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Quantum Field Theory I examination - Formulae Sheet

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- **Lorentz transformations**

  Lorentz transformations are defined by

$$\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}g_{\rho\sigma} = g_{\mu\nu}\,. \tag{1}$$

- **Quantum Lorentz transformations**

  Scalar fields ϕ transform under Lorentz transformations Λ as

$$U(\Lambda)\phi(x)U^{\dagger}(\Lambda) = \phi(\Lambda x)\,. \tag{2}$$

- **Noether's theorem**

  For a Lagrangian consisting of generic fields $\phi^{(i)}$, Noether's theorem reads:

  (i) for an *internal* symmetry, the associated conserved current J^{μ} is given by

$$J^{\mu} = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{(i)})}\delta\phi^{(i)} - X^{\mu}\,, \tag{3}$$

  (ii) for a *space-time* symmetry, the associated conserved current J^{μ} is given by

$$J^{\mu} = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{(i)})}\delta_*\phi^{(i)} + \mathcal{L}\delta x^{\mu}\,. \tag{4}$$

- **Real scalar field**

  The canonical commutation relations for a real Klein Gordon field ϕ read

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{5}$$
$$[\phi(\vec{x},t),\phi(\vec{y},t)] = 0 = [\pi(\vec{x},t),\pi(\vec{y},t)]\,. \tag{6}$$

  The field ϕ can be written as a superposition of plane wave states:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}\left(a(\vec{p})e^{-ip\cdot x} + a^{\dagger}(\vec{p})e^{ip\cdot x}\right)\,. \tag{7}$$

  Similarly,

$$[a(p),a^{\dagger}(k)] = (2\pi)^3(2\omega_p)\delta^{(3)}(\vec{p}-\vec{k})\,, \tag{8}$$

  where $a^{\dagger}$ and a are the creation and annihilation operators, respectively.

- **Energy-momentum tensor**

  For a Lagrangian consisting of generic fields $\phi^{(i)}$, the energy-momentum tensor is

$$T^{\mu\nu} = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^{(i)})}(\partial^{\mu}\phi^{(i)}) - \mathcal{L}g^{\mu\nu}\,. \tag{9}$$

  For energy-momentum conserving systems, $T^{\mu\nu}$ satisfies

$$\partial_\nu T^{\mu\nu} = 0\,. \tag{10}$$

- **Pauli matrices**

  The 2×2 Pauli matrices read

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

  which can collectively be denoted by $\sigma^\mu = (\sigma^0, \sigma^i)$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$.

- **Gamma matrices in $D = 4$ dimensions**

  (i) Dirac basis:

$$\gamma_D^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma_D^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad \gamma_D^5 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad j = 1, 2, 3, \tag{12}$$

$$\gamma_D^0 \equiv \sigma^0 \otimes \sigma^3, \qquad \gamma_D^j \equiv \sigma^j \otimes i\sigma^2, \quad j = 1, 2, 3. \tag{13}$$

  (ii) Weyl basis:

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_W^5 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}. \tag{14}$$

  The "fifth" gamma matrix γ^5 is defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{15}$$

- **Spinor representation of the Lorentz group**

  The Lorentz generators $S^{\mu\nu}$ in the spinor representation are given by

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{16}$$

  The gamma matrices satisfy the following similarity transformation,

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu, \tag{17}$$

  where

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{18}$$

- **Dirac equation**

  The four-component Dirac spinors u_s and v_s satisfy the following equations of motion:

$$(\not{p} - m)u_s(p) = 0, \tag{19}$$

$$(\not{p} + m)v_s(p) = 0. \tag{20}$$

- **Spinors**

  The solutions to the Dirac equations in a generic frame for the Dirac spinors u_s and v_s are given by

$$u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}, \tag{21}$$

  and

$$v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}. \tag{22}$$

- **Spin summation relations**

  The completeness relations for the spinors u_s and v_s read:

$$\sum_s u_s^a(p)\bar{u}_s^b(p) = (\not{p} + m\mathbb{1})^{ab}\,, \tag{23}$$

$$\sum_s v_s^a(p)\bar{v}_s^b(p) = (\not{p} - m\mathbb{1})^{ab}\,. \tag{24}$$

- **Polarisation summation relation**

  The completeness relation for the polarisation vectors $\epsilon_\mu^{(\lambda)}$ with polarisation index λ are given by

$$\sum_\lambda \epsilon_\mu^{(\lambda)}(\vec{p})\epsilon_\nu^{*\,(\lambda)}(\vec{p}) = -g^{\mu\nu}\,. \tag{25}$$

- **Perturbation theory**

  The n-point Green's function in an interacting theory consisting of generic fields $\phi^{(i)}$, $i = 1, \ldots, n$ is given by

$$\langle\Omega|T\Big[\phi^{(1)}(x_1)\phi^{(2)}(x_2)\cdots\phi^{(n)}(x_n)\Big]|\Omega\rangle = \lim_{\tau=\infty(1-i\epsilon)} \frac{\langle 0|T\left[\phi_I^{(1)}(x_1)\phi_I^{(2)}(x_2)\cdots\phi_I^{(n)}(x_n)\exp\Big(-i\int_{-\tau}^{\tau} V_I(t)dt\Big)\right]|0\rangle}{\langle 0|T\left[\exp\Big(-i\int_{-\tau}^{\tau} V_I(t)dt\Big)\right]|0\rangle}\,. \tag{26}$$

  In the above, T denotes the time ordering operator, V_I is the interaction Hamiltonian, and $|\Omega\rangle$ is the vacuum in the interacting theory.