^{(+)}\right\rangle \tag{5.114}$$

$$Q\left|\Psi_n^{(+)}\right\rangle = +n\left|\Psi_n^{(+)}\right\rangle \tag{5.115}$$

Repeated application of the $b^\dagger$ operator on the vacuum, builds states of positive charge:

$$\left|\Psi_n^{(-)}\right\rangle \equiv \left(\prod_{i=1}^{n} b^\dagger(p_i)\right), \tag{5.116}$$

with

$$\hat{N}\left|\Psi_n^{(-)}\right\rangle = n\left|\Psi_n^{(-)}\right\rangle \tag{5.117}$$

$$\vec{P}\left|\Psi_n^{(-)}\right\rangle = \left(\sum_{i=1}^{n}\vec{p_i}\right)\left|\Psi_n^{(-)}\right\rangle \tag{5.118}$$

$$H\left|\Psi_n^{(-)}\right\rangle = \left(\sum_{i=1}^{n}\sqrt{\vec{p_i}^2+m^2}\right)\left|\Psi_n^{(-)}\right\rangle \tag{5.119}$$

$$Q\left|\Psi_n^{(-)}\right\rangle = -n\left|\Psi_n^{(-)}\right\rangle. \tag{5.120}$$

To summarize the main results of this section, mass degeneracy resulted to a new $O(2)$ symmetry of the Lagrangian. This gives rise to a new conserved quantity, the charge Q. A particle state, is now characterized by its momentum, the mass of the particle (or equivalently the energy), and its charge which can be either positive or negative. We have just demonstrated a method to describe physical systems of particles which are accompanied by their anti-particles.

5.3.2 Two real Klein-Gordon fields = One complex Klein-Gordon field

We can recast the original expressions for the operators whose eigenvalues characterize particle $(\hat{N}, \vec{P}, H, Q)$ in terms of the charge ladder operators a, b, rather than a_1, a_2. Substituting,

$$a_1 = \frac{a+b}{\sqrt{2}}, \qquad a_2 = \frac{a-b}{\sqrt{2}i}, \tag{5.121}$$

we find

$$\hat{N} = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right], \tag{5.122}$$

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\frac{\vec{k}}{2}\left[\{a^\dagger, a\} + \{b^\dagger, b\}\right] \tag{5.123}$$

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\frac{\omega_k}{2}\left[\{a^\dagger, a\} + \{b^\dagger, b\}\right]. \tag{5.124}$$

We can now re-write the field operators in terms of a and b. We find that

$$\phi_1 = \frac{\phi+\phi^\dagger}{\sqrt{2}}, \qquad \phi_2 = \frac{\phi+\phi^\dagger}{\sqrt{2}i}, \tag{5.125}$$

with

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left[a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x}\right], \qquad k^0 = \omega_k = \sqrt{\vec{k}^2+m^2}. \tag{5.126}$$

We can cast the Lagrangian representing the fields in terms of ϕ and $\phi^\dagger$, rather than ϕ_1 and ϕ_2. We find,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^T\left(\partial_\mu \vec{\phi}\right) - \frac{1}{2}m^2 \vec{\phi}^T \phi = \left(\partial_\mu \phi\right)\left(\partial_\mu \phi^\dagger\right) - m^2 \phi\phi^\dagger, \tag{5.127}$$

where

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \text{ and } \phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}. \tag{5.128}$$

The Lagrangian in the complex field representation, is more suggestive to the fact that it yields particle and anti-particle states. We could compute the Hamiltonian and momentum operators directly in terms of ϕ and $\phi^\dagger$ arriving to the same expressions as in the representation with two real fields. To compute the charge Q, we would need to identify what is the symmetry of this new Lagrangian. It is perhaps already obvious that the Lagrangian is invariant under a field phase-redefinition,

$$\phi \to e^{-i\alpha}\phi, \quad \phi^\dagger \to e^{+i\alpha}\phi^\dagger. \tag{5.129}$$

This is the equivalent of the rotation symmetry transformation that we have found earlier, in the complex field representation. Let us verify this, by performing a rotation on the vector $\vec{\phi}$,

$$\begin{aligned} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &\to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ \rightsquigarrow \begin{pmatrix} \phi_1 \\ i\phi_2 \end{pmatrix} &\to \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \phi_1 \\ i\phi_2 \end{pmatrix} \\ \rightsquigarrow \phi_1 + i\phi_2 &\to e^{-i\theta}\left(\phi_1 + i\phi_2\right) \\ \rightsquigarrow \phi \quad &to \quad e^{-i\theta}\phi. \end{aligned} \tag{5.130}$$

Exercise: *We have just observed that the U(1) group of unitary transformations of* 1×1 *complex matrices is isomorphic to the SO(2) group of special rotations. Prove that for the following groups there exists a* $2 \to 1$ *homomorphic mapping:*

- *SU(2)* $\to$ *SO(3),*
- *SU(2)* $\times$ *SU(2)* $\to$ *SO(4).*

5.4 Conserved Charges as generators of symmetry transformations

It may occur that we know a conserved charge but we do not know the symmetry transformation from which it originates. It turns out, that we can use the conserved charge in order to uncover the unknown symmetry.

Let us assume that a Lagrangian is symmetric under a generic field transformation,

$$\phi_i \to \phi_i + \delta\phi_i \tag{5.131}$$

leading to a conserved charge

$$Q = \int d^3x \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi_i\right)}\delta\phi_i = \int d^3x \sum_i \pi_i \delta\phi_i. \tag{5.132}$$

The commutator of the charge operator Q with a field operator ϕ_j is,

$$\begin{aligned}[Q,\phi_j(x)] &= \int d^3y \sum_i [\pi_i(y)\delta\phi_i(y),\phi_j(x)] \\ &= \int d^3y \sum_i ([\pi_i(y),\phi_j(x)]\,\delta\phi_i(y)+\pi_i(y)\,[\delta\phi_i(y),\phi_j(x)]) \\ &= \int d^3y \sum_i \left(i\delta_{ij}\delta^{(3)}(\vec{y}-\vec{x})\delta\phi_j(x)+0\right) \\ &\rightsquigarrow [Q,\phi_j(x)] = i\delta\phi_j(x). \end{aligned} \tag{5.133}$$

Exercise: *Verify the above equation for the commutators* $[Q,\phi]$ *and* $\left[Q,\phi^\dagger\right]$ *of the complex Klein-Gordon field.*

Exercise: *In the case of the real Klein-Gordon field we found that the number operator* $\hat{N}$ *is time independent. How does the Klein-Gordon field transform under the corresponding symmetry transformation?*

5.5 Can the Klein-Gordon field be an one-particle wave-function?

In the previous chapter, we performed the quantisation of the Schrödinger field. We found out that field quantisation, although an elegant formalism, was not indispensable. In fact, we connected the results of quantum field theory with a traditional quantum mechanics in which Schrödinger "field" is the wave-function. It is a question worth answering whether field quantisation is truly necessary for a relativistic field or it is just an alternative and perhaps superfluous formalism.

An one-particle wave-function, as a probability amplitude, is in general complex. Let us then interpret the complex Klein-Gordon field as a wave-function. Recall that the Lagrangian is written as,

$$\mathcal{L}_{KG} = (\partial_\mu\psi)(\partial_\mu\psi^*) - m^2\psi\psi^*. \tag{5.134}$$

We denote the field with ψ in order to emphasize our intention to consider it a wave-function. As such, ψ^* is its complex conjugate and not a hermitian conjugate operator.

The complex Klein-Gordon field Lagrangian and the Schrödinger field Lagrangian,

$$\mathcal{L}_S = \psi^* i\partial_t\psi + \frac{|\nabla\psi|^2}{2m}, \tag{5.135}$$

are both symmetric under a phase transformation, as we have discussed in detail. In the Schrödinger case, this lead to a conserved charge, which we identified it with the probability for the particle to be anywhere in space,

$$\text{Total Probability } = \int d^3\vec{x}\,|\psi(\vec{x},t)|^2 = \text{constant}. \tag{5.136}$$

Therefore, we could also interpret the quantity,

$$\rho \equiv |\psi(\vec{x},t)|^2, \tag{5.137}$$

as a probability density.

An attempt to repeat the same steps for the case of the Klein-Gordon field fails badly. The "total probability" integral is now the charge,

$$Q = \int d^3\vec{x} \left[\psi(\vec{x},t)\dot{\psi}^*(\vec{x},t) - \dot{\psi}(\vec{x},t)\psi^*(\vec{x},t) \right], \tag{5.138}$$

with a "density"

$$\rho \equiv \psi(\vec{x},t)\dot{\psi}^*(\vec{x},t) - \dot{\psi}(\vec{x},t)\psi^*(\vec{x},t), \tag{5.139}$$

which is not positive definite, and thus physically unacceptable.

A second embarrassment from the Klein-Gordon equation as a relativistic wave-function equation, is that it predicts particles with a negative energy. This may be accepted for a free particle, if we assume that for whatever reason we can only "see" particles with positive energy. However, if we couple these particles to photons, as it is required for all particles with electromagnetic interactions, spontaneous emission of photons will turn all particles into negative energy particles. Clearly, a single particle wave-function interpretation of the Klein-Gordon field is wrong and it cannot describe physical particles. Quantum Field Theory is the only consistent approach.

Chapter 6

Quantisation of the free electromagnetic field

We will now study the quantisation of the electromagnetic field in empty space. One of the predictions of quantum field theory will be that electromagnetic waves are packaged in photons which travel with the speed of light. Imposing quantisation conditions will not be as straightforward as in the klein-Gordon field due to the transversality of the polarisation of the photons. We will then need to define carefully what are states of physical polarisations. In this chapter we will first review the salient features of classical electrodynamics and then proceed to the quantum theory.

6.1 Maxwell Equations and Lagrangian formulation

The equations of Maxwell are:

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{6.1}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{6.2}$$

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{6.3}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}. \tag{6.4}$$

We now introduce the four-vector,

$$A^\mu \equiv \left(\phi, \vec{A}\right), \tag{6.5}$$

where the components are chosen such as,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{6.6}$$

and

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi. \tag{6.7}$$

Then, the first two Maxwell equations (Eq. 6.1, Eq. 6.2) are automatically satisfied,

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0, \tag{6.8}$$

and

$$\vec{\nabla} \times \left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \right) + \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{\nabla}\phi \right) = \epsilon_{ijk} \partial_i \partial_j \phi = 0. \tag{6.9}$$

We now define the tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{6.10}$$

The components of $F^{\mu\nu}$ are,

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \left(\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\phi \right)^{(i)} = -E^i, \tag{6.11}$$

and

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon^{ijk} B^k. \tag{6.12}$$

All other components are zero, since $F^{\mu\nu}$ is anti-symmetric. In a matrix form we have,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{6.13}$$

This is called the *Electromagnetic field tensor.*

Define now the current four-vector,

$$j^\nu \equiv \left(\rho, \vec{j} \right). \tag{6.14}$$

We can easily verify that the remaining Maxwell equations are given by the compact equation,

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{6.15}$$

For $\nu = 0$,

$$\begin{aligned} & \partial_\mu F^{\mu 0} = \rho \\ \rightsquigarrow \quad & \partial_0 F^{00} + \sum_{i=1}^{3} \partial_i F^{i0} = \rho \\ \rightsquigarrow \quad & \vec{\nabla} \cdot \vec{E} = \rho. \end{aligned} \tag{6.16}$$

For $\nu = 1$,

$$\begin{aligned} & \partial_\mu F^{\mu 1} = j^1 \\ \rightsquigarrow \quad & \partial_0 F^{01} + \sum_{i=1}^{3} \partial_i F^{i1} = j^1 \\ \rightsquigarrow \quad & -\frac{\partial E^1}{\partial t} + \frac{\partial B^3}{\partial x_2} - + \frac{\partial B^2}{\partial x_3} = j^1, \end{aligned} \tag{6.17}$$

which is the first component of the vector equation

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \vec{j}. \tag{6.18}$$

Maxwell equations, expressed in terms of the vector potential A^μ, take the form,

$$\begin{aligned} & \partial_\mu F^{\mu\nu} = 0 \\ \rightsquigarrow \quad & \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0 \\ \rightsquigarrow \quad & \partial^2 A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = 0. \end{aligned} \tag{6.19}$$

6.1.1 Classical gauge invariance and gauge-fixing

Maxwell equations, Eq. 6.15, are invariant under *gauge transformations*, where the vector potential transforms as

$$A^\mu \to A^{\mu\prime} = A^\mu + \partial^\mu \chi, \tag{6.20}$$

or, in components,

$$\phi \to \phi' = \phi + \dot{\chi} \tag{6.21}$$

$$\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\chi. \tag{6.22}$$

χ is an arbitrary function of space-time coordinates. Indeed, the field tensor $F^{\mu\nu}$ is invariant under this transformation,

$$\begin{aligned} F^{\mu\nu} \to F^{\mu\nu\prime} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu \\ &= \partial^\mu \left(A^\nu + \partial^\nu \chi\right) - \partial^\nu \left(A^\mu + \partial^\mu \chi\right) \\ &= F^{\mu\nu}. \end{aligned} \tag{6.23}$$

We can often simplify our calculations if we exploit gauge invariance. Suppose that we are given a field A^μ satisfying Maxwell equations Eq. 6.19. We can "fix the gauge", i.e. perform a special gauge transformation, $A^\mu \to A^{\mu\prime} = A^\mu + \partial_\mu \chi$, such that the second term in Eq. 6.19 disappears,

$$0 = \partial_\mu A^{\mu\prime} = \partial_\mu A^\mu + \partial^2 \chi \rightsquigarrow \partial^2 \chi = -\partial_\mu A^\mu. \tag{6.24}$$

This particular choice of gauge is known as the "Lorentz gauge". In the Lorentz gauge, Maxwell equations take the very simple form (dropping the prime)

$$\partial^2 A^\mu = j^\mu. \tag{6.25}$$

In the vacuum, $j^\nu = 0$, we have

$$\partial^2 A^\mu = 0. \tag{6.26}$$

This equation is particularly easy to solve as plane-waves, and we write the general real classical field solution as an integral

$$A_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_{\lambda=0} \epsilon_\mu^{(\lambda)}(k) \left[a_\lambda(k) e^{-ik\cdot x} + a_\lambda^*(k) e^{ik\cdot x}\right], \tag{6.27}$$

with $k^\mu = \left(\omega_k, \vec{k}\right)$ and $\omega_k = \left|\vec{k}\right|$. We have introduced a basis of four-vectors which determine the direction of the electromagnetic field (polarisation). For example, these vectors can be taken as:

$$\epsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.28}$$

As we can easily verify, the polarisation vectors are normalised as,

$$g^{\mu\nu}\epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda')} = g^{\lambda\lambda'}. \quad g_{\lambda\lambda'}\epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda')} = g_{\mu\nu}. \tag{6.29}$$

Exercise: *Verify the above equations.*

6.1.2 Lagrangian of the electromagnetic field

The following Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{6.30}$$

gives rise to the Maxwell equations as classical equations of motion. Explicitly,

$$\mathcal{L} = -\frac{1}{2}\left[\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right)\right]. \tag{6.31}$$

It is easy to verify that the Euler-Lagrange equations are (**exercise**)

$$\begin{aligned} &\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \\ \rightsquigarrow\ &\partial^2 A^\nu - \partial^\nu\left(\partial_\mu A^\mu\right) = 0. \end{aligned} \tag{6.32}$$

6.2 Quantisation of the Electromagnetic Field

The conjugate momentum for the field components A^μ are,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\mu\right)} = F^{\mu 0}. \tag{6.33}$$

One would like to impose the bosonic quantisation condition,

$$\left[A^\mu(x), \pi^\nu(x')\right] = i g^{\mu\nu}\delta^{(3)}\left(\vec{x} - \vec{x}'\right). \tag{6.34}$$

Notice the presence of $g^{\mu\nu}$, which is needed in order to maintain covariance on the indices μ, ν of the quantisation condition. We immediately come across a problem. For $\mu = 0$,

$$\pi^0 = F^{00} = 0,$$

we encounter an inconsistency

$$0 = \left[A^0(x), \pi^0(x')\right] = i\delta^{(3)}(\vec{x} - \vec{x}') \neq 0. \tag{6.35}$$

Our first attempt to quantising the classical Lagrangian of the electromagnetic field has failed. However, it is possible to quantise a different Lagrangian which nevertheless gives rise to the same physics as the Lagrangian of Eq. 6.30. We can exploit the fact that classical physics is the same in every gauge for A^μ.

Consider a modified Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2. \tag{6.36}$$

This yields the following Euler-Lagrange equations (which are not invariant under gauge transformations):

$$\partial^2 A^\mu - (1-\lambda)\,\partial^\mu\left(\partial_\nu A^\nu\right) = 0. \tag{6.37}$$

Notice that for a special value $\lambda = 1$, the equations of motion are identical to Maxwell equations, $\partial^2 A^\mu = 0$, in the Lorentz gauge. For the new Lagrangian and with $\lambda = 1$, we find

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu, \tag{6.38}$$

which is not zero, as long as we do not impose the Lorentz condition $\partial_\mu A^\mu = 0$. This is at first irreconcilable with classical physics, however there is a way out. We may think of the classical electromagnetic field as the expectation value of a quantum field in a physical quantum state $|\psi\rangle$, i.e.

$$A^\mu_{\text{classic}} \sim \langle\psi|\, A^\mu_{\text{operator}}\,|\psi\rangle\,. \tag{6.39}$$

We must guarantee that the expectation value of $\partial_\mu A^\mu$ on physical states $|\psi\rangle$ vanishes,

$$\langle\psi|\,\partial_\mu A^\mu\,|\psi\rangle = 0, \tag{6.40}$$

which may be achieved in a way such that $\partial_\mu A^\mu \neq 0$ as an operator identity.

We can now carry on with the quantisation of the Lagrangian of Eq. 6.36. The quantisation conditions of Eq. 6.34, substituting in the expressions of the conjugate momenta $\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}$, yield

$$\left[\dot{A}_\mu\left(\vec{x},t\right), A_\nu\left(\vec{x}',t\right)\right] = i g_{\mu\nu}\delta^{(3)}\left(\vec{x}-\vec{x}'\right). \tag{6.41}$$

We now substitute the solution of Eq. 6.27 into Eq. 6.42. We obtain,

$$\left[a_\lambda(k), a^\dagger_{\lambda'}(k')\right] = -g^{\lambda\lambda'} 2k_0\left(2\pi\right)^3 \delta^{(3)}\left(\vec{k}-\vec{k}'\right). \tag{6.42}$$

We can safely interpret the $a_i(k)$ and $a_i^\dagger(k)$ for $i = 1,2,3$ as annihilation and creation operators correspondingly, of photons with space-like polarisations $\epsilon^\mu_{(i)}$. But this is problematic for the operators $a_0(k), a_0^\dagger(k)$, which satisfy

$$\left[a_0(k), a^\dagger_0(k')\right] = -2k_0\left(2\pi\right)^3 \delta^{(3)}\left(\vec{k}-\vec{k}'\right). \tag{6.43}$$

Consider a general one-photon state with a time-like polarisation,

$$|1,\lambda=0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3(2k_0)} f(k) a_0^\dagger(k)\,|0\rangle\,. \tag{6.44}$$

The norm of the state is,

$$\langle 1,\lambda=0|\,1,\lambda=0\rangle = -\int \frac{d^3\vec{k}}{(2\pi)^3(2k_0)}\,|f(k)|^2\,\langle 0|\,0\rangle\,, \tag{6.45}$$

which is negative. In general, a state with n_t photons with a time-like polarisation has a norm

$$\langle n_t,\lambda=0|\,n_t,\lambda=0\rangle = (-1)^{n_t}. \tag{6.46}$$

This is a serious problem since it essentially gives rise to negative probabilities for transition amplitudes among states with time-like polarisations.

We need Eq. 6.40 to hold for any physical state. We will now describe how this can be achieved following the prescription of Gupta and Bleuler. We separate the electromagnetic field into a positive and a negative frequency component,

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x), \tag{6.47}$$

with

$$A^{(+)\mu} = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} \sum_\lambda \epsilon_\lambda^\mu a_\lambda(k) e^{-ik\cdot x}, \tag{6.48}$$

and

$$A^{(-)\mu} = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} \sum_\lambda \epsilon_\lambda^\mu a_\lambda^\dagger(k) e^{+ik\cdot x}. \tag{6.49}$$

Physical states $|\psi_{\text{phys}}\rangle$ are required that they satisfy

$$\partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle = 0. \tag{6.50}$$

This is satisfied automatically by the vacuum state, $|\psi_{\text{phys}}\rangle = |0\rangle$, since $A^{(+)\mu}$ contains only annihilation operators. We also find that,

$$\begin{aligned}
& \langle\psi_{\text{phys}}| \partial_\mu A^\mu |\psi_{\text{phys}}\rangle \\
= & \langle\psi_{\text{phys}}| \partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle + \langle\psi_{\text{phys}}| \partial_\mu A^{(-)\mu} |\psi_{\text{phys}}\rangle \\
= & \langle\psi_{\text{phys}}| \partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle + \langle\psi_{\text{phys}}| \partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle^* \\
= & 2\text{Re}\, \langle\psi_{\text{phys}}| \partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle \\
= & 0.
\end{aligned} \tag{6.51}$$

The Bleuler-Gupta condition takes the explicit form

$$\partial_\mu A^{(+)\mu} |\psi_{\text{phys}}\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} e^{-ik\cdot x} \sum_{\lambda=0}^{3} k\cdot\epsilon_\lambda(k) a_\lambda(k) |\psi_{\text{phys}}\rangle = 0. \tag{6.52}$$

It is easy to verify that the above is not satisfied for photon-states with a purely longitudinal polarisation, $|\psi\rangle = |1, \lambda = 0\rangle$, as in Eq. 6.44.

What is a physical state for a single photon? Consider the general one-photon state with a momentum $p^\mu = (1, 0, 0, 1)$, choosing its momentum along the z-axis. This state can be written as,

$$|\gamma\rangle = \sum_{\lambda=0}^{3} c_\lambda a_\lambda^\dagger(p) |0\rangle, \tag{6.53}$$

where c_λ are arbitrary coefficients determining the mixture of polarisations in the photon physical state. We must have,

$$\begin{aligned}
0 &= \partial_\mu A^{(+)\mu} |\gamma\rangle \\
\rightsquigarrow 0 &= \left(\sum_{\lambda\lambda'} c_\lambda p\cdot\epsilon'_\lambda g^{\lambda\lambda'}\right) |0\rangle \\
\rightsquigarrow 0 &= \left(\sum_{\lambda\lambda'} c_\lambda p\cdot\epsilon'_\lambda g^{\lambda\lambda'}\right)
\end{aligned} \tag{6.54}$$

From the expressions for the vector polarisations in Eq. 6.28 and the momentum $p = (1, 0, 0, 1)$ we derive that,

$$p \cdot \epsilon_1 = p \cdot \epsilon_2 = 0 \tag{6.55}$$

and

$$p \cdot \epsilon_0 = -p \cdot \epsilon_3. \tag{6.56}$$

Then Eq. 6.54 becomes

$$c_0 + c_3 = 0 \rightsquigarrow c_0 = -c_3. \tag{6.57}$$

Therefore, component of the photon state with a time-like polarisation must be accompanied with an "opposite" component with a longitudinal polarisation. Therefore a general physical photon state with momentum $\vec{p}$ can be of the form:

$$|\gamma, \vec{p}\rangle = c\left[a_0^\dagger(p) - a_3^\dagger(p)\right] + c_1 a_1^\dagger(p) + c_2 a_2^\dagger(p). \tag{6.58}$$

Notice that the norm of the state is

$$\begin{aligned} \langle\gamma, \vec{p}|\, \gamma, \vec{p}\rangle &= |c|^2 \left(\langle \vec{p}, \lambda = 0|\, \vec{p}, \lambda = 0\rangle + \langle \vec{p}, \lambda = 3|\, \vec{p}, \lambda = 3\rangle\right) \\ &\quad + |c_1|^2 \langle \vec{p}, \lambda = 1|\, \vec{p}, \lambda = 1\rangle + |c_1|^2 \langle \vec{p}, \lambda = 2|\, \vec{p}, \lambda = 2\rangle \\ &= \left\{|c|^2(-1+1) + |c_1|^2 1 + |c_2|^2 1\right\} (2\pi)^3 2|\vec{p}|\delta^{(3)}(0) \\ &= \left(|c_1|^2 + |c_2|^2\right) (2\pi)^3 2|\vec{p}|\delta^{(3)}(0). \end{aligned} \tag{6.59}$$

It receives contributions only from the transverse polarisations, since the contributions of the time-like and longitudinal components cancel each other [1] Let us calculate the energy in an one-photon monochromatic physical state. After a standard, by now, calculation, we find that the Hamiltonian is given by the expression:

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3 2k_0} k_0 \left[\sum_{\lambda=1}^{3} a_\lambda^\dagger(k) a_\lambda(k) - a_0^\dagger(k) a_0(k)\right] \tag{6.60}$$

with $k_0 = \left|\vec{k}\right| > 0$. We can prove easily that $|\gamma, \vec{p}\rangle$ is an eigenstate of the Hamiltonian:

$$H\,|\gamma, \vec{p}\rangle = |\vec{p}|\,|\gamma, \vec{p}\rangle \tag{6.61}$$

We also find:

$$\langle\gamma|\, H\,|\gamma\rangle = |\vec{p}|\,\langle\gamma|\, \gamma\rangle \tag{6.62}$$

The energy of the state is

$$E_{|\gamma, vecp\rangle} = \frac{\langle\gamma|\, H\,|\gamma\rangle}{\langle\gamma|\, \gamma\rangle} = |\vec{p}|. \tag{6.63}$$

The contributions of the time-like and longitudinal components cancel each other in all physical quantities. We shall see realistic examples of this cancelation for scattering amplitudes in Quantum Electrodynamics.

6.3 Massive photons: The Higgs mechanism*

[1]The infinite magnitude normalisation ($\delta^{(3)}(0)$) is an artefact due to considering a monochromatic state. It is finite for realistic states which are superpositions of photons: $|\gamma\rangle = \int \frac{d^3\vec{p}}{2p_0(2\pi)^3} f(\vec{p})\,|\gamma, \vec{p}\rangle$.

Chapter 7

The Dirac Equation

Dirac identified the source of the problems in interpreting the Klein-Gordon equation as a wave -function equation the fact that it was a second order differential equation in time. The correspondence,

$$i\partial_t \to E,$$

resulted to a second order polynomial equation for the energy of a free-particle obeying, which of course has both a positive and a negative solution. Dirac's idea was then to find a linear equation, whose differential operator represented a sort of a "square-root" of the differential operator in the Klein-Gordon equation. Explicitly, Dirac's linear equation is,

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0. \tag{7.1}$$

Multiplying the Dirac equation with the differential operator $(-i\gamma^\mu \partial_\mu - m)$ from the left,

$$\begin{aligned}
&\rightsquigarrow \quad \left(-i\gamma^\nu \partial_\nu - m\right)\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \\
&\rightsquigarrow \quad \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2\right)\psi = 0 \\
&\rightsquigarrow \quad \left(\frac{1}{2}\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \frac{1}{2}\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2\right)\psi = 0 \\
&\rightsquigarrow \quad \left(\frac{1}{2}\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \frac{1}{2}\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu + m^2\right)\psi = 0 \\
&\rightsquigarrow \quad \left(\frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\}\partial_\mu \partial_\nu + m^2\right)\psi = 0,
\end{aligned} \tag{7.2}$$

which for

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu}, \tag{7.3}$$

gives the Klein-Gordon equation.

In other words, if we can construct objects γ^μ which satisfy the algebra of Eq. 7.3, known as Clifford algebra, then every solution $\psi(\vec{x}, t)$ of the Dirac equation, will also be a solution of the Klein-Gordon equation.

The objects γ^μ are anti-commuting and they cannot be just numbers. They have to be matrices. The first dimensionality for which we can find four non-commuting matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ is four. In the following we shall construct the so called "gamma-matrices" in four dimensions.

7.1 Mathematical interlude

It is useful to review here the properties of the 2×2 Pauli matrices, which we shall use as building blocks for constructing γ-matrices by taking their Kronecker product.

7.1.1 Pauli matrices and their properties

The Pauli matrices are,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.4}$$

The Pauli matrices satisfy the following commutation and anti-commutation relations,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \qquad (\epsilon_{123} = 1), \tag{7.5}$$

and

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times 2}. \tag{7.6}$$

Adding the two equations together, we find that the product of two Pauli matrices is,

$$\sigma_i \sigma_j = \delta_{ij} I_{2\times 2} + i\epsilon_{ijk}\sigma_k. \tag{7.7}$$

For example,

$$\sigma_i^2 = I_{2\times 2}, \sigma_1\sigma_2 = i\sigma_3, \sigma_3\sigma_2 = -i\sigma_1, \ldots \tag{7.8}$$

The determinant and trace of Pauli matrices are,

$$\det(\sigma_i) = -1, \tag{7.9}$$

and

$$\mathrm{Tr}(\sigma_i) = 0. \tag{7.10}$$

As it can be explicitly verified, the Pauli matrices are hermitian:

$$\sigma_i^\dagger = \sigma_i. \tag{7.11}$$

The Pauli matrices, together with the unit matrix $I_{2\times 2}$ constitute a basis of general 2×2 matrices $M_{2\times 2}$. We write,

$$M_{2\times 2} = a_0 I_{2\times 2} + \sum_{i=1}^{3} a_i \sigma_i. \tag{7.12}$$

From the above decomposition we have,

$$\mathrm{Tr}(M) = a_0 \mathrm{Tt}(I_{2\times 2}) + \sum_{i=1}^{3} a_i \mathrm{Tr}(\sigma_i) = 2a_0, \tag{7.13}$$

and (**exercise**),

$$\mathrm{Tr}(\sigma_k M) = 2a_k. \tag{7.14}$$

We can therefore write,

$$M_{2\times 2} = \frac{\mathrm{Tr}(M)}{2} I_{2\times 2} + \sum_{i=1}^{3} \frac{\mathrm{Tr}(M\sigma_i)}{2}\sigma_i. \tag{7.15}$$

7.1.2 Kronecker product of 2×2 matrices

Let us define the Kronecker product of two 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{7.16}$$

as

$$A \otimes B \equiv \begin{pmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{12}a_{11} & b_{12}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{12}a_{21} & b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{21}a_{21} & b_{21}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{pmatrix}. \tag{7.17}$$

The following properties can be proved (**exercise**) explicitly,

$$\det(A \otimes B) = (\det A)^2 (\det B)^2, \tag{7.18}$$

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B), \tag{7.19}$$

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger, \tag{7.20}$$

$$(A + C) \otimes B = A \otimes B + C \otimes B, \tag{7.21}$$

$$A \otimes (B + C) = A \otimes B + A \otimes C, \tag{7.22}$$

and

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2. \tag{7.23}$$

7.2 Dirac representation of γ-matrices

The matrices

$$\gamma^0 = I_{2\times2} \otimes \sigma_3, \tag{7.24}$$

and

$$\gamma^j = \sigma_j \otimes (i\sigma_2), \qquad (j = 1, 2, 3) \tag{7.25}$$

satisfy the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_{4\times4}. \tag{7.26}$$

We find,

$$\begin{aligned} \left(\gamma^0\right)^2 &= (I_{2\times2} \otimes \sigma_3)(I_{2\times2} \otimes \sigma_3) \\ &= I_{2\times2}^2 \otimes \sigma_3^2 = I_{2\times2} \otimes I_{2\times2} = I_{4\times4} \\ \rightsquigarrow \left\{\gamma^0, \gamma^0\right\} &= 2g^{00} I_{4\times4}. \end{aligned} \tag{7.27}$$

Also, for the anti-commutator $\left\{\gamma^j, \gamma^k\right\}$ with $j, k = 1, 2, 3$ we have

$$\begin{aligned} \left\{\gamma^j, \gamma^k\right\} &= \gamma^j\gamma^k + \gamma^k\gamma^j \\ &= (\sigma_j \otimes (i\sigma_2))(\sigma_k \otimes (i\sigma_2)) + (\sigma_k \otimes (i\sigma_2))(\sigma_j \otimes (i\sigma_2)) \\ &= \sigma_j\sigma_k \otimes (i\sigma_2)^2 + \sigma_k\sigma_j \otimes (i\sigma_2)^2 \\ &= \{\sigma_j, \sigma_k\} \otimes (-I_{2\times2}) = (2\delta_{jk} I_{2\times2}) \otimes (-I_{2\times2}) \\ \rightsquigarrow \left\{\gamma^j, \gamma^k\right\} &= -2\delta^{jk} I_{4\times4} \\ \rightsquigarrow \left\{\gamma^j, \gamma^k\right\} &= 2g^{jk} I_{4\times4}. \end{aligned} \tag{7.28}$$

Finally, the anti-commutator $\{\gamma^0, \gamma^j\}$ with $j = 1, 2, 3$ is

$$\begin{aligned}
\{\gamma^0, \gamma^j\} &= \gamma^0\gamma^j + \gamma^j\gamma^0 \\
&= (I_{2\times2} \otimes \sigma_3)(\sigma_j \otimes (i\sigma_2)) + (\sigma_j \otimes (i\sigma_2))(I_{2\times2} \otimes \sigma_3) \\
&= \sigma_j \otimes (i\sigma_2\sigma_3) + \sigma_j \otimes (i\sigma_3\sigma_2) \\
&= \sigma_j \otimes (i\{\sigma_2, \sigma_3\}) \\
&= \sigma_j \otimes \left(i2\delta^{23} I_{2\times2}\right) = \sigma_j \otimes 0_{2\times2} = 0_{4\times4} \\
\rightsquigarrow \{\gamma^0, \gamma^j\} &= 2g^{0j} I_{4\times4}.
\end{aligned} \tag{7.29}$$

We have therefore found a set of γ- matrices satisfying the Clifford algebra. This set is not unique. Let us define new matrices,

$$\gamma^{\mu\prime} = M_{4\times4}^{-1} \gamma^\mu M_{4\times4}, \tag{7.30}$$

with $M_{4\times4}$ any 4×4 matrix with an inverse. Then,

$$\begin{aligned}
\{\gamma^{\mu\prime}, \gamma^{\nu\prime}\} &= M_{4\times4}^{-1} \{\gamma^\mu, \gamma^\nu\} M_{4\times4} \\
&= 2g^{\mu\nu} M_{4\times4}^{-1} I_{4\times4} M_{4\times4} \\
&= 2g^{\mu\nu} I_{4\times4}.
\end{aligned} \tag{7.31}$$

The matrices $\gamma^{\mu\prime}$ satisfy the same algebra, furnishing a different representation.

Let us now compute the hermitian conjugate of the $\gamma-$matrices in the Dirac representation. We have

$$\gamma^{0\dagger} = (I_{2\times2} \otimes \sigma_3)^\dagger = I_{2\times2}^\dagger \otimes \sigma_3^\dagger = I_{2\times2} \otimes \sigma_3 = \gamma^0 = \gamma^0\gamma^0\gamma^0. \tag{7.32}$$

and

$$\gamma^{j\dagger} = (\sigma_j \otimes (i\sigma_2))^\dagger = \sigma_j^\dagger \otimes (-i\sigma_2^\dagger) = \sigma_j \otimes (-i\sigma_2) = -\gamma^j = -\gamma^0\gamma^0\gamma^i = \gamma^0\gamma^j\gamma^0. \tag{7.33}$$

In all cases, we have found:

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \tag{7.34}$$

which can also be written as,

$$\begin{aligned}
\gamma^0\gamma^\mu\gamma^0 &= \gamma^0 \{\gamma^\mu, \gamma^0\} - \left(\gamma^0\right)^2 \gamma^\mu \\
&= 2g^{\mu 0}\gamma^0 - \gamma^\mu.
\end{aligned} \tag{7.35}$$

Explicitly, γ-matrices are hermitian for $\mu = 0$,

$$\gamma^{0\dagger} = \gamma^0, \tag{7.36}$$

while they are anti-hermitian for $\mu = j = 1, 2, 3$

$$\gamma^{j\dagger} = -\gamma^j. \tag{7.37}$$

Eq. 7.34 is of course valid for every other representation of gamma matrices which can be obtained from the Dirac representation via a hermitian similarity transformation.

7.3 Traces of $\gamma-$ matrices

We first observe that the trace of a single gamma matrix is zero:

$$\mathrm{Tr}(\gamma^\mu) = 0. \tag{7.38}$$

This is an immediate consequence of the fact that gamma matrices are constructed out of Pauli matrices in the Dirac representation. But the result is more general. We can prove that $\gamma-$matrices are traceless in every representation. Let us start with

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^\nu(\gamma^\mu)^2\right) = \mathrm{Tr}\left(\gamma^\nu g^{\mu\mu}\right) = g^{\mu\mu}\mathrm{Tr}\left(\gamma^\nu\right) \tag{7.39}$$

(No summation over the index μ)
This is also equal to

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^\mu\left\{\gamma^\nu,\gamma^\mu\right\}\right) - \mathrm{Tr}\left((\gamma^\mu)^2\gamma^\nu\right) = 2g^{\mu\nu}\mathrm{Tr}\left(\gamma^\mu\right) - g^{\mu\mu}\mathrm{Tr}\left(\gamma^\nu\right). \tag{7.40}$$

For $\mu \neq \nu$, we then have

$$g^{\mu\mu}\mathrm{Tr}\left(\gamma^\nu\right) = -g^{\mu\mu}\mathrm{Tr}\left(\gamma^\nu\right) \rightsquigarrow \mathrm{Tr}(\gamma^\nu) = 0. \tag{7.41}$$

Exercise: Prove that

1. $$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{7.42}$$

2. $$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0. \tag{7.43}$$

3. the trace of a product of an odd number of gamma matrices is zero.

4. $$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \tag{7.44}$$

5. Write a computer program to prove that:

$$\begin{aligned}
&\frac{1}{4}\mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_6}) = \\
&g^{\mu_1\mu_2}g^{\mu_3\mu_4}g^{\mu_5\mu_6} - g^{\mu_1\mu_2}g^{\mu_3\mu_5}g^{\mu_4\mu_6} + g^{\mu_1\mu_2}g^{\mu_3\mu_6}g^{\mu_4\mu_5} \\
&-g^{\mu_1\mu_3}g^{\mu_2\mu_4}g^{\mu_5\mu_6} + g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6} - g^{\mu_1\mu_3}g^{\mu_2\mu_6}g^{\mu_4\mu_5} \\
&+g^{\mu_1\mu_4}g^{\mu_2\mu_3}g^{\mu_5\mu_6} - g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6} + g^{\mu_1\mu_4}g^{\mu_2\mu_6}g^{\mu_3\mu_5} \\
&-g^{\mu_1\mu_5}g^{\mu_2\mu_3}g^{\mu_4\mu_6} + g^{\mu_1\mu_5}g^{\mu_2\mu_4}g^{\mu_3\mu_6} - g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_3\mu_4} \\
&+g^{\mu_1\mu_6}g^{\mu_2\mu_3}g^{\mu_4\mu_5} - g^{\mu_1\mu_6}g^{\mu_2\mu_4}g^{\mu_3\mu_5} + g^{\mu_1\mu_6}g^{\mu_2\mu_5}g^{\mu_3\mu_4}.
\end{aligned} \tag{7.45}$$

7.4 $\gamma-$matrices as a basis of 4×4 matrices

The Clifford algebra restricts the number of independent matrices that we can construct by taking products or combinations of products of $\gamma-$matrices in a given representation. First, it imposes that the square of a $\gamma-$matrix is proportional to the unit 4×4 matrix,

$$(\gamma^\mu)^2 = \frac{1}{2}\left\{\gamma^\mu,\gamma^\mu\right\} = g^{\mu\mu}I_{4\times4} = \begin{cases} +I_{4\times4}, & \mu = 0 \\ -I_{4\times4}, & \mu = j = 1,2,3 \end{cases} \tag{7.46}$$

This limits the number of terms in independent products of $\gamma-$matrices to at most four.

- The only independent product of four $\gamma-$matrices is,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{7.47}$$

Exercise: Show that you can write the matrix γ^5 as

$$\gamma_5 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \qquad (\epsilon^{0123} = 1). \tag{7.48}$$

- There are four independent products of three $\gamma-$matrices, $\gamma^0\gamma^1\gamma^2$, $\gamma^1\gamma^2\gamma^3$, $\gamma^2\gamma^3\gamma^1$ and $\gamma^3\gamma^1\gamma^0$. These can be written as a linear combination of

$$\gamma^5\gamma^\mu, \tag{7.49}$$

and products of two only $\gamma-$matrices, by using the Clifford algebra.

- Due to the Clifford algebra, only the six antisymmetric combinations of products of two $\gamma-$matrices are independent. These are the combinations $\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}$ with

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{7.50}$$

In the Dirac representation γ_5 takes the form,

$$\begin{aligned}
\gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= i\left(I_{2\times 2}\otimes\sigma_3\right)\left(\sigma_1\otimes(i\sigma_2)\right)\left(\sigma_2\otimes(i\sigma_2)\right)\left(\sigma_3\otimes(i\sigma_2)\right) \\
&= i\left(\sigma_1\sigma_2\sigma_3\right)\otimes\left(-i\sigma_3\sigma_2\right) \\
&= i\left(i\sigma_3^2\right)\otimes\left(-\sigma_1\right) = I_{2\times 2}\otimes\sigma_1 \\
\rightsquigarrow \gamma_5 &= \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix}.
\end{aligned} \tag{7.51}$$

Exercise: Show that in the Dirac representation,

$$\sigma^{0j} = i\begin{pmatrix} 0_{2\times 2} & \sigma_j \\ \sigma_j & 0_{2\times 2} \end{pmatrix}, \ \textbf{and}\ \sigma^{jk} = i\epsilon_{jkl}\begin{pmatrix} \sigma_l & 0_{2\times 2} \\ 0_{2\times 2} & \sigma_l \end{pmatrix} \tag{7.52}$$

with $j, k = 1, 2, 3$.

A general 4×4 matrix has 16 independent components. From a representation of $\gamma-$matrices we can construct 15 independent 4×4 matrices. Together with the unit matrix $I_{4\times 4}$, they constitute a basis. In other words, a general 4×4 matrix M can be written as,

$$M_{4\times 4} = a_0 I_{4\times 4} + a_\mu\gamma^\mu + a_{5\mu}\gamma_5\gamma^\mu + a_{\mu\nu}\sigma^{\mu\nu}. \tag{7.53}$$

Exercise: Find the coefficients $a_0, a_{5\mu}, a_{\mu\nu}$ for a given 4×4 matrix M.

7.5 Lagrangian for the Dirac field

After this survey of the properties of $\gamma-$matrices, we return to the Dirac equation,

$$\left(i\partial\!\!\!/ - mI_{4\times 4}\right)\psi = 0. \tag{7.54}$$

We have introduced the "slash" notation:

$$\not{a} \equiv \gamma^\mu a_\mu, \qquad \not{\partial} \equiv \gamma^\mu \partial_\mu. \tag{7.55}$$

The Dirac differential operator $i\not{\partial} - m$ is a 4×4 matrix. The object ψ is a four dimensional "Dirac spinor" and it has its own Lorentz transformation, which we shall derive in detail later.

$$\psi(\vec{x}, t) \equiv \psi_a(\vec{x}, t) = \begin{pmatrix} \psi_1(\vec{x}, t) \\ \psi_2(\vec{x}, t) \\ \psi_3(\vec{x}, t) \\ \psi_4(\vec{x}, t) \end{pmatrix}. \tag{7.56}$$

It is very important not to confuse a spinor ψ with a four dimensional vector, which has truly different Lorentz transformations.

We can now take the Hermitian conjugate of the Dirac equation,

$$\begin{aligned} & \left(i\not{\partial} - mI_{4\times 4}\right)\psi = 0 \\ \rightsquigarrow \quad & -i\left(\gamma^\mu \partial_\mu \psi\right)^\dagger - m\psi^\dagger = 0 \\ \rightsquigarrow \quad & -i\left(\partial_\mu \psi^\dagger\right)\gamma^{\mu\dagger} - m\psi^\dagger = 0 \\ \rightsquigarrow \quad & -i\left(\partial_\mu \psi^\dagger\right)\gamma^0\gamma^\mu\gamma^0 - m\psi^\dagger = 0 \end{aligned} \tag{7.57}$$

Multiplying this equation with γ^0 from the left, we obtain

$$\begin{aligned} & -i\left(\partial_\mu \psi^\dagger\right)\gamma^0\gamma^\mu(\gamma^0)^2 - m\psi^\dagger\gamma^0 = 0 \\ \rightsquigarrow \quad & -i\left(\partial_\mu(\psi^\dagger\gamma^0)\right)\gamma^\mu - m(\psi^\dagger\gamma^0) = 0. \end{aligned} \tag{7.58}$$

We now define,

$$\bar{\psi} \equiv \psi^\dagger \gamma^0. \tag{7.59}$$

Then, the Hermitian conjugate of the Dirac equation reads

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0. \tag{7.60}$$

We can obtain the Dirac equation and its conjugate from a Lagrangian density,

$$\mathcal{L} = \bar{\psi}\left(i\not{\partial} - m\right)\psi. \tag{7.61}$$

Exercise: Derive the Euler-Lagrange equations for the spinor field components ψ_a and $\bar{\psi}_a$.

This is a very important piece of information if we want to quantize the Dirac field ψ. We will postpone the quantization of the Dirac field until after we gain a deeper understanding of the origin of the Lagrangian, and the Lorentz transformation properties of spinor fields.

Chapter 8

Lorentz symmetry and free Fields

A fundamental requirement for physical laws is that they must be valid for all relativistic observers. Assume that the fields $\phi_i(x^\mu)$ describe a physical system in a certain coordinate reference frame. Consider now a different relativistic observer. Space-time coordinates in the frame of the two observers are related with a Lorentz transformation,

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu_{\ \nu} x^\nu. \tag{8.1}$$

The new observer will have to transform appropriately both fields and coordinates when computing the Lagrangian,

$$\phi_i(x) \to \phi_i'(x'), \qquad \mathcal{L}(x) \to \mathcal{L}'(x'). \tag{8.2}$$

We will require that the action is the same for both observers. In this way, if it is an extremum for the physical values of $\phi_i(x)$ in the first reference frame, it will also be an extremum for the physical values of the field $\phi_i'(x')$ in the second relativistic frame. For proper Lorentz transformations ($\det(\Lambda^\mu_{\ \nu}) = 1$) we have

$$d^4x' = d^4x \det\left(\frac{\partial x^{\mu\prime}}{\partial x^\nu}\right) = d^4x \det(\Lambda^\mu_{\ \nu}) = d^4x. \tag{8.3}$$

Then

$$S = \int d^4x \mathcal{L}(x) = \int d^4x \mathcal{L}'(x') = S'. \tag{8.4}$$

Therefore, the action is relativistically invariant if the Lagrangian density is a scalar, i.e. it does not transform under a Lorentz transformation:

$$\mathcal{L}'(x') = \mathcal{L}(x), \qquad x^\mu \to x^{\mu\prime} = \Lambda^\mu_\nu x^\nu. \tag{8.5}$$

How many such Lagrangian densities can we find? Obviously, Lorentz symmetry is not sufficient on its own to determine all physical laws. We can restrict the problem to Lagrangians which describe free relativistic particles, that do not interact with each other. The Lagrangians that we are after may contain up to one only parameter, which is the mass m of the free particles. In addition, the equations of motion and field quantization should lead to the correct relativistic energy-momentum relation,

$$p^2 = m^2. \tag{8.6}$$

A Lagrangian of free particles must contain derivatives of the fields $\partial_\mu \phi_i$. They are needed to generate the momentum $\partial_\mu \phi_i \to p_\mu$ in the energy-momentum relation. We include a generic label i, allowing for the possibility of a particle with a mass m and multiple degrees of freedom, such as charge, spin, etc. Our question has now become more specific: *How can we combine the fields ϕ_i and their derivatives $\partial_\mu \phi_i$ into a scalar Lagrangian with one only mass parameter?* We need two ingredients:

- The transformation of derivatives $\partial_\mu \to \partial'_\mu$ under $x^\mu \to \Lambda^\mu_\nu x^\nu$,
- The fields transformations $\phi_i(x) \to \phi'_i(x')$ under $x^\mu \to \Lambda^\mu_\nu x^\nu$.

It is straightforward to find the transformation of derivatives,

$$\begin{aligned} \partial_\mu \to \partial'_\mu &= \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}} \frac{\partial}{\partial x^\nu} \\ \rightsquigarrow \partial_\mu \to \partial'_\mu &= \left(\Lambda^{-1}\right)^\nu_{\ \mu} \partial_\nu. \end{aligned} \tag{8.7}$$

Similarly,

$$\partial^\mu \to \partial^{\mu\prime} = \left(\Lambda^{-1}\right)^\mu_{\ \nu} \partial^\nu. \tag{8.8}$$

8.1 Field transformations and representations of the Lorentz group

In this section, we will examine the allowed possibilities for linear field transformations [1],

$$\phi_i(x) \to \phi'_i(x') = M(\Lambda)_{ij} \phi_j(x) \tag{8.9}$$

under a space-time Lorentz transformation

$$x^\mu \to \Lambda^\mu_{\ \nu} x^\nu. \tag{8.10}$$

Lorentz transformations are a group, i.e if we perform two successive Lorentz transformations,

$$x \to x' = \Lambda x,$$

and

$$x' \to x'' = \Lambda' x',$$

this is equivalent to performing a direct Lorentz transformation

$$x \to x'' = \Lambda'' x,$$

with

$$\Lambda'' = \Lambda' \Lambda. \tag{8.11}$$

How about the matrices $M(\Lambda)_{ij}$? Consider again the transformation $x \to x'' = \Lambda'' x$, where the field is transformed as,

$$\phi(x) \to \phi''(x'') = M(\Lambda'')\phi(x). \tag{8.12}$$

We can arrive to the frame x'' by performing successive Lorentz transformations, $x \to x' = \Lambda x \to x'' = \Lambda' x' = \Lambda' \Lambda x$. Then we have that,

$$\phi''(x'') = M(\Lambda')\phi'(x') = M(\Lambda')M(\Lambda)\phi(x). \tag{8.13}$$

[1] Non-linear transformations "can be made" linear, and are not necessary to consider (eg BRST symmetry in QFTII)

For the last two equations to be consistent with each other, the field transformations $M(\Lambda)$ must obey:

$$M(\Lambda'\Lambda) = M(\Lambda')M(\Lambda). \tag{8.14}$$

In group theory terminology, this means that the matrices $M(\Lambda)$ furnish a representation of the Lorentz group. Field Lorentz transformations are therefore not random, but they can be found if we find all (finite dimension) representations of the Lorentz group.

8.1.1 Scalar representation $M(\Lambda) = 1$

The simplest representation $M(\Lambda)$ is the scalar representation, where a field $\phi(x)$ does not change under Lorentz transformations,

$$\phi(x) \to \phi'(x') = \phi(x). \tag{8.15}$$

A Lagrangian of a scalar field and its derivatives must have the transformations of the derivatives canceled among themselves. A single derivative term,

$$\partial_\mu \phi$$

is not a scalar. The minimum number of derivatives that are needed for a scalar is two,

$$(\partial_\mu \phi)(\partial^\mu \phi), \qquad \phi \partial^2 \phi.$$

The second term is actually equivalent to the first up to a total divergence. Let us assume that a Lagrangian contains this term,

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) + \dots \tag{8.16}$$

Then we can find the mass dimension of the field ϕ, by requiring that the action is dimensionless (in natural units where $c = \hbar = 1$).

$$[\text{mass}]^0 = [S] = \left[\int d^4x \mathcal{L}\right] = [x]^4 [\mathcal{L}] = [\text{mass}]^{-4} [\mathcal{L}] \tag{8.17}$$

which means that all Lagrangian densities must have mass dimensionality 4,

$$[\mathcal{L}] = [\text{mass}]^4 . \tag{8.18}$$

From this we find that a scalar field ϕ has a mass dimensionality one,

$$\begin{aligned} & [(\partial_\mu \phi)(\partial^\mu \phi)] = [\text{mass}]^4 \\ \rightsquigarrow \quad & [\partial_\mu]^2 [\phi]^2 = [\text{mass}]^4 \\ \rightsquigarrow \quad & [\text{mass}]^2 [\phi]^2 = [\text{mass}]^4 \\ \rightsquigarrow \quad & [\phi] = [\text{mass}]^1 . \end{aligned} \tag{8.19}$$

A relativistically invariant Lagrangian may include terms with no derivatives, of the type

$$\phi^n.$$

Notice that the Klein-Gordon equation

$$\mathcal{L}_{\text{Klein-Gordon}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2, \tag{8.20}$$

is, indeed, invariant under Lorentz transformations. In addition, the Klein-Gordon Lagrangian has the reflection symmetry $\phi to \leftrightarrow -\phi$.

8.1.2 Vector representation $M(\Lambda) = \Lambda$

Another representation of the Lorentz group which is easy to figure out is the vector representation, corresponding to

$$M(\Lambda^\mu_{\ \nu}) = \Lambda^\mu_{\ \nu}. \tag{8.21}$$

A field in the vector representation transforms as,

$$A^\mu(x) \to A^{\mu\prime}(x') = \Lambda^\mu_{\ \nu} A^\nu(x), \qquad x^{\mu\prime} = \Lambda^\mu_{\ \nu} x^\nu. \tag{8.22}$$

Terms where all Lorentz indices are contracted are invariant under Lorentz transformations, **Exercise:***Show that the following terms are invariant under Lorentz transformations*

$$(\partial_\mu A^\mu), (\partial_\mu A_\nu)(\partial^\mu A^\nu), (\partial_\mu A_\nu)(\partial^\nu A^\mu). \tag{8.23}$$

Lorentz invariance is not the only symmetry that can be required for free fields. For, example, the Lagrangian for the electromagnetic field is

$$\mathcal{L}_{\text{EM}} = -\frac{1}{2}\left[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right], \tag{8.24}$$

where only a certain combination of two dimension four operators with two derivatives is present in the Lagrangian. This is because of the additional symmetry under gauge transformations which constrains further the Lagrangian form.

8.2 Generators of field representations of Lorentz symmetry transformations

Are the scalar and the vector representation the only representations that we can find? How do we find the rest of them? This question simplifies if we look at small Lorentz transformations and their properties.

A small Lorentz transformation is,

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}, \tag{8.25}$$

where the parameters of the transformation satisfy [2]

$$\omega_{\mu\nu} = -\omega_{\mu\nu}. \tag{8.26}$$

Any $n \times n$ matrix representation of this transformation on fields,

$$M_{ab}(\Lambda^\mu_{\ \nu}) = M_{ab}(\delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}), \qquad a, b = 1 \dots n \tag{8.27}$$

can be expanded in the small parameters $\omega^\mu_{\ \nu}$,

$$M_{ab}(\Lambda^\mu_{\ \nu}) = M_{ab}(\delta^\mu_{\ \nu}) + \sum_{\mu<\nu} \omega_{\mu\nu} \left.\frac{\partial M_{ab}(\Lambda)}{\partial \omega_{\mu\nu}}\right|_{\omega=0} + \mathcal{O}(\omega^2). \tag{8.28}$$

In the above, we restrict the summation to $\mu < \nu$ so that we do not include twice the contribution of the same parameter $\omega_{\mu\nu}$ via its antisymmetric form $\omega_{\nu\mu}$. The $n-$dimensional representation $M_{ab}(\delta^\mu_{\ \nu})$ of the unit transformation is the $n \times n$ unit matrix,

$$M_{ab}(\delta^\mu_{\ \nu}) = I_{n\times n}. \tag{8.29}$$

[2] Notice that generally $\omega^\mu_{\ \nu} \neq -\omega^\nu_{\ \mu}$. Rather, Eq. (8.26) implies that $\omega^\mu_{\ \nu} = -\omega_\nu^{\ \mu}$.

The derivatives of the representation matrices M with respect to the small transformation parameters define the so called generators of the representation,

$$(J_M^{\mu\nu})_{ab} \equiv i \left.\frac{\partial M_{ab}(\Lambda^\mu_\nu)}{\partial \omega_{\mu\nu}}\right|_{\omega=0} . \tag{8.30}$$

The generators form a basis, such that **the representation of every small Lorentz** transformation can be written as a **linear** combination of the generators plus the unit,

$$M_{ab}(\Lambda) = I_{n\times n} - i\omega_{\mu\nu} J_M^{\mu\nu}. \tag{8.31}$$

Finite Lorentz transformations are also generated by exponentiating linear combinations of the generators. If Λ^μ_ν represents large boosts and/or rotations it cannot be expanded directly in small parameters. Then, we can use the representation property,

$$M(\Lambda) = M(\Lambda_1)\dots M(\Lambda_N), \qquad \text{with } \Lambda = \Lambda_1\Lambda_2\dots\Lambda_N \tag{8.32}$$

and construct these large Lorentz transformations exactly from an infinite product of infinitesimal Lorentz transformations with small parameters ω_{ij}. At the end, every Lorentz transformation can be written in the exponential form,

$$M(\Lambda) = e^{-i\sum_{\mu<\nu}\omega_{\mu\nu}J_M^{\mu\nu}}. \tag{8.33}$$

The generators of a group representation is all we ever need in order to determine completely the representation.

8.2.1 Generators of the scalar representation

Let us now study the generators of the scalar representation of the Lorentz symmetry transformation. In this representation, under a Lorentz transformation

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu, \tag{8.34}$$

a scalar field remains invariant

$$\phi'(x^{\mu\prime}) = \phi(x^\mu) \rightsquigarrow \phi'(\Lambda^\mu{}_\nu x^\nu) = \phi(x^\mu). \tag{8.35}$$

The complete transformation of a scalar field is rather trivial. We can recall that a small symmetry transformation of the field can be decomposed into two parts, the "variation at a point" and the variation from one point to another one (of a very close proximity in the linear approximation),

$$\Delta\phi(x) \equiv \phi'(x^{\mu\prime}) - \phi(x^\mu) = \delta_*\phi(x) + \delta\phi(x) = 0, \tag{8.36}$$

where

$$\delta\phi(x) = \phi(\Lambda x) - \phi(x) \tag{8.37}$$

and

$$\delta_*\phi(x) = \phi'(x) - \phi(x) \tag{8.38}$$

The variation at a point $\delta_*\phi$, we can compute as follows. Setting $x \to \Lambda^{-1}x$ in Eq. (8.35) we have,

$$\phi'(x^\mu) = \phi(\Lambda^{-1\mu}{}_\nu x^\nu), \tag{8.39}$$

which for small Lorentz transformations becomes

$$\begin{aligned}
\phi'(x^\mu) &= \phi(x^\mu - \omega^\mu_{\ \nu} x^\nu) \\
&= \phi(x^\mu) - \omega_{\mu\nu} x^\nu \partial^\mu \phi(x^\mu) + \mathcal{O}(\omega^2) \\
&= \phi(x^\mu) - \frac{1}{2}\left(\omega_{\mu\nu} - \omega_{\nu\mu}\right) x^\nu \partial^\mu \phi(x^\mu) + \mathcal{O}(\omega^2) \\
&= \phi(x^\mu) + \omega_{\mu\nu} \frac{1}{2}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \phi(x^\mu) + \mathcal{O}(\omega^2) \\
\rightsquigarrow \phi'(x) &= \left(1 - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{\text{scalar}}\right) \phi(x) \qquad (8.40)
\end{aligned}$$

with the generators of the Lorentz group in the scalar representation being:

$$J^{\mu\nu}_{\text{scalar}} = -i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right). \qquad (8.41)$$

An explicit calculation (**exercise**) yields the identity,

$$\begin{aligned}
&[J^{\mu_1\mu_2}_{\text{scalar}}, J^{\mu_3\mu_4}_{\text{scalar}}] = \\
&i\left(g^{\mu_2\mu_3} J^{\mu_1\mu_4}_{\text{scalar}} + g^{\mu_1\mu_4} J^{\mu_2\mu_3}_{\text{scalar}} - g^{\mu_2\mu_4} J^{\mu_1\mu_3}_{\text{scalar}} - g^{\mu_1\mu_3} J^{\mu_2\mu_4}_{\text{scalar}}\right). \qquad (8.42)
\end{aligned}$$

The above commutation relations of the generators form what is known as the "Lie algebra" of the Lorentz group. Here, we derived it from the scalar field representation of the symmetry transformation. However, as we shall see explicitly, it is satisfied by the generators of every other representation of the symmetry group.

Finally, let us remark that by multiplying an infinite product of small transformations, given the existence of Eq. (8.42), Eq. (8.40) turns into an exponentiated form. We have,

$$\phi'(x) = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{\text{scalar}}} \phi(x). \qquad (8.43)$$

8.2.2 Generators of the vector representation

We can see that the Lie algebra is universal to all generator representations by repeating the same analysis for the generators of the vector representation. A vector field $A^\alpha(x^\mu)$, transforms as

$$\begin{aligned}
A^\alpha(x^\rho) &\to A^{\alpha\prime}(x^{\rho\prime}) = \Lambda^\alpha_{\ \beta} A^\beta(x^\rho) \\
&= \left(\delta^\alpha_\beta + \omega^\alpha_{\ \beta}\right) A^\beta(x^\rho) + \mathcal{O}(\omega^2) \\
&= A^\alpha(x^\rho) + \omega^\alpha_{\ \beta} A^\beta(x^\rho) + \mathcal{O}(\omega^2) \\
\rightsquigarrow A^{\alpha\prime}(x^{\rho\prime}) &= \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}_{\text{vector}}\right)^\alpha_{\ \beta}\right) A^\beta(x^\rho), \qquad (8.44)
\end{aligned}$$

with

$$\left(J^{\mu\nu}_{\text{vector}}\right)_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right). \qquad (8.45)$$

We can now compute explicitly (**exercise**) the commutators of the generators of the vector field representation,

$$\begin{aligned}
&[J^{\mu_1\mu_2}_{\text{vector}}, J^{\mu_3\mu_4}_{\text{vector}}]_{\alpha\beta} = \\
&i\left[g^{\mu_2\mu_3}\left(J^{\mu_1\mu_4}_{\text{vector}}\right)_{\alpha\beta} + g^{\mu_1\mu_4}\left(J^{\mu_2\mu_3}_{\text{vector}}\right)_{\alpha\beta} - g^{\mu_2\mu_4}\left(J^{\mu_1\mu_3}_{\text{vector}}\right)_{\alpha\beta} - g^{\mu_1\mu_3}\left(J^{\mu_2\mu_4}_{\text{vector}}\right)_{\alpha\beta}\right]. \\
&\qquad (8.46)
\end{aligned}$$

We see indeed that the commutators of the vector representation satisfy the same Lie algebra as in the scalar representation (Eq. 8.42). This is of course what we expect from the theory of Lie groups, which we will review in the follwowing subestion.

A Lorentz transformation on a vector field changes its orientation by mixing the indices of A^α. The generators of the transformation for the vector orientation are given in Eq. 8.45. In an exponentiated form, which can be also applied to large transformations, we have

$$A^{\alpha\prime}(x') = e^{-\frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}_{\text{vector}}\right)^\alpha_{\ \beta}} A^\beta(x). \tag{8.47}$$

Exercise: *Prove that a Lorentz transformation of a vector field "at a point" is*

$$A^{\alpha\prime}(x) = e^{-\frac{i}{2}\omega_{\mu\nu}\left[\left(J^{\mu\nu}_{\text{vector}}\right)^\alpha_{\ \beta}+\delta^\alpha_\beta J^{\mu\nu}_{\text{scalar}}\right]} A^\beta(x). \tag{8.48}$$

8.2.3 Lie algebra of continuous groups

Our discussion of the Lorentz symmetry is more general and it applies to any continuous symmetry transformation

$$T(\theta^a)$$

parameterized by a set of continuous parameters

$$\{\theta^a\}, \quad a = 1 \dots N$$

The symmetry transformations form a Lie group and the product of two such transformations is also a symmetry transformation belonging to the group:

$$T(\theta_1^a)T(\theta_2^a) = T\left(f^a(\theta_1^a, \theta_2^a)\right). \tag{8.49}$$

We arrange so that for the unit element of the symmetry group corresponds to all values of the parameters being zero, $\theta^a = 0$,

$$T(\theta^a = 0) = 1. \tag{8.50}$$

The product rule of Eq. 8.49 and our definition of the unit transformation of Eq. 8.50 yield:

$$f^a(0, \theta^a) = f^a(\theta^a, 0) = \theta^a \tag{8.51}$$

Restricting ourselves to small transformations around the unity transformation, we perform a Taylor expansion around $\theta^a = 0$:

$$\begin{aligned} f^a(\theta_1^a, \theta_2^a) &= f^a(0,0) + \frac{\partial f^a}{\partial \theta_1^b}\theta_1^b + \frac{\partial f^a}{\partial \theta_2^b}\theta_2^b \\ &+ \frac{1}{2}\frac{\partial^2 f^a}{\partial\theta_1^b\partial\theta_1^c}\theta_1^b\theta_1^c + \frac{1}{2}\frac{\partial^2 f^a}{\partial\theta_2^b\partial\theta_2^c}\theta_2^b\theta_2^c + \frac{\partial^2 f^a}{\partial\theta_1^b\partial\theta_2^c}\theta_1^b\theta_2^c + \dots \end{aligned} \tag{8.52}$$

Imposing the requirement of Eq 8.51 leads to the form:

$$f^a(\theta_1^a, \theta_2^a) = \theta_1^a + \theta_2^a + f^a_{bc}\theta_1^b\theta_2^c + \dots \tag{8.53}$$

For an ordinary representation $U(T)$ of the symmetry group,

$$U(T(\theta_1^a))U(T(\theta_2^a)) = U\left(T(f^a(\theta_1^a, \theta_2^a))\right). \tag{8.54}$$

For small parameters θ^a, we can expand the representation of the transformation as follows:

$$U(T(\theta^a)) = 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \dots \tag{8.55}$$

where $1, t_a, t_{bc}$ in the above equation are matrices of the same dimensionality as $U(T(\theta^a))$ and the matrix t_{bc} is symmetric in b and c:

$$t_{bc} = t_{cb}. \tag{8.56}$$

Expanding, using Eq. 8.55, the terms in Eq. 8.54 we have

$$\begin{aligned}&\left(1 + i\theta_1^a t_a + \frac{1}{2}\theta_1^b\theta_1^c t_{bc} + \dots\right)\left(1 + i\theta_2^a t_a + \frac{1}{2}\theta_2^b\theta_2^c t_{bc} + \dots\right)\\ =&1 + i\left(\theta_1^a + \theta_2^a + f_{bc}^a\theta_1^b\theta_2^c + \dots\right)t_a + \frac{1}{2}\left(\theta_1^b + \theta_2^b + \dots\right)\left(\theta_1^c + \theta_2^c + \dots\right)t_{bc}\end{aligned}$$

Matching the powers of θ_1^a, θ_2^a, we find that

$$t_{bc} = -t_b t_c - i f_{bc}^a t_a. \tag{8.57}$$

From the symmetry of t_{bc}, we obtain:

$$0 = t_{bc} - t_{cb} = -(t_b t_c - t_c t_b) - i(f_{bc}^a - f_{cb}^a)t_a \tag{8.58}$$

This leads to the commutation relation for the matrices t^a:

$$[t_b, t_c] = iC_{bc}^a t_a. \tag{8.59}$$

where the constants

$$C_{bc}^a = f_{cb}^a - f_{bc}^a \tag{8.60}$$

are antisymmetric in $c \leftrightarrow b$ and are called the *structure constants* of the Lie-group. As you may observe, the structure constants do not depend on the specific representation $U(T(\theta^a))$. Instead, the matrices t_a are specific to the representation and have the same dimensionality. These matrices are known as the *generators* of the representation. Knowing the generators, we can construct the representation of an arbitrary symmetry transformation by using the product property of Eq. 8.54 to assemble large symmetry transformations from many small ones where the expansion of Eq 8.55 is valid. The common commutation relation of Eq. 8.59 satisfied by the generators of all representations of the symmetry group is known as a *Lie algebra*.

Requiring that the representation of a symmetry transformation is unitary, leads to the conclusion that the generators of the transformation are Hermitian operators:

$$UU^\dagger = 1 \rightsquigarrow t_a^\dagger = t_a. \tag{8.61}$$

Therefore, generators are good candidates for physical observables.

In the case of the Lorentz group, we have:

$$[J_M^{\mu\nu}, J_M^{\rho\sigma}] = iC_{\mu\nu\rho\sigma}^{\alpha\beta} J_M^{\alpha\beta}. \tag{8.62}$$

The coefficients $C_{\mu\nu\rho\sigma}^{\alpha\beta}$ are the structure constants of the Lorentz group.

We now have a better strategy in order to find representations of the Lorentz group:

- First, we will use any of the representations that we already know, the scalar or vector representation, in order to find the structure constants $C^{\alpha\beta}_{\mu\nu\rho\sigma}$ of the Lorentz group.
- Then we shall attempt to find matrices $J^{\mu\nu}_{n\times n}$ for each finite dimension n which satisfy the Lie algebra of Eq. 8.62. If we succeed, then we have found an $n \times n$ representation of the generators, and by exponentiation, an $n \times n$ representation of the group.

8.3 Spinor representation

We can find all possible generators for the representations of the Lorentz with a finite dimension in a systematic manner. Of particular interest to us is the Dirac or spinor representation of the Lorenz group, which has the same dimensionality 4×4 s the vector representation. We can verify, after some algebra, that the 4×4 matrices,

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right], \tag{8.63}$$

satisfy the same commutation relation as in Eq. 8.42. Explicitly,

$$\begin{aligned}&[S^{\mu_1\mu_2}, S^{\mu_3\mu_4}] = \\ &\quad i\left(g^{\mu_2\mu_3}S^{\mu_1\mu_4} + g^{\mu_1\mu_4}S^{\mu_2\mu_3} - g^{\mu_2\mu_4}S^{\mu_1\mu_3} - g^{\mu_1\mu_3}S^{\mu_2\mu_4}\right).\end{aligned} \tag{8.64}$$

Therefore, they are generators of a new representation for the Lorentz group, which is neither the scalar nor the vector representation. We call this new representation "spinor" representation. A spinor field

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \tag{8.65}$$

transforms under as,

$$\psi'(x') = \Lambda_{\frac{1}{2}}\psi(x), \tag{8.66}$$

with a finite spinor transformation being

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}. \tag{8.67}$$

We find that (**exercise**) the commutator of the generator $S^{\mu\nu}$ with a $\gamma-$matrix is

$$[\gamma^\mu, S^{\rho\sigma}] = i\left(\delta^\rho_\mu\gamma^\sigma - \delta^\sigma_\mu\gamma^\rho\right), \tag{8.68}$$

or, in the more suggestive form,

$$[\gamma^\mu, S^{\rho\sigma}] = (J^{\rho\sigma}_{\text{vector}})^\mu_{\ \nu}\,\gamma^\nu. \tag{8.69}$$

The above equation can be exponentiated into a general result,

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_{\ \nu}\gamma^\nu. \tag{8.70}$$

Exercise: Prove Eq. 8.70 in the case of very small Lorentz transformations, by expanding in the transformation parameters $\omega_{\mu\nu}$ both sides of the equation and using Eq 8.69.

In the same manner, we verify that:

$$\Lambda_{\frac{1}{2}}\gamma^\mu\Lambda^{-1}_{\frac{1}{2}} = (\Lambda^{-1})^\mu_{\ \nu}\gamma^\nu. \tag{8.71}$$

8.4 Lorentz Invariance of the Dirac Lagrangian

It is easy to verify that both terms, $\bar{\psi}\psi$ and $\bar{\psi}\partial\!\!\!/\psi$, in the Dirac Lagrangian are invariant under Lorentz transformations. As in many previous instances, we shall use the mathematical properties of Lie groups, which allow us to find transformation rules for representations for small Lorentz transformations, and simply exponentiate the results at the end in order to write down the general transformations.

Starting from the transformation of a spinor field, we have:

$$\begin{aligned} \psi &\rightarrow \Lambda_{\frac{1}{2}}\psi \\ \rightsquigarrow \psi^\dagger &\rightarrow \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \\ &\simeq \psi^\dagger \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^\dagger \\ &\simeq \psi^\dagger \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\dagger\mu\nu}\right) \end{aligned} \tag{8.72}$$

We also have,

$$\begin{aligned} S^{\dagger\mu\nu} = \left(\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]\right)^\dagger &= -\frac{i}{4}\left[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\right] = \frac{i}{4}\left[\gamma^0\gamma^\mu\gamma^0, \gamma^0\gamma^\nu\gamma^0\right] \\ &\rightsquigarrow S^{\mu\nu\dagger} = \gamma^0 S^{\mu\nu}\gamma^0. \end{aligned} \tag{8.73}$$

Thus, we have for the transformation of the conjugate spinor field,

$$\psi^\dagger \rightarrow \psi^\dagger\gamma^0 \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^0, \tag{8.74}$$

and for finite Lorentz transformations,

$$\psi^\dagger \rightarrow \psi^\dagger\gamma^0\Lambda_{\frac{1}{2}}^{-1}\gamma^0. \tag{8.75}$$

We observe that $\psi^\dagger$ does not really transform with the inverse transformation of ψ. However, $\bar{\psi}$ does. Multiplying from the left with γ^0 both sides of the above transformation rule, we obtain

$$\begin{aligned} \psi^\dagger\gamma^0 &\rightarrow \psi^\dagger\gamma^0\Lambda_{\frac{1}{2}}^{-1}\left(\gamma^0\right)^2 \\ \rightsquigarrow \bar{\psi} &\rightarrow \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}. \end{aligned} \tag{8.76}$$

It is now obvious that the mass term in the Dirac Lagrangian is Lorentz invariant,

$$m\bar{\psi}\psi \rightarrow m\bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi = m\bar{\psi}\psi. \tag{8.77}$$

The derivative term of the Dirac equation is also Lorentz invariant,

$$\begin{aligned} \bar{\psi}\gamma^\mu\partial_\mu\psi &\rightarrow \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\left(\Lambda^{-1}\right)_\mu^\nu\Lambda_{\frac{1}{2}}\partial_\nu\psi \\ &= \bar{\psi}\left(\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\left(\Lambda^{-1}\right)_\mu^\nu\partial_\nu\psi \\ &= \bar{\psi}\gamma^\rho\Lambda_\rho^\mu\left(\Lambda^{-1}\right)_\mu^\nu\partial_\nu\psi \\ &= \bar{\psi}\gamma^\rho\delta_\rho^\nu\partial_\nu\psi \\ &= \bar{\psi}\partial\!\!\!/\psi. \end{aligned} \tag{8.78}$$

We see that the Dirac Lagrangian contains only the two simplest scalar terms that may be formed out of fields transforming in the spinor representation of the Lorentz group. The relative sign between these two terms is easy to determine by requiring that the equations of motion will also satisfy the Klein-Gordon equation, which gives the relativistically correct energy momentum relations, $p^2 = m^2$.

8.5 General representations of the Lorentz group

The scalar, vector, and spinor representations are the simplest representations of the Lorentz group. However, there is an infinite number of $n \times n$ representations. All of them can be found easily by observing that the Lie algebra of the Lorentz group is the same as for an $SU(2) \times SU(2)$ group. Let us consider the generators in an arbitrary representation, which satisfy the commutation relationship,

$$\begin{aligned}[J^{\mu_1\mu_2}, J^{\mu_3\mu_4}] &= \\ & i\left(g^{\mu_2\mu_3}J^{\mu_1\mu_4} + g^{\mu_1\mu_4}J^{\mu_2\mu_3} - g^{\mu_2\mu_4}J^{\mu_1\mu_3} - g^{\mu_1\mu_3}J^{\mu_2\mu_4}\right).\end{aligned} \tag{8.79}$$

We now define the linear combinations,

$$K_i \equiv J^{0i}, \tag{8.80}$$

and

$$L_i \equiv \frac{1}{2}\epsilon_{ijk}J^{jk}. \tag{8.81}$$

We can change the basis of generators comprising from the six [3] generators $J^{\mu\nu}$ with the linear combinations $\{L^i, K^i\}$. The generators L^i are generators of rotations around the axis$-i$, and the generators K^i are generators of boosts in the direction$-i$ **(exercise: demonstrate this for the generators in the vectorial representation)**. Rotations are a subgroup of the Lorentz group, which satisfy an $SU(2)$ algebra. We find the commutation relations,

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{8.82}$$

However, boosts do not form a subgroup of the Lorentz group. Two boosts in two different directions are not a boost in a third direction, but rather a rotation. We find that the commutators of boost generators are,

$$[K_i, K_j] = -i\epsilon_{ijk}L_k. \tag{8.83}$$

The commutator of a boost and a rotation generator is,

$$[L_i, K_j] = i\epsilon_{ijk}K_k. \tag{8.84}$$

Define now the linear combinations,

$$J_i^{\pm} \equiv \frac{L_i \pm iK_i}{2}. \tag{8.85}$$

Then the Lie algebra breaks into two independent $SU(2)$ algebras,

$$\left[J_i^+, J_j^+\right] = i\epsilon_{ijk}J_k^+, \tag{8.86}$$

[3]they are antisymmetric

$$\left[J_i^-, J_j^-\right] = i\epsilon_{ijk} J_k^-, \tag{8.87}$$

and

$$\left[J_i^+, J_j^-\right] = 0. \tag{8.88}$$

We have already learned about the representations of the $SU(2)$ group in Quantum Mechanics. These can be labeled according to a "spin" number,

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

Each representation acts on $(2j+1)$ objects ψ_m with

$$m = -j, -j+1, \ldots j-1, j, \tag{8.89}$$

transforming them into each other under $SU(2)$.

A general representation of the Lorentz group acts on $(2j^+ + 1)(2j^- + 1)$ objects ψ_{m^+, m^-}, and is labeled by the numbers (j^+, j^-), e.g.

$$(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (0, 1), (\frac{1}{2}, \frac{1}{2}), \ldots. \tag{8.90}$$

The massive Dirac field transforms in the mixed representation,

$$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}).$$

8.6 Weyl spinors

As we have already noted, $\gamma-$matrices are not unique, and there are many equivalent representations. We introduce here a representation which shows better the decomposition of the Lorentz group into two $SU(2)$ subgroups. This is called the chiral or Weyl representation, defined as

$$\gamma^0 = I_{2\times 2} \otimes \sigma_1, \qquad \gamma^i = \sigma_i \otimes (i\sigma_2). \tag{8.91}$$

As an exercise, you can prove that indeed this set of gamma-matrices satisfies the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_{4\times 4}. \tag{8.92}$$

In the chiral representation, the generators of the spinorial Lorentz group representation

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right], \tag{8.93}$$

take a block-diagonal form,

$$S^{0j} = -\frac{i}{2}\sigma_j \otimes \sigma_3 = -\frac{i}{2}\begin{pmatrix} \sigma_j & 0_{2\times 2} \\ 0_{2\times 2} & -\sigma_j \end{pmatrix}, \tag{8.94}$$

and

$$S^{jk} = \frac{1}{2}\epsilon_{jki}\sigma_l \otimes I_{2\times 2} = \frac{1}{2}\epsilon_{jkl}\begin{pmatrix} \sigma_l & 0_{2\times 2} \\ 0_{2\times 2} & \sigma_l \end{pmatrix}. \tag{8.95}$$

A Lorentz transformation for a spinor is

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}, \tag{8.96}$$

and the exponent takes the explicit form,

$$\begin{aligned} -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} &= -\frac{i}{2}\omega_{0j}S^{0j} - \frac{i}{2}\omega_{j0}S^{j0} - \frac{i}{2}\omega_{jk}S^{jk} \\ &= -\vec{\beta}\cdot\left(\frac{\vec{\sigma}}{2}\otimes\sigma_3\right) - i\vec{\theta}\cdot\left(\frac{\vec{\sigma}}{2}\otimes I_{2\times 2}\right), \end{aligned} \tag{8.97}$$

with

$$\beta_j = \frac{\omega_{0j}}{2} \text{ and } \theta_l = \frac{1}{2}\epsilon_{jkl}\omega^{jk}. \tag{8.98}$$

Under a small Lorentz transformation a spinor ψ transforms as,

$$\begin{aligned} \psi \to \psi' &= e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi \\ &\simeq \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi \\ &= \begin{pmatrix} \left[1 - \frac{i}{2}\left(\vec{\theta} - i\vec{\beta}\right)\cdot\vec{\sigma}\right]\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \\ \\ \left[1 - \frac{i}{2}\left(\vec{\theta} + i\vec{\beta}\right)\cdot\vec{\sigma}\right]\begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix} \end{pmatrix} \end{aligned} \tag{8.99}$$

The two-dimensional objects,

$$\psi_L \equiv \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \text{ and } \psi_R \equiv \begin{pmatrix}\psi_3 \\ \psi_4\end{pmatrix}, \tag{8.100}$$

are called "Weyl spinors" and transform independently under a Lorentz transformation,

$$\psi_L \to \psi'_L = \left[1 - \frac{i}{2}\left(\vec{\theta} - i\vec{\beta}\right)\cdot\vec{\sigma}\right]\psi_L, \tag{8.101}$$

and

$$\psi_R \to \psi'_R = \left[1 - \frac{i}{2}\left(\vec{\theta} + i\vec{\beta}\right)\cdot\vec{\sigma}\right]\psi_R. \tag{8.102}$$

In an exponential form,

$$\psi_L \to \psi'_L = \Lambda_L\psi_L \text{ and } \psi_L \to \psi'_R = \Lambda_R\psi_R, \tag{8.103}$$

with

$$\Lambda_L = e^{-\frac{i}{2}\left(\vec{\theta} - i\vec{\beta}\right)\cdot\vec{\sigma}}, \tag{8.104}$$

and

$$\Lambda_R = e^{-\frac{i}{2}\left(\vec{\theta} + i\vec{\beta}\right)\cdot\vec{\sigma}}. \tag{8.105}$$

We now write the Dirac equation in terms of Weyl spinors ψ_L and ψ_R,

$$\begin{aligned} 0 &= \left(i\partial\!\!\!/ - m\right)\psi \\ &= \left(i\gamma^0\partial_0 + i\gamma^j\partial_j - m1_{4\times 4}\right)\psi \\ &= \left(i(I_{2\times 2}\otimes\sigma_1)\partial_0 - i\left(\vec{\sigma}\otimes(i\sigma_2)\right)\vec{\nabla} - m1_{2\times 2}\otimes 1_{2\times 2}\right)\psi \\ &\rightsquigarrow \begin{pmatrix} -m & i\left(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}\right) \\ i\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0. \end{aligned} \tag{8.106}$$

The mass m mixes the two Weyl spinors ψ_L, ψ_R. However, for a massless particle the Dirac equation is diagonalized into two independent equations for the Weyl spinors,

$$\partial_0 \psi_R = -\vec{\sigma} \cdot \vec{\nabla} \psi_R, \tag{8.107}$$

and

$$\partial_0 \psi_L = \vec{\sigma} \cdot \vec{\nabla} \psi_L. \tag{8.108}$$

These are known as Weyl equations. For plane wave solutions (which correspond to particles of definite momentum),

$$\psi_{L,R}(\vec{x}, t) = u_{L,R} e^{-ip\cdot x}, \tag{8.109}$$

Weyl equations turn into:

$$\frac{\vec{p} \cdot \vec{\sigma}}{|\vec{p}|} \psi_L = -\psi_L \tag{8.110}$$

and

$$\frac{\vec{p} \cdot \vec{\sigma}}{|\vec{p}|} \psi_R = +\psi_R. \tag{8.111}$$

Therefore, the $\psi_{L,R}$ plane wave solutions are eigenstates of the "helicity operator"

$$\hat{h} \equiv \frac{\vec{p} \cdot \vec{\sigma}}{|\vec{p}|} = \hat{p} \cdot \sigma, \tag{8.112}$$

which measures the projection of the spin of the particle on the direction of the motion. The spin of the "right-handed" spinors ψ_R is aligned to the momentum of the particle, while it points to the opposite direction for "left-handed" spinors ψ_L. Notice that only massless particles can have definite helicity, while for massive particles the value of the helicity is frame dependent.

Let us now return to the general case of massive particles. We have:

$$\bar{\psi} = \psi^\dagger \gamma^0 = \left(\psi_L^\dagger, \psi_R^\dagger\right) \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 0_{2\times 2} \end{pmatrix} = \left(\psi_R^\dagger, \psi_L^\dagger\right) \tag{8.113}$$

Then, the Dirac Lagrangian in terms of Weyl spinors becomes (**exercise**):

$$\begin{aligned} \mathcal{L}_{Dirac} &= \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \\ &= i\psi_L^\dagger \left(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}\right)\psi_L + i\psi_R^\dagger \left(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}\right)\psi_R - m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right) \end{aligned} \tag{8.114}$$

We observe that it is only the mass term which mixes the left-handed and right-handed spinors. For a massless theory, the left-handed and right-handed fields are agnostic and oblivious of each other.

Finally, we note (**exercise**) that the Dirac Lagrangian is symmetric under a U(1) pgase-transformation,

$$\psi \to \psi' = e^{i\theta}\psi \leftrightarrow \psi_{L,R} \to \psi'_{L,R} = e^{i\theta}\psi_{L,R} \tag{8.115}$$

8.7 Majorana equation

Exercise: *Let's prove the identity,*

$$\sigma_2 \vec{\sigma}^* = -\vec{\sigma}\sigma_2. \tag{8.116}$$

Then , the spinor

$$\psi_L^c \equiv \sigma_2 \psi_L^*, \tag{8.117}$$

transforms as,

$$\begin{aligned}
\sigma_2 \psi_L^* &\to \sigma_2 (1 - \frac{i}{2}\left(\vec{\theta} - i\vec{\beta}\right)\cdot\vec{\sigma})^* \psi_L^* \\
&= \sigma_2 (1 + \frac{i}{2}\left(\vec{\theta} + i\vec{\beta}\right)\cdot\vec{\sigma}^*) \psi_L^* \\
&= (1 - \frac{i}{2}\left(\vec{\theta} + i\vec{\beta}\right)\cdot\vec{\sigma}) \sigma_2 \psi_L^* \\
&\leadsto \psi_L^c \to \Lambda_R \psi_L^c
\end{aligned} \tag{8.118}$$

Similarly,

$$\psi_R^c \to \Lambda_L \psi_R^c. \tag{8.119}$$

We found a somewhat surprising result, that the spinor ψ_L^C (which is made out of the components of a left-handed Weyl spinor) transforms as a right-handed Weyl-spinor. The Dirac Lagrangian for a massive spin-$\frac{1}{2}$ fermion is therefore not the only one which is Lorentz invariant. We could write a Lagrangian made exclusively out of left-handed spinors, replacing ψ_R with ψ_L^c in the mass term of the Dirac Lagrangian of Eq. 8.114. The Lagrangian we obtain in this way is due to Majorana:

$$\begin{aligned}
\mathcal{L}_{Maj} &= i\psi_L^\dagger \left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)\psi_L - \frac{m}{2}\left(\psi_L^\dagger \psi_L^c + \psi_L^{c\dagger}\psi_L\right) \\
&= i\psi_L^\dagger \left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)\psi_L - \frac{m}{2}\left[\psi_L^T \sigma_2 \psi_L + \psi_L^\dagger \sigma_2 \psi_L^*\right]
\end{aligned} \tag{8.120}$$

Exercise: *What is the Majorana Lagrangian for a right-handed Weyl spinor?*

Notice, that unlike the Dirac Lagrangian, the Majorana Lagrangian is not invariant under a U(1) transformation. In particular, under

$$\psi_L \to e^{i\theta}\psi_L,$$

the mass term depends on theta, since

$$\psi_L^T (i\sigma_2)\psi_L \to e^{2i\theta}\psi_L^T (i\sigma_2)\psi_L, \quad \psi_L^\dagger (i\sigma_2)\psi_L^* \to e^{-2i\theta}\psi_L^\dagger (i\sigma_2)\psi_L^*$$

The absense of a U(1) symmetry means that there is no conserved quantity which we can assign to the electric charge. Therefore, the Majorana equation could not be used to describe spin-$\frac{1}{2}$ electrically charged particles, such as electrons.

For a long time, it was believed that only the Dirac equation had a physical realisation and that the Majorana equation was a mere theoretical possibility. However, it has now been shown experimentally that neutrinos, which have no electric charge and electromagnetic interactions, have a mass. It is believed that the Majorana equation can be used to give a mass contribution to neutrinos. It is still an open experimental question whether the mass of the neutrinos is "Dirac" or "Majorana".

Let us now focus on the mass term of the Lagrangian. If we write

$$\psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{8.121}$$

then

$$\psi_L^c = \sigma_2 \psi_L^* = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = i \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix} \tag{8.122}$$

and then

$$\psi_l^\dagger \psi_L^c = i \left(-\psi_1^* \psi_2^* + \psi_2^* \psi_1^* \right). \tag{8.123}$$

If psi_1, ψ_2 are customary classical fields which commute with each other, then the above expression is zero and therefore the mass term does not exist. However, there is another possibility. If we impose that the ψ_i's are anticommuting (so called, *Grassmann variables*) then

$$\psi_1^* \psi_2^* = -\psi_2^* \psi_1^*. \tag{8.124}$$

8.7.1 Majorana Lagrangian and Majorana equation in a four-dimensional spinor notation*

The equations of motion give the Majorana equation

$$\left(\partial_0 - \vec{\sigma} \cdot \vec{\nabla} \right) \psi_L = m \psi_L^c \tag{8.125}$$

Exercise: *Prove that in the massless limit:*

$$\hat{h} \psi_L^c = +\psi_L^c, \quad \hat{h} \psi_R^c = -\psi_R^c.$$

Chapter 9

Classical solutions of the Dirac equation

In this chapter, we shall solve classically the equations of motion for the Dirac field, i.e. the classical Dirac equation. As we have already demonstrated, a field ψ which satisfies the Dirac equation it also satisfies the Klein-Gordon equation. So, it must admit a plane-wave solution. We consider first the general case

$$\psi(x) = u(p)e^{-ip\cdot x}, \tag{9.1}$$

with

$$p^2 = m^2. \tag{9.2}$$

For now, we shall restrict ourselves to the cases with positive energy solutions, $p^0 > 0$. Substituting into the Dirac equation, we obtain additional restrictions for the spinor $u(p)$,

$$(i\partial\!\!\!/ - m)\, u(p)e^{-ip\cdot x} = 0$$

$$\leadsto \quad (p\!\!\!/ - m)\, u(p) = 0. \tag{9.3}$$

Using the Weyl representation for $\gamma-$matrices, $\gamma^0 = I_{2\times2} \otimes \sigma_1, \gamma^j = \sigma_j \otimes (i\sigma_2)$, we obtain

$$\begin{pmatrix} -mI_{2\times2} & p^0 I_{2\times2} - \vec{\sigma}\cdot\vec{p} \\ p^0 I_{2\times2} + \vec{\sigma}\cdot\vec{p} & -mI_{2\times2} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9.4}$$

where we write the four-component spinor $u(p)$ in terms of two Weyl-spinors ξ, η with two components each:

$$u(p) \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{9.5}$$

It is of course possible to find solutions of the above equation directly. However, we may also want to use some cleverness, and exploit the fact that the field ψ transforms as a spinor. Consequently,

$$\psi'(x') = \Lambda_{\frac{1}{2}}\psi(x), \tag{9.6}$$

and equivalently

$$\begin{aligned} u'(p')e^{-ip'\cdot x'} &= \Lambda_{\frac{1}{2}} u(p) e^{-ip\cdot x} \\ \rightsquigarrow u'(\Lambda p)e^{-i(\Lambda p)\cdot(\Lambda x)} &= \Lambda_{\frac{1}{2}} u(p) e^{-ip\cdot x} \\ \rightsquigarrow u'(\Lambda p)e^{-ip\cdot x} &= \Lambda_{\frac{1}{2}} u(p) e^{-ip\cdot x} \\ \\ u'(\Lambda p) &= \Lambda_{\frac{1}{2}} u(p). \end{aligned} \tag{9.7}$$

and $u(p)$ also transforms as a spinor, under a Lorentz transformation,

$$p^\mu \to p^{\mu\prime} = \Lambda^\mu_{\ \nu} p^\nu. \tag{9.8}$$

The strategy for finding a general solution $u(p)$ is then simple. We first solve the Dirac equation for an especially simple value of the momentum p^μ, and then we perform a Lorentz transformation to obtain the result for a general value of the momentum.

9.1 Solution in the rest frame

The Dirac equation (Eq. 9.4) assumes its simplest form in the rest frame,

$$p^\mu = m(1, \vec{0}). \tag{9.9}$$

yielding,

$$\begin{pmatrix} -mI_{2\times 2} & mI_{2\times 2} \\ mI_{2\times 2} & -mI_{2\times 2} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{9.10}$$

and, equivalently,

$$\xi = \eta. \tag{9.11}$$

The general solution for the spinor u in the rest frame is,

$$u\left(p^\mu = (m, \vec{0})\right) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}. \tag{9.12}$$

The factor $\sqrt{m}$ is a convenient normalization for future purposes, and the Weyl spinor ξ is arbitrary. We can choose it to be a linear combination,

$$\xi = \sum_{s=+,-} c_s \xi^s, \tag{9.13}$$

with

$$\xi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \xi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.14}$$

Notice that this basis diagonalizes σ_3, with

$$\sigma_3 \xi^\pm = \pm \xi^\pm. \tag{9.15}$$

From this solution of the Dirac equation in the rest frame, we can go to any other momentum value with a boost.

9.2 Lorentz boost of rest frame Dirac spinor along the z-axis

First, we boost any vector along the z-axis, with a very small Lorentz transformation,

$$p^\mu \to p^{\mu\prime} = \Lambda^\mu_{\ \nu} p^\nu \simeq p^\mu + \omega^\mu_{\ \nu} p^\nu, \tag{9.16}$$

where,

$$\omega_{\mu\nu} = 0, \text{ except } \omega_{03} = -\omega_{30} = Y. \tag{9.17}$$

Notice that,

$$\omega^0_{\ 3} = g^{00}\omega_{03} = Y, \qquad \omega^3_{\ 0} = g^{33}\omega_{30} = -(-Y) = Y. \tag{9.18}$$

The transformed momenta are,

$$p^{0\prime} = p^0 + Y p^3, p^{1\prime} = p^1, p^{2\prime} = p^2, p^{3\prime} = p^3 + Y p^0, \tag{9.19}$$

which is conveniently written as,

$$\begin{pmatrix} p^{0\prime} \\ p^{3\prime} \end{pmatrix} = (I_{2\times 2} + Y\sigma_1) \begin{pmatrix} p^0 \\ p^3 \end{pmatrix}. \tag{9.20}$$

For a finite boost of a vector along the z-direction, we have

$$\begin{pmatrix} p^{0\prime} \\ p^{3\prime} \end{pmatrix} = e^{Y\sigma_1} \begin{pmatrix} p^0 \\ p^3 \end{pmatrix}. \tag{9.21}$$

We can compute the exponential easily, noting that $\sigma_1^2 = I_{2\times 2}$,

$$\begin{aligned} e^{Y\sigma_1} &= \sum_{n=0}^{\infty} \frac{Y^n}{n!}\sigma_1^n \\ &= \sum_{k=0}^{\infty} \frac{Y^{2k}}{(2k)!}\sigma_1^{2k} + \sum_{k=0}^{\infty} \frac{Y^{2k+1}}{(2k+1)!}\sigma_1^{2k+1} \\ &= I_{2\times 2}\sum_{k=0}^{\infty} \frac{Y^{2k}}{(2k)!} + \sigma_1 \sum_{k=0}^{\infty} \frac{Y^{2k+1}}{(2k+1)!} \\ \rightsquigarrow e^{Y\sigma_1} &= I_{2\times 2}\cosh Y + \sigma_1 \sinh Y. \end{aligned} \tag{9.22}$$

We apply this transformation for the momentum vector at rest,

$$\begin{aligned} \begin{pmatrix} m \\ 0 \end{pmatrix} \to \begin{pmatrix} E \\ p^3 \end{pmatrix} &= (I_{2\times 2}\cosh Y + \sigma_1 \sinh Y) \begin{pmatrix} m \\ 0 \end{pmatrix} \\ \rightsquigarrow \begin{pmatrix} E \\ p^3 \end{pmatrix} &= \begin{pmatrix} m\cosh Y \\ m \sinh Y \end{pmatrix}. \end{aligned} \tag{9.23}$$

We can then invert the above equations and compute the parameter Y which generates the Lorentz boost, known as *rapidity*, in terms of the components of the boosted momentum. We find (**exercise:**)

$$e^Y = \frac{E + p^3}{m}, \quad e^{-Y} = \frac{E - p^3}{m}, \qquad Y = \frac{1}{2}\ln\frac{E+p^3}{E-p^3}. \tag{9.24}$$

Let us now find the transformation of a spinor, under the same Lorentz boost. The transformation matrix is,

$$\begin{aligned}
\Lambda_{\frac{1}{2}} &= e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} = e^{-\frac{i}{2}\omega_{03}S^{03}-\frac{i}{2}\omega_{30}S^{30}} = e^{-\frac{Y}{2}\sigma_3\otimes\sigma_3} \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}\left(\sigma_3\otimes\sigma_3\right)^n \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}\sigma_3^n\otimes\sigma_3^n \\
&= \sum_{n=\text{odd}}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}\sigma_3^n\otimes\sigma_3^n + \sum_{n=\text{even}}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}\sigma_3^n\otimes\sigma_3^n \\
&= \sum_{n=\text{odd}}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}\sigma_3\otimes\sigma_3 + \sum_{n=\text{even}}^{\infty}\frac{\left(-\frac{Y}{2}\right)^n}{n!}I_{2\times 2}\otimes I_{2\times 2} \\
\rightsquigarrow \Lambda_{\frac{1}{2}} &= \cosh\left(\frac{Y}{2}\right) I_{2\times 2}\otimes I_{2\times 2} - \sinh\left(\frac{Y}{2}\right)\sigma_3\otimes\sigma_3. \qquad (9.25)
\end{aligned}$$

Applying this spin boost transformation to our solution of the Dirac equation at the rest frame, we obtain

$$\begin{aligned}
u(p) &= \Lambda_{\frac{1}{2}}\sqrt{m}\begin{pmatrix}\xi^s \\ \xi^s\end{pmatrix} \\
\rightsquigarrow u^s(p) &= \begin{pmatrix}\left[\sqrt{E+p^3}\frac{(1-\sigma_3)}{2} + \sqrt{E-p^3}\frac{(1+\sigma_3)}{2}\right]\xi^s \\ \left[\sqrt{E+p^3}\frac{(1+\sigma_3)}{2} + \sqrt{E-p^3}\frac{(1-\sigma_3)}{2}\right]\xi^s\end{pmatrix} \qquad (9.26)
\end{aligned}$$

where we have used the expression for the rapidity, derived in Eq. 9.24. We observe that,

$$\left(\frac{1\pm\sigma_3}{2}\right)^2 = \frac{1\pm\sigma_3}{2}, \qquad (9.27)$$

or, equivalently,

$$\frac{1\pm\sigma_3}{2} = \sqrt{\frac{1\pm\sigma_3}{2}}, \qquad (9.28)$$

and that

$$\frac{1+\sigma_3}{2}\cdot\frac{1-\sigma_3}{2} = 0. \qquad (9.29)$$

We can then collect all terms in Eq. 9.26 under a common square root,

$$u^s(p) = \begin{pmatrix}\sqrt{(E+p^3)\frac{(1-\sigma_3)}{2} + (E-p^3)\frac{(1+\sigma_3)}{2}}\,\xi^s \\ \sqrt{(E+p^3)\frac{(1+\sigma_3)}{2} + (E-p^3)\frac{(1-\sigma_3)}{2}}\,\xi^s\end{pmatrix} \qquad (9.30)$$

which simplifies to

$$u^s(p) = \begin{pmatrix}\sqrt{E-p^3\sigma_3}\,\xi^s \\ \sqrt{E+p^3\sigma_3}\,\xi^s\end{pmatrix} \qquad (9.31)$$

This is the general solution for the spinor $u(p)$, when $p^\mu = (E, 0, 0, p^3)$.

9.3 Solution for an arbitrary vector

We can obtain the spinor solution $u(p)$ of the Dirac equation for an arbitrary vector p^μ, by performing a rotation of the solution that we found for $p^\mu = (E, 0, 0, p^3)$. In fact, things are simpler now and there is not much work to do, if we can write down the previous result in a covariant form.

We define,

$$\sigma^\mu \equiv (1, \vec{\sigma}), \tag{9.32}$$

and

$$\bar{\sigma}^\mu \equiv (1, -\vec{\sigma}). \tag{9.33}$$

Then the expression for the spinor $u^s(p)$, Eq. 9.31, is written as

$$u^s(p) = \begin{pmatrix} \sqrt{\sigma \cdot p}\, \xi^s \\ \sqrt{\bar{\sigma} \cdot p}\, \xi^s \end{pmatrix} \tag{9.34}$$

You can verify (**exercise**) that this is actually a solution of the Dirac equation for an arbitrary vector p^μ. For this, you will need the generally useful identity,

$$(\sigma \cdot p)(\bar{\sigma} \cdot p) = p^2 = m^2, \tag{9.35}$$

and to rewrite the Dirac differential operator as

$$\not{p} - m = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} \tag{9.36}$$

in Eq. 9.4.

9.4 A general solution

We have found a general solution for the Dirac equation, of the form

$$\psi(x) = u^s(p) e^{-ip\cdot x}, \text{ with } p^0 > 0. \tag{9.37}$$

Another possibility is,

$$\psi(x) = v(p) e^{+ip\cdot x}, \text{ with } p^0 > 0. \tag{9.38}$$

Substituting into the Dirac equation we find that the spinor $v^s(p)$ satisfies

$$(\not{p} + m)\, v(p) = 0. \tag{9.39}$$

Repeating the same steps as for $u(p)$, we find this equation is satisfied by spinors

$$v^s(p) = \begin{pmatrix} \sqrt{\sigma \cdot p}\, \xi^s \\ -\sqrt{\bar{\sigma} \cdot p}\, \xi^s \end{pmatrix}. \tag{9.40}$$

A general solution for the classical Dirac equation is,

$$\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^2 2\omega_p} \left[a_s(p) u^s(p) e^{-ip\cdot x} + b_s^*(p) v^s(p) e^{+ip\cdot x} \right], \tag{9.41}$$

where the $a_s(p)$ and $b_s(p)$ are arbitrary coefficients, and

$$\omega_p = p^0 = \sqrt{\vec{p}^2 + m^2}. \tag{9.42}$$

We have not discussed explicitly solutions with $p^0 < 0$. At the classical level, these are included in the above integral, as you can easily verify by repeating here the analysis of Section 5.1.1. Dirac interpreted these negative energy solutions with an ingenious theory (Dirac sea), which lead to the discovery of antimatter. We do not need to bother with this interpretation, except out of historic interest. As with the Klein-Gordon equation, field quantization gives consistently rise to physical states which have a positive energy.

Chapter 10

Quantization of the Dirac Field

To quantize the Dirac field, we promote the field into an operator,

$$\psi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \sum_s \left[a_s(p) u_s(p) e^{-ip\cdot x} + b_s^\dagger(p) v_s(p) e^{ip\cdot x} \right]. \tag{10.1}$$

The conjugate momentum of the field is,

$$\pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger(\vec{x},t), \tag{10.2}$$

where the Lagrangian of the Dirac field is,

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{10.3}$$

The conjugate field is given by,

$$\psi^\dagger(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \sum_s \left[a_s^\dagger(p) u_s^\dagger(p) e^{+ip\cdot x} + b_s(p) v_s^\dagger(p) e^{ip\cdot x} \right]. \tag{10.4}$$

We also find,

$$\bar{\psi}(\vec{x},t) \equiv \psi^\dagger(\vec{x},t)\gamma^0 = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \sum_s \left[a_s^\dagger(p) \bar{u}_s(p) e^{+ip\cdot x} + b_s(p) \bar{v}_s^\dagger(p) e^{ip\cdot x} \right]. \tag{10.5}$$

In terms of ψ and $\bar{\psi}$, the operators a_a, b_s are written as,

$$a_s(p) = \int d^3\vec{x}\,\bar{u}_s(p) e^{ip\cdot x}\gamma^0\psi(x), \tag{10.6}$$

and

$$b_s(p) = \int d^3\vec{x}\,\bar{\psi}(x)\gamma^0 e^{-ip\cdot x} v_s(p). \tag{10.7}$$

To prove the above you will need the spinor summations,

$$\sum_s u_s(p)\bar{u}_s(p) = p\!\!\!/ + m, \text{ and } \sum_s v_s(p)\bar{v}_s(p) = p\!\!\!/ - m, \tag{10.8}$$

which can be derived easily from the explicit expressions for $u_s(p), v_s(p)$ of the last chapter.

Using Noether's theorem, we can compute the Hamiltonian of the Dirac field, which is

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_k)}\omega_k \sum_s \left[a_s^\dagger(k)a_s(k) - b_s(k)b_s^\dagger(k)\right]. \tag{10.9}$$

In the above, we have been careful to preserve the order of operators as they appear.

We can now proceed with the quantization procedure. It is left as an exercise to show that imposing commutation relations, leads to particle states $b_s^\dagger(p)\left|0\right\rangle$ which have a negative energy. This is due to the minus sign in the second term of the Hamiltonian integral in Eq. 10.9. To obtain positive energy eigenvalues for such states, we are forced to quantize by requiring anti-commutation relations,

$$\{\psi(\vec{x}_1,t), \pi(\vec{x}_2,t)\} = i\delta^{(3)}(\vec{x}_1 - \vec{x}_2), \tag{10.10}$$

and

$$\{\psi(\vec{x}_1,t), \psi(\vec{x}_2,t)\} = \{\pi(\vec{x}_1,t), \pi(\vec{x}_2,t)\} = 0. \tag{10.11}$$

For the operators a_s, b_s this results to the anticommutation relations,

$$\left\{a_s(\vec{p}_1), a_{s'}^\dagger(\vec{p}_2)\right\} = \left\{b_s(\vec{p}_1), b_{s'}^\dagger(\vec{p}_2)\right\} = \delta_{s's}(2\pi)^3(2\omega_{p_1})\delta^{(3)}(\vec{p}_2 - \vec{p}_1), \tag{10.12}$$

$$\{a_s(\vec{p}_1), a_{s'}(\vec{p}_2)\} = \{b_s(\vec{p}_1), b_{s'}(\vec{p}_2)\} = 0, \tag{10.13}$$

$$\left\{a_s^\dagger(\vec{p}_1), a_{s'}^\dagger(\vec{p}_2)\right\} = \left\{b_s^\dagger(\vec{p}_1), b_{s'}^\dagger(\vec{p}_2)\right\} = 0, \tag{10.14}$$

$$\left\{a_s(\vec{p}_1), b_{s'}^\dagger(\vec{p}_2)\right\} = \left\{b_s(\vec{p}_1), a_{s'}^\dagger(\vec{p}_2)\right\} = 0. \tag{10.15}$$

Using the anticommutation relation of Eq. 10.12, we cast the Hamiltonian of Eq. 10.9 as,

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_k)}\omega_k \sum_s \left[a_s^\dagger(k)a_s(k) + b_s^\dagger(k)b_s(k)\right] + \left\langle 0\right| H \left|0\right\rangle, \tag{10.16}$$

with

$$\left\langle 0\right| H \left|0\right\rangle = -\delta^{(3)}(0)\int d^3\vec{k}\omega_k. \tag{10.17}$$

Exercise: *Notice that the sign of the vacuum energy is now negative.*

1. *What would be the Casimir force in a gedanken experiment where we could impose that the Dirac field vanishes on two-parallel plates?*

2. *In supersymmetric theories, for each fermion field exists a symmetric boson field, with the same mass. What is the Casimir force?*

As in the Klein-Gordon field, we can redefine the field Hamiltonian to have zero vacuum expectation value,

$$: H := H - \left\langle 0\right| H \left|0\right\rangle. \tag{10.18}$$

The subtraction of the vacuum energy is implemented at the operator level with normal ordering, which in the case of anticommuting fields requires an additional (-) sign,

$$:\ b_s(k)b_s^\dagger(k)\ := -b_s^\dagger(k)b_s(k). \tag{10.19}$$

Again with the use of Noether's theorem, we find that the momentum is conserved and takes the form,

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_k)}\,\vec{k}\sum_s \left[a_s^\dagger(k)a_s(k) + b_s^\dagger(k)b_s(k)\right]. \tag{10.20}$$

10.1 One-particle states

It is left as an exercise to repeat the steps that we followed in the case of the Schrödinger and the Klein-Gordon fields, and to to prove that

- there exists a vacuum state with

$$a_s(p)|0\rangle = b_s(p)|0\rangle = 0, \tag{10.21}$$

- the operators $a_s(p), b_s(p), a_s^\dagger(p), b_s^\dagger(p)$ ladder operators.

We can build two types of one-particle states, using either the $a_s^\dagger$ creation operator

$$|p, s\rangle_a \equiv a_s^\dagger(p)|0\rangle, \tag{10.22}$$

or the $b_s^\dagger$ creation operator

$$|p, s\rangle_b \equiv b_s^\dagger(p)|0\rangle. \tag{10.23}$$

These states are degenerate eigenstates of the Hamiltonian and momentum operators. We define the field momentum four-vector,

$$P^\mu \equiv (:\ H\ :,\ \vec{P}). \tag{10.24}$$

Then

$$P^\mu |p, s\rangle_a = p^\mu |p, s\rangle_a, \tag{10.25}$$

and

$$P^\mu |p, s\rangle_b = p^\mu |p, s\rangle_b, \tag{10.26}$$

with

$$p^\mu = \left(\sqrt{\vec{p}^2 + m^2}, \vec{p}\right). \tag{10.27}$$

These are states of positive energy, and the correct relativistic energy-momentum relation,

$$p^2 = m^2. \tag{10.28}$$

10.1.1 Particles and anti-particles

The Dirac Lagrangian is symmetric under the $U(1)$ transformation,

$$\psi(x) \to \psi'(x) = e^{i\theta}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger\prime}(x) = e^{-i\theta}\psi^\dagger(x). \tag{10.29}$$

For small values of θ, we find

$$\delta\psi = \psi' - \psi = i\theta\psi, \tag{10.30}$$

and

$$\delta\psi^\dagger = \psi^{\dagger\prime} - \psi^\dagger = -i\theta\psi^\dagger, \tag{10.31}$$

or, equivalently,

$$\delta\bar{\psi} = \bar{\psi}' - \bar{\psi} = -i\theta\bar{\psi}. \tag{10.32}$$

Using Noether's theorem, we find that the current,

$$\begin{aligned} J^\mu &= \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi, \\ J^\mu &= \bar{\psi}\gamma^\mu\psi, \end{aligned} \tag{10.33}$$

is conserved, $\partial_\mu J^\mu = 0$. The corresponding conserved charge is,

$$Q = \int d^3\vec{x} J^0 = \int d^3\vec{x} \bar{\psi}\gamma^0\psi = \int d^3\vec{x}\psi^\dagger\psi. \tag{10.34}$$

We can express Q in terms of creation and annihilation operators,

$$Q = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s \left[a_s^\dagger(k)a_s(k) + b_s(k)b_s^\dagger(k)\right]. \tag{10.35}$$

We calibrate the charge of the vacuum state to be zero with normal ordering,

$$\begin{aligned} :Q: &= Q - \langle 0| Q |0\rangle \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s \left[: a_s^\dagger(k)a_s(k) : + : b_s(k)b_s^\dagger(k) :\right] \\ :Q: &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s \left[a_s^\dagger(k)a_s(k) - b_s^\dagger(k)b_s(k)\right]. \end{aligned} \tag{10.36}$$

The states and $|p,s\rangle_a$ and $|p,s\rangle_b$ are non-degenerate eigenstates of the charge operator. We find,

$$:Q: |p,s\rangle_a = (+1)\,|p,s\rangle_a\,, \tag{10.37}$$

and

$$:Q: |p,s\rangle_b = (-1)\,|p,s\rangle_b\,. \tag{10.38}$$

Therefore the operators $a_s^\dagger(p)$ give rise to particles with momentum p^μ and charge $(+1)$, while the operators $b_s^\dagger(p)$ give rise to anti-particles with the same momentum p^μ and opposite charge (-1). To emphasize this, we shall display explicitly the charge of one-particle states,

$$|p,s,+\rangle \equiv |p,s\rangle_a = a_s^\dagger(p)\,|0\rangle\,, \text{ and } |p,s,-\rangle \equiv |p,s\rangle_b = b_s^\dagger(p)\,|0\rangle\,. \tag{10.39}$$

10.1.2 Particles and anti-particles of spin-$\frac{1}{2}$

We have demonstrated in previous chapters that the Dirac Lagrangian is invariant under Lorentz transformations. The spinor field transforms "at a point" as,

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x). \tag{10.40}$$

The variation of the field "at a point" is

$$\begin{aligned} \delta_*\psi &\equiv \psi'(x) - \psi(x) \\ &= \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) - \psi(x) \\ &= \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x^\mu - \omega^\mu_\nu x^\nu) - \psi(x^\mu) + \mathcal{O}(\omega^2) \\ &= \psi(x^\mu - \omega^\mu_\nu x^\nu) - \psi(x^\mu) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\psi(x^\mu - \omega^\mu_\nu x^\nu) + \mathcal{O}(\omega^2). \end{aligned} \tag{10.41}$$

We can expand,

$$\psi(x^\mu - \omega^\mu_\nu x^\nu) = \psi(x^\mu) - \omega_{\mu\nu}x^\nu\partial_\mu\psi = \psi^\mu(x) - i\frac{\omega_{\mu\nu}}{2}J^{\mu\nu}_{\text{scalar}}\psi(x^\nu), \tag{10.42}$$

where we remind that,

$$J^{\mu\nu}_{\text{scalar}} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right). \tag{10.43}$$

Eq. 10.41 becomes,

$$\delta_*\psi = -\frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}_{\text{scalar}} + S^{\mu\nu}\right)\psi. \tag{10.44}$$

Noether's theorem, Eq. 3.56, leads to the result that the quantity

$$J^{\mu\nu} = \int d^3\vec{x}\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\left(J^{\mu\nu}_{\text{scalar}} + S^{\mu\nu}\right)\psi, \tag{10.45}$$

is conserved. Explicitly,

$$J^{\mu\nu} = \int d^3\vec{x}\,\bar{\psi}\gamma^0\left(J^{\mu\nu}_{\text{scalar}} + S^{\mu\nu}\right)\psi. \tag{10.46}$$

and, equivalently,

$$J^{\mu\nu} = \int d^3\vec{x}\,\psi^\dagger\left(J^{\mu\nu}_{\text{scalar}} + S^{\mu\nu}\right)\psi. \tag{10.47}$$

This is very close to what we find for the total angular momentum of a scalar field. But not exactly. The $S^{\mu\nu}$ is novel, and it arises here for the first time. This term is responsible to giving spin, an intrinsic angular momentum, to Dirac particle states.

How can we figure out if a particle has spin? We can do the following test. We put the particle at rest, and measure its total angular momentum. If we find it zero, then the particle has an angular momentum only when it moves and therefore has no intrinsic spin. If it is not zero besides having a zero momentum, then it is all due to the intrinsic spin of the particle.

First, we would like to use normal ordering, as with every conserved quantity, to make sure that the vacuum has a zero total angular momentum:

$$:\ J^{\mu\nu}\ :|0\rangle = 0. \tag{10.48}$$

Let us consider a state of a positively charged Dirac particle which has a zero momentum.

$$|0, s, +\rangle = a^\dagger_s(0)\,|0\rangle\,. \tag{10.49}$$

Does it have any angular momentum? The action of the total-angular momentum on the state is,

$$:\ J^{\mu\nu}\ :|0,s,+\rangle = \left\{:\ J^{\mu\nu}\ :, a^\dagger_s(0)\right\}|0\rangle - a^\dagger_s(0)\ :\ J^{\mu\nu}\ :|0\rangle = \left\{:\ J^{\mu\nu}\ :, a^\dagger_s(0)\right\}|0\rangle\,. \tag{10.50}$$

We now specialize on a rotation around the $z-$axis,

$$\omega_{12} = -\omega_{21} = \theta,\ \text{otherwise}\ \omega_{\mu\nu} = 0. \tag{10.51}$$

After some algebra, we find that

$$\begin{aligned} J^3\,|0,s,+\rangle &\equiv\ :\ J^{\mu\nu}\ :|0,s,+\rangle = \left\{:\ J^{\mu\nu}\ :, a^\dagger_s(0)\right\}|0\rangle \\ &= \sum_r\left(\xi^{s\dagger}\frac{\sigma_3}{2}\xi^r\right)|0,r,+\rangle \\ &= \xi^{s\dagger}\frac{\sigma_3}{2}\xi^{(1)}\,|0,(1),+\rangle + \xi^{s\dagger}\frac{\sigma_3}{2}\xi^{(2)}\,|0,(2),+\rangle\,. \end{aligned} \tag{10.52}$$

where, we choose the basis

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.53}$$

which diagonalizes σ_3,

$$\sigma_3 \xi^{(1)} = +\xi^{(1)}, \quad \sigma_3 \xi^{(2)} = -\xi^{(2)}. \tag{10.54}$$

Then we have,

$$J^3 |0, s, +\rangle = \left(\delta_{s,(1)} \frac{1}{2} - \delta_{s,(2)} \frac{1}{2} \right) |0, s, +\rangle . \tag{10.55}$$

We now have a physical interpretation of the quantum number s. A particle at rest with charge + and $s = (1)$ has total angular momentum $+\frac{1}{2}$ in the $z-$direction,

$$J^3 |0, s, +\rangle = \left(\delta_{s,(1)} \frac{1}{2} - \delta_{s,(2)} \frac{1}{2} \right) |0, s, +\rangle . \tag{10.56}$$

A particle at rest with charge + and $s = (2)$ has total angular momentum $-\frac{1}{2}$ in the $z-$direction. We have then discovered that a general superposition of $\sum_s c_s |p, s, +\rangle$ describes a particle with total spin-$\frac{1}{2}$. Repeating the same steps, we also conclude that a general anti-particle state $\sum_s c_s |p, s, -\rangle$ is also a state with spin-$\frac{1}{2}$.

In summary, the quantization of the Dirac field yields quantum states with spin$-\frac{1}{2}$. Spin emerged simply as a component of the total angular momentum, due to the Dirac fields transforming in the spinorial representation of the Lorentz group. This theory predicts also the existence of both matter and anti-matter. Of course, the theory is highly successful. The electron and positron are states of the quantized Dirac field. The same is valid for other known spin-$\frac{1}{2}$ particles, the muon and tau leptons, the quarks and their antiparticles.

10.2 Fermions

Particles of the Dirac field are fermions. A state with two positively charged particles is

$$\begin{aligned} |p_1, s_1, +; p_2, s_2, +\rangle &\equiv a^\dagger_{s_1}(p_1) a^\dagger_{s_2}(p_2) |0\rangle \\ \rightsquigarrow |p_1, s_1, +; p_2, s_2, +\rangle &= \frac{1}{2} \left[a^\dagger_{s_1}(p_1) a^\dagger_{s_2}(p_2) - a^\dagger_{s_1}(p_1) a^\dagger_{s_2}(p_2) \right] |0\rangle \text{ using Eq. 10.15} \end{aligned} \tag{10.57}$$

The state where all quantum numbers of the two particles are identical $p_1 = p_2$ and $s_1 = s_2$ is forbidden,

$$|p, s, +; p, s, +\rangle = \frac{1}{2} \left[a^\dagger_s(p) a^\dagger_s(p) - a^\dagger_s(p) a^\dagger_s(p) \right] |0\rangle = 0. \tag{10.58}$$

which is Pauli's exclusion principle for fermions. Using the anti-commutation of creation operators we can write a state of N identical particles or anti-particles as,

$$|p_1, s_1, \pm; p_2, s_2, \pm; \ldots; p_N, s_N, \pm;\rangle = \frac{1}{n!} \sum_{\sigma \text{in} S_n} \text{sgn}(\sigma) a^\dagger_{s_{\sigma(1)}} a^\dagger_{s_{\sigma(2)}} \ldots a^\dagger_{s_{\sigma(N)}} |0\rangle , \tag{10.59}$$

which is manifestly anti-symmetric.

The quantization of the Dirac field explains another mystery of quantum mechanics., i.e. that elementary particles with spin$-\frac{1}{2}$ are fermions.

10.3 Quantum symmetries

There is a powerful theorem, due to Wigner, which states that quantum representations of symmetry transformations are either unitary and linear

$$\langle U\psi|\, U\chi\rangle = \langle\psi|\, \chi\rangle\,, \quad U(a_1 |\psi_1\rangle + a_2 |\psi_2\rangle = a_1 U |\psi_1\rangle + a_2 U |\psi_2\rangle\,, \tag{10.60}$$

or anti-unitary

$$\langle U\psi|\, U\chi\rangle = \langle\chi|\, \psi\rangle\,, \quad U(a_1 |\psi_1\rangle + a_2 |\psi_2\rangle = a_1^* U |\psi_1\rangle + a_2^* U |\psi_2\rangle \tag{10.61}$$

In this way, the probability for a transition from a state $|\phi\rangle$ to a state $|\psi\rangle$ is preserved under a symmetry transformation. For example, for an antiunitary transformation:

$$P(|\phi\rangle \to |\psi\rangle) = |\langle\psi|\, \phi\rangle|^2 \to |\langle U\psi|\, U\phi\rangle|^2 = |\langle\phi|\, \psi\rangle|^2 = P(|\phi\rangle \to |\psi\rangle). \tag{10.62}$$

The unity transformation $U = 1$ is a unitary transformations. All transformations which are connected to the unity with a continuous parameter (such as Lorentz and gauge symmetry transformations) are therefore unitary. Antilinear transformations concern discrete symmetries and in particular the ones which involve time reversal.

Requiring that the normalization of a state is invariant under a symmetry transformation, we obtain that

$$||\psi\rangle|^2 = |U\,|\psi\rangle|^2 \rightsquigarrow U^\dagger = U^{-1}. \tag{10.63}$$

Measurable observables are obtained from matrix-elements of the type,

$$\langle \mathrm{STATE}_1|\, \hat{O}\, |\mathrm{STATE}_2\rangle\,. \tag{10.64}$$

For them to be invariant under a symmetry transformation, we must have that if

$$|\mathrm{STATE}\rangle \to |\mathrm{STATE}'\rangle\, U\, |\mathrm{STATE}\rangle$$

then

$$\hat{O} \to \hat{O}' = U\hat{O}U^{-1}. \tag{10.65}$$

10.4 Lorentz transformation of the quantized spinor field

A spinor field transforms classically, as

$$\psi(x) \to \psi'(x') = \Lambda_{\frac{1}{2}}\psi(x). \tag{10.66}$$

What is the transformation of the corresponding quantum field and the quantum states? It does not have to be the same, and in fact it is not, as for the classical field. Relativistic invariance at the classical level is cast as the requirement that the Lagrangian is a scalar,

$$\mathcal{L}'(x') = \mathcal{L}(x).$$

This is sufficient to guarantee that observers in different reference frames find the same minimum value for the action. However, this is not the same criterion that we must introduce for Lorentz invariance of the quantum system.

We consider a quantum one-particle state, e.g.

$$|p, s, +\rangle .$$

Under a Lorentz transformation Λ, a state transforms

$$|p, s, +\rangle \to U(\Lambda)\,|p, s, +\rangle . \tag{10.67}$$

What is the new state, i.e. what is the transformation representation $U(\Lambda)$? It is logical to require that the action of $U(\Lambda)$ yields a particle state with a momentum $\Lambda p \equiv \Lambda^{\mu}{}_{\nu} p^{\nu}$,

$$|p, s, +\rangle \to U(\Lambda)\,|p, s, +\rangle = N(\Lambda, p)\,|\Lambda p, s, +\rangle . \tag{10.68}$$

We express Eq. 10.68 in terms of creation operators,

$$\begin{aligned} U(\Lambda)\,|p, s, +\rangle &= N(\Lambda, p)\,|\Lambda p, s, +\rangle \\ \rightsquigarrow U(\Lambda) a_s^{\dagger}(p)\,|0\rangle &= N(\Lambda, p) a_s^{\dagger}(\Lambda p) U(\Lambda)\,|0\rangle \end{aligned} \tag{10.69}$$

where $N(\Lambda, p)$ is a normalization factor that we need to determine. Conjugating Eq. 10.69 we find that,

$$\langle 0|\, U(\Lambda) a_s(p) U^{-1}(\Lambda) = N(\Lambda, p)^* \,\langle 0|\, a_s(\Lambda p). \tag{10.70}$$

Particle states are orthogonal,

$$\begin{aligned} \langle p_1, s_1, +|\, p_2, s_2, +\rangle &= \langle 0|\, a_{s_1}(p_1) a_{s_2}^{\dagger}(p_2)\,|0\rangle \\ &= \langle 0|\, \left\{ a_{s_1}(p_1), a_{s_2}^{\dagger}(p_2) \right\} |0\rangle - \langle 0|\, a_{s_2}^{\dagger}(p_2) a_{s_1}(p_1)\,|0\rangle \\ &= (2\pi)^3 2\omega_{p_1} \delta^{(3)}(p_2 - p_1) \delta_{s_1 s_2} \,\langle 0|\, 0\rangle - 0 \end{aligned}$$

$$\rightsquigarrow \langle p_1, s_1, +|\, p_2, s_2, +\rangle = (2\pi)^3 2\omega_{p_1} \delta^{(3)}(p_2 - p_1) \delta_{s_1 s_2}. \tag{10.71}$$

Two different particles in a reference frame should remain distinct in any other reference frame. If we perform a Lorentz transformation, Eq. 10.68, we have

$$\begin{aligned} \langle p_1, s_1, +|\, U^{-1} U\,|p_2, s_2, +\rangle &= \langle 0|\, a_{s_1}(p_1) U^{-1} U a_{s_2}^{\dagger}(p_2)\,|0\rangle \\ &= \langle 0|\, U a_{s_1}(p_1) U^{-1} U a_{s_2}^{\dagger}(p_2) U\,|0\rangle \\ &= |N(\Lambda, p_1)|^2 (2\pi)^3 2\omega_{\Lambda p_1} \delta^{(3)}(\Lambda p_2 - \Lambda p_1) \delta_{s_1 s_2} \end{aligned} \tag{10.72}$$

We can now prove that

$$2\omega_{\Lambda p_1} \delta^{(3)}(\Lambda p_1 - \Lambda p_2) = 2\omega_{p_1} \delta^{(3)}(p_1 - p_2). \tag{10.73}$$

Proof: Consider a scalar function $f(\vec{p})$. Then,

$$f(\vec{p}') = \int \frac{d^3\vec{p}}{2\omega_p} 2\omega_p \delta^{(3)}(\vec{p} - \vec{p}') f(\vec{p}). \tag{10.74}$$

The measure

$$\int \frac{d^3\vec{p}}{2\omega_p} = \int d^4 p\, \delta(p^2 - m^2) \Theta\left(p^0 > 0\right)$$

is invariant under Lorentz transformations. Given that f is a scalar, we must have that

$$\omega_p \delta^{(3)}(\vec{p} - \vec{p}')$$

is also a scalar, which proves Eq. 10.73. The orthogonality of the states is therefore preserved under Lorentz transformations. Eq. 10.72 becomes

$$\langle p_1, s_1, +| U^{-1} U |p_2, s_2, +\rangle = |N(\Lambda, p_1)|^2 (2\pi)^3 2\omega_{p_1} \delta^{(3)}(p_2 - p_1)\delta_{s_1 s_2}, \tag{10.75}$$

which for $N(\Lambda, p) = 1$, guarantees that

$$\langle p_1, s_1, +| U^{-1} U |p_2, s_2, +\rangle = \langle p_1, s_1, +| \, p_2, s_2, +\rangle . \tag{10.76}$$

10.4.1 Transformation of the quantized Dirac field

We now proceed heuristically in order to derive the transformation properties of the Dirac field. A more rigorous and detailed exposition can be found in Weinberg Vol. I (chapter 2). Motivated from Eq. 10.69, we infer that the transformation of the creation and annihilation operators is:

$$U(\Lambda) a_s^\dagger(p) U^{-1}(\Lambda) = a_s^\dagger(\Lambda p), \tag{10.77}$$

$$U(\Lambda) a_s(p) U^{-1}(\Lambda) = a_s(\Lambda p), \tag{10.78}$$

$$U(\Lambda) b_s^\dagger(p) U^{-1}(\Lambda) = b_s^\dagger(\Lambda p), \tag{10.79}$$

$$U(\Lambda) b_s(p) U^{-1}(\Lambda) = b_s(\Lambda p). \tag{10.80}$$

What is then the quantum $\psi'(x')$

$$\psi(x) \to \psi'(x') = \psi'(x') = U(\Lambda)\psi(x)U^{-1}(\Lambda)? \tag{10.81}$$

We can compute the right hand side, knowing the transformation of the creation and annihilation operators from Eq. 10.77,

$$\begin{aligned} U\psi(x)U^{-1} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s U \left[a_s(k)u(k)e^{-ik\cdot x} + b_s^\dagger(k)v(k)e^{+ik\cdot x}\right] U^{-1} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_s \left[a_s(\Lambda k)u(k)e^{-ik\cdot x} + b_s^\dagger(\Lambda k)v(k)e^{+ik\cdot x}\right]. \end{aligned} \tag{10.82}$$

Now we use that,

$$\frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^3\Lambda k}{(2\pi)^3 2\omega_{\Lambda k}}, \quad k\cdot x = \Lambda k \cdot \Lambda x, \tag{10.83}$$

and

$$u(k) = \Lambda_{\frac{1}{2}}^{-1} u(\Lambda k), \quad \text{and } v(k) = \Lambda_{\frac{1}{2}}^{-1} v(\Lambda k), \tag{10.84}$$

which yields, setting $p = \Lambda k$,

$$\begin{aligned} U\psi(x)U^{-1} &= \Lambda_{\frac{1}{2}}^{-1} \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_s \left[a_s(p)u(p)e^{-ip\cdot\Lambda x} + b_s^\dagger(p)v(p)e^{+ip\cdot\Lambda x}\right] \\ &= \Lambda_{\frac{1}{2}}^{-1} \psi(\Lambda x). \end{aligned} \tag{10.85}$$

We have found the transformation of the quantum Dirac field. It reads,

$$\psi(x) \to \psi'(x') = U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x). \tag{10.86}$$

This is different than the representation of the classical field: $\psi'(x') = \Lambda_{\frac{1}{2}}\psi(x)$. Therefore, the classical field and the quantum operator have inverse Lorentz transformations.

10.5 Parity

We now derive the transformation properties of a spinor field under parity. This is a discrete transformation,

$$(t, \vec{x}) \to (t, -\vec{x}). \tag{10.87}$$

Under parity, the momentum of a particle transforms as,

$$\vec{p} \to -\vec{p} \tag{10.88}$$

The representation of the parity transformation on an one-particle state should be, for example,

$$U_{\mathcal{P}} |\vec{p}, s, +\rangle = \lambda |-\vec{p}, s, +\rangle, \tag{10.89}$$

where λ is such that

$$1 = \langle \vec{p}, s, +| U_{\mathcal{P}}^{\dagger} U_{\mathcal{P}} |\vec{p}, s, +\rangle = |\lambda|^2 \langle -\vec{p}, s, +| -\vec{p}, s, +\rangle \rightsquigarrow |\lambda|^2 = 1. \tag{10.90}$$

This yield the following transformation for the creation and annihilation operators,

$$U_{\mathcal{P}} a_s(\vec{p}) U_{\mathcal{P}}^{-1} = \lambda_a a_s(-\vec{p}), \tag{10.91}$$

and

$$U_{\mathcal{P}} b_s(\vec{p}) U_{\mathcal{P}}^{-1} = \lambda_b b_s(-\vec{p}). \tag{10.92}$$

Under a parity transformation, a Dirac quantum field transforms as

$$\begin{aligned} U_{\mathcal{P}}\psi(\vec{x}, t)U_{\mathcal{P}}^{-1} &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s U_{\mathcal{P}} \left[a_s(\vec{k})u(\vec{k})e^{-ik\cdot x} + b_s^{\dagger}(\vec{k})v(\vec{k})e^{+ik\cdot x}\right] U_{\mathcal{P}}^{-1} \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \sum_s \left[\lambda_a a_s(-\vec{k})u(k)e^{-ik\cdot x} + \lambda_b^* b_s^{\dagger}(-\vec{k})v(\vec{k})e^{+ik\cdot x}\right] \end{aligned} \tag{10.93}$$

Now we use that the measure is invariant under a parity transformation,

$$\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} = \frac{d^3(-\vec{k})}{(2\pi)^3 2\omega_{-k}}, \tag{10.94}$$

and we rewrite,

$$k \cdot x = Et - \vec{k}\vec{x} = Et - (-\vec{k})(-\vec{x}). \tag{10.95}$$

Eq. 10.93 becomes

$$\begin{aligned} U_{\mathcal{P}}\psi(\vec{x}, t)U_{\mathcal{P}}^{-1} &= \\ \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \sum_s &\left[\lambda_a a_s(\vec{p})u(\vec{p})e^{-i(Et-\vec{p}\cdot(-x))} + \lambda_b^* b_s^{\dagger}(\vec{p})v(-\vec{p})e^{+i(Et-\vec{p}\cdot(-x))}\right], \end{aligned} \tag{10.96}$$

where we set $\vec{p} = -\vec{k}$. Recall now that, in the Weyl representation,

$$u_s(p) = \begin{pmatrix} \sqrt{E - \vec{p}\vec{\sigma}} \\ \sqrt{E + \vec{p}\vec{\sigma}} \end{pmatrix} \text{ and } \gamma^0 = \begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix}. \tag{10.97}$$

We then have

$$\gamma^0 u_s(\vec{p}) = u_s(-\vec{p}), \tag{10.98}$$

and similarly,

$$\gamma^0 v_s(\vec{p}) = -v_s(-\vec{p}). \tag{10.99}$$

We can now fix the arbitrary normalization phases λ_a, λ_b to

$$\lambda_a = -\lambda_b^* = 1. \tag{10.100}$$

Then, Eq. 10.96 yields a simple parity transformation for the field,

$$U_{\mathcal{P}} \psi(\vec{x}, t) U_{\mathcal{P}}^{-1} = \gamma^0 \psi(-\vec{x}, t). \tag{10.101}$$

It is easy to show that we can take $U_{\mathcal{P}}$ to be its own inverse. We have,

$$\begin{aligned} U_{\mathcal{P}} \psi(-\vec{x}, t) U_{\mathcal{P}}^{-1} &= \gamma^0 \psi(\vec{x}, t) \\ \rightsquigarrow \psi(-\vec{x}, t) &= \gamma^0 U_{\mathcal{P}}^{-1} \psi(\vec{x}, t) U_{\mathcal{P}} \\ \rightsquigarrow \gamma^0 \psi(-\vec{x}, t) &= (\gamma^0)^2 U_{\mathcal{P}}^{-1} \psi(\vec{x}, t) U_{\mathcal{P}} \\ \\ \rightsquigarrow U_{\mathcal{P}} \psi(\vec{x}, t) U_{\mathcal{P}}^{-1} &= U_{\mathcal{P}}^{-1} \psi(\vec{x}, t) U_{\mathcal{P}}, \end{aligned} \tag{10.102}$$

which is satisfied for

$$U_{\mathcal{P}} = U_{\mathcal{P}}^{-1}. \tag{10.103}$$

We can study the transformation of composite operators made up from two Dirac fields. The combination $\bar{\psi}\psi$ which transforms as a scalar under Lorentz transformations, transforms under parity as a scalar too.

$$\begin{aligned} \bar{\psi}(t, \vec{x}) \psi(t, \vec{x}) &\to \bar{\psi}'(t, -\vec{x}) \psi'(t, -\vec{x}) = U_{\mathcal{P}} \bar{\psi}(t, \vec{x}) U_{\mathcal{P}} U_{\mathcal{P}} \psi(t, \vec{x}) U_{\mathcal{P}} \\ &= \bar{\psi}(t, -\vec{x}) \gamma^0 \gamma^0 \psi(t, -\vec{x}, t) \\ &= \bar{\psi}(t, -\vec{x}) \psi(t, -\vec{x}). \end{aligned} \tag{10.104}$$

However, there is a combination $\bar{\psi}\gamma^5\psi$ which transforms as a scalar under Lorentz transformations (**exercise: prove that it is indeed so**), but transforms differently under parity.

$$\begin{aligned} \bar{\psi}(t, \vec{x}) \gamma^5 \psi(t, \vec{x}) &\to \bar{\psi}'(t, -\vec{x}) \gamma^5 \psi'(t, -\vec{x}) = U_{\mathcal{P}} \bar{\psi}(t, \vec{x}) U_{\mathcal{P}} \gamma^5 U_{\mathcal{P}} \psi(t, \vec{x}) U_{\mathcal{P}} \\ &= \bar{\psi}(t, -\vec{x}) \gamma^0 \gamma^5 \gamma^0 \psi(t, -\vec{x}, t) \\ &= -\bar{\psi}(t, -\vec{x}) \gamma^5 \gamma^0 \gamma^0 \psi(t, -\vec{x}, t) \\ &= -\bar{\psi}(t, -\vec{x}) \gamma^5 \psi(t, -\vec{x}). \end{aligned} \tag{10.105}$$

This is called a "pseudoscalar".

The combination $\bar{\psi}\gamma^\mu\psi$ which transforms as a vector under Lorentz transformations (**exercise**), transforms also as a vector under parity.

$$
\begin{aligned}
\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x}) &\rightarrow \bar{\psi}'(t,-\vec{x})\gamma^\mu\psi'(t,-\vec{x}) = U_{\mathcal{P}}\bar{\psi}(t,\vec{x})U_{\mathcal{P}}\gamma^\mu U_{\mathcal{P}}\psi(t,\vec{x})U_{\mathcal{P}} \\
&= \bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x},t) \\
&= \bar{\psi}(t,-\vec{x})\left(2g^{0\mu}\gamma^0 - \gamma^\mu(\gamma^0)^2\right)\psi(t,-\vec{x}) \\
&= \left(2g^{0\mu}-1\right)\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x}) \qquad (10.106)
\end{aligned}
$$

For $\mu = 0$,

$$\bar{\psi}(t,\vec{x})\gamma^0\psi(t,\vec{x}) \rightarrow \bar{\psi}(t,-\vec{x})\gamma^0\psi(t,-\vec{x}). \qquad (10.107)$$

For $\mu = i = 1,2,3$,

$$\bar{\psi}(t,\vec{x})\gamma^i\psi(t,\vec{x}) \rightarrow \bar{\psi}(t,-\vec{x})\gamma^i\psi(t,-\vec{x}). \qquad (10.108)$$

This is exactly how a vector $x^\mu = (t,\vec{x}) \rightarrow (t,-\vec{x})$ transforms under parity.

The combination $\bar{\psi}\gamma^\mu\gamma^5\psi$ which transforms as a scalar under Lorentz transformations (**exercise: prove that it is indeed so)**, transforms oppositely under parity.

$$
\begin{aligned}
\bar{\psi}(t,\vec{x})\gamma^m u\gamma^5\psi(t,\vec{x}) &\rightarrow \bar{\psi}'(t,-\vec{x})\gamma^\mu\gamma^5\psi'(t,-\vec{x}) = U_{\mathcal{P}}\bar{\psi}(t,\vec{x})U_{\mathcal{P}}\gamma^\mu\gamma^5 U_{\mathcal{P}}\psi(t,\vec{x})U_{\mathcal{P}} \\
&= \bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(t,-\vec{x},t) \\
&= -\bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\gamma^5\psi(t,-\vec{x},t) \\
&= -\bar{\psi}(t,-\vec{x})\left(2g^{0\mu}\gamma^0 - \gamma^\mu(\gamma^0)^2\right)\gamma^5\psi(t,-\vec{x}) \\
&= -\left(2g^{0\mu}-1\right)\bar{\psi}(t,-\vec{x})\gamma^\mu\gamma^5\psi(t,-\vec{x}) \qquad (10.109)
\end{aligned}
$$

This is called a "pseudo-vector".

The Dirac Lagrangian,

$$i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi,$$

is composed from true scalar terms, and it transforms as a scalar under parity.

Exercise: Prove that the classical Dirac field has the same parity transformation as the quantum Dirac field.

10.6 Other discrete symmetries

Exercise: Prove the following,

1. **Under time-reversal** $(t,\vec{x}) \rightarrow (-t,\vec{x})$**,**

$$U_{\mathcal{P}}a_s(\vec{p})U_{\mathcal{P}} = a_{-s}(-\vec{p}), \qquad \xi^{-s} = -i\sigma_2(\xi^s)^*. \qquad (10.110)$$

and

$$U_{\mathcal{P}}\psi(t,\vec{x})U_{\mathcal{P}} = \psi(-t,\vec{x}). \qquad (10.111)$$

2. **Under charge conjugation, exchanging a particle with its antiparticle,**

$$U_C a_s(\vec{p})U_C = b_s(\vec{p}), \qquad U_C b_s(\vec{p})U_C = a_s(\vec{p}), \qquad (10.112)$$

and

$$U_C\psi(t,\vec{x})U_C = -i(\bar{\psi}(t,\vec{x})\gamma^0\gamma^2)^T. \qquad (10.113)$$

Chapter 11

Propagation of free particles

In previous chapters we studied the particle states from the quantization of free fields, such as the Schrödinger, the Klein-Gordon, the spinor, and the electromagnetic field. We are now ready to study the simplest transition amplitude, for the transition of a free particle from a space-time position $y^\mu \equiv (y^0, \vec{y})$ to a position $x^\mu \equiv (x^0, \vec{x})$, with $x^0 > y^0$. The transition amplitude is given by,

$$M_{x \to y} = \langle x | \, y \rangle |_{x^0 > y^0} \,. \tag{11.1}$$

where $|x\rangle \,, |y\rangle$ are states of a single particle at positions x, y correspondingly.

11.1 Transition amplitude for the Schrödinger field

We shall compute the transition amplitude for the case of the Scrödinger field. This will give us an opportunity to compare with other theories, which are relativistic, and appreciate the merits of the latter. We recall that the Schrödinger field is given by,

$$\psi(x^0, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} a(k) e^{-i\left(E_k x^0 - \vec{k}\cdot\vec{x}\right)}, \tag{11.2}$$

where

$$E_k = \frac{\vec{k}^2}{2m}, \tag{11.3}$$

and the operator $a(k)$ satisfies the quantization condition,

$$\left[a(k), a^\dagger(k')\right] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'). \tag{11.4}$$

A state of a particle at a position x is given by the action of a field on the vacuum (see Section 4.6),

$$|x\rangle = \psi^\dagger(x^0, \vec{x}) \, |0\rangle \,, \qquad |y\rangle = \psi^\dagger(y^0, \vec{y}) \, |0\rangle \,. \tag{11.5}$$

The transition amplitude of Eq. 11.1 is then,

$$\begin{aligned} M_{y \to x} &= \langle 0 | \, \psi(x^0, \vec{x}) \psi^\dagger(y^0, \vec{y}) \, |0\rangle \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\left(E_k \Delta t - \vec{k}\cdot\Delta\vec{x}\right)}, \end{aligned} \tag{11.6}$$

with

$$\Delta t = x^0 - y^0, \qquad \Delta\vec{x} = \vec{x} - \vec{y}. \tag{11.7}$$

We can compute the above integral exactly. We first change to spherical coordinates,

$$d^3\vec{k} = dkk^2\, dcos\theta\, d\phi, \tag{11.8}$$

and write

$$\vec{k}\Delta\vec{x} = k\,\Delta x\,\cos\theta. \tag{11.9}$$

Then it is easy to perform successively the integrations in ϕ, $\cos\theta$ and k. The result reads,

$$M_{y\to x} = \left(\frac{m}{2\pi i\Delta t}\right)^{\frac{3}{2}} e^{-i\Delta t\left[\frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)^2\right]}. \tag{11.10}$$

This is an oscillatory function which is defined for arbitrary values of the particle speed $\frac{\Delta x}{\Delta t}$ even when its value is not smaller than the speed of light. Particles which are quanta of the Schrödinger field, can propagate at space-time intervals anywhere outside the light-cone, which is in blatant disagreement with special relativity.

11.2 Transition amplitude for the real Klein-Gordon field

We write the real Klein-Gordon field as a sum of a "positive frequency" and a "negative frequency" term,

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{11.11}$$

with

$$\phi^{(+)}(x) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\Theta(k^0)e^{-ik\cdot x}a(k), \tag{11.12}$$

and

$$\phi^{(-)}(x) = \int \frac{d^4k}{(2\pi)^3}\delta(k^2 - m^2)\Theta(k^0)e^{+ik\cdot x}a^\dagger(k). \tag{11.13}$$

We notice that,

$$\phi^{(+)}(x)\,|0\rangle = 0, \tag{11.14}$$

and

$$\langle 0|\,\phi^{(-)}(x) = 0. \tag{11.15}$$

The operator $a(k)$ satisfies the quantization condition,

$$\left[a(k), a^\dagger(k')\right] = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{k}'), \tag{11.16}$$

and

$$\omega_k = \sqrt{\vec{k}^2 + m^2}. \tag{11.17}$$

The amplitude for the transition $y^\mu \to x^\mu$, with $x^0 > y^0$, is then

$$\begin{aligned}
M_{y\to x} &= \langle 0|\,\phi^\dagger(x)\phi(y)\,|0\rangle \\
&= \langle 0|\,\phi(x)\phi(y)\,|0\rangle \\
&= \langle 0|\left(\phi^{(+)}(x) + \phi^{(-)}(x)\right)\left(\phi^{(+)}(y) + \phi^{(-)}(y)\right)|0\rangle \\
&= \langle 0|\,\phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(x)\phi^{(-)}(y)\,|0\rangle \\
&= \langle 0|\,\phi^{(+)}(x)\phi^{(-)}(y)\,|0\rangle \\
&= \langle 0|\left[\phi^{(+)}(x), \phi^{(-)}(y)\right] + \phi^{(-)}(x)\phi^{(+)}(y)\,|0\rangle \\
&= \langle 0|\left[\phi^{(+)}(x), \phi^{(-)}(y)\right]|0\rangle\,.
\end{aligned} \tag{11.18}$$

The commutator gives,

$$\left[\phi^{(+)}(x), \phi^{(-)}(y)\right] = \int \frac{d^4k}{(2\pi)^3} \Theta(k^0) e^{-ik(x-y)} \delta(k^2 - m^2), \tag{11.19}$$

which is a $c-$number. We then have for the transition amplitude,

$$M_{y\to x} = \langle 0| \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] |0\rangle = \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] \langle 0|\, 0\rangle = \left[\phi^{(+)}(x), \phi^{(-)}(y)\right], \tag{11.20}$$

and explicitly,

$$M_{y\to x} = \int \frac{d^4k}{(2\pi)^3} \Theta(k^0) e^{-ik(x-y)} \delta(k^2 - m^2). \tag{11.21}$$

We now integrate out the energy component k^0 and use spherical coordinates for $\vec{k}$, yielding:

$$M_{y\to x} = \int_0^\infty \frac{dk}{(2\pi)^2} \int_0^\pi d\cos\theta \frac{k^2}{2\sqrt{k^2+m^2}} e^{-i\sqrt{k^2+m^2}\Delta t + ik\Delta x \cos\theta} \tag{11.22}$$

with $\Delta t = x^0 - y^0$ and $\Delta x = |\vec{x} - \vec{y}|$. We can also perform the integration over the angle θ. This yields:

$$\begin{aligned} M_{y\to x} &= \int_0^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{2\sqrt{k^2+m^2}} e^{-i\sqrt{k^2+m^2}\Delta t} \frac{e^{+ik\Delta x} - e^{-ik\Delta x}}{2ik\Delta x} \\ &= -\frac{i}{2(2\pi)^2 \Delta x} \int_{-\infty}^\infty dk \frac{k}{\sqrt{k^2+m^2}} e^{-i\sqrt{k^2+m^2}\Delta t + ik\Delta x} \end{aligned} \tag{11.23}$$

We would like to investigate whether the above expression for the transition amplitude respects the expectation from special relativity, which intuitively we expect it to restrict particles to transitions within the light-cone. It is now a bit more tedious to evaluate the integral of Eq. 11.23. For simplicity, we shall evaluate it for two special transitions,

- **Transition A:** where we take $\Delta x > 0$ and $\Delta t = 0$, which is a space-like transition (outside the light-cone).
- **Transition B:** where we take $\Delta x = 0$ and $\Delta t > 0$, which is a time-like transition (within the light-cone)

Space-like transition A

In this case, the transition amplitude becomes

$$M_{y\to x} = \frac{-i}{2(2\pi)^2(\Delta x)} \int_{-\infty}^{+\infty} dk \frac{k e^{ik(\Delta x)}}{\sqrt{k^2+m^2}}. \tag{11.24}$$

The above integral has branch cuts at $k = \pm im$, as in Figure 11.1. For $(\Delta x) > 0$ we can wrap the contour of integration around the upper branch cut. The integral is then becoming, after setting $k = i\rho$,

$$M_{y\to x} = \frac{1}{4\pi^2(\Delta x)} \int_m^\infty d\rho \frac{\rho e^{-\rho(\Delta x)}}{\sqrt{\rho^2 - m^2}}, \tag{11.25}$$

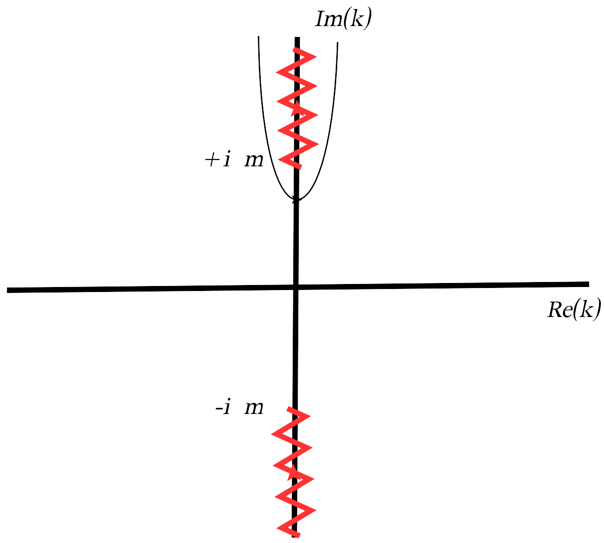


Figure 11.1: Branch cuts of the transition amplitude integral for a space-like transition

or equivalently,

$$
\begin{aligned}
M_{y\to x} &= \frac{1}{4\pi^2(\Delta x)}\int_m^\infty d\rho \frac{e^{-\rho(\Delta x)}}{\sqrt{1-\frac{m^2}{\rho^2}}} \\
&= \frac{1}{4\pi^2(\Delta x)}\int_m^\infty d\rho e^{-\rho(\Delta x)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)}{n!}m^{2n}\rho^{-2n}.
\end{aligned} \tag{11.26}
$$

We can perform the ρ integration in terms of the incomplete Gamma function,

$$\Gamma(a,z) = \int_z^\infty dt e^{-t}t^{a-1}, \tag{11.27}$$

which, as $z \to \infty$, has the asymptotic behaviour:

$$\Gamma(a,z) \sim z^{a-1}e^{-z}. \tag{11.28}$$

Asymptotically, for a large space-like interval $\Delta x \to \infty$, the transition integral becomes

$$\lim_{(\Delta x)\to\infty} M_{y\to x} \sim e^{-m(\Delta x)}. \tag{11.29}$$

We find that the transition amplitude vanishes exponentially the further away we move from the light-cone ($\Delta x = 0$). Is this a contradiction with special relativity? Actually it is not when we remember that our theory is in addition a quantum theory. Essentially, a particle can escape from the light-cone by a distance Δx which is not observable to us due to the uncertainty principle. The transition amplitude is only significant for

$$m(\Delta x) \sim 1, \tag{11.30}$$

which is the uncertainty we expect for determining its position for a momentum uncertainty $\Delta p \sim m$. Essentially, to observe that a particle of a certain mass has escaped the light-cone we must first measure the momentum of the particle with an accuracy as good as its mass (in order to identify the particle from its mass). Then, the uncertainty in its position will be as large as the distance that we expect a finite probability for the particle to escape away from the light-cone.

Time-like transition B

For large times, $\Delta t \to \infty$, we can prove (**exercise**) that

$$\lim_{\Delta t \to \infty} \sim e^{-im\Delta t}. \tag{11.31}$$

The transition amplitude for a particle to remain within the light-cone is finite.

11.3 Time Ordering and the Feynman-Stückelberg propagator for the real Klein-Gordon field

In order for the "correlation function",

$$\langle 0| \phi(x)\phi(y) |0\rangle ,$$

to be a transition amplitude where a particle is created at point y^μ and it is destroyed at a point x^μ, the time x^0 must occur after y^0, i.e. $x^0 > y^0$. These are the transitions which have a physical realisation. Keeping track of the time variable is cumbersome in a relativistic theory where space and time coordinates get mixed up with Lorentz transformations. We need a simple method to do so.

It has been noticed that there is a simple way to demand a physical flow of time in transition amplitudes (propagators) when they are appropriately combined. Consider the Feynman-Stückelberg propagator which is defined to account for both cases, $x^0 > y^0$ and $x^0 < y^0$ in a transition in between two arbitrary space-time points x^μ, y^μ:

$$\begin{aligned}\langle 0| T\phi(x)\phi(y) |0\rangle &= \Theta(x^0 - y^0) \langle 0| \phi(x)\phi(y) |0\rangle \\ &\quad + \Theta(y^0 - x^0) \langle 0| \phi(y)\phi(x) |0\rangle .\end{aligned} \tag{11.32}$$

The "time-ordering" symbol T is defined to order the operators following it from the later to the earlier times. The Feynman-Stückelberg propagator has a very nice integral representation. Explicitly, from the above definition and the result of Eq. 11.23, we find

$$\begin{aligned}\langle 0| T\phi(x)\phi(y) |0\rangle &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2 + m^2}} \left[\Theta(x^0 - y^0)e^{-ik\cdot(x-y)} + \Theta(y^0 - x^0)e^{-ik\cdot(y-x)}\right] \\ &= \lim_{\delta \to 0} \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)}.\end{aligned} \tag{11.33}$$

We can verify that the four-dimensional integral of the last line is equal to the line before explicitly, by using Cauchy's theorem. In Fig. 11.2 we plot the poles of the integrand on the energy k^0 complex-plane. These are located at $k^0 = -\omega + i\delta$ and $k^0 = \omega - i\delta$, with $\omega = \sqrt{\vec{k}^2 + m^2}$. At $k^0 = \pm i\infty$, the exponential in the integrand behaves as,

$$e^{-ik\cdot(x-y)} \to e^{+i\vec{k}(\vec{x}-\vec{y})} e^{\pm\infty(x^0 - y^0)}. \tag{11.34}$$

If $x^0 > y^0$ we can close the contour of integration to the lower complex half-plane, guaranteeing that the integrand vanishes at $-i\infty$. If $x^0 < y^0$ we must close the contour to the upper half-plane. In each case, we pick up one of the residue at $k^0 = \pm\omega$, recovering the two terms of the first line in Eq. 11.33.

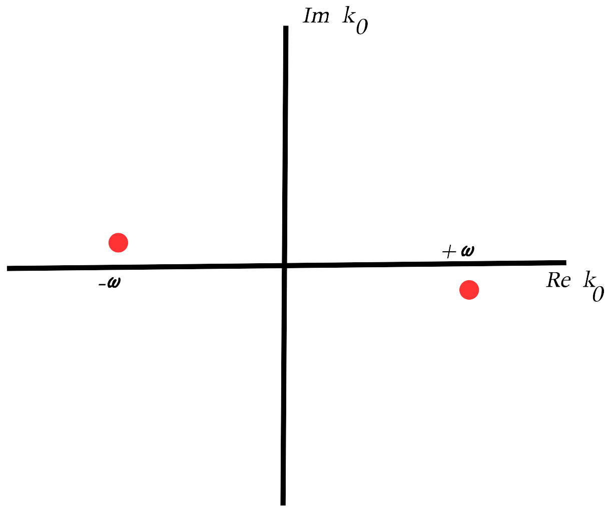


Figure 11.2: Poles of the integrand for the Feynman-Stückelberg propagator

The expression for the Feynman propagator,

$$\langle 0| T\phi(x)\phi(y) |0\rangle = \lim_{\delta \to 0} \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)}, \tag{11.35}$$

is of paramount importance for the computation of generic transition amplitudes in quantum field theory and it is simple. It accounts for the correct time-flow with the $i\delta$ *prescription* in the denominator. An important property of the Feynman propagator is that it is a **Green's function** of the Klein-Gordon equation. We can act on it with the differential operator of the Klein-Gordon equation,

$$\begin{aligned} \left(\partial^2 + m^2\right) \langle 0| T\phi(x)\phi(y) |0\rangle &= \left(\partial^2 + m^2\right) \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)} \\ &= -i \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \\ \rightsquigarrow \left(\partial^2 + m^2\right) \langle 0| T\phi(x)\phi(y) |0\rangle &= -i\delta^4(x-y). \end{aligned} \tag{11.36}$$

11.4 Feynman propagator for the complex Klein-Grodon field

We will now compute the transition amplitude, or the Feynman propagator, for the complex Klein-Gordon field,

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \left[a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x}\right]. \tag{11.37}$$

Notice that the hermitian conjugate of the field acting on the vacuum creates a particle with charge $+1$ at a certain position x,

$$|\text{particle at position } x\rangle = \phi^\dagger(\vec{x}, t) |0\rangle, \tag{11.38}$$

while a field acting on the vacuum creates an anti-particle with charge -1,

$$|\text{anti-particle at position } x\rangle = \phi(\vec{x},t)\,|0\rangle\,. \tag{11.39}$$

Consider the correlation function,

$$\langle 0|\,\phi(y)\phi^\dagger(x)\,|0\rangle \tag{11.40}$$

which we can interprete it as a causal sequence where a particle is first created at $x^\mu \equiv (x^0, \vec{x})$ and it is destroyed at $y^\mu \equiv (y^0, \vec{y})$. We must then have $y^0 > x^0$. Similarly, the correlation function

$$\langle 0|\,\phi^\dagger(x)\phi(y)\,|0\rangle \tag{11.41}$$

can be interpreted an anti-particle which is first created at $y^\mu \equiv (y^0, \vec{y})$ and it is destroyed at $x^\mu \equiv (x^0, \vec{x})$. We must then have $x^0 > y^0$.

We define the Feynman propagator to account for the two equivalent possibilities for the forward in time propagation of either a particle or an anti-particle.

$$\begin{aligned} &\langle 0|\,T\phi(x)\phi^\dagger(y)\,|0\rangle \\ &= \Theta(x^0 - y^0)\,\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle + \Theta(y^0 - x^0)\,\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle \end{aligned} \tag{11.42}$$

We can repeat the same computation as for the Feynman propagator of the real Klein-Gordon field. The result is identical:

$$\langle 0|\,T\phi(x)\phi^\dagger(y)\,|0\rangle \;=\; \lim_{\delta\to 0}\int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\delta}e^{-ik\cdot(x-y)}. \tag{11.43}$$

11.5 Feynman propagator for the Dirac field

We now recall the expression of the Dirac field operator,

$$\psi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\sum_s \left[a_s(p)u_s(p)e^{-ip\cdot x} + b_s^\dagger(p)v_s(p)e^{ip\cdot x}\right]. \tag{11.44}$$

The operators $a_s^\dagger(\vec{p})$ acting on the vacuum create particles of momentum $\vec{p}$ with a spin index s, while the operators $b_s^\dagger(\vec{p})$ create antiparticles. The state:

$$\bar{\psi}(x^\mu)\,|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\sum_s \;\bar{u}_s(p)e^{+ip\cdot x}a_s^\dagger(p)\,|0\rangle\,,$$

is the Fourier transform of one-particle momentum eigenstates. They can therefore be interpreted as particle states of a certain position x^μ. Let us now compute the correlation function of two Dirac spinor fields,

$$\begin{aligned} \langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle &= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\sum_s u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\,(\not{p} + m)_{ab}\,e^{-ip\cdot(x-y)} \\ \leadsto \langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle &= \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} + m\mathbf{1}\right)_{ab}\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}e^{-ip\cdot(x-y)} \end{aligned} \tag{11.45}$$

Similarly, we find (**exercise**)

$$\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle \;\;=\;\; -\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m\mathbf{1}\right)_{ab} \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} e^{-ip\cdot(y-x)} \tag{11.46}$$

Notice the overall (-) sign for the transition amplitude of an anti-particle with respect to the corresponding amplitude in the case of a particle.

We can combine the two forward in time propagations by including this relative minus sign into the definition of the time ordering for fermions.

$$\langle 0|\,T\psi_a(x)\bar{\psi}_b(y)\,|0\rangle = \Theta(x^0-y^0)\,\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle - \Theta(y^0-x^0)\,\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle \tag{11.47}$$

From the above, we find for:

$$\begin{aligned}\langle 0|\,T\psi_a(x)\bar{\psi}_b(y)\,|0\rangle \;\;&=\;\; \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m\mathbf{1}\right)_{ab} \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left[\Theta(y^0-x^0)e^{-ip\cdot(y-x)}\right. \\ &\qquad\qquad \left. +\Theta(x^0-y^0)e^{-ip\cdot(x-y)}\right] \\ &= \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} + m\mathbf{1}\right)_{ab} \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\delta} e^{-ik\cdot(x-y)},\end{aligned} \tag{11.48}$$

which yields for the fermion Feynman propagator the integral representation,

$$\langle 0|\,T\psi_a(x)\bar{\psi}_b(y)\,|0\rangle = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4} \frac{i\,(\not{k}+m\mathbf{1})_{ab}}{k^2-m^2+i\delta} e^{-ik\cdot(x-y)}. \tag{11.49}$$

Exercise: Show that

$$\langle 0|\,T\psi_a(x)\psi_b(y)\,|0\rangle = \langle 0|\,T\bar{\psi}_a(x)\bar{\psi}_b(y)\,|0\rangle = 0. \tag{11.50}$$

11.6 Feynman propagator for the photon field

The photon Feynman propagator is defined as,

$$\begin{aligned}\langle 0|\,TA^\mu(x)A^\nu(y)\,|0\rangle \;\;&=\;\; \langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle\,\Theta(x^0-y^0) \\ &\quad + \langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle\,\Theta(y^0-x^0)\end{aligned} \tag{11.51}$$

This is worked out in the exercise tutorials. We find that

$$\langle 0|\,TA^\mu(x)A^\nu(y)\,|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\left[g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right]}{p^2+i\delta} e^{-ip(x-y)}. \tag{11.52}$$

11.7 Wick's theorem: time-ordering, normal-ordering and propagation

We decompose the free scalar field ϕ into a term with a creation operator and a term with an annihilation operator,

$$\phi(x) = \phi_+(x) + \phi_-(x), \tag{11.53}$$

with

$$\phi_-(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} e^{-ik\cdot x} a(k), \tag{11.54}$$

and

$$\phi_+(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} e^{+ik\cdot x} a^\dagger(k), \tag{11.55}$$

The time-ordered product of two scalar fields is,

$$\begin{aligned}
& T\left\{\phi(x)\phi(y)\right\} = \Theta(x^0 - y^0)\phi(x)\phi(y) + (x^\mu \leftrightarrow y^\mu) \\
= \; & \Theta(x^0 - y^0)\Bigg\{ \phi_+(x)\phi_+(y) + \phi_-(x)\phi_-(y) + \phi_+(x)\phi_-(y) \\
& +\phi_-(x)\phi_+(y) \Bigg\} + (x^\mu \leftrightarrow y^\mu) \\
= \; & \Theta(x^0 - y^0)\Bigg\{ \phi_+(x)\phi_+(y) + \phi_-(x)\phi_-(y) + \phi_+(x)\phi_-(y) \\
& +\phi_+(y)\phi_-(x) + [\phi_+(x), \phi_-(y)] \Bigg\} + (x^\mu \leftrightarrow y^\mu) \\
= \; & \left[\Theta(x^0 - y^0) + \Theta(y^0 - x^0)\right] \Bigg\{ \phi_+(x)\phi_+(y) + \phi_-(x)\phi_-(y) \\
& +\phi_+(x)\phi_-(y) + \phi_+(y)\phi_-(x) \Bigg\} \\
& + [\phi_-(x), \phi_+(y)]\, \Theta(x^0 - y^0) + [\phi_-(y), \phi_+(x)]\, \Theta(y^0 - x^0)
\end{aligned} \tag{11.56}$$

The sum of the theta functions in the first term is equal to one. We observe that in the curly bracket all field products appear with creation operators preceding annihilation operators. The curly bracket is then just the normal ordering of the product of the two field operators

$$\Bigg\{ \dots \Bigg\} =: \phi(x)\phi(y) : \tag{11.57}$$

The sum of the two terms in the last line is a known object, the Feynman-Stückelberg propagator, which is not an operator but a $c-$number. Using that the relation

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{k}'), \tag{11.58}$$

we find that

$$[\phi_-(x), \phi_+(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)}. \tag{11.59}$$

Combining the two terms together we have

$$\begin{aligned}
& [\phi_-(x), \phi_+(y)]\, \Theta\left(x^0 - y^0\right) + [\phi_-(y), \phi_+(x)]\, \Theta\left(y^0 - x^0\right) \\
= \; & \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik(x-y)}\Theta(x^0 - y^0) + e^{-ik(y-x)}\Theta(y^0 - x^0) \right] \\
= \; & \langle 0 | \, T\left\{\phi(x)\phi(y)\right\} |0\rangle \, .
\end{aligned} \tag{11.60}$$

We can then write that,

$$T\left\{\phi(x)\phi(y)\right\} =: \phi(x)\phi(y) : + \left\langle 0\right| T\left\{\phi(x)\phi(y)\right\}\left|0\right\rangle . \tag{11.61}$$

We have proved that the time-ordered product of two free-field operators is the normal-ordering of the same product plus a $c-$function which is the propagator of the two-fields.

This result generalizes easily to the time-ordered product of an arbitrary number of field operators. For three fields we have

$$\begin{aligned} T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\right\} &= \ :\phi(x_1)\phi(x_2)\phi(x_3): \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_2)\right\}\left|0\right\rangle \phi(x_3) \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_3)\right\}\left|0\right\rangle \phi(x_2) \\ &+ \ \left\langle 0\right| T\left\{\phi(x_2)\phi(x_3)\right\}\left|0\right\rangle \phi(x_1) \end{aligned} \tag{11.62}$$

For four fields we have

$$\begin{aligned} T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right\} &= \ :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_2)\right\}\left|0\right\rangle :\phi(x_3)\phi(x_4): \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_3)\right\}\left|0\right\rangle :\phi(x_2)\phi(x_4): \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_4)\right\}\left|0\right\rangle :\phi(x_2)\phi(x_3): \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_2)\right\}\left|0\right\rangle \left\langle 0\right| T\left\{\phi(x_3)\phi(x_4)\right\}\left|0\right\rangle \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_3)\right\}\left|0\right\rangle \left\langle 0\right| T\left\{\phi(x_2)\phi(x_4)\right\}\left|0\right\rangle \\ &+ \ \left\langle 0\right| T\left\{\phi(x_1)\phi(x_4)\right\}\left|0\right\rangle \left\langle 0\right| T\left\{\phi(x_2)\phi(x_3)\right\}\left|0\right\rangle . \end{aligned} \tag{11.63}$$

In general, Wick's theorem states that

$$\begin{aligned} & T\left\{\phi(x_1)\ldots\phi(x_n)\right\} = \\ = & \ :\phi(x_1)\ldots\phi(x_n) + \text{ all contractions }: \end{aligned} \tag{11.64}$$

where a "contraction" means to replace one or more pairs of fields with their propagator. The theorem can be proved easily by induction.

Notice that the normal ordered products in the expressions produced with Wick's theorem are vanishing when bracketed with the vacuum,

$$\left\langle 0\right| : \prod_j \phi(x_j) \ : \left|0\right\rangle = 0, \tag{11.65}$$

given that creation operators are placed before annihilation operators in the normal ordering and

$$\left\langle 0\right| a^\dagger(k) = a(k)\left|0\right\rangle = 0.$$

From Eqs 11.61-11.63 we derive the tautology

$$\left\langle 0\right| T\left\{\phi(x)\phi(y)\right\}\left|0\right\rangle = \left\langle 0\right| T\left\{\phi(x)\phi(y)\right\}\left|0\right\rangle , \tag{11.66}$$

and the more informative equations,

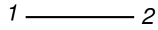
$$\left\langle 0\right| T\left\{\phi(x_1)\phi(x_2)\phi(x_3)\right\}\left|0\right\rangle \ = \ 0, \tag{11.67}$$

and

$$
\begin{aligned}
\langle 0| T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle &= \langle 0| T\{\phi(x_1)\phi(x_2)\}|0\rangle \langle 0| T\{\phi(x_3)\phi(x_4)\}|0\rangle \\
&+ \langle 0| T\{\phi(x_1)\phi(x_3)\}|0\rangle \langle 0| T\{\phi(x_2)\phi(x_4)\}|0\rangle \\
&+ \langle 0| T\{\phi(x_1)\phi(x_4)\}|0\rangle \langle 0| T\{\phi(x_2)\phi(x_3)\}|0\rangle .
\end{aligned}
\tag{11.68}
$$

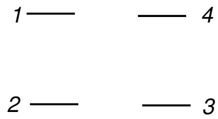
Such equations admit a graphical representation. Let us represent the "two-point" correlation function with a straight line

$$\langle 0| T\{\phi(x_1)\phi(x_2)\}|0\rangle = \tag{11.69}$$



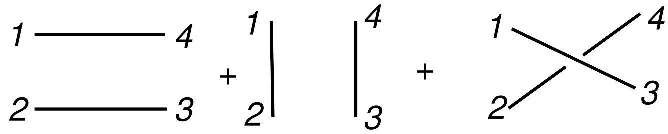
The four point function is represented as

$$\langle 0| T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle =$$



contracted

$$= \tag{11.70}$$



It is then very easy to apply Wick's theorem pictorially, simply by drawing all possible pairing of the points which appear in $\langle 0| T\{\phi(x_1)\phi(x_2)\ldots\}|0\rangle$.

11.7.1 Wick's theorem for Dirac fermion fields*

Consider a free Dirac field,

$$\psi_b(x) = \psi_b^-(x) + \psi_b^+(x), \tag{11.71}$$

where the index b is a spinor index, and

$$\psi_-^b(x) = \sum_s \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} u_s^b(k) a_s(k) e^{-ik\cdot x}, \tag{11.72}$$

$$\psi_+^b(x) = \sum_s \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} v_s^b(k) b_s^\dagger(k) e^{+ik\cdot x}, \tag{11.73}$$

are the terms with annihilation and creation operators respectively. The spin-sum is over all the solutions s of the Dirac equation for the spinors u, v. We remind the following identities

$$\sum_s u_s^a(p)\bar{u}_s^b(p) = (\not{p} + m\mathbf{1})_{ab}\,, \tag{11.74}$$

$$\sum_s v_s^a(p)\bar{v}_s^b(p) = (\not{p} - m\mathbf{1})_{ab}\,. \tag{11.75}$$

Let us now work out the time-ordered product

$$T\psi_a(x)\bar{\psi}_b(y)$$

Following the steps of the previous section and recalling the minus signs in the definitions of the normal ordering and the time-ordering, we find that (**exercise**):

$$\begin{aligned}T\psi_a(x)\bar{\psi}_b(y) &= \, :\psi_a(x)\bar{\psi}_b(y): +\Theta(x^0-y^0)\left\{\psi_a^-(x), \overline{\psi_b^-}(y)\right\} \\ &\quad -\Theta(y^0-x^0)\left\{\overline{\psi_b^+}(y), \psi_a^+(x)\right\} \end{aligned} \tag{11.76}$$

The anticommutators above are complex numbers and, without any harm, we can take them to be equal to their vacuum expectation values,

$$\begin{aligned}\left\{\psi_a^-(x), \overline{\psi_b^-}(y)\right\} &= \langle 0| \left\{\psi_a^-(x), \overline{\psi_b^-}(y)\right\} |0\rangle \\ &= \langle 0| \psi_a^-(x)\overline{\psi_b^-}(y) |0\rangle = \langle 0| \psi_a(x)\bar{\psi}_b(y) |0\rangle \end{aligned} \tag{11.77}$$

and similarly,

$$\left\{\overline{\psi_b^+}(y), \psi_a^+(x)\right\} = \langle 0| \bar{\psi}_b(y)\psi_a(x) |0\rangle \,. \tag{11.78}$$

We can then cast Eq 11.76 in the form:

$$T\psi_a(x)\bar{\psi}_b(y) =: \psi_a(x)\bar{\psi}_b(y): + \langle 0| T\psi_a(x)\bar{\psi}_b(y) |0\rangle \,. \tag{11.79}$$

Let us now study the vacuum expectation value of products of four fields. To facilitate the manipulations we shall introduce the shorthand notation:

$$\psi_{a_i}(x_i) \equiv i, \quad \langle 0| \ldots |0\rangle \equiv \langle \ldots \rangle \,. \tag{11.80}$$

Assuming that $x_1^0 > x_2^0 > x_3^0 > x_4^0$. Then, after a little algebra, we find:

$$\begin{aligned}\langle T\bar{1}2\bar{3}4\rangle &= \langle \bar{1}2\bar{3}4\rangle \\ &= \langle \bar{1}_+(2_- + 2_+)(\bar{3}_- + \bar{3}_+)4_+\rangle \\ &= \langle [\{\bar{1}_+, 2_+\} + \bar{1}_+2_-]\,[\{\bar{3}_+, 4_+\} + \bar{3}_-4_+]\rangle \\ &= \langle \bar{1}2\rangle \langle \bar{3}4\rangle + \langle \bar{1}_+2_-\bar{3}_-4_+\rangle \\ &= \langle \bar{1}2\rangle \langle \bar{3}4\rangle + \langle \bar{1}_+\{2_-, \bar{3}_-\}\,4_+\rangle - \langle \bar{1}_+\bar{3}_-2_-4_+\rangle \\ &= \langle \bar{1}2\rangle \langle \bar{3}4\rangle + \langle \bar{1}_+4_+\rangle \{2_-, \bar{3}_-\} + \langle \bar{3}_-\bar{1}_+2_-4_+\rangle \\ &= \langle \bar{1}2\rangle \langle \bar{3}4\rangle + \langle \bar{1}4\rangle \langle 2\bar{3}\rangle + 0 \\ &= \langle T\bar{1}2\rangle \langle T\bar{3}4\rangle + \langle T\bar{1}4\rangle \langle T2\bar{3}\rangle \,. \end{aligned} \tag{11.81}$$

Let us now assume that $x_1^0 > x_2^0 > x_4^0 > x_3^0$. We have

$$\begin{aligned}\langle T\bar{1}2\bar{3}4\rangle &= -\langle \bar{1}24\bar{3}\rangle \\ &\ldots \\ &= -\left(\langle \bar{1}2\rangle \langle 4\bar{3}\rangle - \langle \bar{1}4\rangle \langle 2\bar{3}\rangle\right) \\ &= -\left(-\langle T\bar{1}2\rangle \langle T\bar{3}4\rangle - \langle T\bar{1}4\rangle \langle T2\bar{3}\rangle\right) \\ &= \langle T\bar{1}2\rangle \langle T\bar{3}4\rangle + \langle T\bar{1}4\rangle \langle T2\bar{3}\rangle \,. \end{aligned} \tag{11.82}$$

You can verify explicitly that all possible time-orderings give the same result give the result of Eq. 11.81. We can compute similarly other expectation values of time-ordered products of fermion fields. For example,

$$\langle T\bar{1}\bar{2}34\rangle = -\langle T\bar{1}3\rangle\langle T\bar{2}4\rangle + \langle T\bar{1}4\rangle\langle T\bar{2}3\rangle \tag{11.83}$$

The construction of expressions such as Eqs. 11.81-11.83 is algorithmic. Consider the expectation value of a generic time-ordered product of fermionic fields:

$$\langle T f_1 f_2 \dots f_n\rangle .$$

In the above, f_i can be either ψ or $\bar{\psi}$. Now take the first field in the product and "contract" it (form a propagator) with every other field:

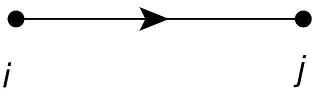
$$\begin{aligned}\langle T f_1 f_2 \dots f_n\rangle \;=\;& (+\langle T f_1 f_2\rangle)\langle T f_3 f_4 f_5 \dots f_n\rangle \\ &+ (-\langle T f_1 f_3\rangle)\langle T f_2 f_4 f_5 \dots f_n\rangle \\ &+ (+\langle T f_1 f_4\rangle)\langle T f_2 f_3 f_5 \dots f_n\rangle \\ &+ \dots\end{aligned} \tag{11.84}$$

The sign of the contracted terms in the rhs can be positive or negative In is equal to $(-1)^{n_P}$ where n_P is the number of permutations which are needed in order to bring the two contracted fields which form a propagator in adjacent positions, starting from the ordering in the expression of the lhs. Some of these contractions may vanish, since $\langle ji\rangle = \langle \bar{i}\bar{j}\rangle = 0$ and only $\langle \bar{i}j\rangle = -\langle j\bar{i}\rangle \neq 0$. Eq. 11.84 can be applied recursively until no more propagators can be formed. For example, for the expectation value of the time-ordered product of six fields we get:

$$\begin{aligned}\langle T\bar{1}2\bar{3}4\bar{5}6\rangle &= +\langle T\bar{1}2\rangle\langle T\bar{3}4\bar{5}6\rangle + \langle T\bar{1}4\rangle\langle T2\bar{3}\bar{5}6\rangle + \langle T\bar{1}6\rangle\langle T2\bar{3}4\bar{5}\rangle \\ &= +\langle T\bar{1}2\rangle\left(\langle T\bar{3}4\rangle\langle T\bar{5}6\rangle + \langle T\bar{3}6\rangle\langle T4\bar{5}\rangle\right) \\ &\quad + \langle T\bar{1}4\rangle\left(\langle T2\bar{3}\rangle\langle T\bar{5}6\rangle - \langle T2\bar{5}\rangle\langle T\bar{3}6\rangle\right) \\ &\quad + \langle T\bar{1}6\rangle\left(\langle T2\bar{3}\rangle\langle T4\bar{5}\rangle + \langle T2\bar{5}\rangle\langle T\bar{3}4\rangle\right) \\ &= -\langle T2\bar{1}\rangle\langle T4\bar{3}\rangle\langle T6\bar{5}\rangle + \langle T2\bar{1}\rangle\langle T4\bar{5}\rangle\langle T6\bar{3}\rangle \\ &\quad + \langle T2\bar{3}\rangle\langle T4\bar{1}\rangle\langle T6\bar{5}\rangle - \langle T2\bar{3}\rangle\langle T4\bar{5}\rangle\langle T6\bar{1}\rangle \\ &\quad + \langle T2\bar{5}\rangle\langle T4\bar{3}\rangle\langle T6\bar{1}\rangle - \langle T2\bar{5}\rangle\langle T4\bar{1}\rangle\langle T6\bar{3}\rangle\end{aligned} \tag{11.85}$$

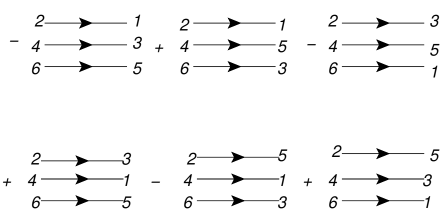
We can represent Eq. 11.85 graphically. Let us define the Dirac fermion propagator as an arrowed line:

$$\langle i\bar{j}\rangle \equiv \langle 0|\, T\psi_{a_i}(x_i)\bar{\psi}_{a_j}(x_j)\,|0\rangle \equiv \tag{11.86}$$



The arrow leaves a field ψ and points to a field $\bar{\psi}$. Then, Eq. 11.85 can be depicted as:

$$\langle \bar{1}2\bar{3}4\bar{5}6\rangle = \tag{11.87}$$



This accounts for all possibilities that we can connect the six points with propagators (bearing in mind that only ψ and $\bar{\psi}$ can be contracted, while two ψ's or two $\bar{\psi}$'s cannot). The sign of each graph is determined by the number of permutations which are required in order to bring the contracted fields next to each other, starting from the ordering in the left hand side of the equation.

A useful application of what we have learnt is in computing probability amplitudes in perturbative computations in quantum electrodynamics (QED) and in the Standard Model of particle physics. One encounters expectation values of the type:

$$\frac{1}{n!}\int d^4x_1 \dots d^4x_n \, \langle TV(x_1)\dots V(x_n)\rangle \,, \tag{11.88}$$

with

$$V(x) \equiv \bar{\psi}(x)\Gamma\psi(x) = \Gamma_{ab}\bar{\psi}_a(x)\psi_b(x). \tag{11.89}$$

Γ_{ab} is a generic 4×4 matrix which can be of course written in a basis of $1, \gamma^\mu, \gamma_5, \gamma_5\gamma^\mu, [\gamma^\mu, \gamma^\nu]$.
for next time:

- *Actually, state Wick's theorem for Dirac fermions*
- *Derive the Feynman rules in momentum space*
- *Derive minus sign in loops and when crossing fermion lines.*

11.7.2 Wick's theorem for Majorana fermions*

Chapter 12

Scattering Theory (S-matrix)

We have studied the quantum field theories of free elementary particles with spin-0 (Klein-Gordon field), spin-$\frac{1}{2}$ (Dirac spinor field), and spin-1 (Electromagnetic photon field). In all these cases, we could identify a field four momentum $P^\mu = \left(H, \vec{P}\right)$ which was a conserved quantity. It was also relatively easy to find a complete set of eigenstates for P^μ. In all these "free field theories" the Hamiltonian eigenstates happened to be, in addition, eigenstates of a time-independent "number operator". States with definite energy and momentum were, therefore, also states of a conserved number of indestructible particles [1].

In reality, however, particles interact and may also be destroyed or created in a scattering process. Realistic field theories contain additional terms in their Lagrangian which do not make it possible to find a constant particle-number operator. In a free field theory, conservation of energy and momentum is a consequence of the fact that the theory is invariant under time and space translations. This is a property which will also hold for interacting field theories. Conservation of the particle number was, on the other hand, only a consequence of an "accident" that a field free theory was rather simple. This "accident" does not occur anymore in an interacting field theory, where we expect it to describe successfully particle creation and annihilation.

In this chapter we will consider interacting field theories in general, without specifying the details of their Lagrangian. We will only require that they yield physical laws which are invariant under space-time translations, and therefore their is a conserved field momentum operator. We will also make one further assumption, which we will explain later in its detail, that particle interactions happen at short distances, and that particles when are at far distances they behave as if they were free. These two considerations will lead us to general expressions for the probability amplitude of a scattering process.

12.1 Propagation in a general field theory

We start our analysis by studying the propagator of fields in a general theory, as we have specified it above. We shall also assume that there is a ground state, which we denote by

$$|\Omega\rangle \,.$$

For simplicity we consider here the propagator of a scalar (interacting) field $\phi(x)$, which we write as

$$\langle\Omega| \, T\left\{\phi(x)\phi(y)\right\} |\Omega\rangle = \langle\Omega| \, T\left\{\phi(x)\mathbf{1}\phi(y)\right\} |\Omega\rangle \,. \tag{12.1}$$

[1] In mathematical parlance, the Hilbert space of states had a Fock-state representation.

Let us form a unit operator **1** out of a complete set of states. Space-time translation invariance dictates that $P^\mu \equiv (H, \vec{P})$, the field four momentum of the Lagrangian system is a conserved quantity. Notice that we do not need the explicit expression for P^μ; Noether's theorem guarantees that such an operator does exist. We denote with $|p\rangle$ the corresponding eigenstates of the momentum-operator,

$$P^\mu |p\rangle = p^\mu |p\rangle, \tag{12.2}$$

or, in components,

$$H |p\rangle = E_p |p\rangle, \qquad \vec{P} |p\rangle = \vec{p} |p\rangle, \tag{12.3}$$

with

$$p^\mu = (E_p, \vec{p}). \tag{12.4}$$

As we have remarked, the states $|p\rangle$, are not , necessarily, states with a definite number of particles. To simplify the discussion, we also assume here that all momentum eigenvalues are not light-like, having $p^2 \neq 0$. Then, for each state $|p\rangle$ with momentum p^μ, we can perform a Lorentz boost which renders $\vec{p} = 0$. Conversely, all states $|p\rangle$ can be produced by applying Lorentz transformations to states $|\lambda_0\rangle$ with

$$\vec{P} |\lambda_0\rangle = 0. \tag{12.5}$$

Let $|\lambda_p\rangle$ a boosted state from $|\lambda_0\rangle$. Then we can write,

$$\mathbf{1} = |\Omega\rangle \langle\Omega| + \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} |\lambda_p\rangle \langle\lambda_p|. \tag{12.6}$$

The sum in λ runs over all possible states with a zero space momentum $|\lambda_0\rangle$. Let us now assume that $x^0 > y^0$. The propagator is then written as,

$$\begin{aligned}
& \langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle|_{x^0>y^0} = \\
& \langle\Omega| \phi(x) \left(|\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} |\lambda_p\rangle\langle\lambda_p| \right) \phi(y) |\Omega\rangle \\
= \; & \langle\Omega| \phi(x) |\Omega\rangle \langle\Omega| \phi(y) |\Omega\rangle \\
+ \; & \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \langle\Omega| \phi(x) |\lambda_p\rangle \langle\lambda_p| \phi(y) |\Omega\rangle.
\end{aligned} \tag{12.7}$$

Consider now the matrix-element

$$\langle\Omega| \phi(x) |\lambda_p\rangle$$

, and recall that the system is invariant under space-time translations $x^\mu \to x^\mu + \epsilon^\mu$. The field four-momentum is the generator of space-time translations. If we know the value of the quantum field at a point y^μ, then we can find its value at a different space-time point by applying,

$$\phi(x^\mu) = e^{iP\cdot(x-y)} \phi(y) e^{-iP(x-y)}. \tag{12.8}$$

Exercise: *Prove the above using the theorem of Section 5.4*
It is very convenient to express the field in the matrix-element in terms of the value of the quantum field at the origin,

$$\langle\Omega| \phi(x) |\lambda_p\rangle = \langle\Omega| e^{iP\cdot x} \phi(0) e^{-iP\cdot x} |\lambda_p\rangle. \tag{12.9}$$

The ground state is invariant under space-time translations,

$$e^{-iP\cdot x}\left|\Omega\right\rangle = \left|\Omega\right\rangle . \tag{12.10}$$

Also, a state $\left|\lambda_p\right\rangle$ has a definite momentum $p^\mu = (E_p, \vec{p})$. Thus,

$$e^{-iP\cdot x}\left|\lambda_p\right\rangle = \left|\lambda_p\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_p} . \tag{12.11}$$

We can then write,

$$\left\langle\Omega\right|\phi(x)\left|\lambda_p\right\rangle = \left\langle\Omega\right|\phi(0)\left|\lambda_p\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_p} . \tag{12.12}$$

We now recall that the state $\left|\lambda_p\right\rangle$ is produced from a state with a zero space-momentum with a Lorentz boost,

$$\left|\lambda_p\right\rangle = U\left(\Lambda_p\right)\left|\lambda_0\right\rangle . \tag{12.13}$$

We can also exploit that the vacuum state is invariant under Lorentz transformations,

$$U\left(\Lambda_p\right)\left|\Omega\right\rangle = U^{-1}\left(\Lambda_p\right)\left|\Omega\right\rangle = \left|\Omega\right\rangle , \tag{12.14}$$

and cast Eq. 12.12 as

$$\left\langle\Omega\right|\phi(x)\left|\lambda_p\right\rangle = \left\langle\Omega\right|U^{-1}\left(\Lambda_p\right)\phi(0)U\left(\Lambda_p\right)\left|\lambda_0\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_p} . \tag{12.15}$$

For a scalar field,

$$U\left(\Lambda_p\right)\phi(x^\mu)U^{-1}\left(\Lambda_p\right) = \phi\left(\Lambda^\mu_\nu x^\nu\right) , \tag{12.16}$$

which for a position at the origin $x^\mu = 0$ yields

$$U\left(\Lambda_p\right)\phi(0)U^{-1}\left(\Lambda_p\right) = U^{-1}\left(\Lambda_p\right)\phi(0)U\left(\Lambda_p\right) = \phi(0). \tag{12.17}$$

We then find that the space-time dependence of the matrix-element in Eq. 12.15 is a simple exponential,

$$\left\langle\Omega\right|\phi(x)\left|\lambda_p\right\rangle = \left\langle\Omega\right|\phi(0)\left|\lambda_0\right\rangle e^{-ip\cdot x}\Big|_{p^0=E_p} . \tag{12.18}$$

Similarly,

$$\left\langle\lambda_p\right|\phi(y)\left|\Omega\right\rangle = \left\langle\lambda_0\right|\phi(0)\left|\Omega\right\rangle e^{+ip\cdot y}\Big|_{p^0=E_p} . \tag{12.19}$$

Finally,

$$\left\langle\Omega\right|\phi(y)\left|\Omega\right\rangle = \left\langle\Omega\right|e^{iP\cdot y}\phi(0)e^{-iP\cdot y}\left|\Omega\right\rangle = \left\langle\Omega\right|\phi(0)\left|\Omega\right\rangle . \tag{12.20}$$

Using Eqs 12.18-12.20, we find that the propagator in Eq. 12.7 becomes,

$$\begin{aligned}\left\langle\Omega\right|T\left\{\phi(x)\phi(y)\right\}\left|\Omega\right\rangle\Big|_{x^0>y^0} &= \left\langle\Omega\right|\phi(0)\left|\Omega\right\rangle^2 \\ + \sum_\lambda \left|\left\langle\Omega\right|\phi(0)\left|\lambda_0\right\rangle\right|^2 &\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\, e^{-ip\cdot(x-y)}\Big|_{p^0=E_p} .\end{aligned} \tag{12.21}$$

In the same fashion, we obtain for the second time-ordering possibility

$$\begin{aligned}\left\langle\Omega\right|T\left\{\phi(x)\phi(y)\right\}\left|\Omega\right\rangle\Big|_{y^0>x^0} &= \left\langle\Omega\right|\phi(0)\left|\Omega\right\rangle^2 \\ + \sum_\lambda \left|\left\langle\Omega\right|\phi(0)\left|\lambda_0\right\rangle\right|^2 &\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\, e^{-ip\cdot(y-x)}\Big|_{p^0=E_p} .\end{aligned} \tag{12.22}$$

The propagator is,

$$\begin{aligned}\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle &= \langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle|_{x^0>y^0}\Theta(x^0-y^0) \\ &+ \langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle|_{x^0>y^0}\Theta(x^0-y^0),\end{aligned} \tag{12.23}$$

which yields,

$$\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle^2 + \sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \times$$
$$\times \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[e^{-ip\cdot(x-y)}\Theta(x^0-y^0) + e^{-ip\cdot(y-x)}\Theta(y^0-x^0)\right]_{p^0=E_p}. \tag{12.24}$$

We recognize that the above integral is the Feynman-Stückelberg propagator of Eq. 11.33, of the free scalar field theory. We denote by m_λ^2 the eigenvalue of the squared field-momentum P^2 corresponding to the state $|\lambda_0\rangle$,

$$P^2|\lambda_0\rangle = m_\lambda^2|\lambda_0\rangle. \tag{12.25}$$

Then, for the boosted state $|\lambda_p\rangle$, we have

$$P^2|\lambda_p\rangle = \left(H^2 - \vec{P}^2\right)|\lambda_p\rangle = \left(E_p^2 - \vec{p}^2\right)|\lambda_p\rangle. \tag{12.26}$$

Given that $|\lambda_p\rangle = U(\Lambda_p)$, and P^2 is invariant under a Lorentz transformation, we obtain that

$$E_p(\lambda)^2 - \vec{p}^2 = m_\lambda^2. \tag{12.27}$$

Then, using the result of Eq. 11.33, we write

$$\begin{aligned}&\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle^2 \\ &+\sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\delta}e^{-ip\cdot(x-y)}\end{aligned} \tag{12.28}$$

This is a general result for the propagator of any scalar field in any quantum field theory which is symmetric under Poincare' (Lorentz and translation) transformations.

Let us now assume that there exists a density of states $|\lambda_0\rangle$, with $P^2|\lambda_0\rangle = m_\lambda^2|\lambda_0\rangle$. We can replace the sum in the above expression with an integral,

$$\begin{aligned}&\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle^2 \\ &+\int dM^2\rho(M^2)\,|\langle\Omega|\phi(0)|\lambda_0(M)\rangle|^2 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-M^2+i\delta}e^{-ip\cdot(x-y)}\end{aligned} \tag{12.29}$$

This is the Källen-Lehmann representation for the scalar field propagator. It is an amazing result for its simplicity, given its generality. Up to constants ($\langle\Omega|\phi(0)|\Omega\rangle^2$, $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$), the propagator is determined fully by the density of the energy eigenstates with zero space-momentum.

12.1.1 A special case: free scalar field theory

Our result for the Feynman-Stückelberg can be now rederived as a special case of the Källen-Lehmann propagator of Eq. 12.29 when considering the free Klein-Gordon field. In this case, we are able to find a simple expression for the quantum field $\phi(x)$,

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^2 2\omega_p} \left[a(p) e^{-ip\cdot x} + a^\dagger(p) e^{+ip\cdot x} \right], \tag{12.30}$$

with $\omega_p = \sqrt{\vec{p}^2 + m^2}$. The vacuum state is,

$$|\Omega\rangle = |0\rangle, \tag{12.31}$$

and it is annihilated by the ladder operator $a(p)$,

$$a(p)|0\rangle = \langle 0| a^\dagger(p) = 0. \tag{12.32}$$

We can easily see that

$$\langle 0| \phi(0) |0\rangle = 0. \tag{12.33}$$

We also need to compute the density of states $|\lambda_0\rangle$, with $H|\lambda_0\rangle = m_\lambda |\lambda_0\rangle$ and $\vec{P}|\lambda_0\rangle = 0$. In the case of the free Klein-Gordon field we can identify a number of particles for each eigenstate of the P^μ operator. We can build states of higher and higher energy with $\vec{p} = 0$, by applying repeatedly the creation operator $a^\dagger(0)$ on the vacuum. We have

$$\begin{aligned}
H\left(a^\dagger(0)|0\rangle\right) &= m\left(a^\dagger(0)|0\rangle\right), \\
H\left(a^\dagger(0)^2|0\rangle\right) &= 2m\left(a^\dagger(0)^2|0\rangle\right), \\
&\cdots \\
H\left(a^\dagger(0)^n|0\rangle\right) &= nm\left(a^\dagger(0)^n|0\rangle\right), \\
&\cdots
\end{aligned}$$

The density is then

$$\rho(M^2) = \sum_{n=1}^{\infty} \delta(M^2 - n^2 m^2). \tag{12.34}$$

Finally, we require the constants

$$\langle 0| \phi(0) |\lambda_0\rangle = \langle 0| \phi(0) \left(a^\dagger(0)^n |0\rangle \right) = \delta_{n,1}. \tag{12.35}$$

Exercise: prove the above.

We now substitute Eqs 12.33, 12.34 and Eq. 12.35 into Eq. 12.29. We recover

$$\langle 0| T\{\phi(x)\phi(y)\} |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\delta} e^{-ip\cdot(x-y)}, \tag{12.36}$$

which is the familiar Feynman-Stückelberg propagator for the free Klein-Gordon field.

12.1.2 "Typical" interacting scalar field theory

It is not possible to compute the density $\rho(M^2)$ of zero-momentum $\vec{p} = 0$ states $|\lambda_0\rangle$ in realistic field theories where the Lagrangian contains additional potential terms to the ones of the free scalar Klein-Gordon field. We can make, however, some reasonable assumptions which agree with observations on the systems that we would like to describe.

We have emphasized several times that in interacting theories it is not, in general, possible to identify a number of particles corresponding to a certain energy-momentum eigenstate. Nevertheless, we do observe particles which act as if they were free and isolated. Such a state is characterized by the mass m of the particle. If we now consider a state of two "approximately free" particles the energy of the state when the combined momentum of the two particles is $\vec{p} = 0$ is also "approximately" twice the mass of one-particle $2\,m$. However, their interaction potential will not allow for this exact value, and in general, their energy will be larger. States other than one-particle states form a continuum of masses M^2 which is larger than the minimum energy (corresponding to free particles) required to get two particles in the same state. An exception to this occurs when the two-particles form a bound state, where their binding energy is negative and their effective mass is generally smaller than $2\,m$.

Although a very important issue, we will not analyze in this course bound state problems. Then, a typical density is,

$$\rho\left(M^2\right) = \delta(M^2 - m^2_{1-\text{particle}}) + \Theta(M^2 - 4m^2_{1-\text{particle}})\rho_{\text{cont}}(M^2). \tag{12.37}$$

where ρ_{cont} a continuous function. Substituting this density functional form into the general expression of Eq. 12.29 for the scalar propagator we find,

$$\begin{aligned}
&\langle\Omega| T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle = \\
&|\langle\Omega|\,\phi(0)\,|1_{\text{particle}}\rangle|^2 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2_{1_{\text{particle}}} + i\delta}e^{-ip\cdot(x-y)} \\
&+ \int dM^2 \rho_{\text{cont}}(M^2)\,|\langle\Omega|\,\phi(0)\,|\lambda_0(M)\rangle|^2 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - M^2 + i\delta}e^{-ip\cdot(x-y)} \\
&\quad + \langle\Omega|\,\phi(0)\,|\Omega\rangle^2
\end{aligned} \tag{12.38}$$

The above result is very important, stating that one-particle states are responsible for poles in the propagator of an interacting field. If a theory has states which behave as isolated then we anticipate to find poles in the propagator for squared momenta $p^2 = m_{1_{\text{particle}}}$ which correspond to the physical mass of the particles. Suggestive to the above, it is then worth to remember in practice that the scalar propagator is

$$\langle\Omega| T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i\tilde{Z}_\phi e^{-ip\cdot(x-y)}}{p^2 - m^2_{\text{phys}} + i\delta} + \text{continuum}, \tag{12.39}$$

where

$$\tilde{Z}_\phi = |\langle\Omega|\,\phi(0)\,|1_{\text{particle}}\rangle|^2\,, \tag{12.40}$$

a constant, and m_{phys} the physical mass of a particle.

12.2 Spectral assumptions in scattering theory

We are now ready to study using quantum field theory the scattering of elementary particles. Scattering processes occur in Lagrangian systems with interaction potentials, i.e. additional terms in the Lagrangian other than the ones that we have found in the quantization of free fields. It has never been possible to find an exact solution for the Hamiltonian of any field theory which describes realistic particle interactions in four space-time dimensions. However, it is possible under some reasonable assumptions to develop a theory for scattering probability amplitudes. For simplicity, we shall carry out our analysis for the case of interacting scalar fields. This is sufficient to demonstrate the salient features of the scattering theory.

Our basic set of assumptions is the following,

- The theory has a ground state $|\Omega\rangle$
- There are "first excited" states $|p\rangle$ with

$$P^2 |p\rangle = m_{phys}^2 |p\rangle , \tag{12.41}$$

  corresponding intuitively to "single-particle" states. We also assume that $m_{phys}^2 \geq 0$, which means that there are no tachyons (particles which travel faster than the light)

- All remaining eigenstates $|q\rangle$ of the field momentum operator have

$$P^2 |q\rangle = M^2 |q\rangle , \tag{12.42}$$

  with

$$M^2 \geq 4m_{phys}^2. \tag{12.43}$$

  These states form a continuum, and they correspond, intuitively, to multi-particle excitations.

- The expectation value of the field operator in the ground state is zero,

$$\langle\Omega| \phi(x) |\Omega\rangle = 0. \tag{12.44}$$

Roughly, $\phi(x) |\Omega\rangle$ should correspond to an one-particle excited state. A single particle should not disappear spontaneously into a state with no particles $|\Omega\rangle$. The corresponding probability amplitude should therefore vanish,

$$\langle\Omega| \mathbf{1}_{\text{particle}}\rangle \sim \langle\Omega| \phi(x) |\Omega\rangle = 0.$$

12.3 "In" and "Out" states

As we have discussed earlier, states in interacting field theories do not longer describe systems of particles with a necessarily constant number of them. Free field theories have a Hilbert-space of states with states of definite particle number, but Hamiltonian eigenstates in interacting theories do not. Nevertheless, we do observe particles which behave as free (e.g. free electrons and photons) when they are sufficiently separated. By the uncertainty principle, such states describing particles which are isolated at certain times must be wave-packets.

Two classes of wave-packets are interesting in order to describe the scattering of freely moving particles.

- Wave-packets which are isolated at a time very far in the past, $t = -\infty$, but may be overlapping at some finite time t
- Wave-packets which are isolated at in the far future, $t = +\infty$, but may be overlapping at some finite time t.

We should remember that in our development of the field theory formalism, we are using the Heisenberg picture for time evolution. In this picture, field operators depend on time, while states are time independent [2]. Time-independent states are perhaps counter-intuitive, and can mislead us to believe that they describe non-evolving systems. But this is of course not true. Measurements of expectation values of field operators on a state are time-dependent, and in this sense, states encapsulate the time-evolution of the physical system that they describe. We define a state as an $\underline{|\text{in}\rangle \text{ state}}$, if it describes a system of freely moving particles in the very far past, which may or may not be interacting at a later time. We define a state as an $\underline{|\text{out}\rangle \text{ state}}$, if it describes a system of particles which may or may not be interacting at a finite time but they are freely moving in the far future.

Consider an $|\text{in}\rangle$ state which describes a single particle with momentum p in the far past. We will assume that free particles are indestructible, stable, in the absence of other particles. Then this state will also describe a free single particle in the far future, and it will also be an state $|\text{in}\rangle$:

$$|\text{in, single} - \text{particle}, p\rangle = |\text{out, single} - \text{particle}, p\rangle . \tag{12.45}$$

More complicated "in" and "out" states containing many free particles in the very far past or future respectively, allow for particle interactions at a finite time. What are the possible "in" and "out" states? We will make the following assumption:
" Every possible state is a linear combination of either $|\text{in}\rangle$ or $|\text{out}\rangle$ states."
In physical terms, this means that every physical system will behave as a system of freely moving particles which are getting increasingly isolated if we wait long enough. Conversely, if we look further back in the far past, every physical system originates from a system of freely moving isolated particles.

The above assumption is in accordance with the particle interactions occur when particles are brought very close together. Short range interactions cannot keep particles together perpetually. There is a striking important "exception" to this assumption: elementary particles may form bound states. In practice, we know from experimentation which are the bound states that may be formed (hadrons, atoms, molecules, etc). These are stable, and we can account for them by including them in a generalized definition of $|\text{in}\rangle$ and $|\text{out}\rangle$ states.

We adopt the following notation. We denote by

$$|\{p_i\}, \text{in}\rangle$$

an "in" state where the i^{th} particle has momentum p_i. We denote by

$$|\{k_j\}, \text{out}\rangle$$

[2] For example, in the free scalar field case the field has a solution which is explicitly time-dependent,

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^2 2\omega_p} \left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{+ip\cdot x}\right],$$

while states, $|0\rangle, |p\rangle = a^\dagger(p)\,|0\rangle, \ldots$, are manifestly time-independent.

an "out" state where the j^{th} particle has momentum k_j. These states are complete. Every state can be written as a linear combination of either of them, and we have:

$$\sum |\{p_i\}, \text{in}\rangle \langle \text{in}, \{p_i\}| = \sum |\{k_j\}, \text{out}\rangle \langle \text{out}, \{k_j\}| = \mathbf{1} \tag{12.46}$$

12.4 Scattering Matrix-Elements

Obviously, an $|\{p_i\}, \text{in}\rangle$ state can be written as a superposition of $|\{k_j\}, \text{out}\rangle$ states. This physically means that a system of originally free particles will evolve to a system of ultimately free particles. If these particles never interact, the "in" state is the same as the "out" state. If they do, then we may have isolated particles after the interaction with different momenta or even a different number of them. We write,

$$|\{p_i\}, \text{in}\rangle = \sum_{\{k_j\}} c\left(\{k_j\}, \{p_i\}\right) |\{k_j\}, \text{out}\rangle \,. \tag{12.47}$$

The probability amplitude for the transition of a system of $i = 1 \dots N$ incoming particles with momenta $\{p_i\}$ to a system of $j = 1 \dots M$ outgoing particles with momenta $\{k_j\}$ is known as the "scattering matrix-element",

$$S_{ji} \equiv c\left(\{k_j\}, \{p_i\}\right) |\{k_j\}\rangle = \langle \text{out}, \{k_j\}| \, \{p_i\}, \text{in}\rangle \tag{12.48}$$

The scattering probability is

$$P\left(\{p_i\} \to \{k_j\}\right) = |S_{ji}|^2 = |\langle \text{out}, \{k_j\}| \, \{p_i\}, \text{in}\rangle|^2 \,. \tag{12.49}$$

The scattering matrix-elements constitute the S-matrix, which is a matrix acting on the space of "in" states and transforms them into "out" states,

$$S \equiv S_{ij}, \qquad |\alpha, \text{out}\rangle = S_{\alpha\beta} |\beta, \text{in}\rangle \,. \tag{12.50}$$

The S-matrix is unitary,

$$\begin{aligned} \left(SS^\dagger\right)_{\alpha\gamma} &= \sum_\beta S_{\alpha\beta} S^\dagger_{\beta\gamma} = \sum_\beta \langle \alpha, \text{in}| \, \text{out}, \beta\rangle \langle \beta, \text{out}| \, \gamma, \text{in}\rangle \\ &= \langle \alpha, \text{in}| \left(\sum_\beta |\beta, \text{out}\rangle \langle \text{out}, \beta| \right) |\gamma, \text{in}\rangle = \delta_{\alpha\gamma}. \end{aligned} \tag{12.51}$$

We are usually not interested in computing the probability of not having a scattering. The matrix-elements of the S-matrix which are relevant for a true scattering are the non-diagonal. We then define the so called "transition-matrix" T, which is given by

$$S \equiv \mathbf{1} + \mathbf{iT}. \tag{12.52}$$

In the rest of this chapter we shall prove a fundamental relation of S-matrix elements to expectation values of field operators in the ground state of the interacting theory.

12.5 S-matrix and Green's functions

In the last section, we established very few properties of the S-matrix which, at a first sight, offer little help in computing it explicitly. It is possible, however, to extract S-matrix elements from Green's functions, which are defined as

$$G(x_1, x_2, \ldots, x_N) \equiv \langle\Omega| T\{\phi(x_1)\phi(x_2)\ldots\phi(x_N)\} |\Omega\rangle . \tag{12.53}$$

In the simplest case of one space-time point,

$$\langle\Omega| \phi(x) |\Omega\rangle = 0, \tag{12.54}$$

we have assumed that the corresponding Green's function vanishes, as part of our spectral assumptions for an interacting field theory.

The two-point Green's function is our familiar propagator, for which we have found the Källen-Lehmann representation

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\tilde{Z}_\phi e^{-ip\cdot(x-y)}}{p^2 - m_{phys}^2 + i\delta} + \text{continuum}. \tag{12.55}$$

Consider the explicit case with $x^0 > \tau > -\tau > y^0$ with $\tau \to \infty$, for two space-time points where one is very far in the future and the second very far in the past. Then,

$$\begin{aligned}
G(x,y)\big|_{\substack{x^0\to+\infty \\ y^0\to-\infty}} &= \langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle \\
&= \langle\Omega| \{\phi(x)\phi(y)\} |\Omega\rangle \\
&= \langle\Omega| \{\phi(x)\mathbf{1}\,\mathbf{1}\,\phi(y)\} |\Omega\rangle \\
&= \langle\Omega| \phi(x) \left(\sum_{\text{out}} |\text{out}\rangle \langle\text{out}|\right) \left(\sum_{\text{in}} |\text{in}\rangle \langle\text{in}|\right) \phi(y) |\Omega\rangle \\
\rightsquigarrow G(x,y)\big|_{\substack{x^0\to+\infty \\ y^0\to-\infty}} &= \sum_{\text{in}}\sum_{\text{out}} \langle\Omega| \phi(x) |\text{out}\rangle \langle\text{out}| \text{in}\rangle \langle\text{in}| \phi(y) |\Omega\rangle
\end{aligned} \tag{12.56}$$

We have defined an $|\text{out}\rangle$ as any state which behaves as a state of any number of free isolated particles in the far future. The the sum over all "out" states corresponds to

$$\begin{aligned}
&\sum_{out} \langle\Omega| \phi(x) |\text{out}\rangle \langle\text{out}| \to \langle\Omega| \phi(x) |\Omega\rangle \langle\Omega| \\
+ &\int \frac{d^3\vec{p}_1}{(2\pi)^3 2\omega_{p_1}} \langle\Omega| \phi(x) |p_1;\text{out}\rangle \langle p_1;\text{out}| \\
+ &\int \frac{d^3\vec{p}_1}{(2\pi)^3 2\omega_{p_1}} \frac{d^3\vec{p}_2}{(2\pi)^3 2\omega_{p_2}} \langle\Omega| \phi(x) |p_1,p_2;\text{out}\rangle \langle p_1,p_2;\text{out}| \\
+ &\ldots
\end{aligned} \tag{12.57}$$

We now make use the property of "out" states behaving as states of free particles at large times $x^0 > \tau$, and compute $\langle\Omega| \phi(x) |\text{out}\rangle$ in free field theory. In free field theory, the state $\phi(x)\Omega$ corresponds to a state with exactly one-particle, and it has no overlap with states of a different particle multiplicity. We then have,

$$\langle\Omega| \phi(x) |\Omega\rangle = \langle\Omega| \phi(x) |p_1,p_2;\text{out}\rangle = \langle\Omega| \phi(x) |p_1,p_2,p_3;\text{out}\rangle = \ldots = 0, \tag{12.58}$$

and only

$$\langle\Omega|\,\phi(x)\,|p_1;\text{out}\rangle \neq 0. \tag{12.59}$$

We have already computed this non-vanishing already up to a constant. From Eq. 12.18 and Eq. 12.40 we have

$$\langle\Omega|\,\phi(x)\,|p;\text{out}\rangle = \sqrt{\tilde{Z}_\phi}e^{-ip\cdot x} \tag{12.60}$$

We compute the sum over "in" states in Eq. 12.56 in an identical fashion. Then we arrive to a simple result

$$G(x,y)|_{\substack{x^0\to+\infty\\ y^0\to-\infty}} = \tilde{Z}_\phi \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\frac{d^3\vec{q}}{(2\pi)^3 2\omega_q} e^{-i(p\cdot x - q\cdot y)} \langle p;\text{out}|\,q;\text{in}\rangle\,. \tag{12.61}$$

Eq. 12.61 relates the Green's function function for two well separated in time points to the transition amplitude from an "in" single-particle state to an "out" single-particle state. How about more complicated Green's functions and transition amplitudes? We can repeat the same procedure for a general Green's function

$$G(x_1,\ldots,y_1\ldots)|_{\substack{x_i^0\to+\infty\\ y_i^0\to-\infty}} = \langle\Omega|\,T\left\{\phi(x_1)\ldots\phi(y_1)\ldots\right\}|\Omega\rangle_{\substack{x_i^0\to+\infty\\ y_i^0\to-\infty}}\,. \tag{12.62}$$

with N_{out} points at far future times x_i^0 and N_{in} points y_j at far past times. We find that

$$\begin{aligned} G(x_1,\ldots,y_1\ldots)|_{\substack{x_i^0\to+\infty\\ y_i^0\to-\infty}} &= \int \left(\prod_{i=1}^{N_{\text{out}}} \frac{d^3\vec{p}_i\; e^{-ip_i\cdot x_i}}{(2\pi)^3 2\omega_{p_i}}\right)\left(\prod_{j=1}^{N_{\text{in}}} \frac{d^3\vec{q}_j\; e^{+iq_j\cdot y_j}}{(2\pi)^3 2\omega_{q_j}}\right) \\ &\times \tilde{Z}_\phi^{\frac{N_{\text{in}}+N_{\text{out}}}{2}} \langle p_1,\ldots,p_{N_{\text{out}}};\text{out}|\,q_1,\ldots q_{N_{\text{in}}};\text{in}\rangle \end{aligned} \tag{12.63}$$

We arrive to a very useful result. Green's functions with $N_{\text{in}} + N_{\text{out}}$, with N_{in} of them chosen in the far past and N_{out} in the far future are, up to an overall constant, some sort of a Fourier transform of S-matrix elements for the scattering of N_{in} particles to N_{out} particles. In the next section we shall use Eq. 12.63 to compute the matrix-elements.

12.6 The LSZ reduction formula

We will compute scattering matrix elements by inverting Eq. 12.63. This would require a type of a Fourier transformation on the space-time coordinates x_i^μ and y_j^μ. The integration over the space components, $\vec{x}_i$ and $\vec{y}_j$, can be unrestricted (from $-\infty$ to $+\infty$), but we need to be careful concerning the time integrations, since we have made assumptions about x_i^0 being far future and y_j^0 far past times.

Let us try to integrate x_i^μ and y_j^μ over the maximum space-time region where we do not contradict our time-ordering assumptions. Consider the integral,

$$\begin{aligned} I\left[\{k_i\},\{l_j\}\right] &\equiv \left(\prod_i \int_\tau^\infty dx_i^0 \int d^3\vec{x}_i e^{ik_i\cdot x_i}\right)\left(\prod_j \int_{-\tau}^{-\infty} dy_j^0 \int d^3\vec{y}_j e^{-il_j\cdot y_j}\right) \\ &\times G(x_1,\ldots,y_1,\ldots). \end{aligned} \tag{12.64}$$

We now substitute the expression for the Green's function that we have found in Eq. 12.63, and perform the space integrations using the identity

$$\int d^3\vec{x} e^{i\vec{k}\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k}). \tag{12.65}$$

The integral of Eq. 12.64 becomes,

$$
\begin{aligned}
I\left[\{k_i\},\{l_j\}\right] &\equiv \left(\prod_i \int_\tau^\infty dx_i^0 \frac{e^{i(k_i^0-\omega(\vec{k}_i))x_i^0}}{2\omega(\vec{k}_i)}\right)\left(\prod_j \int_{-\tau}^{-\infty} dy_j^0 \frac{e^{-i(l_j^0-\omega(\vec{l}_j))y_j^0}}{2\omega(\vec{l}_j)}\right) \\
&\quad \times \tilde{Z}_\phi^{\frac{N_{\rm in}+N_{\rm out}}{2}} \left\langle k_1,\ldots,k_{N_{\rm out}};{\rm out}\right| l_1,\ldots l_{N_{\rm in}};{\rm in}\rangle
\end{aligned} \tag{12.66}
$$

The integrals over the time variables are also easy to perform,

$$
\int_\tau^\infty dt e^{it\left(l^0-\omega(\vec{l})\right)} = \frac{-i}{l^0-\omega(\vec{l})}\left[\lim_{T\to\infty} e^{iT\left(l^0-\omega(\vec{l})\right)} - e^{i\tau\left(l^0-\omega(\vec{l})\right)}\right] \tag{12.67}
$$

The above is problematic due to the term which requires a not well defined limit at infinity. Suppose now, that we modify the time integration in our Fourier transformations and make it slightly complex:

$$
\lim_{\delta\to 0}\int_\tau^\infty dt e^{it(1+i\delta)\left(l^0-\omega(\vec{l})\right)} = \frac{i}{l^0-\omega(\vec{l})} e^{i\tau\left(l^0-\omega(\vec{l})\right)} \tag{12.68}
$$

Then, we can write

$$
\begin{aligned}
I\left[\{k_i\},\{l_j\}\right] &\equiv \left(\prod_i \frac{-i}{(k_i^0-\omega(\vec{k}_i)2\omega(\vec{k}_i)} e^{i\tau\left(k_i^0-\omega(\vec{k}_i)\right)}\right)\left(\prod_j \frac{-i}{(l_j^0-\omega(\vec{l}_j)2\omega(\vec{l}_j)} e^{i\tau\left(l_j^0-\omega(\vec{l}_j)\right)}\right) \\
&\quad \times \tilde{Z}_\phi^{\frac{N_{\rm in}+N_{\rm out}}{2}} \left\langle k_1,\ldots,k_{N_{\rm out}};{\rm out}\right| l_1,\ldots l_{N_{\rm in}};{\rm in}\rangle
\end{aligned} \tag{12.69}
$$

We have now achieved to relate $S-$matrix elements for the probability of a transition of a system with $N_{\rm in}$ particles with momenta $\{l_j^\mu\}$ to a system with $N_{\rm out}$ particles with momenta $\{k_i\}$ to a Fourier-type integrals over Green's functions.

Let us analyze Eq. 12.69 further. We observe that a "Fourier transformed" Green's function on the lhs has poles at

$$
l_i^0 = \omega(\vec{l}_i) = \sqrt{\vec{l}_i^2 + m_{phys}^2},
$$

due to every particle in the "in" and "out" states. We also notice that the poles are independent of the time τ, since the τ dependent exponential becomes one at the pole position $l_0=\omega$. Indeed , we can expand

$$
\frac{e^{i\tau(l_0-\omega)}}{2\omega(l_0-\omega)} = \frac{1}{2\omega(l_0-\omega)} + \frac{i\tau}{2\omega} + \mathcal{O}((l_0-\omega)\tau^2) \tag{12.70}
$$

This means that had we decided to put no restrictions in the time integrations for $I\left[\{k_i\},\{l_j\}\right]$ the coefficient of the poles (their residue) would be unchanged. We can then write,

$$
\begin{aligned}
& \left(\text{ Regular at } l_i^0, k_j^0 \to \omega(\vec{l}_i), \omega(\vec{l}_i)\right) \\
+ & \left(\prod \frac{i\tilde{Z}_\phi^{\frac{1}{2}}}{2\omega(\vec{l}_i)(l_i^0-\omega(\vec{l}_i))}\right)\left(\prod \frac{i\tilde{Z}_\phi^{\frac{1}{2}}}{2\omega(\vec{k}_j)(k_j^0-\omega(\vec{k}_j))}\right)\left\langle k_1,\ldots,k_{N_{\rm out}};{\rm out}\right| l_1,\ldots l_{N_{\rm in}};{\rm in}\rangle \\
= & \ \tilde{G}\left(k_1,\ldots,k_{N_{\rm out}};l_1,\ldots l_{N_{\rm in}}\right),
\end{aligned} \tag{12.71}
$$

where the Green's function $\tilde{G}$ in momentum space is defined as a normal Fourier transform of a Green's function in $x-$space,

$$\begin{aligned}\tilde{G}\left(k_1, \ldots, k_{N_{\rm out}}; l_1, \ldots l_{N_{\rm in}}\right) &= \int_{-\infty}^{+\infty} d^4x_1 \ldots d^4x_{N_{\rm out}} d^4y_1 \ldots d^4y_{N_{\rm in}} e^{i\left[\sum_i x_i\cdot k_i - \sum_j y_j \cdot l_j\right]} \\ &\times \langle\Omega| T\left\{\phi(x_1)\ldots\phi(x_{N_{\rm out}})\phi(y_1)\ldots\phi(y_{N_{\rm in}})\right\}|\Omega\rangle \end{aligned} \tag{12.72}$$

In the above integration, the exponential factors must be slightly deformed [3] in the complex plane,

$$e^{ik\cdot x} \to e^{ik^0x^0(1+i\delta)-i\vec{k}\vec{x}}. \tag{12.75}$$

We now multiply Eq. 12.71 with a factor

$$\frac{l^2 - m^2_{phys}}{i\sqrt{\tilde{Z}_\phi}}$$

for each "in" state and "out" state particle, and take the limit $l^2 - m^2_{phys} = (l^0-\omega)(l^0+\omega) \to 0$. The regular term vanishes in this limit from the lhs of the equation. We are left with the LSZ reduction formula,

$$\begin{aligned}\langle k_1, \ldots, k_{N_{\rm out}}; {\rm out}|\, l_1, \ldots l_{N_{\rm in}}; {\rm in}\rangle &= \lim_{\substack{l_i^2 = m^2_{phys}\\ k_j^2 = m^2_{phys}}} \left[\prod_i \frac{l_i^2 - m^2_{phys}}{i\sqrt{\tilde{Z}_\phi}}\right]\left[\prod_j \frac{k_j^2 - m^2_{phys}}{i\sqrt{\tilde{Z}_\phi}}\right] \\ &\times \tilde{G}\left(k_1, \ldots, k_{N_{\rm out}}; l_1, \ldots l_{N_{\rm in}}\right). \end{aligned} \tag{12.76}$$

which is due to Lehmann, Symanzik and Zimmermann.

12.7 Truncated Green's functions

Let us recall here the Kählen-Lehman representation of the propagator of Eq. 12.39, which we can invert by applying a Fourier transformation. We find

$$\tilde{G}(p) = \frac{i\tilde{Z}_\phi}{p^2 - m^2_{phys}} + \left(\text{Regular at } p^2 \to m^2_{phys}\right) \tag{12.77}$$

where we have defined the propagator in momentum space as

$$\tilde{G}(p) \equiv \int d^4x e^{ip\cdot x} \langle\Omega| T\left\{\phi(x)\phi(0)\right\}|\Omega\rangle. \tag{12.78}$$

[3] Alternatively, we can leave intact the exponentials and perform a real time-integration in Eq. 12.72, but then we have to evaluate the kernel Green's function at deformed space-time points,

$$\begin{aligned}\tilde{G}\left(k_1, \ldots, k_{N_{\rm out}}; l_1, \ldots l_{N_{\rm in}}\right) &= \int_{-\infty}^{+\infty} d^4x_1 \ldots d^4x_{N_{\rm out}} d^4y_1 \ldots d^4y_{N_{\rm in}} e^{i\left[\sum_i x_i\cdot k_i - \sum_j y_j \cdot l_j\right]} \\ &\times \langle\Omega| T\left\{\phi(\tilde{x}_1)\ldots\phi(\tilde{x}_{N_{\rm out}})\phi(\tilde{y}_1)\ldots\phi(\tilde{y}_{N_{\rm in}})\right\}|\Omega\rangle \end{aligned} \tag{12.73}$$

where

$$\tilde{x}^\mu = (x^0(1-i\delta), \vec{x}). \tag{12.74}$$

We can then cast the S-matrix from the LSZ formula as

$$\langle k_1, \ldots, k_{N_{\text{out}}}; \text{out}|\, l_1, \ldots l_{N_{\text{in}}}; \text{in}\rangle = \tilde{Z}_\phi^{\frac{N_{\text{in}}+N_{\text{out}}}{2}} \left. \frac{\tilde{G}\left(k_1, \ldots, k_{N_{\text{out}}}; l_1, \ldots l_{N_{\text{in}}}\right)}{\left(\prod_i \tilde{G}(k_i)\right)\left(\prod_j \tilde{G}(l_j)\right)} \right|_{\substack{k_i^2 \to m_{phys}^2 \\ l_j^2 \to m_{phys}^2}} \tag{12.79}$$

A Green's function which is divided by a propagator for each external particle in the "in" and "out" states is called truncated:

$$\tilde{G}_{\text{trunc}}\left(k_1, \ldots, k_{N_{\text{out}}}; l_1, \ldots l_{N_{\text{in}}}\right) \equiv \frac{\tilde{G}\left(k_1, \ldots, k_{N_{\text{out}}}; l_1, \ldots l_{N_{\text{in}}}\right)}{\left(\prod_i \tilde{G}(k_i)\right)\left(\prod_j \tilde{G}(l_j)\right)} \tag{12.80}$$

The LSZ formula takes the form

$$\langle k_1, \ldots, k_{N_{\text{out}}}; \text{out}|\, l_1, \ldots l_{N_{\text{in}}}; \text{in}\rangle = \tilde{Z}_\phi^{\frac{N_{\text{in}}+N_{\text{out}}}{2}} \left. \tilde{G}_{\text{trunc}}\left(k_1, \ldots, k_{N_{\text{out}}}; l_1, \ldots l_{N_{\text{in}}}\right) \right|_{\substack{k_i^2 \to m_{phys}^2 \\ l_j^2 \to m_{phys}^2}} \tag{12.81}$$

We now have a relation of probability amplitudes for the scattering of free particles in Quantum Field theory, in terms of truncated Green's functions. As we have discussed already, there is no realistic field theory in four dimensions where all such probabilities can be computed exactly. In the next chapter, we shall resort to perturbation theory, which turns to be an amazingly powerful tool.

12.8 Cross-sections*

Chapter 13

Perturbation Theory and Feynman Diagrams

We have been able to solve exactly Hamiltonian systems for the simplest quantum field theories which describe free particles. For these, we could find all eigenstates and eigenvalues of the Hamiltonian. However, in general, when interactions among particles are present, we must resort to perturbation theory.

As a concrete example, we consider a simple new term in the Hamiltonian of a real Klein-Gordon field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{13.1}$$

The Hamiltonian density is,

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}, \quad \phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \tag{13.2}$$

The Hamiltonian is then found to be

$$H = H_0 + H', \tag{13.3}$$

where

$$H_0 = \int d^3\vec{x} \frac{\pi^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2}{2}, \tag{13.4}$$

is the Hamiltonian a free real Klein-Gordon field, and

$$H' = \int d^3\vec{x} \frac{\lambda}{4!}\phi^4, \tag{13.5}$$

is a new term.

For a generic $\lambda \neq 0$, it is not possible to find an exact expression for the quantum field ϕ. In this chapter, we will learn how to construct a solution using perturbation theory, by expanding around the known result for $\lambda = 0$.

It is only necessary to know the field $\phi(\vec{x}, t)$ at a specific time value $t = \tau$. We can use symmetry under time translations in order to determine the field at any other moment t. The generator of space-time translations $x^\mu \to x^\mu + \epsilon^\mu$ is the field momentum $P^\mu = \left(H, \vec{P}\right)$. Specifically for time translations, the generator is the Hamiltonian H. Fields at two different times are related via,

$$\phi(x, t) = e^{+iH(t-\tau)}\phi(\vec{x}, \tau)e^{-iH(t-\tau)} \tag{13.6}$$

(this is an exponentiated form of an equation such as Eq. 5.133, with $Q = P^\mu \delta x_\mu, \delta x_i = 0$). As you may have noticed field operators evolve with time, while states do not. This is usually termed as the Heisenberg picture.

Let us now compute the time evolution of the field $\phi(\vec{x}, \tau)$ in the hypothetical case that there is no interaction ($\lambda = 0$). We find that

$$\phi_I(x,t) = e^{+iH_0(t-\tau)} \phi(\vec{x},\tau) e^{-iH_0(t-\tau)}, \tag{13.7}$$

and solving for $\phi(\vec{x}, \tau)$, we find

$$\phi(x,\tau) = e^{-iH_0(t-\tau)} \phi_I(\vec{x},t) e^{+iH_0(t-\tau)}. \tag{13.8}$$

$\phi_I(\vec{x}, \tau)$ is the field in the "interaction picture". The fields in the interaction picture and the Heisenberg picture are identical if $H = H_0$.

Combining Eq 13.6 and Eq. 13.8 we find the identity

$$\phi(\vec{x},t) = U^{-1}(t,\tau) \phi_I(\vec{x},t) U(t,\tau) \tag{13.9}$$

where we have defined the time-evolution operator in the interaction picture

$$U(t,\tau) \equiv e^{+iH_0(t-\tau)} e^{-iH(t-\tau)}. \tag{13.10}$$

We now make a very important assumption. We require that there is a time, for example in the far past, for which the field of the full Hamiltonian H is a solution of the free Hamiltonian H_0. This is a reasonable approximation if we can identify a time that particles are far from each other and their interaction can be neglected. We can formally implement this by requiring for example that the interaction switches on at a certain time.

Given the existence of such a special time, the interaction field $\phi_I(\vec{x}, t)$ will continue to be a solution of the free Hamiltonian at any time t. Then, Eq. 13.9 is a transformation of the field ϕ_I in the free theory (H_0) to the field ϕ in the full "interacting" theory (H).

13.1 Time evolution operator in the interaction picture

The calculation of the field operator in the full theory proceeds through the evaluation of the time-evolution operator $U(t, \tau)$ given that we can determine the field operator in the free theory.

We shall first derive a differential equation for $U(t, \tau)$. Differentiating with respect to time, we have:

$$\begin{aligned} i\frac{\partial U(t,\tau)}{\partial t} &= i\left[iH_0 e^{iH_0(t-\tau)} e^{-iH(t-\tau)} - i e^{iH_0(t-\tau)} H e^{-iH(t-\tau)}\right] \\ &= e^{iH_0(t-\tau)} (H - H_0) e^{-iH(t-\tau)} \\ &= e^{iH_0(t-\tau)} (H - H_0) e^{-iH_0(t-\tau)} e^{iH_0(t-\tau)} e^{-iH(t-\tau)} \end{aligned} \tag{13.11}$$

We write the above in the form,

$$i\frac{\partial U(t,\tau)}{\partial t} = V_I(t-\tau) U(t,\tau), \tag{13.12}$$

with the interaction potential defined as,

$$V_I(t-\tau) \equiv e^{iH_0(t-\tau)}(H-H_0)e^{-iH_0(t-\tau)}. \tag{13.13}$$

We can cast the general solution of the differential equation Eq. 13.12 as a time ordered exponential (as with an ordinary Schrödinger equation). Integrating both sides of the equation with respect to time, we obtain

$$U(t,\tau) = U(\tau,\tau) + \frac{1}{i}\int_\tau^t dt_1 V_I(t_1-\tau)U(t_1,\tau). \tag{13.14}$$

Let us rewrite the above equation, replacing t with t_1 and t_1 with t_2. We have,

$$U(t_1,\tau) = U(\tau,\tau) + \frac{1}{i}\int_\tau^t dt_2 V_I(t_2-\tau)U(t_2,\tau). \tag{13.15}$$

We can substitute Eq. 13.15 into Eq. 13.14, obtaining

$$\begin{aligned} U(t,\tau) &= \left[1+\frac{1}{i}\int_\tau^t dt_1 V_I(t_1-\tau)\right]U(\tau,\tau) \\ &+ \left(\frac{1}{i}\right)^2 \int_\tau^t dt_1 \int_\tau^{t_1} dt_2 V_I(t_1-\tau)V_I(t_2-\tau)U(t_2,\tau). \end{aligned} \tag{13.16}$$

Obviously, we can repeat inserting Eq. 13.14 to itself as many times as we wish. After an infinite number of iterations we obtain

$$\begin{aligned} U(t,\tau) &= \Bigg[1+\sum_{n=1}^{\infty}\left(\frac{1}{i}\right)^n \int_\tau^t dt_1 \int_\tau^{t_1} dt_2 \ldots \int_\tau^{t_{n-1}} dt_n \\ &\qquad V_I(t_1-\tau)V_I(t_2-\tau)\ldots V_I(t_n-\tau)\Bigg]U(\tau,\tau). \end{aligned} \tag{13.17}$$

Time-Ordering

Consider one of the simplest integrals in the above series,

$$I[t,\tau] = \int_\tau^t dt_1 \int_\tau^{t_1} dt_2 V_I(t_1-\tau)V_I(t_2-\tau). \tag{13.18}$$

It can be written as

$$I[t,\tau] = \int_\tau^t dt_1 \int_\tau^{t} dt_2 V_I(t_1-\tau)V_I(t_2-\tau)\Theta(t_1-t_2). \tag{13.19}$$

By changing integration variables, $t_1 \to t_2$ and $t_2 \to t_1$, we obtain

$$I[t,\tau] = \int_\tau^t dt_1 \int_\tau^{t} dt_2 V_I(t_2-\tau)V_I(t_1-\tau)\Theta(t_2-t_1). \tag{13.20}$$

Adding Eq. 13.19 and Eq. 13.20, we obtain

$$\int_\tau^t dt_1 \int_\tau^{t_1} dt_2 V_I(t_1-\tau)V_I(t_2-\tau) = \frac{1}{2!}\int_\tau^t dt_1 dt_2 T\left\{V_I(t_1-\tau)V_I(t_2-\tau)\right\}, \tag{13.21}$$

where the symbol $T\{\ldots\}$ is our familiar time-ordering symbol, ordering operators from the largest to the smallest times,

$$T\left\{V_I(t_1-\tau)V_I(t_2-\tau)\right\} = V_I(t_1-\tau)V_I(t_2-\tau)\Theta(t_1-t_2)$$

$$+V_I(t_2-\tau)V_I(t_1-\tau)\Theta(t_2-t_1). \tag{13.22}$$

It is not hard to convince ourselves that in general,

$$\int_\tau^t dt_1 V_I(t_1-\tau)\int_\tau^{t_1} dt_2 V_I(t_2-\tau)\ldots\int_\tau^{t_{n-1}} dt_n V_I(t_n-\tau)$$

$$= \frac{1}{n!}\int_\tau^t dt_1\ldots dt_n T\left\{V_I(t_1-\tau)\ldots V_I(t_n-\tau)\right\}. \tag{13.23}$$

Time-Ordered Exponentiated Integral

We define a time-ordered exponentiated integral as the time-ordering of its Taylor series expansion,

$$\begin{aligned} T e^{\int_\tau^t dt' A(t')} &\equiv T\sum_{n=0}^{\infty}\frac{\left[\int_\tau^t dt' A(t')\right]^n}{n!} \\ &= 1+\frac{1}{n!}\sum_{n=1}^{\infty}\int_\tau^t dt_1\ldots dt_n T\left\{A(t_1)\ldots A(t_n)\right\}. \end{aligned} \tag{13.24}$$

We can now write a compact expression for the time-evolution operator of Eq. 13.17 as a time-ordered exponential, by using Eq. 13.23 and the definition of Eq. 13.24. We obtain that,

$$U(t,\tau) = T e^{-i\int_\tau^t dt' V_I(t'-\tau)} U(\tau,\tau). \tag{13.25}$$

To determine the operator $U(t,\tau)$ at an arbitrary time, we require a boundary value $U(\tau,\tau)$ at a time τ. This is identically the unit operator.

$$U(\tau,\tau) = 1, \tag{13.26}$$

Recall that τ is selected to be a time in the far past, when indeed

$$\phi(\vec{x},\tau) = \phi_I(\vec{x},\tau). \tag{13.27}$$

13.2 Field operators in the interacting and free theory

Let us summarize here what we have achieved.

- The field in the full theory $\phi(\vec{x},t)$ is related to the field in the free theory $\phi_I(\vec{x},t)$ at any time t, via a simple relation,

$$\phi(\vec{x},t) = U(t,\tau)^{-1}\phi_I(\vec{x},t)U(t,\tau). \tag{13.28}$$

- The time evolution operator, $U(t,\tau)$, is a time ordered exponential given by Eq. 13.25.

A time-ordered product of fields in the full theory can be written as,

$$T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} = \\ T\left\{U^{-1}(x_1^0,\tau)\phi_I(x_1)\tilde{U}(x_1^0,x_2^0)\phi_I(x_2)\tilde{U}(x_2^0,x_3^0)\dots\tilde{U}(x_{n-1}^0,x_n^0)\phi_I(x_n)U(x_n^0,\tau)\right\}, \tag{13.29}$$

where we define the operator

$$\tilde{U}(t_2,t_1) \equiv U(t_2,\tau)U^{-1}(t_1,\tau). \tag{13.30}$$

The above operator is independent of the reference time τ. It is given by (**exercise**):

$$\tilde{U}(t_2,t_1) = Te^{i\int_{t_1}^{t_2} dt' V_I(t')}, \tag{13.31}$$

where $t_2 > t_1$. We can also prove (**exercise**) that:

$$\tilde{U}(t_3,t_2)\tilde{U}(t_2,t_1) = \tilde{U}(t_3,t_1), \tag{13.32}$$

with $t_3 > t_1$. The inverse is then

$$\tilde{U}(t_2,t_1)\tilde{U}(t_1,t_2) = 1 \rightsquigarrow U^{-1}(t_2,t_1) = \tilde{U}(t_1,t_2) = U^\dagger(t_2,t_1). \tag{13.33}$$

13.3 The ground state of the interacting and the free theory

Our goal is to develop a formalism for the evaluation of Green's functions

$$\langle\Omega| T\{\phi(x_i)\dots\phi(x_n)\}|\Omega\rangle,$$

in the interacting theory. As we have seen, their Fourier transforms will give us, with the LSZ reduction formula, the probability amplitudes for physical scattering processes. We remind that the points x_i^μ in the above need to have time components which are slightly complex (Eq. 12.74).

From the discussion of the previous section we have that

$$\langle\Omega| T\{\phi(x_i)\dots\phi(x_n)\}|\Omega\rangle = \\ \langle\Omega| T\left\{U^{-1}(x_1^0,\tau)\tilde{U}(x_1^0,x_n^0)\phi_I(x_i)\dots\phi_I(x_n)U(x_n^0,\tau)\right\}|\Omega\rangle \tag{13.34}$$

where the time-ordering allowed us to combine together all the $\tilde{U}$ operators.

In general, we anticipate that the ground state $|\Omega\rangle$ of the interacting theory $H = H_0 + H'$ is a different state from the vacuum state $|0\rangle$ of the free theory H_0. Let us assume that the Hamiltonian H has a spectrum $|\psi_n\rangle$ with

$$H|\psi_n\rangle = E_n|\psi_n\rangle, \tag{13.35}$$

and $|\psi_0\rangle \equiv |\Omega\rangle$. Eigenstates of the full Hamiltonian form a complete set,

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_{n\neq 0}|\psi_n\rangle\langle\psi_n|. \tag{13.36}$$

Let us now act with an operator to the vacuum state $|0\rangle$ of the free theory H_0,

$$e^{-iH(\tau+T)(1-i\delta)}e^{+iH_0(\tau+T)(1-i\delta)}\,|0\rangle\,,$$

choosing T a large time in the future and δ a very small "dumbing" parameter $\delta \to 0_+$. This is in anticipation of needing to evaluate Green's functions with slightly complex times. We have,

$$\begin{aligned} e^{-iH(\tau+T)(1-i\delta)}e^{+iH_0(\tau+T)(1-i\delta)}\,|0\rangle &= e^{-iH(\tau+T)(1-i\delta)}e^{+i0(\tau+T)(1-i\delta)}\,|0\rangle \\ &= e^{-iH(\tau+T)(1-i\delta)}\,|0\rangle \end{aligned} \tag{13.37}$$

We now use the completeness of the Hamiltonian eigenstates for the full theory.

$$\begin{aligned} & e^{-iH(\tau+T)(1-i\delta)}e^{+iH_0(\tau+T)(1-i\delta)}\,|0\rangle = e^{-iH(\tau+T)(1-i\delta)}\mathbf{1}\,|0\rangle \\ = \;& e^{-iH(\tau+T)(1-i\delta)}\left(|\Omega\rangle\langle\Omega| + \sum_{n\neq 0}|\psi_n\rangle\langle\psi_n|\right)|0\rangle \\ = \;& |\Omega\rangle\, e^{-iE_0(\tau+T)-E_0(\tau+T)\delta}\,\langle\Omega|\,0\rangle + \sum_{n\neq 0}|\psi_n\rangle\, e^{-iE_n(\tau+T)-E_n(\tau+T)\delta}\,\langle\psi_n|\,0\rangle\,. \end{aligned} \tag{13.38}$$

Assuming a hierarchy $E_n > E_0$ for the first energy level of the full Hamiltonian with respect to the ground energy, the factor $e^{-E_n\delta(\tau+T)}$ vanishes faster than $e^{-E_0\delta(\tau+T)}$, as $\delta \to 0_+$. In this limit, we can write a simple relation between the ground state in the full theory and the vacuum in the free theory,

$$e^{-iH(\tau+T)(1-i\delta)}e^{+iH_0(\tau+T)(1-i\delta)}\,|0\rangle = |\Omega\rangle\, e^{-iE_0(1-i\delta)(\tau+T)}\,\langle\Omega|\,0\rangle\,. \tag{13.39}$$

We now observe that the operator on the left side of the above equation is nothing else than $U^{-1}(-T(1-i\delta),\tau(1-i\delta))$. We then have

$$|\Omega\rangle = \mathcal{N}U^{-1}\left(-T(1-i\delta),\tau(1-i\delta)\right)|0\rangle\,, \tag{13.40}$$

$$\langle\Omega| = \mathcal{N}^*\,\langle 0|\,U\left(T(1-i\delta),\tau(1-i\delta)\right), \tag{13.41}$$

where $\mathcal{N}$ is a normalization constant, which we determine from the normalization condition $\langle\Omega|\,\Omega\rangle = 1$,

$$|\mathcal{N}|^2 = \langle\Omega|\,\tilde{U}\left(T(1-i\delta),-T(1-i\delta)\right)|\Omega\rangle \tag{13.42}$$

Substituting into Eq. 13.34, we have that

$$\begin{aligned} & \langle\Omega|\,T\left\{\phi(x_i)\ldots\phi(x_n)\right\}|\Omega\rangle = |\mathcal{N}|^2 \\ & \langle 0|\,U(T,\tau) \\ & T\left\{U^{-1}(x_1^0,\tau)\tilde{U}(x_1^0,x_n^0)\phi_I(x_i)\ldots\phi_I(x_n)U(x_n^0,\tau)\right\} \\ & U^{-1}(-T,\tau)\,|0\rangle \end{aligned} \tag{13.43}$$

Given that the times $T > x_i^0 > -T$, we can put all operators under the time-ordering symbol. We then obtain, the final result for the Green's function

$$\begin{aligned} \langle\Omega|\,T\left\{\phi(x_i)\ldots\phi(x_n)\right\}|\Omega\rangle = & \lim_{\substack{\delta\to 0\\ T\to\infty}} \\ & \frac{\langle 0|\,T\phi_I(x_1)\ldots\phi_I(x_n)e^{-i\int_{-T}^{T}dt' V_I(t')}\,|0\rangle}{\langle 0|\,Te^{-i\int_{-T}^{T}dt' V_I(t')}\,|0\rangle}, \end{aligned} \tag{13.44}$$

where the points x_i^μ are computed in slightly complex times $x_i^0(1 - i\delta)$. Notice what we have achieved. On the lhs we require a Green's function for the fields $\phi(x)$ of the full theory. On the rhs this is expressed purely in terms of fields ϕ_I and the vacuum state $|0\rangle$ in the free field theory.

Eq. 13.44 is an exact result, but it can serve as our basis for perturbation theory. The exponential can be written as a function of the perturbation Lagrangian with free fields. For our example interaction

$$e^{-i\int_{-\infty}^{\infty} dt' V_I(t')} = e^{-i\int d^4x \frac{\lambda}{4!}\phi_I(x)^4}, \tag{13.45}$$

and it can be expanded as a Taylor series in λ.

13.4 Feynman Diagrams for ϕ^4 theory

In the previous section, we found that Green's functions can be cast in a form suitable for applying the method of perturbation theory. For the example Hamiltonian of Eq. 13.5, we can write the result,

$$\langle\Omega| T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \lim_{\substack{\delta\to 0\\ T\to\infty}} \frac{\langle 0| T\phi_I(x_1)\ldots\phi_I(x_n)\sum_{n=0}^{\infty}\left(\frac{-i\lambda}{4!}\int d^4x\phi_I(x)^4\right)^n|0\rangle}{\langle 0|\sum_{n=0}^{\infty}\left(\frac{-i\lambda}{4!}\int d^4x\phi_I(x)^4\right)^n|0\rangle}, \tag{13.46}$$

To evaluate the terms of the right hand side of Eq. 13.46, we need to be able to compute expectation values in the free-field theory of time-ordered products of field operators. This can be achieved by means of Wick's theorem of Section 11.7.

The pictorial application of Wick's theorem yields to the representation of Green's functions in the full theory as a perturbative expansion in terms of Feynman diagrams. As a concrete example, we shall consider the two-point function in the full theory, through order $\mathcal{O}(\lambda)$ in the coupling parameter λ. From Eq. 13.46 we have

$$\begin{aligned}
&\langle\Omega| T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle \\
&= \frac{\langle 0| T\phi_I(x_1)\phi_I(x_2)\left(1+\frac{(-i\lambda)}{4!}\int d^4x\phi_I(x)^4\right)|0\rangle + \mathcal{O}(\lambda^2)}{\langle 0| T\left(1+\frac{(-i\lambda)}{4!}\int d^4x\phi_I(x)^4\right)|0\rangle + \mathcal{O}(\lambda^2)} \\
&= \langle 0| T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle + \frac{(-i\lambda)}{4!}\int d^4x\,\langle 0| T\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x)^4\right\}|0\rangle \\
&-\frac{(-i\lambda)}{4!}\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\}|0\rangle\int d^4x\,\langle 0|\phi_I(x)^4|0\rangle + \mathcal{O}(\lambda^2)
\end{aligned} \tag{13.47}$$

Let us compute pictorially the second term in the expansion,

$$\frac{(-i\lambda)}{4!}\int d^4x\,\langle 0| T\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x)^4\right\}|0\rangle$$

The corresponding Green's function is equal to the contractions of the graph in Fig 13.4. We find two different types of contractions which lead to two different "Feynman diagrams".

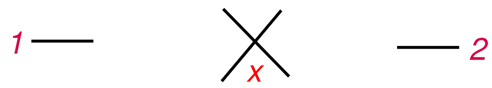
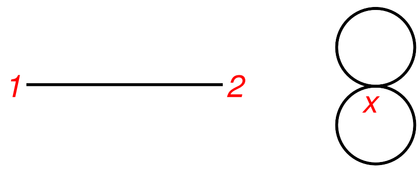


Figure 13.1: The term $\langle 0| T \left\{ \phi_I(x_1)\phi_I(x_2)\phi_I(x)^4 \right\} |0\rangle$ as the sum of all contractions of the graph above

- **Diagram I:** The two fields defined at the "external" points x_1 and x_2 get contracted with each other. The four fields defined at the "internal" point x contract then among themselves.

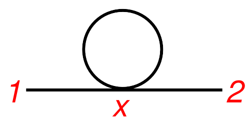


This Feynman diagram is classified as "disconnected", meaning that the internal point x is not connected to any of the external points x_1 and x_2.

This configuration occurs in more than one ways. The external fields at x_1 and x_2 can be contracted in 1 way. Take now one of the four fields at the internal point x. This field can be contracted with any of the remaining internal fields with 3 ways. Finally, the remaining two internal fields can be contracted in 1 way only. Therefore, this Feynman diagram appears

$$1 \times 3 \times 1 \text{ times.}$$

- **Diagram II:** The two external fields defined at x_1 and x_2 contract with internal fields at x.



This is a "connected" diagram, meaning that internal points are connected to external points by following the lines of the graph.

We can now compute the multiplicity of the Feynman diagram. There are 4 ways that we can contract the field at the first external point with one of the fields at the internal point x. The second external point can then be contracted to another internal field in three ways. Finally, two remaining internal fields can be contracted with each other in one way. Therefore, this Feynman diagram occurs

$$4 \times 3 \times 1 = \frac{4!}{2} \text{ times.}$$

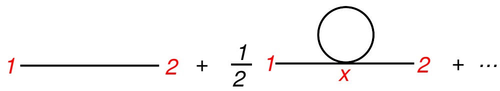
Now we examine the last term

$$-\frac{(-i\lambda)}{4!} \langle 0| T \left\{ \phi_I(x_1)\phi_I(x_2) \right\} |0\rangle \int d^4x \, \langle 0| \phi_I(x)^4 |0\rangle$$

in Eq. 13.47, which is the contribution of the denominator in Eq. 13.46 to the λ Taylor expansion. It is easy to convince ourselves that this term yields the same Feynman diagram as the disconnected **Diagram I**. The two external fields are contracted together. The four internal fields are also contracted among themselves. However, this terms carries an overall minus sign. As a result, the disconnected Feynman diagram drops out from the result. This observation holds at all orders in perturbation theory. *Disconnected Feynman diagrams do not contribute to the perturbative expansion of Green's functions* [1]. These factorize in the numerator of Eq. 13.46 and cancel exactly against the denominator.

In summary, the perturbative expansion of the two-point Green's function through order $\mathcal{O}(\lambda)$ can be represented very simply with the following Feynman diagrams.

$$\langle\Omega|\, T\left\{\phi(x_1)\phi(x_2)\right\}|\Omega\rangle \quad = \tag{13.48}$$

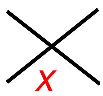


We can obtain a concrete mathematical expression from the above diagrams by following very simple rules, pioneered by Feynman:

1. We associate a propagator to each line

$$\rightarrow \langle 0|\, T\left\{\phi_I(x_1)\phi_I(x_2)\right\}|0\rangle \tag{13.49}$$

2. We associate a factor of $(-i\lambda)$ and a space-time integration to each vertex



$$\rightarrow (-i\lambda)\int d^4x \tag{13.50}$$

These rules give us a simple pictorial representation in terms of Feynman diagrams of the perturbative series for any Green's function $G(x_1, x_2, \ldots, x_N)$. We draw all possible graphs with N external and with no more vertices than the maximum order in the perturbative expansion λ^n that we require. Then we translate the Feynman diagrams into mathematical expressions using the rules above. One difficulty in this procedure is to determine the combinatorial rational factor that multiplies the diagram, as for example the factor 1/2 in the second diagram of Eq. 13.48. This number can always be obtained following the procedure in Eq. 13.48, which is a rather brute force method. For everything practical, this method is sufficient and for more complicated cased we can easily program it in a computer code. The combinatorial factor, also known as symmetry factor, can be determined cleverly as the inverse of the independent exchange symmetry operations of the Feynman diagram (exchanging vertices and propagators) which leave it intact. This is a nice method for everyone who is confident in spotting all symmetries without double counting equivalent ones. It is, however, prone to human errors.

[1]The general proof of this statement is easier with the path integral formalism, and we postpone it for QFTII.

13.5 Feynman rules in momentum space

The LSZ reduction formula expresses scattering amplitudes as truncated Green's functions in momentum space. We can develop simple Feynman rules for computing them directly.

We start with the Fourier transform of the propagator from the origin to a point x.

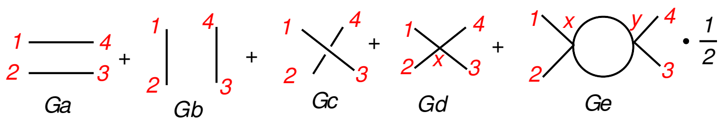
$$\tilde{G}(p) = \int d^4x e^{ip\cdot x} \langle\Omega| T\{\phi(x)\phi(0)\}|\Omega\rangle \tag{13.51}$$

At leading order in perturbation theory we have,

$$\begin{aligned}\tilde{G}(p) &= \int d^4x e^{ip\cdot x} \langle 0| T\{\phi_I(x)\phi_I(0)\}|0\rangle + \mathcal{O}(\lambda) \\ &= \int d^4x e^{ip\cdot x} \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\delta}e^{-ikx} + \mathcal{O}(\lambda) \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\delta}\int d^4x e^{i(p-k)\cdot x} + \mathcal{O}(\lambda) \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\delta}(2\pi)^4\delta^{(4)}(p-k) + \mathcal{O}(\lambda) \\ \rightsquigarrow \tilde{G}(p) &= \frac{i}{p^2-m^2+i\delta} + \mathcal{O}(\lambda) \end{aligned}\tag{13.52}$$

Let us now look at the scattering of four particles. The Green's function we require is

$$G(x_1, x_2, x_3, x_4) =$$



$$+\ldots \tag{13.53}$$

Let us now compute the Green's function in momentum space,

$$\tilde{G}(p_3, p_4; p_1, p_2) = \int d^4x_1 e^{-i(x_1\cdot p_1 + x_2\cdot p_2 - x_3\cdot p_3 - x_4\cdot p_4)} G(x_1, x_2, x_3, x_4), \tag{13.54}$$

where we consider p_1, p_2 in the initial state and p_3, p_4 in the final state. We write,

$$\tilde{G}(p_3, p_4; p_1, p_2) = G_a + G_b + G_c + G_d + G_e + \ldots, \tag{13.55}$$

where $G_{a,b,c}$ are the contributions of the first three diagrams and G_d, G_e are the contributions of the fourth and fifth diagram at order $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$. The first three diagrams give,

$$\begin{aligned} G_a + G_b + G_c &= (2\pi)^4\delta^4(p_1-p_4)\frac{i}{p_1^2-m^2+i\delta}(2\pi)^4\delta^4(p_2-p_3)\frac{i}{p_2^2-m^2+i\delta} \\ &+ (2\pi)^4\delta^4(p_1-p_3)\frac{i}{p_1^2-m^2+i\delta}(2\pi)^4\delta^4(p_2-p_4)\frac{i}{p_2^2-m^2+i\delta} \\ &+ (2\pi)^4\delta^4(p_1+p_2)\frac{i}{p_1^2-m^2+i\delta}(2\pi)^4\delta^4(p_3+p_4)\frac{i}{p_3^2-m^2+i\delta} \\ &+ \end{aligned}\tag{13.56}$$

The fourth diagram is converted into a mathematical expression following again the Feynman rules of the previous section. We have

$$\begin{aligned}
G_d &= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-i(x_1\cdot p_1 + x_2\cdot p_2 - x_3\cdot p_3 - x_4\cdot p_4)} \\
&\times (-i\lambda)\int d^4x \qquad (\text{ vertex }) \\
&\times \langle 0|\, T\{\phi_I(x_1)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_1 \text{ and } x\ ) \\
&\times \langle 0|\, T\{\phi_I(x_2)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_2 \text{ and } x\ ) \\
&\times \langle 0|\, T\{\phi_I(x_3)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_3 \text{ and } x\ ) \\
&\times \langle 0|\, T\{\phi_I(x_4)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_4 \text{ and } x\ ).
\end{aligned} \tag{13.57}$$

Substituting the expression for $\langle 0|\, T\{\phi_I(x_3)\phi_I(x)\}\,|0\rangle$ Performing the x and x_i integrations we find,

$$G_d = (-i\lambda)\frac{i}{p_1^2 - m^2 + i\delta}\frac{i}{p_2^2 - m^2 + i\delta}\frac{i}{p_3^2 - m^2 + i\delta}\frac{i}{p_4^2 - m^2 + i\delta}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4). \tag{13.58}$$

The diagram G_e is a "loop diagram". Let us write the corresponding expression, using the Feynman rules in position space. We have

$$\begin{aligned}
G_e &= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-i(x_1\cdot p_1 + x_2\cdot p_2 - x_3\cdot p_3 - x_4\cdot p_4)} \\
&\times (-i\lambda)\int d^4x d^4y \qquad (\text{ vertices }) \\
&\times \langle 0|\, T\{\phi_I(x_1)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_1 \text{ and } x\ ) \\
&\times \langle 0|\, T\{\phi_I(x_2)\phi_I(x)\}\,|0\rangle \qquad (\text{ contraction of } x_2 \text{ and } x\ ) \\
&\times \langle 0|\, T\{\phi_I(x_3)\phi_I(y)\}\,|0\rangle \qquad (\text{ contraction of } x_3 \text{ and } y\ ) \\
&\times \langle 0|\, T\{\phi_I(x_4)\phi_I(y)\}\,|0\rangle \qquad (\text{ contraction of } x_4 \text{ and } y\ ) \\
&\times \langle 0|\, T\{\phi_I(x)\phi_I(y)\}\,|0\rangle^2 \qquad (\text{ two contractions of } x \text{ and } y\ )
\end{aligned} \tag{13.59}$$

Performing all $x-$space integrations we find,

$$\begin{aligned}
G_e &= \frac{i}{p_1^2 - m^2 + i\delta}\frac{i}{p_2^2 - m^2 + i\delta}\frac{i}{p_3^2 - m^2 + i\delta}\frac{i}{p_4^2 - m^2 + i\delta} \\
&\quad \times(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) \\
&\quad \times(-i\lambda)^2 \\
&\quad \times \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\delta}\frac{i}{(k + p_1 + p_2)^2 - m^2 + i\delta}.
\end{aligned} \tag{13.60}$$

Observe our final expressions for Ga, b, c, Gd and Ge. We can make easy rules (Feynman rules) to produce them from the Feynman diagrams.

- In every vertex we pick up a factor

$$(-i\lambda),$$

- With each propagator comes a factor

$$\frac{i}{p^2 - m^2 + i\delta}.$$

- At each vertex, we have delta functions guaranteeing that momentum is conserved. As a result, for all particles that are connected with each other there is an overal factor

$$(2\pi)^4 \delta^{(4)} \left(\sum_{in} p_{in} - \sum_{out} p_{out} \right),$$

which forces the sum of incoming momenta in a connected diagram to be equal to the sum of the out-coming momenta.

- The loop in diagram G_e introduces an integration over a "loop momentum" and a measure

$$\int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}$$

.

13.6 Truncated Green's functions in perturbation theory

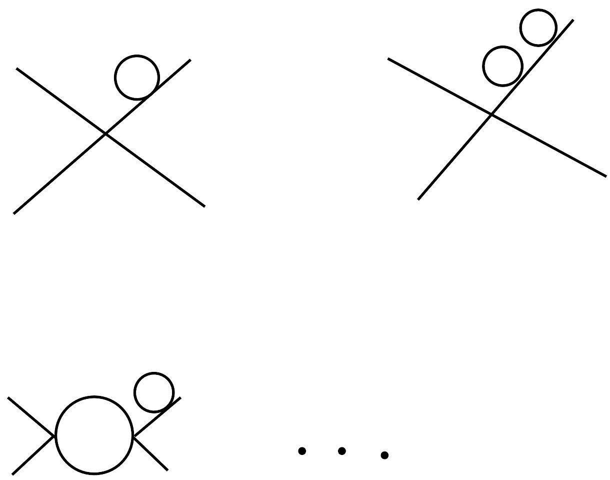


Figure 13.2: Feynman diagrams which are correction to external lines and do not contribute to physical scattering amplitudes.

We are interested in computing amplitudes for physical scattering processes, as given by the LSZ formula. For the scattering of two particles with momenta p_1, p_2 to two particles with momenta p_3, p_4 we then need a truncated Green's function

$$\langle p_3, p_4 | \, p_1, p_2 \rangle = \tilde{Z}_\phi^2 \tilde{G}_{trunc}(p_3, p_4; p_1, p_2) \Big|_{p_i^2 = m_{phys}^2} \tag{13.61}$$

compute for squared external momenta equal to the physical mass of the particles. The truncated Green's function is defined as

$$\tilde{G}_{trunc}(p_3, p_4; p_1, p_2) = \frac{\tilde{G}(p_3, p_4; p_1, p_2)}{\tilde{G}(p_1)\tilde{G}(p_2)\tilde{G}(p_3)\tilde{G}(p_4)}, \tag{13.62}$$

where the Green's function in momentum space is divided with the propagator of each of the particles in the initial and final states.

The Kählen-Lehmann representation for the pole of the propagator is

$$\tilde{G}(p) = \frac{i\tilde{Z}}{p^2 - m_{phys}^2} + \text{regular terms} \tag{13.63}$$

In the previous section we computed the perturbative result,

$$\tilde{G}(p) = \frac{i}{p^2 - m^2} + \mathcal{O}(\lambda). \tag{13.64}$$

At leading order in perturbation theory, we can then identify the physical mass of a particle m_{phys} with the mass parameter m of our Lagrangian, and determine the normalization $\tilde{Z}_\phi = 1$.

We then find that the diagrams G_a, G_b, G_c which do not connect all eternal particles with each other do not contribute to the scattering amplitude $\langle p_3, p_4 | \, p_1, p_2 \rangle$. We find for example that, at leading order in λ,

$$\frac{G_a}{\tilde{G}(p_1)\tilde{G}(p_2)\tilde{G}(p_3)\tilde{G}(p_4)} \sim (p_3^2 - m^2)(p_4^2 - m^2)\delta(p_4 - p_1)\delta(p_3 - p_2)\bigg|_{p_i^2 = m^2} = 0. \tag{13.65}$$

The diagrams G_d and G_e connect all external particles and have poles $\frac{i}{p_i^2 - m^2}$ for each one of them. These contribute to the scattering amplitude.

We can perform the division with the external propagators as it is demanded by the LSZ formula we need three more rules in our set of Feynman rules.

- Lines starting from an external point contribute a factor 1 (and not $i/(p^2 - m^2)$).

We must be, though, a bit more careful. Eq. 13.64 has perturbative corrections of order λ. We need to divide our Feynman diagrams with the full perturbative expansion of the propagator and not just the leading order contribution. We can account for this division entirely with one more clever trick.

- Through away all Feynman diagrams which "correct" external lines.

For example, the diagrams of Fig. 13.2 should all be disregarded when computing truncated Green's functions for physical scattering amplitudes.

Chapter 14

Loop Integrals

In order to compute the perturbative expansion of any Green's function, we must perform unrestricted integrations over the momenta of particles circulating in loops of Feynman diagrams. The exact evaluation of loop integrals is a difficult and often prohibitive task. Many loop integrals are divergent! This is a very unpleasant surprise if we aim towards a realistic description of physical phenomena with finite probabilities. The problem of divergences is a very serious one, and it jeopardizes the mathematical foundation of the perturbative method.

14.1 The simplest loop integral. Wick rotation

Let us consider the simplest loop-integral in field theory,

$$I \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\delta} \tag{14.1}$$

This is a formidable integral, given that it requires four integrations for each space-time dimension. We notice that only the combination $k^2 = k_0^2 - \vec{k}^2$ appears in the integrand. The integral would be easily solvable had we had a Euclidean metric for $k_E^2 = k_0^2 + \vec{k}^2$, by using spherical coordinates in four dimensions. With a Minkowski metric, we ought to treat the time integration specially. We write

$$I = \lim_{\delta \to 0_+} \int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^4} \int_{-\infty}^{+\infty} dk_0 \frac{1}{(k_0 - \omega_k + i\delta)(k_0 + \omega_k - i\delta)}, \tag{14.2}$$

with

$$\omega_k = \sqrt{\vec{k}^2 + m^2}. \tag{14.3}$$

The integrand has two poles, at $k_0 = \omega_k - i\delta$ and $k_0 = -\omega_k + i\delta$, which are found in the lower-right and upper-left quarter planes in the $(\Re k_0, \Im k_0)$ plane We can perform the so called Wick rotation, rotating the integration axis by 90 degrees and integrating over the imaginary axis,

$$I = \int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^4} \int_{-i\infty}^{+i\infty} dk_0 \frac{1}{k_0^2 - \vec{k}^2 - m^2 + i\delta}. \tag{14.4}$$

Setting

$$k_0 = ik_0',$$

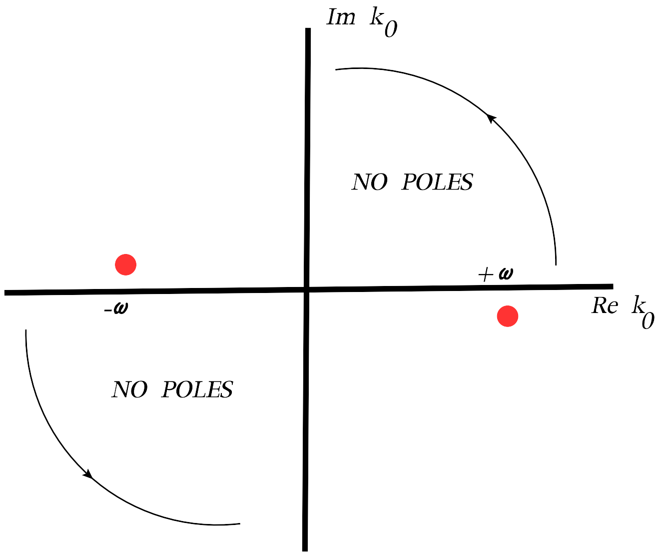


Figure 14.1: Wick's rotation: The integrand has no poles on the upper-right and lower-left k_0 complex plane. We can rotate the integration axis by 90^o, and integrate along the imaginary axis $k_o \in [-i\infty, +i\infty]$.

the integral becomes

$$I = i \int_{-\infty}^{+\infty} \frac{d^3\vec{k}}{(2\pi)^4} \int_{-i\infty}^{+i\infty} dk'_0 \frac{1}{-(k'_0)^2 - \vec{k}^2 - m^2 + i\delta}. \tag{14.5}$$

Notice that after Wick's rotation we can consider the four-vector $k_E \equiv (k'_0, \vec{k})$ in Euclidean space, and express k_E in four-dimensional spherical coordinates:

$$dk_E = d|k|\,|k|^3 d\Omega_4. \tag{14.6}$$

The integral then becomes

$$I = -i\Omega_4 \int_0^{\infty} d|k| \frac{|k|^3}{|k|^2 + m^2 - i\delta}. \tag{14.7}$$

Notice that the integral diverges as $|k| \to \infty$, which is an embarrassment.

Infinities were historically the biggest puzzle in the development of quantum field theory. But, we now know what to do with them for many Lagrangian systems which are realistic descriptions of nature. We will learn later that we can "sweep infinities under the carpet", exploiting certain "freedoms" that we have. We are allowed to absorb infinities in the the normalization constants $\mathcal{Z}_\phi$ of the LSZ formula, or in the definition of physical mass and coupling parameters in terms of the corresponding parameters appearing in the Lagrangian. These are very few "freedoms", and it is amazing that they suffice to solve the problem.

The method of "sweeping infinities under the carpet" is called renormalization. A prerequisite to it is to quantify the infinities with the help of a regulator. We shall introduce a parameter which renders the integrations well defined and finite except at a certain limit. For example, we can restrict the integration in Eq. 14.7 below a certain cut-off value Λ for $|k|$,

$$I(\Lambda) \equiv -i\Omega_4 \int_0^\Lambda d|k| \frac{|k|^3}{|k|^2 + m^2 - i\delta} = -i\Omega_4 \left[\frac{\Lambda^2}{2} + \dots\right] \tag{14.8}$$

The regularized integral $I(\Lambda)$ is equal to the original integral I in the limit of an infinite Λ,

$$\lim_{\Lambda \to \infty} I(\Lambda) = I. \tag{14.9}$$

What is the benefit of introducing a regularization method? It will give us the possibility to compute the divergent parts of all loop-integrals that enter the evaluation of $S-$matrix elements for values of the regulator Λ where they are all well-defined. Then, we can perform "renormalization" and express the parameters of the Lagrangian in terms of physical mass and coupling parameters. Then we shall take the limit of the regulator which corresponds to the original loop integrals. It will turn out that for many field theories the final renormalized S-matrix elements are finite.

14.2 Dimensional Regularization

A method to regularize loop integrals is not unique. A well chosen regularization procedure can be very beneficial for practical computations, since the difficulty in carrying out the integrations may vary significantly. What is more important, some regularization methods may violate some of the symmetries of the theory. For example, a "cut-off" regularization violates Lorentz invariance. When symmetries are broken due to the regularization method it may be required to pay additional efforts in order to relate S-matrix elements to physical observables.

The most elegant regularization method known to date was developed by t'Hooft and Veltman in the 70's. It is called dimensional regularization. It has revolutionized the evaluation of loop integrals for its simplicity. It is also known to preserve most of the symmetries which are found in physical Lagrangian systems.

In dimensional regularization, the regulator is the number of space-time dimensions d. We shall explain how this work, by revisiting the simplest example of a loop integral in Eq. 14.7. Let us now compute this integral in an arbitrary number of dimensions,

$$I_d = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\delta}, \tag{14.10}$$

which after Wick's rotation becomes

$$I_d = i \int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2 - m^2 + i\delta}, \tag{14.11}$$

and

$$k^2 = k_0^2 + \vec{k}^2. \tag{14.12}$$

In spherical coordinates,

$$I_d = \frac{i\Omega_d}{(2\pi)^d} \int_0^\infty dk \frac{k^{d-1}}{-k^2 - m^2 + i\delta}. \tag{14.13}$$

For infinitely large loop momenta it behaves as,

$$I_d \to -\frac{i\Omega_d}{(2\pi)^d}\int^{\infty} dk k^{d-3}. \tag{14.14}$$

We observe that the integral has a finite ultraviolet limit ($k \to \infty$) for values of the dimension $d < 3$. In dimensional regularization we compute the integral for a generic value of the dimension d which is allowed for it in in order to be well-defined. At the end of our computation, we shall perform an "analytic continuation" to a physical dimension value $d = 4$.

14.2.1 Angular Integrations

For $d = 2$, the solid angle is

$$d\Omega_2 = d\phi,$$

for $d = 3$, we have

$$d\Omega_3 = \sin\theta d\theta d\phi,$$

and so on. We are used to defining solid angles for an integer number of dimensions. What is the value of the solid angle Ω_d for an arbitrary real valued dimension d? Our generalization of the dimension from integer to real values for the solid angle is mathematically similar to the generalizing of the factorial to the complex-valued Γ function,

$$n! \to \Gamma(z),$$

and relies on finding a defining recurrence relation.

Let us take a $d-$dimensional vector

$$\vec{k}^{(d)} = \left(\vec{k}^{(d-1)}, k_d\right), \tag{14.15}$$

where k_d is the component corresponding to the d^{th} dimension. We shall assume that we know how to express the $\vec{k}^{(d-1)}$ in terms of $(d-1)$ polar coordinates, i.e. a radial r_{d-1} coordinate and $(d-2)$ angular coordinates. Then

$$\int_{-\infty}^{+\infty} dk_1 \ldots dk_d = \int_{-\infty}^{+\infty} dk_d \int_0^{\infty} dr_{(d-1)} r_{(d-1)}^{d-2} \int d\Omega_{d-1}. \tag{14.16}$$

The $d-$dimensional vector is now expressed in cylindrical coordinates,

$$\vec{k}^{(d)} = \left(k_{d-1}\hat{u}^{(d-1)}\left(\theta_1, \theta_2, \ldots \theta_{d-2}\right), k_d\right), \tag{14.17}$$

where $\hat{u}^{(d-1)}$ is a unit vector. Now we change variables to polar coordinates in $d-$dimensions by performing the transformation,

$$\begin{array}{r} r_{(d-1)} = l\sin\theta_{d-1} \\ k_d = l\cos\theta_{d-1} \end{array}. \tag{14.18}$$

This gives for the integration measure $\int d^d k$,

$$\int_{-\infty}^{+\infty} dk_d \int_0^{\infty} dr_{(d-1)} r_{(d-1)}^{d-2} \int d\Omega_{d-1} = \int_0^{\infty} dl l^{d-1} \int_0^{\pi} d\theta_{d-1} \sin^{d-2}\theta_{d-1} \int d\Omega_{d-1}. \tag{14.19}$$

We have then arrived to the recurrence relation,

$$\int d\Omega_d = \int_0^\pi d\theta_{d-1} \sin^{d-2}\theta_{d-1} \int d\Omega_{d-1}. \tag{14.20}$$

Recall that for $d = 2$,

$$\int d\Omega_2 = \int_0^{2\pi} d\theta_1. \tag{14.21}$$

Using this as a boundary condition for Eq. 14.20, we obtain the solid-angle in arbitrary integer dimensions,

$$\int d\Omega_d = \int_0^\pi d\theta_{d-1} \sin^{d-2}\theta_{d-1} \dots \int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\theta_1. \tag{14.22}$$

All integrals can be performed easily using that,

$$\begin{aligned} \int_0^\pi d\theta \sin^x\theta &= \int_{-1}^1 d(\cos\theta)(1-\cos\theta)^{\frac{x-1}{2}} \\ &= \int_0^1 da a^{\frac{1}{2}-1}(1-a)^{\frac{x+1}{2}-1} \qquad (a = \cos^2\theta) \\ \rightsquigarrow \int_0^\pi d\theta \sin^x\theta &= B\left(\frac{1}{2}, \frac{1+x}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+2}{2}\right)}. \end{aligned} \tag{14.23}$$

The the solid angle in $d-$dimensions is

$$\Omega_d = 2\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{14.24}$$

On the rhs we have a function which is defined, as we shall discuss immediately, to arbitrary complex values of the dimension parameter d, with the exception of non-positive integers. We thus have derived a generalization of the solid angle to an arbitrary such value of d.

A quicker derivation [1] of Eq. 14.24 proceeds as follows. Consider the exponential integral,

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}. \tag{14.25}$$

For d such integrals we have,

$$\pi^{\frac{d}{2}} = \int d^d x e^{-\sum_{i=1}^d x_i^2} = \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2}. \tag{14.26}$$

Setting $t = x^2$, we find

$$\pi^{\frac{d}{2}} = \frac{1}{2}\Omega_d \int_0^\infty dt t^{\frac{d}{2}-1} e^{-t} = \frac{1}{2}\Omega_d \Gamma\left(\frac{d}{2}\right). \tag{14.27}$$

[1] shown in class

14.2.2 Properties of the Gamma function

In the above, the Beta function is defined as

$$B(x,y) = \int_0^2 1 d\xi x i^{x-1}(1-\xi)^{y-1}. \tag{14.28}$$

The $\Gamma-$function is defined through the integral representation,

$$\Gamma(z) \equiv \int_0^\infty dx x^{z-1} e^{-x}. \tag{14.29}$$

By using integration by parts, we can easily show that

$$\Gamma(z+1) = z\Gamma(z), \tag{14.30}$$

which is the same recursion relation as for the factorial of integers. For $z = n$ a positive integer, we have

$$\Gamma(n) = (n-1)! \tag{14.31}$$

The function $\Gamma(z)$ is analytic except for $z = 0, -1, -2, \ldots$. From the integral representation we conclude that the Γ function is convergent for $\mathrm{Re}\, z > 0$. But, the recursion relation of Eq. 14.30 defines it for all complex values of z, with the exception of non-positive integers.

A useful value is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{14.32}$$

We can also prove the identity,

$$\Gamma(x)\Gamma\left(\frac{1}{2}\right) = 2^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right). \tag{14.33}$$

We shall often need to expand a Gamma function around an integer value for its argument. We have

$$\begin{aligned}
\Gamma(1+\epsilon) &= \int_0^\infty x^\epsilon e^{-x} \\
&= \int_0^\infty e^{\epsilon \ln x} e^{-x} \\
&= \int_0^\infty \left(1 + \epsilon \ln x + \frac{\epsilon^2}{2}\ln^2 x + \ldots\right) e^{-x} \\
&= \\
\rightsquigarrow \Gamma(1+\epsilon) &= 1 - \epsilon\gamma + \epsilon^2\left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right) + \mathcal{O}(\epsilon^3).
\end{aligned} \tag{14.34}$$

The "Euler-gamma" constant is defined as

$$\gamma \equiv -\int_0^\infty dx e^{-x} \log(x) \approx 0.544\ldots \tag{14.35}$$

14.2.3 Radial Integrations

We now continue with the remaining integration of Eq. 14.14 over the magnitude of the Euclidean four-momentum. We have

$$I_d = -\frac{i}{2}\frac{\Omega_d}{(2\pi)^d}\int dk \frac{k^{d-2}}{k^2+m^2-i\delta}. \tag{14.36}$$

We perform the change of variables

$$k^2 \to \frac{x}{1-x}\left(m^2-i\delta\right), \tag{14.37}$$

which yields

$$I_d = -\frac{i}{2}\frac{\Omega_d}{(2\pi)^d}\left(m^2-i\delta\right)^{\frac{d}{2}-1}\int_0^1 dx x^{\frac{d}{2}-1}(1-x)^{-\frac{d}{2}} \tag{14.38}$$

which yields the final result,

$$\int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{k^2-m^2+i\delta} = -\Gamma\left(1-\frac{d}{2}\right)\left(m^2-i\delta\right)^{\frac{d}{2}-1} \tag{14.39}$$

Notice that the integral is divergent for $d = 2, 3, 4, \ldots$, but it is well-defined for any other value of d.

Exercise: Prove that

$$\int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{\left[k^2-m^2+i\delta\right]^\nu} = (-1)^\nu\frac{\Gamma\left(\nu-\frac{d}{2}\right)}{\Gamma(\nu)}\left(m^2-i\delta\right)^{\frac{d}{2}-\nu}. \tag{14.40}$$

An interesting case is when $m^2 = 0$. In dimensional regularization, where the dimension parameter is considered non-integer, we have that

$$\lim_{m\to 0} m^d = 0^d = 0. \tag{14.41}$$

We then have that integrals with no mass scales vanish, and in our exemplary case,

$$\int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{\left[k^2+i\delta\right]^\nu} = 0. \tag{14.42}$$

14.3 Feynman Parameters

Consider a more complicated integral. For example,

$$I_2 = \int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{\left[k^2-m^2+i\delta\right]\left[(k+p)^2-m^2+i\delta\right]}. \tag{14.43}$$

The integrand has now four poles for

$$k_0 = \pm\left(\sqrt{\vec{k}^2+m^2}-i\delta\right), \quad \pm\left(\sqrt{(\vec{k}+\vec{p})^2+m^2}-i\delta\right). \tag{14.44}$$

It is now more difficult to perform the integration over k_0, as well as the angular integrations since the denominator depends explicitly on angles,

$$(k+p)^2-m^2 = k^2+p^2-m^2+2k_0p_0-2|\vec{k}||\vec{p}|\cos\theta. \tag{14.45}$$

The method of Feynman parameters allows to integrate out the loop momentum k easily, applying Wick's rotation and integrating over angles in exactly the same manner as for the simplest integral of Eq. 14.40. The price that we have to pay is to introduce new integration variables over "Feynman parameters". It is easy to prove that,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left[xA + (1-x)B\right]^2}. \tag{14.46}$$

We can use such identity in order to combine denominators in loop intergrals under one term. For example, the integral of Eq. 14.43 can be written as

$$\begin{aligned} I_2 &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{\left[x\left((k+p)^2 - m^2\right) + (1-x)\left(k^2 - m^2\right) + i\delta\right]^2} \\ &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{\left[k^2 - m^2 + 2xk\cdot p + xp^2 + i\delta\right]^2} \\ &= \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{\left[(k+px)^2 - m^2 + x(1-x)p^2 + i\delta\right]^2} \end{aligned} \tag{14.47}$$

We perform a shift in the integration variable k, usually called "loop momentum", defining

$$K^\mu = k^\mu + xp^\mu. \tag{14.48}$$

We then have,

$$I_2 = \int_0^1 dx \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{\left[K^2 - (m^2 - x(1-x)p^2 - i\delta)\right]^2} \tag{14.49}$$

The integral over K^μ can be solved using the general result of Eq. 14.40,

$$I_2 = \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 dx \left(m^2 - x(1-x)p^2 - i\delta\right)^{\frac{d}{2}-2} \tag{14.50}$$

The above integral is only one-dimensional. We still have the task of performing the task of integrating over Feynman parameters. Feynman parameter integrals are generalized hypergeometric functions, which are only partially understood in the mathematical science. Said differently, the mathematics of hypergeometric functions is far from sufficient in order to tackle the Feynman integrals that appear in the perturbative expansion of Green's functions. This limits computations to simple cases. Nevertheless, even with the mathematical understanding of loop integrations being in its infancy, we can still derive rather precise theoretical predictions for S-matrix elements of interesting scattering processes.

For physics predictions, we are interested in the value of loop integrals as a Laurent expansions around the physical value of the dimension parameter, $d = 4$. It is customary to write,

$$d = 4 - 2\epsilon, \tag{14.51}$$

and expand around $\epsilon = 0$. The $(-)$ sign is because we usually think of the dimension parameter as being smaller than four, in order to make an integral convergent in the UV limit $|k| \to \infty$. The factor of 2 in front of ϵ is convenient due to that typically the combination $\frac{d}{2}$ emerges as result of loop-integrations. The integral of Eq. 14.50 becomes

$$\begin{aligned} I_2 &= \Gamma(\epsilon) \int_0^1 dx \left(m^2 - x(1-x)p^2 - i\delta\right)^\epsilon \\ & \frac{1}{\epsilon} - \gamma + \int_0^1 dx \log\left[m^2 - x(1-x)p^2 - i\delta\right] + \mathcal{O}(\epsilon). \end{aligned} \tag{14.52}$$

Exercise: Solve the above integral, assuming that $p^2 < 4m^2$. For $p^2 > 4m^2$ the integral develops an imaginary part. Compute it, by performing an analytic continuation of the result for $p^2 < 4m^2$.

The procedure of combining denominators in a single term using Feynman parameters and making complete squares of the loop momenta can be performed in general. For an arbitrary number of denominators in a loop-integral we can use the formula,

$$\frac{1}{A_1^{\nu_1}\dots A_n^{\nu_n}} = \frac{\Gamma\left(\nu_1+\dots\nu_n\right)}{\prod\Gamma(\nu_i)}\int_0^1 dx_1\dots dx_n\delta\left(1-\sum_i x_i\right)\frac{x_1^{\nu_1-1}\dots x_n^{\nu_n-1}}{\left[\sum x_iA_i\right]^{\sum\nu_i}} \tag{14.53}$$

__Proof:__ Let us perform the change of variables

$$x_i = \frac{a_i}{1+a_n} \tag{14.54}$$

in the rhs of Eq. 14.53. The delta function becomes,

$$\delta\left(1-\sum_{i=1}^{n}\frac{a_i}{1+a_n}\right) = \delta\left(\frac{1-\sum_{i=1}^{n-1}a_i}{1+a_n}\right) = (1+a_n)\delta\left(1-\sum_{i=1}^{n-1}a_i\right). \tag{14.55}$$

The parameters a_i are constrained to be in the interval $[0,1]$, due to the δ function for $i = 1\dots n-1$, while the parameter a_n ranges in between $a_n = 0(x_n = 0)$ and $a_n = \infty(x_n = \infty)$. The Jacobian of the transformation is

$$dx_i = \frac{da_i}{1+a_n}, \quad i \neq n, \tag{14.56}$$

and

$$dx_n = \frac{da_n}{(1+a_n)^2}. \tag{14.57}$$

All factors conspire in Eq. 14.53, so that the integrand is almost the same as the original, replacing x_i with a_i. The only difference is that the delta function does not contain a_n anymore, and that the range of this variable is from 0 to infinity. Explicitly,

$$\begin{aligned}\text{rhs of Eq. (14.53)} &= \frac{\Gamma\left(\nu_1+\dots\nu_n\right)}{\prod\Gamma(\nu_i)}\\ &\times\int_0^\infty da_n\int_0^1 da_1\dots da_{n-1}\delta\left(1-\sum_{i=1}^{n-1}a_i\right)\frac{a_1^{\nu_1-1}\dots a_n^{\nu_n-1}}{\left[\sum a_iA_i\right]^{\sum\nu_i}}\end{aligned} \tag{14.58}$$

We now change once more variables to

$$a_n = \frac{x}{1-x}\frac{\sum_{i=1}^{n-1}a_iA_i}{A_n}, \tag{14.59}$$

and perform the integration over x, which ranges from 0 to 1. The result is

$$\begin{aligned}\text{rhs of Eq. (14.53)} &= \frac{1}{A_n^{\nu_n}}\times\\ &\times\frac{\Gamma\left(\nu_1+\dots\nu_{n-1}\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_{n-1})}\int_0^1 dx_1\dots dx_n\delta\left(1-\sum_i x_i\right)\frac{x_1^{\nu_1-1}\dots x_n^{\nu_n-1}}{\left[\sum x_iA_i\right]^{\sum\nu_i}}\end{aligned} \tag{14.60}$$

We have now factorized the term $1/A_n^{\nu_n}$ out of an integral, which is now cast in the same form as the original but with $n-1$ Feynman parameters. Obviously, we can repeat the same procedure as many times as needed in order to factorize all $\frac{1}{A_i^{\nu_i}}$ terms, as in the lhs of Eq. 14.53.

Chapter 15

Quantum Electrodynamics

One of the most amazing successes of Quantum Field theory, is the precision in which it describes the interaction of radiation and matter. The rules governing these interactions are very simple, and are given by the theory of Quantum Electrodynamics (QED). What is also astonishing, emerges naturally by combining two very simple principles: symmetry and locality. The QED Lagrangian is symmetric under phase-redefinitions ($U(1)$ transformations) of the electron field which are local, i.e. we can choose the symmetry transformation parameters differently at each space-time point.

Local symmetry transformations govern all quantum field theories, QED, QCD and the Standard Model of electroweak interactions, which describe the three forces of nature other than gravity. These theories describe physical phenomena extremely well within the experimentally accessible accuracy.

15.1 Gauge invariance

We consider a free electron field:

$$\mathcal{L} = \bar{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x). \tag{15.1}$$

As we have discussed already, this Lagrangian is invariant under a $U(1)$ transformation,

$$\psi(x) \to \psi'(x) = \exp(ie\theta)\psi(x), \tag{15.2}$$

if the phase θ is global, i.e. it is chosen to be the same at every point in space-time $\left(\frac{\partial\theta}{\partial x} = 0\right)$.

Let us now take a bold step and ask whether we can have a a different phase θ at every space-time point x^μ,

$$U(x) = \exp\left(ie\theta(x)\right). \tag{15.3}$$

The Lagrangian is no longer invariant,

$$\bar{\psi}\partial\!\!\!/\psi \to \bar{\psi}\partial\!\!\!/\psi + \bar{\psi}e^{-ie\theta}\left[\partial\!\!\!/ e^{ie\theta}\right]\psi. \tag{15.4}$$

The problem is that the space-time derivative does not transform simply under the local $U(1)$ transformation any more,

$$\partial\!\!\!/\psi \to e^{ie\theta}\partial\!\!\!/\psi + \text{ something else }.$$

If we insist on locality, we shall need to modify the derivative so that it transforms more conveniently. We will look for a new derivative which, under a *local* $U(1)$ transformation transforms as:

$$\not{D}\psi \to U(x)\not{D}\psi. \tag{15.5}$$

The simplest modification we can think of, is to add to the space-time derivative a space-time function,

$$\partial^\mu \to D^\mu = \partial^\mu + \text{function}^\mu \tag{15.6}$$

which transforms in the same way as the space-time derivative under Lorentz transformations, i.e. it transforms as vector. A function of space-time, in field theory, is nothing else but a field. We are then proposing to modify the definition of a derivative by adding to it a vector field. We write a "covariant derivative",

$$D_\mu = \partial_\mu - ieA_\mu(x), \tag{15.7}$$

where the $-ie$ factor is conventional.

The vector field $A^\mu(x)$ must have a very specific trasformation which we can find, in order for the modified derivative to transform simply. We require that

$$\begin{aligned}
& D_\mu\psi(x) \to D'_\mu\psi' = U(x)D_\mu\psi \\
\rightsquigarrow\ & \left(\partial_\mu - ieA'_\mu\right)(U(x)\psi) = U(x)\left(\partial_\mu - ieA_\mu\right)\psi \\
\rightsquigarrow\ & U(x)\partial_\mu\psi + \left[\partial_\mu U(x)\right]\psi - ieA'_\mu U(x)\psi = U(x)\partial_\mu\psi - ieA_\mu U(x)\psi \\
\rightsquigarrow\ & A'_\mu = A_\mu - \frac{i}{e}U^{-1}(x)\partial_\mu U(x)
\end{aligned} \tag{15.8}$$

This is an astonishing result. We find that the vector field transforms as,

$$A'_\mu = A_\mu + e\partial_\mu\theta, \tag{15.9}$$

which is a gauge transformation, the transformation of the photon field. In other words, if we would like that a Lagrangian of electrons is invariant under local $U(1)$ transformations, then the same Lagrangian must describe a photon field.

The covariant derivative transforms as:

$$\begin{aligned}
D_\mu \to D'_\mu &= \partial_\mu - ieA'_\mu \\
&= \partial_\mu - ie\left(A_\mu - \frac{i}{g}U^{-1}\partial\mu U\right) \\
&= \partial_\mu - ieA_\mu - U^{-1}\left(\partial_\mu U\right) \\
&= \partial_\mu - ieA_\mu + U\left(\partial_\mu U^{-1}\right) \\
&= U(x)\left(\partial_\mu - ieA_\mu\right)U^{-1}(x)
\end{aligned} \tag{15.10}$$

Therefore:

$$D_\mu \to D'_\mu = U(x)D_\mu U^{-1}(x) \tag{15.11}$$

We can now replace the free Lagrangian of the spin-1/2 field with a new Lagrangian which is symmetric locally,

$$\begin{aligned}
& \mathcal{L} = \bar{\psi}\left[i\not{D} - m\right]\psi \\
& \to \bar{\psi}U^{-1}U\left[i\not{D} - m\right]U^{-1}U\psi \\
& \bar{\psi}\left[i\not{D} - m\right]\psi.
\end{aligned}$$

If A_μ is a physical field, we need to introduce a kinetic term in the Lagrangian for it. We will insist on constructing a fully gauge invariant Lagrangian. To this purpose, we can use the covariant derivative as a building block. Consider the gauge transformation of the product of two covariant derivatives:

$$\begin{aligned} &D_\mu D_\nu \to D'_\mu D'_\nu = U D_\mu U^{-1} U D_\nu U^{-1} \\ &= U D_\mu D_\nu U^{-1}. \end{aligned}$$

This is not a gauge invariant object. Now look at the commutator:

$$\begin{aligned} &[D_\mu, D_\nu] \to \left[D'_\mu, D'_\nu\right] \\ &= U\left[D_\mu, D_\nu\right] U^{-1} \end{aligned} \tag{15.12}$$

This is gauge invariant. To convince ourselves we write the commutator explicitly:

$$\begin{aligned} & [D_\mu, D_\nu] = (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - [\mu \leftrightarrow \nu] \\ = \;& \partial_\mu\partial_\nu - ie\left(\partial_\mu A_\nu\right) - ieA_\nu\partial_\mu - ieA_\mu\partial_\nu + (ie)^2 A_\mu A_\nu - [\mu \leftrightarrow \nu] \\ = \;& -ie\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right]. \end{aligned} \tag{15.13}$$

Inserting Eq. 15.13 into Eq. 15.12, we find that the commutator of covariant derivatives (in the abelian case) is gauge invariant. We have also found that it is proportional to the field strength tensor of the gauge (photon) field:

$$F_{\mu\nu} = \frac{i}{e}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15.14}$$

We now have invariant terms for a Lagrangian with an "electron" and a "photon" field. The classical Lagrangian for QED reads

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{15.15}$$

This Lagrangian accounts for the majority of the phenomena that we experience in nature. It is a beautiful synthesis of locality and symmetry!

Exercise: *Find the Noether current and conserved charge due to the invariance under the $U(1)$ gauge transformation of the QED Lagrangian.*

15.2 Perturbative QED

Let us take the QED Lagrangian of Eq. 15.15 and substitute the expression for the covariant derivative. We have,

$$\begin{aligned} \mathcal{L} \;&=\; \bar{\psi}\left[i\not{\partial} - m\right]\psi \qquad \text{(free electron field)} \\ &+\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{(free photon field)} \\ &+\; e\bar{\psi}\not{A}\psi \qquad \text{(electron-photon interaction)} \end{aligned} \tag{15.16}$$

The last term, which couples the electron and fermion fields, differentiates the QED Lagrangian from the Lagrangian of two free electron and photon fields. Can we solve the energy eigenstates of the QED Hamiltonian? No, unless we resort to perturbation theory. The strength of the photon-electron interaction is very small. We can consider the last term a small perturbation, and expand probability amplitudes around $e = 0$.

For $e = 0$, we should recover the results from the quantization of the free photon and free electron fields. We then encounter the same problems as when we quantized the free electromagnetic field, which required to fix the gauge for the field A^μ. We then add to the QED Lagrangian a term,

$$\mathcal{L}_{gauge-fix} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 . \tag{15.17}$$

In this way, we can at least be sure that the theory is correctly defined for $e = 0$. It is not obvious that this modification will solve the problem of obtaining a consistently quantized theory at higher orders in perturbation theory. It actually does! But the justification of this statement will be a very important topic of study in QFTII. As a primitive diagnostic, we shall consider ξ an arbitrary parameter in our calculations. We will then check that higher order corrections do not modify it.

We then have,

$$\mathcal{L}_{QED} = \mathcal{L}_e + \mathcal{L}_\gamma + \mathcal{L}_{int}, \tag{15.18}$$

with

$$\mathcal{L}_e = \bar{\psi}\left[i\partial\!\!\!/ - m\right]\psi, \tag{15.19}$$

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{}{2\xi}\left(\partial_\mu A^\mu\right)^2 , \tag{15.20}$$

and

$$\mathcal{L}_{int} = e\bar{\psi}A\!\!\!/\psi. \tag{15.21}$$

We can develop a similar formalism for perturbation theory as in the case of a scalar field theory. This leads to the derivation of Feynman rules for $S-$matrix elements for QED transition amplitudes. Essentially, we can picture QED processes as photons and electrons propagating in between interactions, where a photon is absorbed or emitted by electrons. For every photon absorption or emission we need a factor of e, which is a number representing the strength of the interaction. We then have to include all possibilities for a scattering process to happen, with so many interaction vertices as the maximum order in our perturbative series. The rules are:

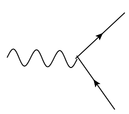
- For the propagation of a photon with momentum p, assign the factor

$$\equiv -iD^0_{\mu\nu}(p) = \frac{i}{p^2 + i\delta}\left[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2 + i\delta}\right]. \tag{15.22}$$

- For the propagation of an electron, assign the factor

$$\equiv iS_0(p) = \frac{i}{p\!\!\!/ - m + i\delta}. \tag{15.23}$$

- For photon absorption or emission, assign a factor



$$-ie\gamma^\mu \tag{15.24}$$

In addition,

- For each loop in a Feynman diagram, we perform an integration over the loop momentum
$$\int \frac{d^d k}{(2\pi)^d}, \tag{15.25}$$
where d is the dimension.

- For each fermion loop we multiply with a factor
$$(-1) \tag{15.26}$$

- We divide each Feynman diagram with a "symmetry factor", accounting for the equivalent field permutations.

The LSZ reduction formula is slightly modified for fermions and photons, where the constants $\tilde{Z}^{1/2}$ are replaced by $\tilde{Z}^{1/2}$ times a "spin factor" which is different from one. For truncated external lines, we have the rules

- multiply with a factor
$$\bar{u}(p),$$
for each outgoing fermion,

- a factor
$$v(p),$$
for each outgoing anti-fermion,

- a factor
$$u(p),$$
for each incoming fermion,

- a factor
$$\bar{v}(p),$$
for each incoming anti-fermion,

- and a factor
$$\epsilon^\mu(p)$$
for each external photon.

15.3 Dimensional regularization for QED

QED is plagued by divergent loop integrals. We shall use dimensional regularization as our regularization method. In this Section, we discuss some new features that arise in dimensional regularization in gauge theories.

First, we consider the QED action integral in arbitrary d dimensions.
$$S = \int d^d x \mathcal{L}_{QED}. \tag{15.27}$$

The action has no mass dimensions,

$$[S] = \left[d^d x \mathcal{L}_{QED}\right] = [mass]^0 . \tag{15.28}$$

Since the space-time coordinates have an inverse mass dimension we find that all terms in the Lagrangian must have mass dimensionality d,

$$[\mathcal{L}_{QED}] = [mass]^d . \tag{15.29}$$

From the kinetic terms of the photon and fermion fields we can determine that their mass dimensionalities ought to be,

$$\left[(\partial_\mu A_\nu)^2\right] = [mass]^d \rightsquigarrow [A_\nu] = [mass]^{\frac{d}{2}-1} , \tag{15.30}$$

and

$$\left[\bar{\psi}\partial\!\!\!/\psi\right] = [mass]^d \rightsquigarrow [\psi] = [mass]^{\frac{d-1}{2}} . \tag{15.31}$$

From the gauge fixing term, we find that the parameter ξ

$$[\xi] = [mass]^0 , \tag{15.32}$$

is dimensionless. Finally, the interaction term gives,

$$\left[e\bar{\psi}A\!\!\!/\psi\right] = [mass]^d \rightsquigarrow [e] = [mass]^{\frac{4-d}{2}} . \tag{15.33}$$

The coupling constant e is a dimensionful parameter in dimensions other than four. In dimensional regularization, we take

$$d = 4 - 2\epsilon,$$

which translates into

$$[e] = [mass]^\epsilon . \tag{15.34}$$

We can then substitute in the Lagrangian and the Feynman rule for photon absorption or emission,

$$e \to e\mu^\epsilon, \tag{15.35}$$

introducing an arbitrary mass scale μ in the Lagrangian. Naively, given that we shall take the $\epsilon \to 0$ after renormalisation, the arbitrary scale μ seems innocuous. In practice, we cannot get rid of μ easily, unless we are able to compute the perturbative series at all orders. Loop integrals will produce poles in ϵ of the form,

$$\frac{\mu^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \log(\mu) + \mathcal{O}(\epsilon). \tag{15.36}$$

The procedure of renormalisation will eliminate the residues of the poles in ϵ in the expressions for scattering matrix-elements. However the $\log(\mu)$ terms can only eliminated by means of a resummation of all perturbative orders.

15.3.1 Gamma-matrices in dimensional regularization

How do we treat γ-matrices in dimensional regularization? The prescription that is followed in conventional dimensional regularization for the Clifford algebra is

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{4\times 4}, \tag{15.37}$$

where the metric is taken in $D = 4 - 2\epsilon$ dimensions,

$$g_\mu^\mu = 4 - 2\epsilon, \tag{15.38}$$

and the dimensionality of the gamma matrices is four by four,

$$tr\mathbf{1}_{4\times 4} = 4. \tag{15.39}$$

Let us do some simple calculations with this algebra.

$$\gamma^\mu \gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = g_\mu^\mu \mathbf{1}_{4\times 4} = (4 - 2\epsilon)\mathbf{1}_{4\times 4}. \tag{15.40}$$

Also,

$$\begin{aligned} \gamma^\mu\gamma^\nu\gamma_\mu &= \{\gamma^\mu, \gamma^\nu\}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu \\ &= 2g^{\mu\nu}\gamma_\mu - (4 - 2\epsilon)\gamma^\nu \\ \rightsquigarrow \gamma^\mu\gamma^\nu\gamma_\mu &= -2(1 - \epsilon)\gamma^\nu. \end{aligned} \tag{15.41}$$

and so on.

A difficulty arises in defining the matrix γ_5 in $4 - 2\epsilon$ dimensions. In four dimensions this is defined as,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{15.42}$$

It is often just fine to use a prescription of an anti-commuting γ_5,

$$\{\gamma^\mu, \gamma_5\} = 0,$$

in performing the gamma-matrix algebra for QED amplitudes. But a discussion of this issue lies beyond the scope of this course.

15.3.2 Tensor loop-integrals

In computing QED scattering amplitude, we need to calculate loop integrals where tensors of the loop-momentum appear in the numerator. These arise due to the Feynman rule for the fermion propagator, which is written as

$$\frac{i}{\not{k} - m} = \frac{i\,(\gamma^\mu k_\mu + m)}{k^2 - m^2}. \tag{15.43}$$

Particles propagating in loops, called virtual, generate loop-momenta in the numerators of loop integrands.

At one-loop we can always express tensor integrals in terms of scalar integrals. The technique is known as Passarino-Veltman technique and we can illustrate it with a couple of examples. Consider the following tensor one-loop integral,

$$I\left[k^\mu\right] \equiv \int d^dk \frac{k^\mu}{k^2(k+p)^2}, \tag{15.44}$$

where the $+i\delta$ term in the denominator is implicit and we take for simplicity $m = 0$. The integrand is a rank one vector in the loop momentum. Due to Lorentz covariance, the result of the integration will also be a rank one tensor. We can then write,

$$I\left[k^\mu\right] = A p^\mu, \tag{15.45}$$

where on the rhs we have written the most general rank-one tensor that we can construct from the momenta in the integral other than the loop momentum. The coefficient A is as yet undetermined. We can express it in terms of integrals which have scalar numerators. We multiply Eq. 15.45 with p_μ and rearrange

$$A = \frac{1}{p^2} I\left[k \cdot p\right]. \tag{15.46}$$

Notice that the scalar product in the numerator of the integral above can be expressed in terms of the denominators of the integral,

$$k \cdot p = \frac{1}{2}(k+p)^2 - \frac{k^2}{2} - \frac{p^2}{2}. \tag{15.47}$$

We then have that,

$$A = \frac{1}{2}\left[-\int d^dk \frac{1}{k^2(k+p)^2} + \frac{1}{p^2}\int d^dk \frac{1}{k^2} - \frac{1}{p^2}\int d^dk \frac{1}{(k+p)^2}\right]. \tag{15.48}$$

Notice, that we only find integrals with a constant numerator, which is independent of the loop momentum. The last two integrals are identical, as we can see by performing the shift $k \to k + p$ in one of them. In this particular example, they cancel against each other, and we have

$$\int d^dk \frac{k^\mu}{k^2(k+p)^2} = -\frac{p^\mu}{2}\int d^dk \frac{1}{k^2(k+p)^2}. \tag{15.49}$$

Consider now a more complicated example,

$$I\left[k^\mu k^\nu\right] \equiv \int d^dk \frac{k^\mu k^\nu}{k^2(k+p)^2}. \tag{15.50}$$

The most general rank two tensor that we can write using the external momenta of the integral is,

$$I\left[k^\mu k^\nu\right] = A_1 g^{\mu\nu} + A_2 \frac{p^\mu p^\nu}{p^2}. \tag{15.51}$$

Contracting with $g_{\mu\nu}$ and $\frac{p_\mu p_\nu}{p^2}$, we obtain the system of equations,

$$\begin{pmatrix} g^{\mu\nu} g_{\mu\nu} & \frac{p^\mu p^\nu}{p^2} g_{\mu\nu} \\ g^{\mu\nu}\frac{p_\mu p_\nu}{p^2} & \frac{p^\mu p^\nu}{p^2}\frac{p_\mu p_\nu}{p^2} \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} I\left[k^2\right] \\ \frac{I\left[(k\cdot p)^2\right]}{(p^2)^2} \end{pmatrix}. \tag{15.52}$$

In arbitrary dimensions,

$$g_{\mu\nu} g^{\mu\nu} = d = 4 - 2\epsilon,$$

and the above system of equations becomes,

$$\begin{pmatrix} 4-2\epsilon & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} I\left[k^2\right] \\ \frac{I\left[(k\cdot p)^2\right]}{(p^2)^2} \end{pmatrix}. \tag{15.53}$$

with a solution,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{3-2\epsilon} \begin{pmatrix} 1 & -1 \\ -1 & 4-2\epsilon \end{pmatrix} \begin{pmatrix} I\left[k^2\right] \\ \frac{I\left[(k\cdot p)^2\right]}{(p^2)^2} \end{pmatrix}. \tag{15.54}$$

The two integrals on the rhs can be expressed in terms of two scalar "master integrals",

$$I_a \equiv \int d^d k \frac{1}{k^2}, \tag{15.55}$$

and

$$I_b \equiv \int d^d k \frac{1}{k^2 (k+p)^2}. \tag{15.56}$$

Specifically, we have

$$\begin{aligned} I\left[k^2\right] &= \int d^d k \frac{k^2}{k^2(k+p)^2} = \int d^d k \frac{1}{(k+p)^2} \\ &= \int d^d (k+p) \frac{1}{(k+p)^2} = \int d^d k \frac{1}{k^2} \\ &= I_a. \end{aligned} \tag{15.57}$$

I_a is a scaleless integral and, as we have discussed, it is zero in dimensional regularization. We let it as an exercise to prove that

$$I\left[(k\cdot p)^2\right] = \frac{p^2}{2} I_a + \frac{(p^2)^2}{4} I_b. \tag{15.58}$$

We can express all one-loop tensor integrals in terms of four master integrals,

$$M_A \equiv \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 - m_1^2 + i\delta}, \tag{15.59}$$

$$M_B \equiv \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{(k^2 - m_1^2 + i\delta)\left[(k+p)^2 - m_2^2 + i\delta\right]}, \tag{15.60}$$

$$M_C \equiv \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{(k^2 - m_1^2 + i\delta)\left[(k+p_1)^2 - m_2^2 + i\delta\right]\left[(k+p_2)^2 - m_3^2 + i\delta\right]}, \tag{15.61}$$

and

$$\begin{aligned} M_D &\equiv \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \times \\ &\times \frac{1}{(k^2 - m_1^2 + i\delta)\left[(k+p_1)^2 - m_2^2 + i\delta\right]\left[(k+p_2)^2 - m_3^2 + i\delta\right]\left[(k+p_2)^2 - m_4^2 + i\delta\right]}. \end{aligned} \tag{15.62}$$

Exercise: Express the following tensor integrals in terms of master integrals,

$$\begin{gathered} \int d^d k \frac{k^\mu}{k^2 - m^2} \\ \int d^d k \frac{k^\mu k^\nu}{k^2 - m^2} \\ \int d^d k \frac{k^\mu}{(k^2 - m_1^2)\left[(k+p)^2 - m_2^2\right]} \\ \int d^d k \frac{k^\mu k^\nu}{(k^2 - m_1^2)\left[(k+p)^2 - m_2^2\right]} \end{gathered} \tag{15.63}$$

The above master integrals are known analytically since many years. Their computation is involved for the general case of arbitrary masses m_i and momenta p_i. We shall need the following two special cases,

$$\int \frac{d^dk}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 - m^2} = -\Gamma(-1+\epsilon) \left(m^2\right)^{1-\epsilon}, \tag{15.64}$$

and

$$\int \frac{d^dk}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 (k+p)^2} = \frac{c_\Gamma}{\epsilon(1-2\epsilon)} \left(-p^2\right)^{-\epsilon}, \tag{15.65}$$

where

$$c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}. \tag{15.66}$$

Exercise: Prove the above using the method of Feynman parameters

15.4 The electron propagator at one-loop

We consider now the one-loop correction to the electron propagator which is depicted by the Feynman diagram of Fig. 15.1. For simplicity, we will set the mass of the electron to

Figure 15.1: The one-loop perturbative correction to the electron propagator

zero $m_e = 0$. Following our Feynman rules, this truncated diagram is:

$$\begin{aligned} -i\Sigma_{1-loop} &= \int \frac{d^dk}{(2\pi)^d} \left(-ie\mu^\epsilon \gamma^\mu\right) \frac{i}{\not{k} + \not{p}} \left(-ie\mu^\epsilon \gamma^\nu\right) \frac{(-i)}{k^2} \left[g_{\mu\nu} - \bar{\xi} \frac{k_\mu k_\nu}{k^2}\right] \\ &= -e^2 \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \left[\frac{\gamma^\mu \not{k}_p \gamma_\mu}{k^2 k_p^2} - \bar{\xi} \frac{\not{k} \not{k}_p \not{k}}{(k^2)^2 k_p^2}\right] \end{aligned} \tag{15.67}$$

where we have introduced the shorthand notation $\bar{\xi} \equiv 1 - \xi$ and $k_p \equiv k + p$. We can simplify the numerator structure with the following gamma-matrix algebra identities (**exercise**) such as:

$$\gamma^\mu \not{k} \gamma_\mu = -2(1-\epsilon)\not{k}, \tag{15.68}$$

etc. We then find the two tensor integrals which can be reduced to scalar integrals with the techniques seen earlier in this Chapter:

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{k^2 k_p^2} = -\frac{p^\mu}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 k_p^2}, \tag{15.69}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2)^2 k_p^2} = -\frac{p^\mu}{2p^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 k_p^2} \left[1 + \frac{p^2}{k^2}\right], \tag{15.70}$$

The two terms in the rhs can be simplified with integration by parts. We have

$$\begin{aligned} 0 &= \int \frac{d^dk}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{k^2 k_p^2} \\ \rightsquigarrow & \quad \int \frac{d^dk}{(2\pi)^d} \frac{p^2}{(k^2)^2 k_p^2} = -(1-2\epsilon) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2) k_p^2}. \end{aligned} \tag{15.71}$$

Therefore,

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^2 k_p^2} = -\epsilon \frac{p^\mu}{p^2} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{k^2 k_p^2}. \tag{15.72}$$

With the tensor reduction at hand we then easily arrive at:

$$-i\Sigma_{1-loop} = e^2 \mu^{2\epsilon} \xi (1-\epsilon) \not{p} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 k_p^2} \tag{15.73}$$

Substituting the expression for the master integral, we obtain:

$$\Sigma_{1-loop} = -\frac{e^2}{(4\pi)^2} \left(-\frac{4\pi\mu^2}{p^2} \right)^\epsilon c_\Gamma \frac{(1-\epsilon)}{\epsilon(1-2\epsilon)} \xi \not{p}. \tag{15.74}$$

This expression is divergent in $d = 4 - 2\epsilon$ dimensions. Expanding in ϵ, we obtain:

$$\Sigma_{1-loop} = \Sigma_{V,1-loop}(p^2) \not{p}, \tag{15.75}$$

with

$$\Sigma_{V,1-loop}(p^2) = -\frac{\frac{e^2}{4\pi}}{\pi} (4\pi)^\epsilon e^{-\gamma\epsilon} \left[\frac{1}{4\epsilon} + \frac{1}{4} \log\left(-\frac{\mu^2}{p^2} \right) + \frac{1}{4} + \mathcal{O}(\epsilon) \right] \xi. \tag{15.76}$$

Notice that the artificial mass μ which we introduced in the Lagrangian to fix the dimensions of the coupling constant in dimensional regularisation combines with the physical scale p^2 in a dimensionless ratio as an argument of a logarithm. To evaluate the logarithm, we should recall that p^2 possesses a small imaginary part $p^2 + i\delta$.

15.5 Electron propagator at all orders

We can classify the loop corrections of the electron propagator into two classes of Feynman diagrams: one-particle-irreducible (1PI) and one-particle-reducible. You can see an example of an one-particle-reducible graph in Fig. 15.2. By removing one line, the graph separates into two graphs. On the contrary, 1PI Feynman diagrams cannot be reduced

Figure 15.2: An example of an one-particle-reducible graph which contributes to the electron propagator

into two graphs by removing any of the lines. A sample of 1PI graphs is presented in Fig. 15.3.

Figure 15.3: Sample of one-particle-irreducible graphs which contribute to the electron propagator

Let us examine first the sum of all truncated (removing propagators from the external legs) 1PI Feynman diagrams, which we conventionally denote by $-i\Sigma(p)$. Obviously, it is not possible to produce an analytic expression for such a sum of an infinite number of graphs. However, we can determine its Lorentz structure as a 4×4 matrix in spin-space. Given that

$$\not{p}^{2n+1} = (p^2)^n \not{p},$$

we can only have up to a linear term when expressing Σ as a polynomial in $\not{p}$:

$$\Sigma(p) = A(p^2)\mathbf{1} + \Sigma_V(p^2)\not{p} \tag{15.77}$$

We can also convince ourselves that the coefficient of the unity $A(p^2)$ must vanish in the limit of a zero electron mass $m \to 0$. We can isolate this coefficient by taking the trace:

$$A(p^2) = \frac{\mathrm{tr}\,(\Sigma(p))}{4}. \tag{15.78}$$

Examining the truncated graphs of Fig. 15.3, we notice that they have an even number of vertices and an odd number of propagators. Each vertex and *massless* propagator contribute a single gamma matrix according to the QED Feynman rules. Therefore, the trace of Σ will always be a sum of traces of an odd number of gamma matrices. This is zero. To reflect this, we can therefore cast Σ in the form:

$$\Sigma(p) = m\Sigma_S(p^2)\mathbf{1} + \Sigma_V(p^2)\not{p} \tag{15.79}$$

Note that the functions Σ_s, Σ_V have also an implicit dependence on the mass of the electron m. In what it follows, we will be interested in their UV divergences. In this limit, the electron mass can be safely taken to zero (after, of course, we have scaled out the mass factor m in the unity coefficient of Eq. 15.79).

Let us now represent the 1PI Feynman diagrams as:

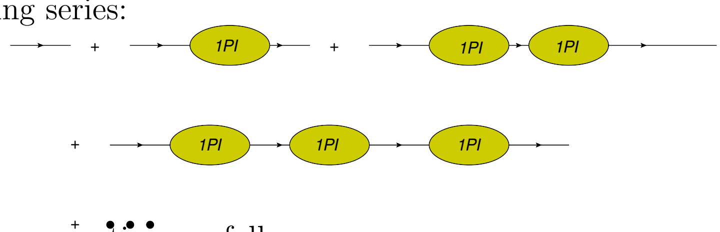
$$= -i\Sigma(p) \tag{15.80}$$

The full propagator:

$$iS(p) \equiv \int d^4x e^{-ip\cdot x} \langle\Omega| T\bar{\psi}(x)\psi(0) |\Omega\rangle\,, \tag{15.81}$$

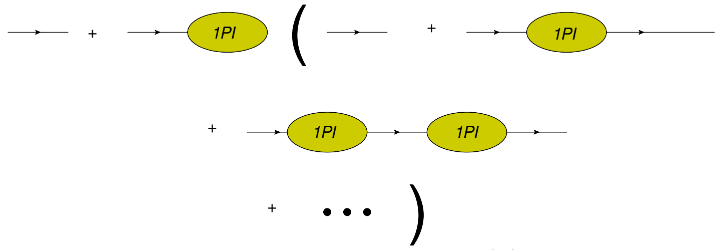
includes includes in addition one-particle-reducible graphs. We can account for all graphs by considering the following series:

$iS(p) =$



We can rearrange the above equations as follows:

$iS(p) =$



In the parenthesis, we recognise the original expression for $iS(p)$. Therefore, the electron propagator $iS(p)$ at all orders can be written in the recursive form:

$$iS(p) = iS_0(p) + iS_0(p)(-i\Sigma(p))iS(p), \tag{15.82}$$

where iS_0 is the propagator of the free electron field field:

$$iS_0 = \frac{i}{\not{p} - m}, \tag{15.83}$$

and $-i\Sigma(p)$ is the sum of all one-particle-irreducible (1PI) graphs at all orders. Multiplying from the right with the inverse of S and from the left with the inverse of S_0, we then obtain:

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) - \Sigma(p) \\ &= \not{p} - m - \not{p}\Sigma_V(p^2) - m\Sigma_S(p^2). \end{aligned} \tag{15.84}$$

Inverting the above, we find that

$$iS(p) = \frac{i}{1 - \Sigma_V(p^2)} \frac{1}{\not{p} - m\dfrac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)}} \tag{15.85}$$

or, equivalently,

$$iS(p) = \frac{i}{1 - \Sigma_V(p^2)} \frac{\not{p} + mf(p^2)}{p^2 - m^2 f^2(p^2)} \tag{15.86}$$

with

$$f(p^2) = \frac{1 + \Sigma_S(p^2)}{1 - \Sigma_V(p^2)} \tag{15.87}$$

Exercise: Show that

$$\Sigma_{S,1-loop}(p^2) = \frac{\frac{e^2}{4\pi}}{\pi} (4\pi)^\epsilon e^{-\gamma\epsilon}(3 + \xi) \left[\frac{1}{4\epsilon} + \mathcal{O}(\epsilon^0)\right]. \tag{15.88}$$

15.5.1 The electron mass

While the pole of the propagator S_0 for a free electron is at $p^2 = m^2$, where m is the electron mass that we inserted in the QED Lagrangian, the location of the pole in Eq. 15.85 is not at the same point. The physical mass of the electron should actually be determined from the full propagator, as the solution of the equation:

$$p^2 - m^2 f^2(p^2) = 0. \tag{15.89}$$

Let us assume that we have solved this equation and found that it is satisfied for $p^2 = m_{\rm phys}^2$, i.e.

$$m_{\rm phys}^2 - m^2 f^2(m_{\rm phys}^2) = 0. \tag{15.90}$$

Substituting the mass m in terms of the physical mass and expanding in $p^2 - m_{\rm phys}^2$ we find that

$$iS(p) = \frac{1}{1 - S_V(p^2 = m_{\rm phys}^2)} \frac{i}{\not{p} - m_{\rm phys}} + \quad \text{continuum.} \tag{15.91}$$

We can compare this result with the Källen-Lehmann formula for the electron propagator:

$$iS(p) = \tilde{Z}_\psi \frac{i}{\not{p} - m_{\rm phys}} + \quad \text{continuum,} \tag{15.92}$$

from which we obtain that:

$$\tilde{Z}_\psi = \frac{1}{1 - S_V(p^2 = m_{\rm phys}^2)}. \tag{15.93}$$

Exercise: *Calculate $m_{\rm phys}$ and $\tilde{Z}_\psi$ through one-loop in perturbation theory. Notice the divergent structure of the results as $\epsilon \to 0$.*

15.6 The photon propagator at one-loop

We consider now the one-loop correction to the photon propagator which is depicted by the Feynman diagram of Fig. 15.4. For simplicity, we will set the mass of the electron to

Figure 15.4: The one-loop perturbative correction to the photon propagator

zero $m_e = 0$. Following our Feynman rules, this truncated diagram is:

$$\begin{aligned} i\Pi^{\mu\nu}_{1-loop} &= (-1)\int \frac{d^dk}{(2\pi)^d} tr\left[\frac{i}{\not k + \not p}(-ie\mu^\epsilon\gamma^\mu)\frac{i}{\not k}(-ie\mu^\epsilon\gamma^\nu)\right] \\ &= -(e\mu^\epsilon)^2\left[\int \frac{d^dk}{(2\pi)^d}\frac{(k_\rho + p_\rho)k_\sigma}{(k+p)^2k^2}\right] tr\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] \end{aligned} \tag{15.94}$$

The trace of gamma matrices is:

$$tr\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] = 4\left[g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\sigma\mu}\right] \tag{15.95}$$

where the indices μ, ν, ρ, σ are $d-$dimensional with $d = 4 - 2\epsilon$. The tensor loop integral can be computed with the techniques of the previous section. We find:

$$\begin{aligned} I_{\rho\sigma} &\equiv \int \frac{d^dk}{(2\pi)^d}\frac{(k_\rho + p_\rho)k_\sigma}{(k+p)^2k^2} \\ &= -\frac{1}{4(d-1)}\left\{g_{\rho\sigma}p^2 + (d-2)p_\rho p_\sigma\right\}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k+p)^2k^2} \end{aligned} \tag{15.96}$$

The expression for the scalar integral has been given in Eq. 15.65. We then find:

$$\Pi^{\mu\nu}_{1-loop} = \Pi_{1-loop}(p^2)\left[p^2g^{\mu\nu} - p^\mu p^\nu\right], \tag{15.97}$$

with

$$\Pi_{1-loop}(p^2) = -\frac{\frac{e^2}{4\pi}}{\pi}\left(-\frac{4\pi\mu^2}{p^2}\right)^\epsilon \frac{c_\Gamma}{\epsilon}\frac{1-\epsilon}{(1-2\epsilon)(3-2\epsilon)} \tag{15.98}$$

We find that the expression has an $1/\epsilon$ pole and it is therefore divergent in exactly $d = 4$ dimensions. Performing an expansion in ϵ, we find:

$$\Pi_{1-loop}(p^2) = -\frac{\frac{e^2}{4\pi}}{\pi}(4\pi)^\epsilon e^{-\gamma\epsilon}\left[\frac{1}{3\epsilon} + \frac{1}{3}\log\left(-\frac{\mu^2}{p^2}\right) + \frac{5}{9} + \mathcal{O}(\epsilon)\right] \tag{15.99}$$

Finally, we see that the one-loop correction $\Pi^{\mu\nu}$ is proportional to the tensor:

$$T^{\mu\nu} \equiv g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}. \tag{15.100}$$

This is a simpler structure than the one that we find at the tree-level Feynman rule for the photon propagator.

$$= \frac{i}{p^2}\left[-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right] = -\frac{i}{p^2}\left[T^{\mu\nu} + \xi L^{\mu\nu}\right], \tag{15.101}$$

where we find a second tensor:

$$L^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2}, \tag{15.102}$$

to multiply the gauge fixing parameter ξ. The tensor $T^{\mu\nu}$ is "transverse" to the momentum:

$$T^{\mu\nu} p_\mu = T^{\mu\nu} p_\nu = 0. \tag{15.103}$$

The tensor $L^{\mu\nu}$ is "parallel" to the momentum:

$$L^{\mu\nu} p_\mu = p^\nu. \tag{15.104}$$

The one-loop diagram corrects only the part of the propagator which is transverse, leaving the gauge-parameter longitudinal part unchanged. As we shall see soon, this observation is more general and holds at all loop orders.

The tensors $T^{\mu\nu}, L^{\mu\nu}$ are projectors. Indeed, we find:

$$T^{\mu\nu} T_\nu^{\ \rho} = T^{\mu\rho}, \quad L^{\mu\nu} L_\nu^{\ \rho} = L^{\mu\rho}, \quad T^{\mu\nu} L_\nu^{\ \rho} = 0. \tag{15.105}$$

We can write any rank two tensor which is made exclusively out of the metric and a single momentum p^μ as:

$$X^{\mu\nu} = X_T(p^2) T^{\mu\nu} + X_L(p^2) L^{\mu\nu}. \tag{15.106}$$

The inverse can be written down easily:

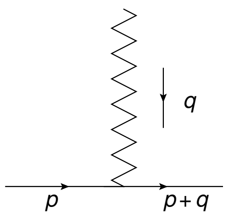
$$\left(X^{-1}\right)^{\mu\nu} = \frac{1}{X_T(p^2)} T^{\mu\nu} + \frac{1}{X_L(p^2)} L^{\mu\nu}. \tag{15.107}$$

Indeed,

$$\begin{aligned} X^{\mu\nu} \left(X^{-1}\right)_\nu^{\ \rho} &= \left(X_T(p^2) T^{\mu\nu} + X_L(p^2) L^{\mu\nu}\right) \left(\frac{1}{X_T(p^2)} T_\nu^{\ \rho} + \frac{1}{X_L(p^2)} L_\nu^{\ \rho}\right) \\ &= T^{\mu\rho} + L^{\mu\rho} = g^{\mu\rho}. \end{aligned} \tag{15.108}$$

15.7 Ward identity*

Consider the tree-level (i.e. zero loops) electron-electron-photon Green's function:

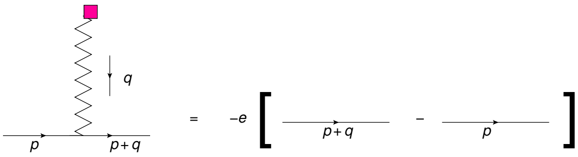


$$\equiv ie\epsilon_\mu V_0^\mu = \epsilon_\mu \frac{i}{\not p + \not q - m} \left(-ie\gamma^\mu\right) \frac{i}{\not p - m} \tag{15.109}$$

where we have truncated the photon external line, but let the electron lines untruncated. Now, let us replace the polarization vector with the momentum of the photon $\epsilon_\mu \to q_\mu$. We find:

$$\begin{aligned} -ieq_\mu V_0^\mu &= \frac{i}{\not p + \not q - m} \left(-ie\not q\right) \frac{i}{\not p - m} \\ &= ie \left[\frac{1}{\not p + \not q - m} \left([\not p + \not q - m] - [\not p - m]\right) \frac{1}{\not p - m}\right] \\ &= -ie \left[\frac{1}{\not p + \not q - m} - \frac{1}{\not p - m}\right] \\ \rightsquigarrow \quad & -ieq_\mu V_0 = -e \left[iS_0(p+q) - iS_0(p)\right] \end{aligned} \tag{15.110}$$

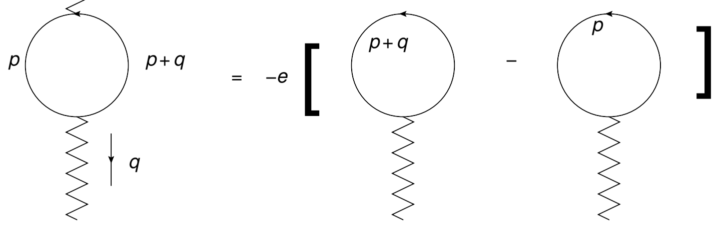
Graphically, we can represent the above equation as:



(15.111)

where the filled square at the start of the incoming photon line represents the operation of substituting the polarisation vector with the momentum vector. This identity has important consequences at all orders in perturbation theory.

Consider the example of the previous section, the one-loop diagram for the photon propagator $\Pi^{\mu\nu}_{1-loop}$, and contract it with the momentum of the photon $q_\mu \Pi^{\mu\nu}_{1-loop}$. We have found, after an explicit calculation, that $q_\mu \Pi^{\mu\nu}_{1-loop} = 0$. We can now elucidate this result by means of Eq. 15.111. Indeed, we can join together the edges of the electron lines of both sides of Eq. 15.111 forming one-loop diagrams. In addition, we can attach a second photon to these loops. The resulting identity is then:

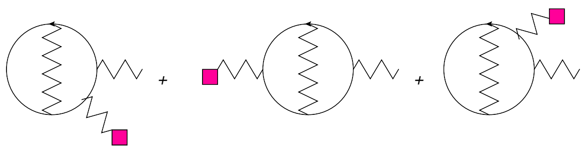


(15.112)

Notice that the diagrams in the rhs are identical except that the momenta $p, p+q$ circulating in the loops of the rhs differ by a shift. However, the loop momentum p is an integration variable and we are allowed (at least, in dimensional regularisation) to make a change $p \to p+q$ in the first of the two terms in the rhs. With this transformation the two integrals in the rhs are shown to be identical and cancel. This is an alternative proof that the one-loop corrections to the propagator cancel.

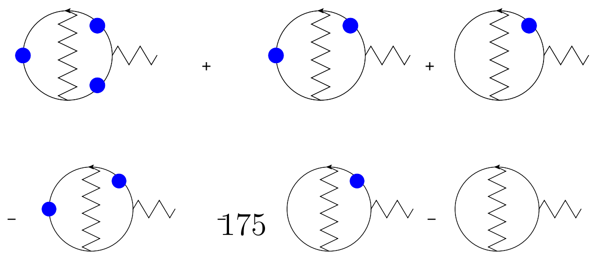
The generalisation of this identity at higher loop orders is straightforward. The two-loop 1PI diagrams which contribute to the photon propagator $q_\mu \Pi^{\mu\nu}_{2-loop}$ are:

$$iq_\mu \Pi^{\mu\nu}_{2-loop} =$$


(15.113)

We can now apply the identity of Eq. 15.111 for the photon lines which are contracted by q_μ. The result is proportional to the following diagrams:

$$iq_\mu \Pi^{\mu\nu}_{2-loop} \propto$$


(15.114)

The blue dot on a line signifies that the momentum of this propagator is shifted by q with respect to the analogous propagator in the diagram without any dots. As you notice, four of these terms cancel in pairs without any further manipulation. The first and the last term differ by a shift of both the loop momenta by q^μ. Therefore, they also cancel as it can be shown with an appropriate change of variables. We have therefore shown without performing the loop integrations that

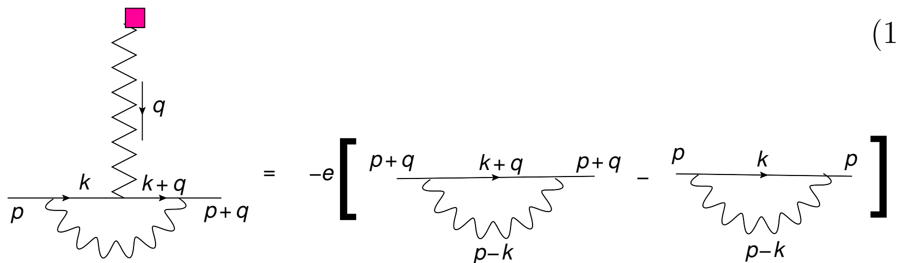
$$iq_\mu \Pi^{\mu\nu}_{2-loop} = 0. \tag{15.115}$$

Exercise: It is not hard to convince yourselves that this result holds at three and higher loop orders as well:

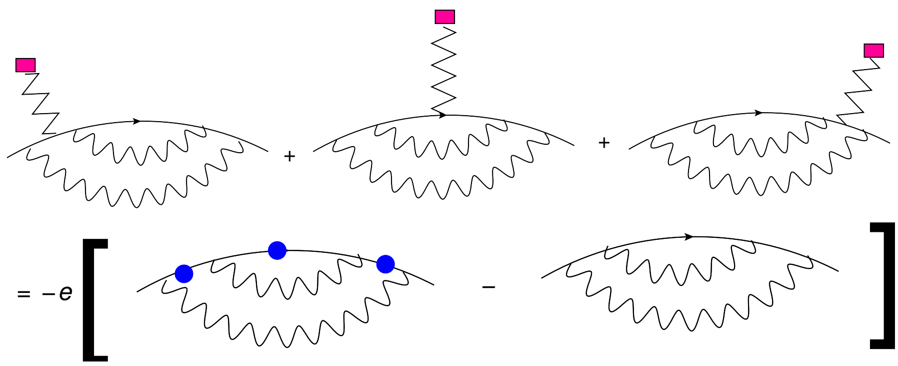
$$q_\mu \Pi^{\mu\nu} = 0. \tag{15.116}$$

Let us now look at the one-loop correction to the three-point function (electron-electron-photon vertex), substituting the photon polarisation with the momentum $\epsilon^\mu \to q^\mu$. Applying the identity of Eq. 15.111, we find that

(15.117)



We observe that by replacing the polarisation with the momentum, the 3-point vertex becomes the difference of propagator diagrams. After cancelations of the type we observed in Eq. 15.114, we find that a similar result holds for all two-loop vertex diagrams.
Exercise: Prove the following:



(where we remind that the dot signifies a propagator momentum shifted by q). This result holds in general. We can write the so called "Ward identity":

$$q_\mu \left(-ie\Gamma^\mu\right) = -ie\left[S^{-1}(p+q) - S^{-1}(p)\right] \tag{15.118}$$

where by $-ie\epsilon_\mu \Gamma^\mu$ we denote the untruncated 3-point electron-electron-photon Green's function.
for next time:

- *show that one can derive many more "Ward identities"*

- *Show that replacing the polarisation vector with the momentum for a truncated Green's function as it enters in the LSZ formula gives zero:*

$$\epsilon_\mu \mathcal{M}^\mu = 0. \tag{15.119}$$

15.8 Photon propagator at all orders

We are now in a position to determine the functional form of the full photon propagator at all-loop orders:

$$-iD_{\mu\nu}(p) \equiv \int d^4x e^{-ip\cdot x} \langle\Omega| TA^\mu(x)A^\nu(0) |\Omega\rangle . \tag{15.120}$$

This contains both one-particle-reducible and one-particle-irreducible Feynman diagrams. Following faithfully the analysis which led to the result of Eq. 15.82, we find in the same way a recursion formula for the photon propagator:

$$-iD_{\mu\nu}(p) = -iD^0_{\mu\nu}(p) + (-iD^0_{\mu\rho}(p))(i\Pi^{\rho\sigma})(-iD_{\sigma\nu}(p)). \tag{15.121}$$

In the above, the propagator of a free photon is given by Eq. 15.22 and $\Pi^{\mu\nu}$ is the sum of all 1PI graphs at all orders in perturbation theory. We can multiply Eq. 15.121 with $(D^{-1})^{\lambda\mu}(D_0^{-1})^{\nu\lambda}$ and recast it in the form:

$$\left(D^{-1}\right)_{\mu\nu} = \left(D^0\right)^{-1}_{\mu\nu} - \Pi_{\mu\nu}. \tag{15.122}$$

As we have seen in the previous section, $\Pi^{\mu\nu}$ is a transverse tensor:

$$\Pi^{\mu\nu} = \Pi(p^2)(g^{\mu\nu}p^2 - p^\mu p^\nu) = \Pi(p^2)p^2T^{\mu\nu}. \tag{15.123}$$

Substituting Eq. 15.22 and Eq. 15.123 into Eq. 15.122 and inverting one more time, we find that the photon propagator at all orders takes the form:

$$-iD^{\mu\nu}(p) = -\frac{i}{p^2}\left[\frac{T^{\mu\nu}}{1-\Pi(p^2)} + \xi L^{\mu\nu}\right] \tag{15.124}$$

Notice that there is a pole at $p^2 = 0$. Therefore, the photon mass is zero also when we included perturbative corrections.

Chapter 16

Renormalisation of QED

Let us for simplicity work with the QED theory in the case of a massless electron. The QED Lagrangian in the Lorentz gauge is given in this case by

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + e\mu^\epsilon\bar{\psi}A\!\!\!/\psi. \tag{16.1}$$

This Lagrangian is built out of the requirement of gauge symmetry and the requirement of not introducing any mass scales (other than the mass of the electron which we set to zero).

Without spoiling our symmetry principle, we are allowed to redefine multiplicatively the fields as well as the parameters e,ξ of the Lagrangian:

$$\psi = Z_\psi^{\frac{1}{2}}\tilde{\psi}, \tag{16.2}$$

$$A^\mu = Z_A^{\frac{1}{2}}\tilde{A}^\mu, \tag{16.3}$$

$$e = Z_\alpha^{\frac{1}{2}}\tilde{e}, \tag{16.4}$$

$$\xi = Z_\xi\tilde{\xi}. \tag{16.5}$$

With these redefinitions, the Lagrangian takes the form:

$$\mathcal{L} = Z_\psi i\bar{\tilde{\psi}}\partial\!\!\!/\tilde{\psi} - \frac{Z_A}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{Z_A}{2Z_\xi\tilde{\xi}}\left(\partial_\mu\tilde{A}^\mu\right)^2 + Z_\alpha^{\frac{1}{2}}Z_A^{\frac{1}{2}}Z_\psi\tilde{e}\mu^\epsilon\bar{\tilde{\psi}}\tilde{A\!\!\!/}\tilde{\psi}. \tag{16.6}$$

Consider now the photon two-point correlation function:

$$\langle\Omega|\,TA^\mu(x)A^\nu(0)\,|\Omega\rangle = Z_A\,\langle\Omega|\,T\tilde{A}^\mu(x)\tilde{A}^\nu(0)\,|\Omega\rangle \tag{16.7}$$

Taking the Fourier transform, we obtain:

$$D^{\mu\nu}(p) = Z_A\tilde{D}^{\mu\nu}(p), \tag{16.8}$$

where $\tilde{D}^{\mu\nu}$ is the propagator for the renormalised field $\tilde{A}^\mu$. Substituting the result of Eq. 15.124, we find:

$$\tilde{D}^{\mu\nu} = \frac{1/}{p^2}\left[\frac{T^{\mu\nu}}{Z_A(1-\Pi(p^2))} + \frac{Z_\xi}{Z_A}\tilde{\xi}L^{\mu\nu}\right]. \tag{16.9}$$

The factor $1-\Pi(p^2)$ contains infinities in the form of $1/\epsilon$ poles as we have observed at the one-loop level (Eq. 15.99). For generic values of Z_A, Z_ξ the expansion in ϵ of the expression above would therefore be divergent. Is there an exception to this?

Notice, that the $1/\epsilon$ pole in the expression of Eq. 15.99 has a coefficient which is independent of p^2 and it consists of constants only. This allows us to cancel the infinity in $Z_A(1-\Pi(p^2))$ with a judicial choice of Z_A. Indeed, let us set:

$$Z_a = 1 + \mathcal{O}(\alpha), \tag{16.10}$$

and

$$Z_A = 1 - \frac{\alpha(\mu_R)}{\pi}\left(\frac{1}{3\epsilon}\right) + \mathcal{O}(\alpha^2), \tag{16.11}$$

where we have defined the renormalised coupling constant $\alpha(\mu_R)$ via:

$$\frac{e^2}{4\pi}e^{-\gamma\epsilon}(4\pi\mu^2)^\epsilon = Z_\alpha \frac{\bar{e}^2}{4\pi}e^{-\gamma\epsilon}(4\pi\mu^2)^\epsilon \equiv Z_\alpha(\alpha(\mu_R))\alpha(\mu_R)\mu_R^{2\epsilon}. \tag{16.12}$$

The mass parameter μ_R is an arbitrary "renormalisation scale" which in effect replaces the scale μ of dimensional regularisation. With these definitions, we find

$$\begin{aligned} Z_A(1-\Pi(p^2)) &= Z_A(1-\Pi_{1-loop}(p^2) + \mathcal{O}(\alpha^2)) \\ &= \left(1 - \frac{\alpha(\mu_R)}{\pi}\left(\frac{1}{3\epsilon}\right) + \mathcal{O}(\alpha^2)\right) \\ &\times \left(1 + \frac{\alpha_s(\mu_R)}{\pi}\left[\frac{1}{3\epsilon} + \frac{5}{9} + \frac{1}{3}\log\left(-\frac{\mu_R^2}{\pi^2}\right)\right] + \mathcal{O}(\alpha^2)\right) \\ &= 1 + \frac{\alpha_s(\mu_R)}{\pi}\left[\frac{5}{9} + \frac{1}{3}\log\left(-\frac{\mu_R^2}{\pi^2}\right)\right] + \mathcal{O}(\alpha^2), \end{aligned} \tag{16.13}$$

which is finite. Therefore, there is a photon field redefinition $A^\mu \to \tilde{A}^\mu$, with $A^\mu = Z_A^{\frac{1}{2}}\tilde{A}^\mu$, which renders the transverse part of the $\tilde{A}^\mu$ photon propagator (Eq. 16.9) finite. In fact, given that loop corrections at all orders are transverse and do not change the coefficient of $L^\mu\nu$ in Eq. 16.9, we can guarantee that also the longitudinal part is finite by setting

$$Z_\xi = Z_A. \tag{16.14}$$

We can repeat the same process for the electron propagator:

$$\langle\Omega| T\bar{\psi}(x)\psi(0)|\Omega\rangle = Z_\psi \langle\Omega| T\bar{\tilde{\psi}}(x)\tilde{\psi}(0)|\Omega\rangle, \tag{16.15}$$

which, by taking the Fourier transform, yields

$$S(p) = Z_\psi \tilde{S}(p). \tag{16.16}$$

As, for the renormalised photon field, we will require that the renormalised electron field $\tilde{\psi}$ yields a propagator which is free of $1/epsilon$ divergences. Substituting Eq. 15.85, we find:

$$\tilde{S}(p) = \frac{1}{Z_\psi\left(1-\Sigma_V(p^2)\right)}\frac{1}{\not{p} - \tilde{m}\dfrac{Z_m Z_\psi\left(1+\Sigma_S(p^2)\right)}{Z_\psi\left(1-\Sigma_V(p^2)\right)}} \tag{16.17}$$

We can fix the factors Z_ψ, Z_m by requiring that the following two quantities are finite:

$$Z_\psi\left(1-\Sigma_V(p^2)\right) = \text{finite}, \quad Z_m Z_\psi\left(1+\Sigma_S(p^2)\right). \tag{16.18}$$

We can achieve this by choosing, for example,

$$Z_\psi = 1 - \frac{\alpha(\mu_R)}{\pi} \frac{\xi}{4\epsilon} + \mathcal{O}(\alpha^2) \tag{16.19}$$

and

$$Z_m = 1 - \frac{\alpha(\mu_R)}{\pi} \frac{3}{4\epsilon} + \mathcal{O}(\alpha^2). \tag{16.20}$$

Finally, the renormalisation constant Z_α serves to cancel the UV divergences in the photon-fermion-fermion vertex. In fact, writing the Ward identity of Eq 15.118 in terms of renormalised quantities, we have

$$q_\mu \Gamma^\mu = Z_\psi^{-1} \left[\tilde{S}^{-1}(p+q) - \tilde{S}^{-1}(p) \right] \tag{16.21}$$

In the rhs, the divergent part is Z_ψ^{-1} (which is a constant independent of p^μ, q^μ). Therefore, the combination

$$Z_\psi \Gamma^\mu \sim \text{ finite }. \tag{16.22}$$

The physical matrix element for an electron-photon interaction is:

$$\tilde{Z}_\psi \tilde{Z}_A^{\frac{1}{2}} e \Gamma^\mu = \tilde{Z}_A^{\frac{1}{2}} Z_\alpha^{\frac{1}{2}} \left(\tilde{Z}_\psi \Gamma^\mu \right), \tag{16.23}$$

where we have included the appropriate factors $\tilde{Z}$ of the LSZ reduction formula of Eq. 12.81. The factors $\tilde{Z}$ and Z they do not need to be the sama. However, comparing the general Källen-Lehmann form of the propagator where $\tilde{Z}$ appear and the conditions we set for determining the corresponding field renormalisation Z, we arrive at the conclusion that $\tilde{Z}$ must have the same divergences as Z (**exercise**). The matrix element of Eq. 16.23 must be finite in order to make physical sense. Given that the combination $\left(\tilde{Z}_\psi \Gamma^\mu \right)$ is finite, we can achieve this by choosing:

$$Z_\alpha = Z_A^{-1}. \tag{16.24}$$

We now have to inspect for infinities one-loop matrix-elements with four or more external particles, as for example in the process $e^+e^- \to \gamma\gamma$, etc. Notice, that we have already determined all Z factors in the renormalised QED Lagrangian of Eq. 16.6. If we find any infinite matrix elements beyond the ones we have already used to fix the Z constants, we will not be able to absorb these infinities into a renormalisation without adding new terms in the Lagrangian. However, in QED, the infinities appear only in the propagators and the electron-photon vertex. It also turns out that the Z factors suffice to absorb the infinities at all loop orders and not just the one-loop case that we have considered here. A proof of the above statements will be the subject of the QFT2 course.

A comment is in order about the choice of the Z constants in Eq 16.11, Eq. 16.14, Eq. 16.19, Eq. 16.20, Eq. 16.24 and the definition of the coupling constant in Eq. 16.12. Clearly, these are not unique. We can change any Z by a finite constant,

$$Z \to Z' = Z + finite, \tag{16.25}$$

achieving the same scope of absorbing the infinities from the propagators and the electron-photon vertex. The choice of finite pieces for the Z factors defines a "renormalisation sceme". Our choice corresponds to the "modified minimal subtraction" *MSbar* renormalisation scheme.

16.1 Running of the QED coupling constant and the electron mass*

Consider Eq. 16.12:

$$\frac{e^2}{4\pi} e^{-\gamma\epsilon} (4\pi\mu^2)^\epsilon = Z_\alpha(\alpha(\mu_R))\alpha(\mu_R)\mu_R^{2\epsilon}. \tag{16.26}$$

The general structure of the Z_α factor through two-loops is:

$$Z_\alpha(\alpha(\mu_R)) = 1 - \frac{\beta_0}{\epsilon}\frac{\alpha(\mu_R)}{\pi} + \left[\frac{\beta_0^2}{\epsilon^2} - \frac{1}{2}\frac{\beta_1}{\epsilon}\right]\left(\frac{\alpha(\mu_R)}{\pi}\right) + \mathcal{O}(\alpha^3) \tag{16.27}$$

The lhs of the above equation does not depend on the scale μ_R. Differentiating both sides of the equation with respect to μ_R, we must have:

$$0 = \frac{d}{d\mu_R^2}\left(Z_\alpha(\alpha(\mu_R))\alpha(\mu_R)\mu_R^{2\epsilon}\right) \tag{16.28}$$

This leads to an equation:

$$\frac{d\log\alpha}{d\log\mu_R^2} = -\epsilon - \beta(\alpha(\mu_R)), \tag{16.29}$$

where the so called "beta" function is given by:

$$\beta(\alpha) = \beta_0\frac{\alpha}{\pi} + \beta_1\left(\frac{\alpha}{\pi}\right)^2 + \beta_2\left(\frac{\alpha}{\pi}\right)^3 + \ldots \tag{16.30}$$

For the next time:

- To explain: importance of sign of β_0
- Plot: alpha as a function of μ_R for QED
- To explain: why μ_R corresponds to energy...
- Renormalisation group

Appendix A

Special Relativity

Special relativity is based on the assumption that the laws of nature are the same for inertial observers where their co-ordinates are related via Lorentz transformations:

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu_{\ \nu} x^\nu + \rho^\mu. \tag{A.1}$$

where

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{A.2}$$

a "four-vector" comprising space-time coordinates with $x^0 = ct$, ρ^μ is a constant four-vector and $\Lambda^\mu_{\ \nu}$ satisfies:

$$\Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} g_{\mu\nu} = g_{\rho\sigma}. \tag{A.3}$$

The 4×4 matrix $g_{\mu\nu}$ is the so-called *metric*, defined as:

$$g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3 \\ 0, & \mu \neq \nu \end{cases} \tag{A.4}$$

In the above we have used Einstein's summation convention. For example, one would write explicitly

$$\Lambda^\mu_{\ \nu} x^\nu = \Lambda^\mu_{\ 0} x^0 + \Lambda^\mu_{\ 1} x^1 + \Lambda^\mu_{\ 2} x^2 + \Lambda^\mu_{\ 3} x^3. \tag{A.5}$$

This is a convention that we will use extensively from now on.

A.1 Proper time

Lorentz transformations leave invariant "proper-time" intervals. These are defined as:

$$d\tau^2 \equiv c^2 dt^2 - d\vec{x}^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{A.6}$$

Indeed, in a different reference frame we have from Eq. A.1:

$$dx^{\mu\prime} = \Lambda^\mu_{\ \nu} dx^\nu. \tag{A.7}$$

A proper-time interval in the new frame is

$$\begin{aligned} d\tau'^2 &= g_{\mu\nu} dx'^\mu dx'^\nu \\ &= g_{\mu\nu} \left(\Lambda^\mu{}_\rho dx^\rho\right) \left(\Lambda^\nu{}_\sigma dx^\sigma\right) \\ &= \left(g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma\right) dx^\rho dx^\sigma \\ &= g_{\rho\sigma} dx^\rho dx^\sigma = d\tau^2. \end{aligned} \tag{A.8}$$

As a consequence of the invariance of proper-time intervals, *the speed of light is the same in all inertial frames.* Indeed, for light we have:

$$\left|\frac{d\vec{x}}{dt}\right| = c \rightsquigarrow d\tau^2 = c^2 dt^2 - d\vec{x}^2 = 0 \tag{A.9}$$

In a new frame,

$$d\tau^{2\prime} = d\tau^2 = 0 \rightsquigarrow \left|\frac{d\vec{x}'}{dt'}\right| = c \tag{A.10}$$

Lorentz transformations are the only non-singular transformations which preserve proper-time intervals:

$$\begin{aligned} d\tau^2 &= d\tau'^2 \\ \rightsquigarrow g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} dx'^\mu dx'^\nu \\ \rightsquigarrow g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma} dx^\rho dx^\sigma, \end{aligned} \tag{A.11}$$

concluding that:

$$g_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma}. \tag{A.12}$$

Differentiating with dx^ϵ, we obtain:

$$0 = g_{\mu\nu} \left[\frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} + \frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho}\right]. \tag{A.13}$$

To this, we add the same equation with $\epsilon \leftrightarrow \rho$ and subtract the same equation with $\epsilon \leftrightarrow \sigma$. We obtain:

$$0 = 2 g_{\mu\nu} \frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \tag{A.14}$$

Assuming that the transformation $x^\mu \to x'^\mu$ is a well behaved differentiable function and that the inverse of the transformation also exists,

$$\frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\nu} = \delta_{\mu\nu}, \tag{A.15}$$

we obtain that

$$\frac{\partial^2 x'^\mu}{\partial x^\epsilon \partial x^\rho} = 0. \tag{A.16}$$

Therefore, the transformation $x^\mu \to x'^\mu$ ought to be linear:

$$x'^\mu = \Lambda^\mu\ \nu + \rho^\mu. \tag{A.17}$$

A.2 Subgroups of Lorentz transformations

The set of all Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + \rho^{\mu}. \quad g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma} \tag{A.18}$$

form a group (**exercise:** prove it), which is known as the group of *inhomogeneous Lorentz group* or the Poincare' group. The subset of transformations with $\rho^{\mu} = 0$ is known as the *homogeneous Lorentz group*.

From

$$g_{\mu\nu}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma} = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have:

$$(\Lambda^{0}_{\ 0})^2 - \sum_{i=1}^{3}(\Lambda^{i}_{\ 0})^2 = 1 \leadsto (\Lambda^{0}_{\ 0})^2 \geq 1. \tag{A.19}$$

Also, in matrix form the definition of the Lorentz transformation becomes:

$$g = \Lambda^T g \Lambda \leadsto \det g = \det(\Lambda^T g \Lambda) \leadsto (\det \Lambda)^2 = 1. \tag{A.20}$$

The subgroup of transformations with

$$\det \Lambda = 1,\ \Lambda^{0}_{\ 0} \geq 1,$$

which contains the unity $\mathbf{1} = \delta^{\mu}_{\nu}$, is known as the proper group of Lorentz transformations. All other transformations are known as *improper Lorentz transformations*. It is impossible with a continuous change of parameters to change

$$\det \Lambda = 1 \to \det \Lambda = -1 \text{ or } \Lambda^{0}_{\ 0} \geq 1 \to \Lambda^{0}_{\ 0} \leq -1.$$

Improper Lorentz transformations involve either space-reflection ($\det \Lambda = -1, \Lambda^{0}_{\ 0} \geq 1$) or time-inversion ($\det \Lambda = 1, \Lambda^{0}_{\ 0} \leq -1$) or both ($\det \Lambda = -1, \Lambda^{0}_{\ 0} \leq -1$).

Proper homogeneous or inhomogeneous Lorentz transformations have a further subgroup: the group of rotations,

$$\Lambda^{0}_{\ 0} = 1,\ \Lambda^{i}_{\ 0} = \Lambda^{0}_{\ i} = 0,\ \Lambda^{i}_{\ j} = R_{ij}, \tag{A.21}$$

with

$$\det R = 1,\ R^T R = 1. \tag{A.22}$$

Thus, for rotations and translations ($x^{\mu} \to x'^{\mu} = x^{\mu} + \rho^{\mu}$) Lorentz transformations are no different than Galilei transformations.

A difference with Galilei transformations arises in *boosts*. Assume a reference frame O in which a certain particle appears at rest, and O' a reference frame where the particle appears to move with a velocity $\vec{v}$. Space-time intervals in the two frames are related via

$$dx'^{\mu} = \Lambda^{\mu}_{\ \nu} dx^{\nu} = \Lambda^{\mu}_{\ 0} c dt, \tag{A.23}$$

given that $d\vec{x} = 0$ in the frame O. For $\mu = 0$, this equation gives

$$dt' = \Lambda^{0}_{\ 0} dt. \tag{A.24}$$

For $\mu = i = 1, 2, 3$ we have:

$$dx'^i = \Lambda^i{}_0 c dt \tag{A.25}$$

Dividing the two, we have

$$v^i \equiv \frac{dx'^i}{dt'} = c\frac{\Lambda^i{}_0}{\Lambda^0{}_0} \rightsquigarrow \Lambda^i{}_0 = \frac{v^i}{c}\Lambda^0{}_0. \tag{A.26}$$

From

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

and for $\rho = \sigma = 0$, we have:

$$(\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1$$

$$\rightsquigarrow \quad \Lambda^0{}_0 = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}. \tag{A.27}$$

and thus

$$\Lambda^i{}_0 = \gamma\frac{v^i}{c}. \tag{A.28}$$

The remaining components are not determined uniquely by knowing the velocity $\vec{v}$ of the particle. Indeed, two Lorentz transformations

$$\Lambda^\mu{}_\nu \quad \text{and} \quad \Lambda^\mu{}_\rho R^\rho{}_\nu$$

where R is a rotation, boost a particle to the same velocity. For coordinate systems O and O' with parallel axes we find that (**exercise**)

$$\Lambda^i{}_j = \delta^i_j + \frac{v^i v^j}{v^2}(\gamma - 1) \tag{A.29}$$

and

$$\Lambda^0{}_j = \gamma\frac{v^j}{c}. \tag{A.30}$$

A.3 Time dilation

Consider an inertial observer O which looks as a clock at rest. Two ticks of the clock correspond to a space-time interval

$$d\vec{x} = 0, \quad dt = \Delta t. \tag{A.31}$$

The proper time interval is

$$d\tau = (c^2 dt^2 - d\vec{x}^2)^{\frac{1}{2}} = c\Delta t. \tag{A.32}$$

A second observer sees the clock with velocity $\vec{v}$. Two ticks of the clock define a space-time interval

$$dt' = \Delta t', \quad d\vec{x}' = \vec{v}dt'. \tag{A.33}$$

The proper-time interval in the new frame is:

$$d\tau' = (c^2 dt'^2 - d\vec{x}'^2)^{\frac{1}{2}} = c\Delta t'\sqrt{1 - \left|\frac{d\vec{x}'}{cdt'}\right|^2} = c\Delta t'\sqrt{1 - \frac{\vec{v}^2}{c^2}}. \tag{A.34}$$

The proper-time is invariant under the change of inertial reference frames. Thus we conclude that

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = \gamma\Delta t \tag{A.35}$$

A.4 Doppler effect

Take our clock to be a source of light with a frequency

$$\omega = \frac{2\pi}{\Delta t}.$$

For an observer where the light-source is moving with velocity $\vec{v}$ this time interval is measured to be

$$dt' = \gamma \Delta t.$$

In the same period, the distance of the observer from the light source increases by

$$v_r dt'$$

where v_r is the component of the velocity of the light-source along the direction of sight of the observer. The time elapsing between the reception of two successive light wave-fronts from the observer is

$$c dt_0 = c dt' + v_r dt'. \tag{A.36}$$

The frequency measured by the observer is

$$\omega' = \frac{2\pi}{dt_0} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v_r}{c}} \omega. \tag{A.37}$$

If the light-source is moving along the line of sight, $v_r = v$, we have

$$\omega' = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \omega. \tag{A.38}$$

If the light-source moves away from the observer, $v_r > 0$, the frequency decreases and the light appears to be more red (red shift). If the source moves towards the observer, the frequency increases (violet shift).
Exercise: Calculate the angle of the direction of motion of the light-source with respect to the line of sight of the observer for which there is no shift in the frequency.

For an application of the Doppler effect in cosmology, read about Hubble's law.

A.5 Particle dynamics

How can we compute the force of a particle which moves with a relativistic velocity $\vec{v}$? We should expect that our classical formulae from Newtonian mechanics need to be modified. Nevertheless, Newtonian expressions for the force should be valid if a particle is at rest. We can always change reference frame with Lorentz transformations to bring a particle at rest and calculate the change in its velocity for a small time interval using Newtonian mechanics. However, we will need to perform these changes of reference frame at every small increase of the velocity of the particle during its acceleration due to the force.

In a more elegant solution to the problem, we define a relativistic force acting on a particle as

$$f^\mu = mc^2 \frac{d^2 x^\mu}{d\tau^2}, \tag{A.39}$$

where m is the mass of a particle [1]. If the particle is at rest, the proper-time interval $d\tau$ coincides with the common time-interval dt

$$d\tau = cdt.$$

Therefore, in the rest frame of the particle, the"space"-components of the force four-vector become

$$f^i_{\text{rest}} = m\frac{d^2x^i}{dt^2} = F^i_{\text{Newton}}, \quad \text{for } i = 1, 23, \tag{A.40}$$

where $\vec{F}_{\text{Newton}}$ is the force-vector as we know it from Newtonian mechanics. The "time" component of the force four-vector vanishes:

$$f^0_{rest} = mc\frac{d^2t}{dt^2} = 0. \tag{A.41}$$

Under a Lorentz transformation, f^μ transforms as

$$f^\mu = mc^2\frac{d^2x^\mu}{d\tau^2} \to mc^2\frac{d^2x'^\mu}{d\tau^2} = mc^2\frac{d^2\left(\Lambda^\mu{}_\nu x^\nu + \rho^\mu\right)}{d\tau^2} = \Lambda^\mu_{\ \nu}\, mc^2\frac{d^2x^\nu}{d\tau^2} \tag{A.42}$$

Therefore,

$$f'^\mu = \Lambda^\mu{}_\nu f^\nu. \tag{A.43}$$

The components of f^μ transform under Lorentz transformations in exactly the same way as the components of space-time coordinates. It is therefore a four-vector as well.

For a specific transformation from the rest frame of a particle to a frame where the particle moves with a velocity $\vec{v}$, we have

$$f^\mu = \Lambda^\mu_\nu(\vec{v})f^\nu_{\text{rest}}. \tag{A.44}$$

where, we have found that,

$$\Lambda^0{}_0(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda^i{}_0(\vec{v}) = \Lambda^0{}_i(\vec{v}) = \gamma\frac{v^i}{c}.$$
$$\Lambda^i{}_j(\vec{v}) = \delta^i_j + \frac{v^iv^j}{v^2}(\gamma - 1) \tag{A.45}$$

Therefore, the force on a moving particle is:

$$\vec{f} = \vec{F}_{\text{Newton}} + (\gamma - 1)\frac{\vec{v}\left(\vec{F}_{\text{Newton}} \cdot \vec{v}\right)}{v^2}, \tag{A.46}$$

and

$$f^0 = \gamma\frac{\vec{v} \cdot \vec{F}_{\text{Newton}}}{c} = \frac{\vec{v}}{c} \cdot \vec{f}. \tag{A.47}$$

In Newtonian mechanics, if the force $\vec{F}$ is given, we can compute the trajectory $\vec{x}(t)$ by solving the second order differential equation:

$$\frac{d^2\vec{x}}{dt^2} = \frac{\vec{F}(\vec{x}, t)}{m}. \tag{A.48}$$

[1] With mass, we mean the mass of a particle as it is measured in its rest-frame. We will refrain from using the "relativistic", velocity dependent, mass.

Similarly, in special relativity, when the relativistic force f^μ is known, the differential equation A.39 can, in principle, be solved to give the space-time coordinates as a function of the proper time τ:

$$x^\mu = x^\mu(\tau). \tag{A.49}$$

To calculate the trajectory, we then need to calculate the proper-time in terms of the time coordinate by inverting

$$x^0 = x^0(\tau) \rightsquigarrow \tau = \tau(x^0), \tag{A.50}$$

which we can use to cast the space components directly as functions of the time-coordinate.

We should not forget a second constrain that must be satisfied for our solutions $x^\mu(\tau)$, namely

$$\Omega \equiv g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1. \tag{A.51}$$

We have for the derivative of Ω with respect to proper-time:

$$\frac{d\Omega}{d\tau} = 2g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau} = \frac{2}{mc^2}g_{\mu\nu}f^\mu\frac{dx^\nu}{d\tau}. \tag{A.52}$$

The rhs is a Lorentz invariant quantity. In a new frame,

$$g_{\mu\nu}f'^\mu\frac{dx'^\nu}{d\tau} = g_{\mu\nu}\left(\Lambda^\mu{}_\rho f^\rho\right)\frac{(\Lambda^\nu{}_\sigma dx^\sigma)}{d\tau} = \left(g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\right)f^\rho\frac{dx^\sigma}{d\tau} = g_{\rho\sigma}f^\rho\frac{dx^\sigma}{d\tau}.$$

We are therefore allowed to compute $\frac{d\Omega}{d\tau}$ in any reference frame we wish. Let us choose the rest frame of the particle, where

$$x^\mu = (ct, \vec{0}), \quad f^\mu = (0, \vec{F}_{\text{Newton}}).$$

We obtain:

$$\frac{d\Omega}{d\tau} = \frac{2}{mc^2}\left(f^0\frac{dx^0}{d\tau} - \vec{f}\cdot\vec{x}\right) = 0. \tag{A.53}$$

Therefore, the quantity Ω is always a constant:

$$\Omega(\tau) = \text{constant}. \tag{A.54}$$

If for some initial value τ_0 we choose the constant to be one, we will have

$$\Omega(\tau) = \Omega(\tau_0) = 1, \qquad \forall\ \tau. \tag{A.55}$$

Exercise: Calculate the trajectory of a particle on which the four-vector force exerted is $f^\mu = 0$.

A.6 Energy and momentum

We can define a relativistic four-vector analogue of momentum as

$$p^\mu = mc\frac{dx^\mu}{d\tau} \tag{A.56}$$

We have that

$$d\tau = \left(c^2 dt^2 - d\vec{x}^2\right)^{\frac{1}{2}} = cdt\left[1 - \left(\frac{d\vec{x}}{cdt}\right)^2\right]^{\frac{1}{2}} = cdt\left[1 - \frac{\vec{v}^2}{c^2}\right]^{\frac{1}{2}} = \frac{cdt}{\gamma}. \tag{A.57}$$

Thus, for the time-component ($\mu = 0$) of the four-momentum we have

$$p^0 = mc\frac{dx^0}{d\tau} = m\gamma c. \tag{A.58}$$

For the space-components ($\mu = i = 1, 2, 3$) we have

$$p^i = mc\frac{dx^i}{d\tau} = m\gamma\frac{dx^i}{dt} = m\gamma v^i. \tag{A.59}$$

For small velocities, we can expand the factor γ as

$$\gamma = \left[1 - \frac{v^2}{c^2}\right]^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right). \tag{A.60}$$

Therefore, for small velocities the space-components of the four-momentum become the classical momentum,

$$p^i \approx mv^i + \ldots, \tag{A.61}$$

while the time-component becomes

$$p^0 \approx mc + \frac{1}{2c}mv^2 + \ldots \tag{A.62}$$

In the second term of the above expansion we recognize the kinetic energy $\frac{1}{2}mv^2$ of the particle. We then identify the relativistic energy of a particle with

$$E = cp^0 = m\gamma c^2. \tag{A.63}$$

Eliminating the velocity $\vec{v}$ from Eqs A.59-A.63, we obtain the relation:

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4} \tag{A.64}$$

A.7 The inverse of a Lorentz transformation

Recall the metric matrix

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{A.65}$$

We define an inverse

$$g^{\mu\nu}: \quad g^{\mu\nu}g_{\nu\rho} = \delta^\mu_{\ \nu}, \tag{A.66}$$

where $\delta^\mu_{\ \nu}$ is the Kronecker delta. It is easy to verify that the inverse of the metric is the metric itself:

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1). \tag{A.67}$$

Now consider a Lorentz transformation $\Lambda^\mu_{\ \nu}$, which satisfies the identity:

$$\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}g_{\mu\nu} = g_{\rho\sigma}. \tag{A.68}$$

We can prove that the matrix

$$\Lambda_\mu^{\ \nu} \equiv g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho_{\ \sigma} \tag{A.69}$$

is the inverse of $\Lambda^\mu_{\ \nu}$. Indeed

$$\Lambda^\mu_{\ \lambda} \Lambda_\mu^{\ \nu} = g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho_{\ \sigma} \Lambda^\mu_{\ \lambda} = g_{\sigma\lambda} g^{\nu\sigma} = \delta^\lambda_{\ \nu}. \tag{A.70}$$

If $\Lambda^\mu_{\ \nu}$ is a velocity $\vec{v}$ boost transformation of Eq. A.45, then

$$\Lambda_0^{\ 0}(\vec{v}) = \gamma = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \Lambda_i^{\ 0}(\vec{v}) = \Lambda_0^{\ i}(\vec{v}) = -\gamma \frac{v^i}{c}.$$

$$\Lambda_i^{\ j}(\vec{v}) = \delta_i^j + \frac{v^i v^j}{v^2}(\gamma - 1) \tag{A.71}$$

We, therefore have that the inverse of a boost is

$$\Lambda_\mu^{\ \nu}(\vec{v}) = \Lambda^\mu_{\ \nu}(-\vec{v}), \tag{A.72}$$

as we also expect physically.

A.8 Vectors and Tensors

It is now time to give officially a definition for vectors in special relativity. We call any set of four components which transform according to the rule:

$$V^\mu \to V'^\mu = \Lambda^\mu_{\ \nu} V^\nu \tag{A.73}$$

a *contravariant* vector. Contravariant vectors transform in the same way as space-time coordinates x^μ do under homogeneous Lorentz transformations.

Not all vectors transform as contravariant vectors. Consider the derivative $\frac{\partial}{\partial x^\mu}$. Under a Lorentz transformation, it transforms as:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho}. \tag{A.74}$$

We have that

$$\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \delta^\rho_{\ \nu} \rightsquigarrow \left(\frac{\partial x^\rho}{\partial x'^\mu}\right) \Lambda^\mu_{\ \nu} = \delta^\rho_{\ \nu}. \tag{A.75}$$

Therefore, $\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$ is the inverse of a Lorentz transformation $\Lambda^\mu_{\ \nu}$:

$$\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu^{\ \nu}. \tag{A.76}$$

Substituting into Eq. A.74, we find:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \Lambda_\mu^{\ \rho} \frac{\partial}{\partial x^\rho}. \tag{A.77}$$

We found that the derivative does not transform according to the Lorentz transformation but according to its inverse. All vectors which transform with the inverse Lorentz transformation:

$$U_\mu = \Lambda_\mu^{\ \nu} U_\nu, \tag{A.78}$$

are called covariant vectors.

For every contravariant vector U^μ there is a dual vector

$$U_\mu = g_{\mu\nu} U^\nu. \tag{A.79}$$

We can invert the above equation multiplying with $g^{\rho\mu}$,

$$g^{\rho\mu} U_\mu = g^{\rho\mu} g_{\mu\nu} U^\nu = \delta^\rho_{\ \nu} U^\nu = U^\rho. \tag{A.80}$$

The dual vector U_μ is a covariant vector. Indeed, under a Lorentz transformation we have

$$U_\mu \to U'_\mu = g_{\mu\nu} U'^\nu = g_{\mu\nu} \Lambda^\nu_{\ \rho} U^\rho = g_{\mu\nu} \Lambda^\nu_{\ \rho} g^{\rho\sigma} U_\sigma = \Lambda_\mu^{\ \sigma} U_\sigma. \tag{A.81}$$

The scalar product of a contravariant and a covariant vector

$$A \cdot B \equiv A^\mu B_\mu = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu \tag{A.82}$$

is invariant under Lorentz transformations. Indeed,

$$A \cdot B \to A' \cdot B' = A'^\mu B'_\mu = \Lambda^\mu_{\ \rho} A^\rho \Lambda_\mu^{\ \sigma} B_\sigma = \delta^\sigma_{\ \rho} A^\rho B_\sigma = A_\rho B_\rho = A \cdot B. \tag{A.83}$$

Let us define for short:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{A.84}$$

and the dual contravariant vector:

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu. \tag{A.85}$$

The D' Alembert operator is the scalar product:

$$\Box \equiv \partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{A.86}$$

Due to it being a scalar product, the D'Alembert operator is invariant under Lorentz transformations.

Finally, we define a tensor with multiple "up" and/or "down" indices to be an object

$$T^{\mu_1 \mu_2 \ldots}_{\nu_1 \nu_2 \ldots} \tag{A.87}$$

which transfroms as:

$$T^{\mu_1 \mu_2 \ldots}_{\nu_1 \nu_2 \ldots} \to \Lambda^{\mu_1}_{\ \rho_1} \Lambda^{\mu_2}_{\ \rho_2} \ldots \Lambda_{\nu_1}^{\ \sigma_1} \Lambda_{\nu_2}^{\ \sigma_2} \ldots T^{\rho_1 \rho_2 \ldots}_{\sigma_1 \sigma_2 \ldots} \tag{A.88}$$

A.9 Currents and densities

Consider a set of particles $\{n\}$ with charged q_n at positions $\vec{r}_n(t)$. The charge and current density are

$$\rho(\vec{x}, t) = \sum_n q_n \delta(\vec{x} - \vec{r}_n(t)), \tag{A.89}$$

$$\vec{j}(\vec{x}, t) = \sum_n q_n \frac{d\vec{r}_n(t)}{dt} \delta(\vec{x} - \vec{r}_n(t)) = \sum_n q_n \frac{d\vec{x}}{dt} \delta(\vec{x} - \vec{r}_n(t)), \tag{A.90}$$

Recall now that for the vector $x^\mu = (ct, \vec{x})$, we have

$$\frac{dx^\mu}{dt} = \frac{d}{dt}(ct, \vec{x}) = \left(c, \frac{d\vec{x}}{dt}\right). \tag{A.91}$$

We can then combine the charge and current densities into one object:

$$j^\mu \equiv \left(c\rho, \vec{j}\right) \tag{A.92}$$

with

$$j^\mu(\vec{x}, t) = \sum_n q_n \frac{dx^\mu}{dt} \delta(\vec{x} - \vec{r}_n(t)), \tag{A.93}$$

We can now cast j^μ in a form which manifestly shows that it is a contravariant four-vector. First, we define a delta function in four dimensions as

$$\delta\left(x^\mu - y^\mu\right) = \delta(x^0 - y^0)\delta(\vec{x} - \vec{y}) = \frac{1}{c}\delta(t_x - t_y)\delta(\vec{x} - \vec{y}). \tag{A.94}$$

Notice that the $\delta-$function of a four-vector is a scalar. Under a Lorentz transformation,

$$\delta\left(U^\mu\right) \to \delta\left(U'^\mu\right) = \delta\left(\Lambda^\mu_{\ \nu} U^\nu\right) = \frac{\delta(U^\nu)}{|\det \Lambda|} = \delta(U^\nu). \tag{A.95}$$

With the use of the delta-function, we can write the current-density four-vector of Eq. A.93 as an integral

$$j^\mu(\vec{x}, t) = \sum_n q_n \int dt' \frac{dx^\mu}{dt'} \delta(\vec{x} - \vec{r}_n(t))\delta(t' - t) = c \sum_n q_n \int dt' \frac{dx^\mu}{dt'} \delta(x^\mu - r_n^\mu(t)), \tag{A.96}$$

where

$$x^\mu \equiv (ct', \vec{x}), \quad r_n^\mu(t) \equiv (ct, \vec{r}_n(t)).$$

Now, we change integration variables from $t' \to \tau$. Recall that

$$dt' = \frac{d\tau}{c}\gamma. \tag{A.97}$$

Then, we have

$$j^\mu(\vec{x}, t) = c \sum_n q_n \int d\tau \frac{dx^\mu}{d\tau} \delta(x^\mu - r_n^\mu(\tau)), \tag{A.98}$$

which shows manifestly that j^μ transforms as $\frac{dx^\mu}{d\tau}$ and it is therefore a contravariant four-vector.

We have already shown that charge-conservation implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{A.99}$$

In relativistic notation, the continuity equation takes the more elegant form:

$$\partial_\mu j^\mu = 0 \tag{A.100}$$

A.10 Energy-Momentum tensor

Consider a collection of particles $\{n\}$ at positions $\vec{r}_n(t)$. The energy density is:

$$\text{energy density} = \sum_n E_n(t)\delta(\vec{x} - \vec{r}_n(t)) \tag{A.101}$$

Changes in the energy density result to a "energy-current density":

$$\text{energy current density} = \sum_n E_n(t)\frac{d\vec{r}_n}{dt}\delta(\vec{x} - \vec{r}_n(t)). \tag{A.102}$$

As with the charge density and its current-density of the last section, we can combine the two together into a single object:

$$\sum_n E_n(t)\frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)). \tag{A.103}$$

Similarly to the energy, the density/current-density for the components of the three-dimensional momentum are:

$$\sum_n p_n^i(t)\frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)), \quad i = 1, 2, 3. \tag{A.104}$$

Collectively, we can form the so called "energy-momentum tensor" which encompasses the density and current-density for all components of the four-momentum:

$$T^{\mu\nu} \equiv \sum_n p_n^\mu \frac{dr_n^\nu}{dt}\delta(\vec{x} - \vec{r}_n(t)) \tag{A.105}$$

or, in the equivalent form:

$$T^{\mu\nu} = \sum_n \int d\tau p_n^\mu \frac{dx_n^\nu}{d\tau}\delta(x^\rho - r_n^\rho(\tau)). \tag{A.106}$$

From the last equation, we can see manifestly that this object is justifiably called a tensor since it transforms as the product of two four-vectors under Lorentz transformations:

$$T'^{\mu\nu} = \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}T^{\rho\sigma} \tag{A.107}$$

The energy momentum tensor is symmetric:

$$T^{\mu\nu} = T^{\nu\mu}. \tag{A.108}$$

To verify this, we recall that

$$p_n^\nu = m_n\frac{dr_n^\nu}{d\tau} = m_n\gamma\frac{dr_n^\nu}{dt} = E_n\frac{dr_n^\nu}{dt}. \tag{A.109}$$

The energy momentum tensor takes then the explicitly symmetric form:

$$T^{\mu\nu} \equiv \sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta(\vec{x} - \vec{r}_n(t)). \tag{A.110}$$

For the charge-density four-vector, we have found a continuity identity

$$\partial_\mu j^\mu = 0.$$

This was a consequence of the conservation of charge. If the total energy and momentum of the system of particles is conserved, we anticipate a similar continuity identity for the energy-momentum tensor:

$$\partial_\nu T^{\mu\nu} = 0.$$

We have:

$$\begin{aligned}
\partial_i T^{\mu i} &= \sum_n p_n^\mu \frac{dr_n^i}{dt} \frac{\partial}{\partial x^i} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\sum_n p_n^\mu \frac{dr_n^i}{dt} \frac{\partial}{\partial r_n^i} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\sum_n p_n^\mu \frac{\partial}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\frac{\partial}{\partial t} \sum_n p_n^\mu \delta\left(\vec{x} - \vec{r}_n\right) + \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= -\frac{\partial}{\partial t} T^{00} + \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right). \qquad \text{(A.111)}
\end{aligned}$$

In the above we have used that

$$\begin{aligned}
\partial_x \delta(x-y) &= 2\pi \partial_x \int_{-\infty}^{\infty} dw e^{-iw(x-y)} = -2i\pi \int_{-\infty}^{\infty} dw\, w e^{-iw(x-y)} \\
&= -2\pi \partial_y \int_{-\infty}^{\infty} dw e^{-iw(x-y)} = -\partial_x \delta(x-y). \qquad \text{(A.112)}
\end{aligned}$$

We have therefore arrived to the equation

$$\partial_\nu T^{\mu\nu} = G^\mu, \qquad \text{(A.113)}$$

where

$$G^\mu = \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) = \sum_n \frac{\partial \tau}{\partial t} f_n^\mu(t) \delta\left(\vec{x} - \vec{r}_n\right) \qquad \text{(A.114)}$$

is the “density of force”.

For free particles, where the energy and momentum of all particles separately is conserved $p_n^\mu = constant$, the energy-momentum tensor satisfies the continuity equation:

$$\partial_\nu T^{\mu\nu} = 0. \qquad \text{(A.115)}$$

The energy momentum tensor is also conserved if the particles interact only at the points where they collide with each other. In that case, the force density is

$$\begin{aligned}
G^\mu &= \sum_n \frac{\partial p_n^\mu}{\partial t} \delta\left(\vec{x} - \vec{r}_n\right) \\
&= \sum_{coll.} \delta(\vec{x} - \vec{x}_{coll}(t)) \frac{d}{dt} \sum_{n \in coll.} p_n^\mu(t). \qquad \text{(A.116)}
\end{aligned}$$

We have grouped the sum over all particles contributing to the force density according to the collision points that they meet at. In each collision point, the sum of the momenta of the colliding particles is conserved

$$\frac{d}{dt}\sum_{n\in coll.} p_n^\mu(t) = 0 \rightsquigarrow \partial_\nu T^{\mu\nu} = 0. \tag{A.117}$$

If the continuity equation is satisfied, then

$$\begin{aligned} 0 &= \partial_\nu T^{\mu\nu} \\ \rightsquigarrow 0 &= \partial_0 T^{\mu 0} + \partial_i T^{\mu i} \\ \rightsquigarrow 0 &= \partial_0 \int d^3\vec{x} T^{\mu 0} + \int d^3\vec{x}\partial_i T^{\mu i} \\ 0 &= \partial_0 \int d^3\vec{x} T^{\mu 0} \end{aligned} \tag{A.118}$$

which means that the vector

$$P^\mu \equiv \int d^3\vec{x} T^{\mu 0} = \text{constant} \tag{A.119}$$

is conserved. We find that the conserved vector is the sum of all the total four-momentum of the particles in the distribution:

$$P^\mu = \sum_n \int d^3\vec{x} p_n^\mu \delta(\vec{x} - \vec{r}_n(t)) = \sum_n p_n^\mu. \tag{A.120}$$

A.11 Relativistic formulation of Electrodynamics

From now on we will set $\epsilon_0 = c = 1$. It will be easy to restore the full dependence on these constants with dimensional analysis when necessary. Maxwell equations are

$$\vec{\nabla}\cdot\vec{E} = \rho \tag{A.121}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \tag{A.122}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{A.123}$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j} \tag{A.124}$$

(A.125)

We construct an "electromagnetic field tensor" $F^{\mu\nu}$ from the components of the electric $\vec{E} \equiv (E^1, E^2, E^3)$ and magnetic field $\vec{B} \equiv (B^1, B^2, B^3)$ as:

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon_{ijk} B^k, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{A.126}$$

(we use $\epsilon_{123} = +1$)
Explicitly,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{A.127}$$

Conversely, If the tensor $F^{\mu\nu}$ is given, we can obtain the magnetic field via (**exercise**):

$$B^i = -\frac{1}{2}\epsilon_{ijk}F^{jk}. \tag{A.128}$$

Exercise: What is the covariant tensor $F_{\mu\nu}$ in matrix form?
Exercise: Write the matrix:

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{A.129}$$

Exercise: Prove that:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{\vec{E}^2 - \vec{B}^2}{2}, \quad \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 8\vec{E}\cdot\vec{B}. \tag{A.130}$$

As we have already noted, the charge and current densities form a four-vector:

$$j^\mu = (\rho, \vec{j}).$$

We can observe that two of Maxwell equations can be combined into one:

$$\partial_\nu F^{\nu\mu} = j^\mu. \tag{A.131}$$

Indeed, for $\mu = 0$, we have

$$\partial_0 F^{00} + \partial_i F^{i0} = j^0 \rightsquigarrow \vec{\nabla}\cdot\vec{E} = \rho.$$

For $\mu = i = 1, 2, 3$ we have

$$\partial_0 F^{0i} + \partial_j F^{ji} = j^i \rightsquigarrow -\partial_0 E^i + \epsilon_{ijk}\partial_j B^k = j^i \rightsquigarrow \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}.$$

The remaining two Maxwell equations tell us, as we have found earlier, that we can derive the electric and magnetic fields by means of the scalar and vector potential $\phi, \vec{A}$, via

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{A.132}$$

We can combine the scalar and vector potential into a single four-vector:

$$A^\mu \equiv (\phi, \vec{A}) = (\phi, A^1, A^2, A^3). \tag{A.133}$$

Then, the above equations take the elegant form:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{A.134}$$

Indeed, for $\mu = 0, \nu = i = 1, 2, 3$ we have

$$F^{0i} = \partial^0 A^i - \partial^i A^0 \rightsquigarrow -\vec{E} = \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\phi$$

For $\mu = i, \nu = j,\ i, j = 1, 2, 3$ we have

$$F^{ij} = \partial^i A^j - \partial^j A^i \rightsquigarrow -\frac{1}{2}\epsilon_{ijk}F^{ij} = \epsilon_{ijk}\partial_i A^j \rightsquigarrow \vec{B} = \vec{\nabla} \times \vec{A}.$$

It is now straightforward to prove the following identities (**exercise**):

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \tag{A.135}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{A.136}$$

Substituting Eq. A.134 into Eq. A.131 we find

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu. \tag{A.137}$$

Explicitly, for $\nu = 0$ and $\nu = i = 1, 2, 3$ we recover the known differential equations for the scalar and vector potentials respectively:

$$\Box\phi - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial\phi}{\partial t}\right) = \rho, \tag{A.138}$$

$$\Box\vec{A} + \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{\partial\phi}{\partial t}\right] = \vec{J}. \tag{A.139}$$

The property of gauge invariance becomes more elegant as well in relativistic notation. The gauge transformations of the vector and scalar potentials which leave Maxwell equations invariant are written as:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi, \tag{A.140}$$

where χ is a scalar function. Indeed, under a gauge transformation the electromagnetic field tensor remains invariant:

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\chi) - \partial_\nu(A_\mu + \partial_\mu\chi) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{A.141}$$

The Lorentz gauge-fixing condition

$$\vec{\nabla} \cdot \vec{A} + \frac{\partial\phi}{\partial t} = 0 \tag{A.142}$$

can be written simply as

$$\partial_\mu A^\mu = 0. \tag{A.143}$$

In the Lorentz gauge, Maxwell equations for the four-vector potential become:

$$\partial^2 A^\mu = j^\mu. \tag{A.144}$$

Earlier, we have found that the vector and scalar potential can be computed via

$$\phi(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{r}' dt' G\left(\vec{r} - \vec{r}'; t - t'\right)\rho(\vec{r}', t') \tag{A.145}$$

and

$$\vec{A}(\vec{r}, t) = \int_{-\infty}^{\infty} d^3\vec{r}' dt' G\left(\vec{r} - \vec{r}'; t - t'\right)\vec{j}(\vec{r}', t') \tag{A.146}$$

where the Green's function can be written in the form:

$$G(\Delta\vec{r}, \Delta t) = \frac{1}{2\pi}\delta\left(\Delta t^2 - \Delta\vec{r}^2\right)\Theta(\Delta t > 0). \tag{A.147}$$

In relativistic notation, the solutions for the four-vector potential take the form:

$$A^\nu(x^\mu) = \frac{1}{2\pi}\int d^4x' j^\nu(x'^\mu)\delta\left((x'^\mu - x^\mu)^2\right)\Theta\left(x^0 > x'^0\right). \tag{A.148}$$

The electromagnetic force acting on a particle with a charge q is:

$$f^\mu = qF^{\mu\nu}\frac{dx_\nu}{d\tau}. \tag{A.149}$$

Indeed, in the rest frame of the particle $d\tau = dt, d\vec{x} = 0$, we get

$$f^\mu_{\text{rest}} = qF^{\mu 0} \rightsquigarrow f^0_{\text{rest}} = 0,\ \vec{f}_{\text{rest}} = q\vec{E}. \tag{A.150}$$

Since our expression for the electromagnetic force is correct in one reference frame, it should hold in every inertial reference frame due to it being written as an equation of four-vectors. In a frame where the charge is moving with a velocity $\vec{v}$, the three-dimensional force is:

$$\begin{aligned} f^i &= qF^{i0}\frac{dx^0}{d\tau} - qF^{ij}\frac{dx^j}{d\tau} = q\gamma\left(E^i + \epsilon_{ijk}B^k v^j\right) \\ \rightsquigarrow \vec{f} &= q\gamma\left(\vec{E} + \vec{v}\times\vec{B}\right). \end{aligned} \tag{A.151}$$

Recall that

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt}. \tag{A.152}$$

We therefore have recovered our familiar expression for Lorentz' force:

$$\frac{d\vec{p}}{dt} = \left(\vec{E} + \vec{v}\times\vec{B}\right). \tag{A.153}$$

A.11.1 Energy-Momentum Tensor in the presence of an electromagnetic field

Consider a number of charges q_n which interact via the electromagnetic field. The energy-momentum tensor is not conserved:

$$\partial_\nu T^{\mu\nu} = G^\mu \tag{A.154}$$

where the force density is given by

$$\begin{aligned} G^\mu &= \sum_n \frac{\partial\tau}{\partial t} f^\mu_n(t)\delta\left(\vec{x} - \vec{r}_n\right) \\ &= \sum_n \frac{\partial\tau}{\partial t} q_n F^\mu{}_\nu \frac{dr^\nu_n}{d\tau}\delta\left(\vec{x} - \vec{r}_n\right) \\ &= F^\mu{}_\nu \sum_n q_n \frac{dr^\nu_n}{dt}\delta\left(\vec{x} - \vec{r}_n\right) \\ &= F^{\mu\nu} j_\nu. \end{aligned} \tag{A.155}$$

Consider the tensor

$$T_{em}^{\mu\nu} \equiv F_{\rho}^{\mu} F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{A.156}$$

This tensor is explicitly

- symmetric
- gauge-invariant.

The components of the tensor are (**exercise**):

$$T_{em}^{00} = \frac{\vec{E}^2 + \vec{B}^2}{2}, \quad T_{em}^{0i} = T_{em}^{i0} = (\vec{E} \times \vec{B})_i \tag{A.157}$$

Exercise: Find the remaining components.

We find

$$\begin{aligned}
\partial_\nu T_{em}^{\mu\nu} &= \partial_\nu \left\{ F^{\mu\rho} F_{\rho}^{\ \nu} + \frac{g^{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma} \right\} \\
&= F^{\mu\rho} \partial_\nu F_{\rho}^{\ \nu} + (\partial^\nu F^{\mu\rho}) F_{\rho\nu} + \frac{1}{2} F_{\rho\sigma} \partial^\mu F^{\rho\sigma} \\
&= -F^{\mu\rho} j_\rho + \frac{1}{2} (\partial^\nu F^{\mu\rho} - \partial^\rho F^{\mu\nu}) F_{\rho\nu} + \frac{1}{2} F_{\rho\sigma} \partial^\mu F^{\rho\sigma} \\
&= -F^{\mu\rho} j_\rho + \frac{1}{2} (\partial^\sigma F^{\mu\rho} + \partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\mu}) F_{\rho\sigma}
\end{aligned} \tag{A.158}$$

which, due to Eq. A.136, yields:

$$\partial_\nu T_{em}^{\mu\nu} = -F^{\mu\nu} j_\nu. \tag{A.159}$$

$T_{em}^{\mu\nu}$ is purely electromagnetic. While neither $T^{\mu\nu}$ nor $T_{em}^{\mu\nu}$ satisfy a continuity equation, but their sum

$$\Theta^{\mu\nu} \equiv T^{\mu\nu} + T_{em}^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\nu}{E_n} \delta(\vec{x} - \vec{r}_n(t)) + F^{\mu\rho} F_{\rho}^{\ \nu} + \frac{g^{\mu\nu}}{4} F_{\rho\sigma} F^{\rho\sigma}. \tag{A.160}$$

does:

$$\partial_\nu \Theta^{\mu\nu} = 0. \tag{A.161}$$

From the continuity equation, we obtain that there is a conserved four-vector:

$$\partial_t P^\mu \equiv \partial_t \int d^3\vec{x} \Theta^{\mu 0} = -\int d^3\vec{x} \partial_i \Theta^{0i} = -\left. \Theta^{0i} \right|_\infty = 0. \tag{A.162}$$

The conserved vector is:

$$P^\mu = \int d^3\vec{x} \Theta^{\mu 0} = \sum_n p_n^\mu + \int d^3\vec{x} T_{em}^{\mu 0}. \tag{A.163}$$

The four-momentum of the charges

$$P_{\text{charges}}^\mu = \sum_n p_n^\mu \tag{A.164}$$

is not conserved on its own. Some momentum

$$P^{\mu}_{em} = \int d^3\vec{x} T^{\mu 0} \tag{A.165}$$

is also carried by the electromagnetic field itself. This is not conserved either. Momentum can be exchanged between the charges and the field, however this is done in such a way so that the total momentum is always the same:

$$P^{\mu} = P^{\mu}_{\text{charges}} + P^{\mu}_{em} = \text{constant}. \tag{A.166}$$

The time-component of the four-vector is the total energy. The energy stored in the electromagnetic field is:

$$E_{em} = \int d^3\vec{x} T^{00}_{em}. \tag{A.167}$$

Therefore, the energy density w of the electromagnetic field is:

$$w = T^{00}_{em} = \frac{\vec{E}^2 + \vec{B}^2}{2}, \quad \text{EM field energy density.} \tag{A.168}$$

Similarly, we find that the three-momentum density $\vec{S}$ of the electromagnetic field is:

$$\vec{S} = T^{0i}_{em} = T^{i0}_{em} = (\vec{E} \times \vec{B})_i \text{ EM field momentum density.} \tag{A.169}$$

The vector

$$\vec{S} \equiv \vec{E} \times \vec{B}, \tag{A.170}$$

is known as the *Poynting vector*.

Setting $\mu = 0$ in Eq. A.159 we have

$$\partial_0 T^{00} + \partial_i T^{i0} = -F^{0i} j_i, \tag{A.171}$$

and equivalently,

$$\frac{\partial w}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{E} \cdot \vec{j}. \tag{A.172}$$