Numerical methods for perturbative computations

In QCD and in the EFT of Large Scale Structure

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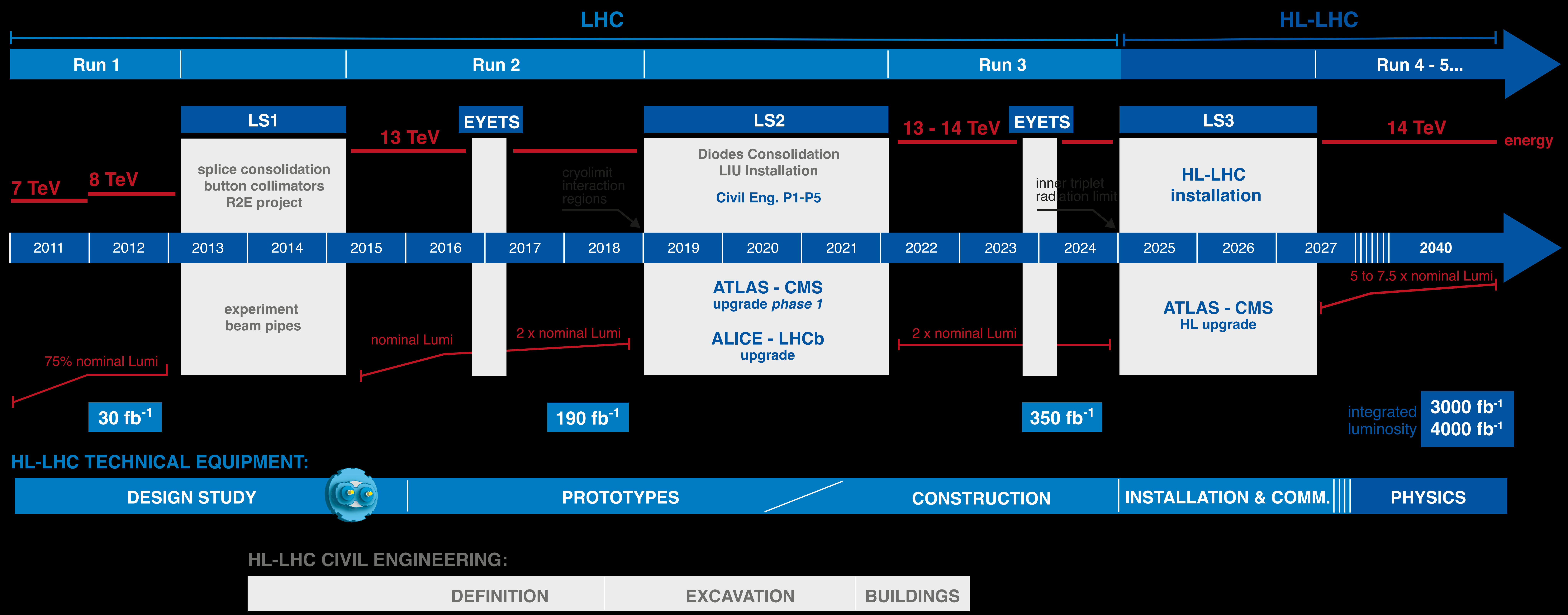
CA, Julia Karlen, George Sterman, Ani Venkata arXiv:2403.13712,
CA, George Sterman, *JHEP* 05 (2023) 242
CA, R. Haindl, G. Sterman, Z. Yang, M. Zeng *JHEP 04 (2021) 222*
CA, G. Sterman *JHEP 07 (2019) 056*

CA, Braganca, Senatore, Zheng,
JHEP 01 (2024) 002
CA, Favorito, Senatore, Zheng,
in progress

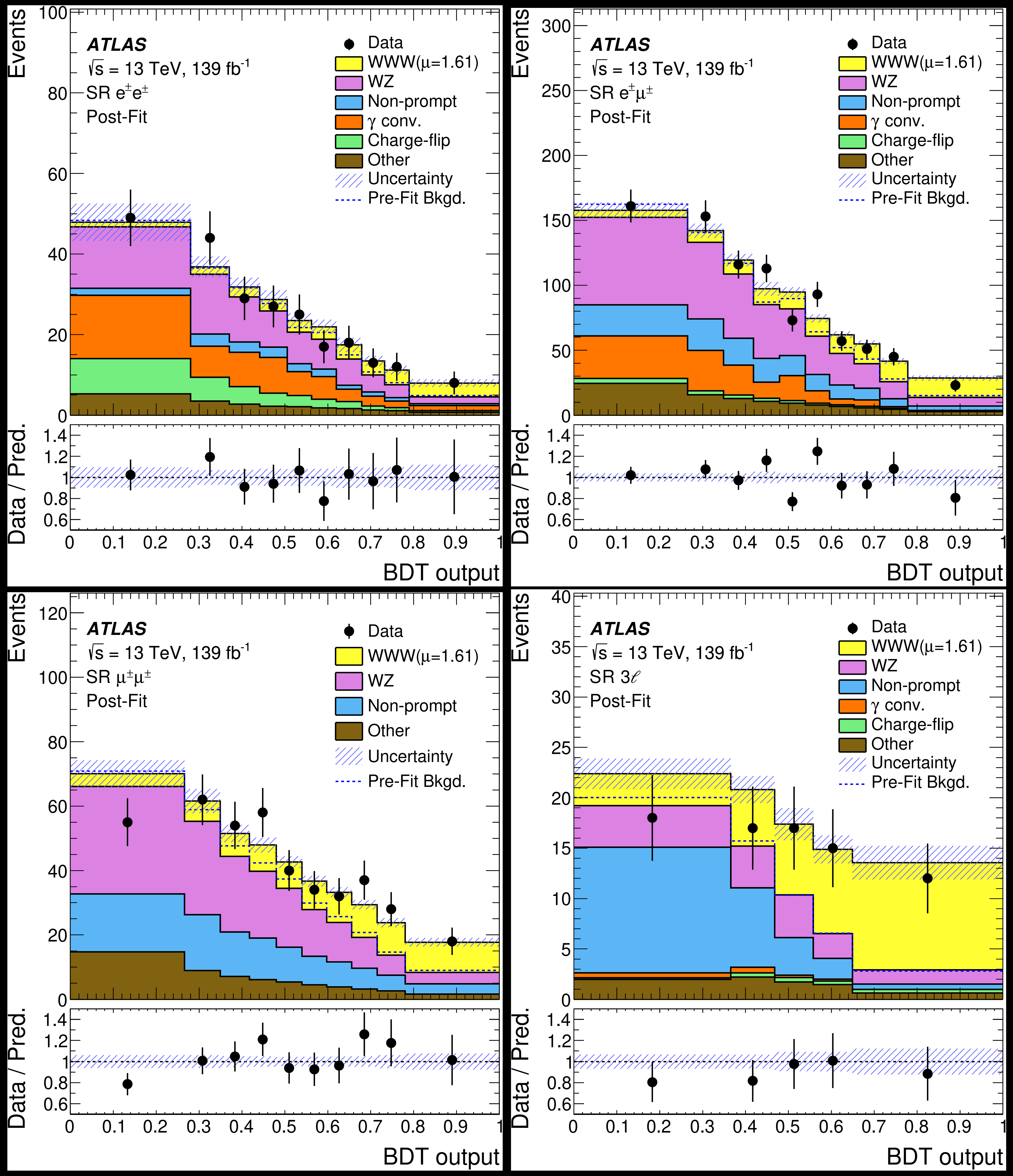
Oxford
January 23, 2025

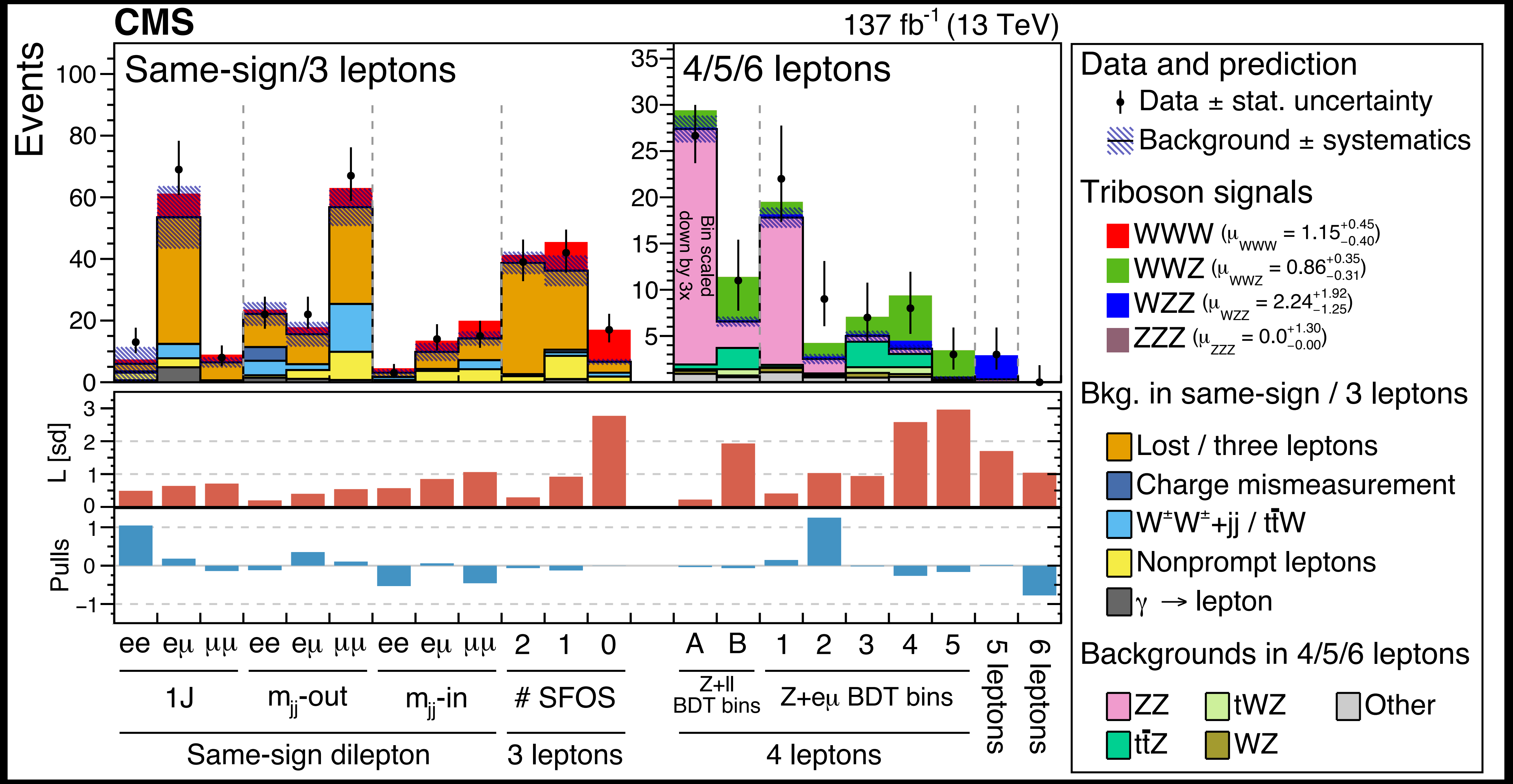
LHC / HL-LHC Plan

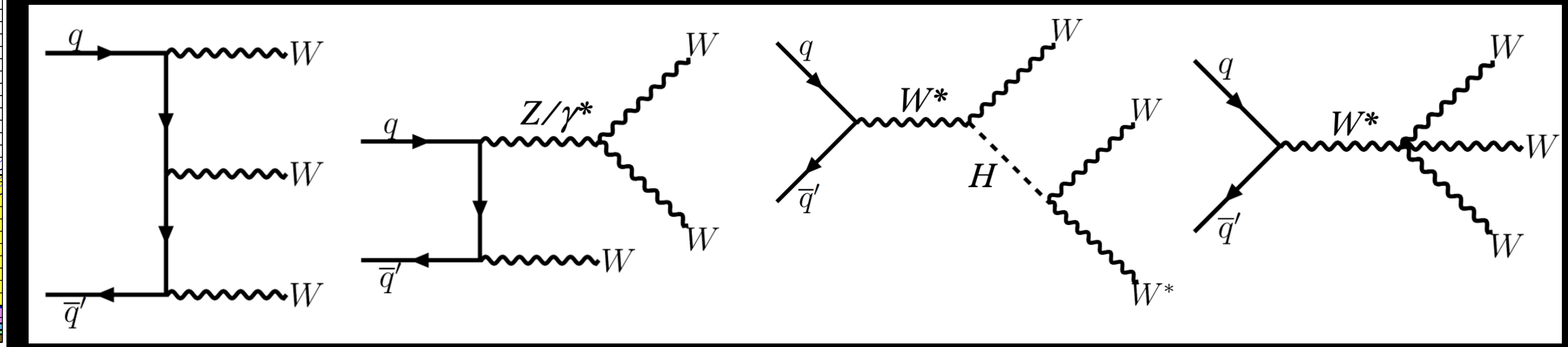


A vivid programme of precision collider physics for the next two decades.

“Rare” $2 \to 3$ processes at the LHC



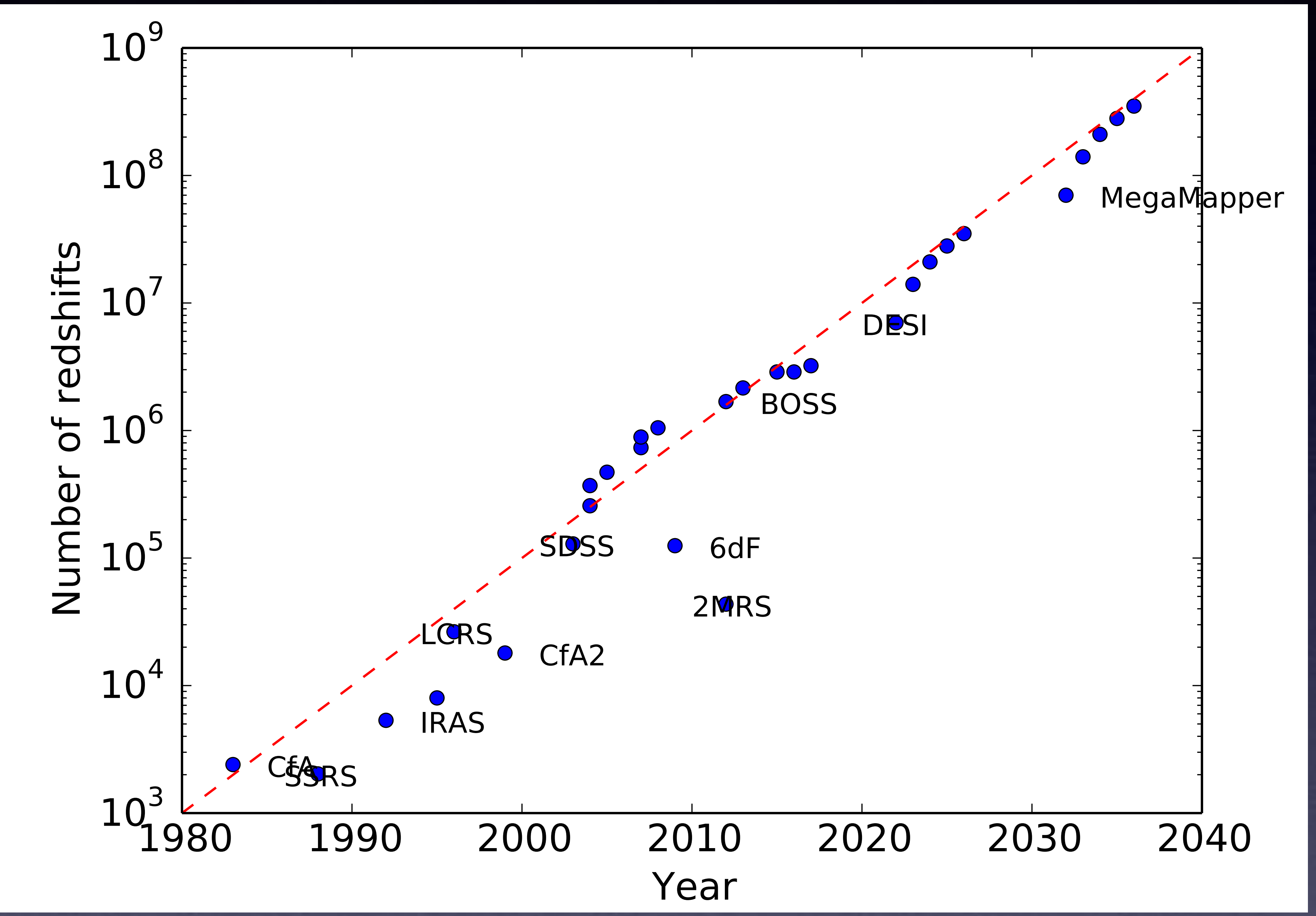




$$\sigma\left[pp \to VVV\right] \qquad V = W, Z$$

A vivid programme to survey the Large Scale Structure

- Abundance of new and future data (BOSS, DESI, MegaMapper, PUMA, …)
- Testing cosmological models stringently
- Precise determinations or limits of Hubble's constant, dark matter density, neutrino masses, primordial non-Gaussianities, curvature and other cosmological parameters
- Competitive or better uncertainties than from CMB measurements.

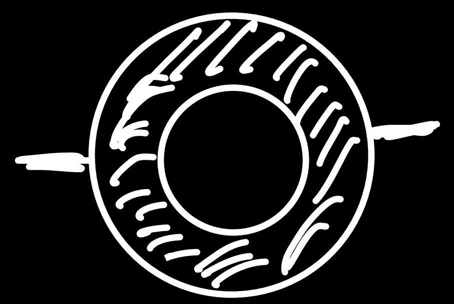


Astro2020 APC White Paper [1907.11171]

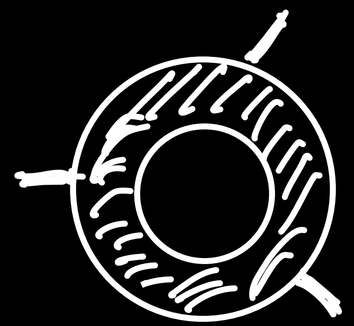
Cosmological correlators of density contrast

- Power-spectrum

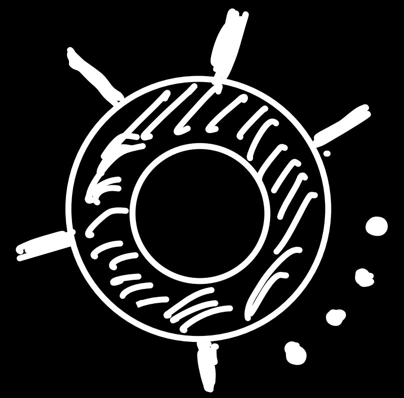
$$\mathscr{P}(k) \equiv \int \frac{d^3\vec{r}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \left\langle \left(\frac{\delta\rho}{\rho}\right)(\vec{x}) \left(\frac{\delta\rho}{\rho}\right)(\vec{x}+\vec{r}) \right\rangle$$

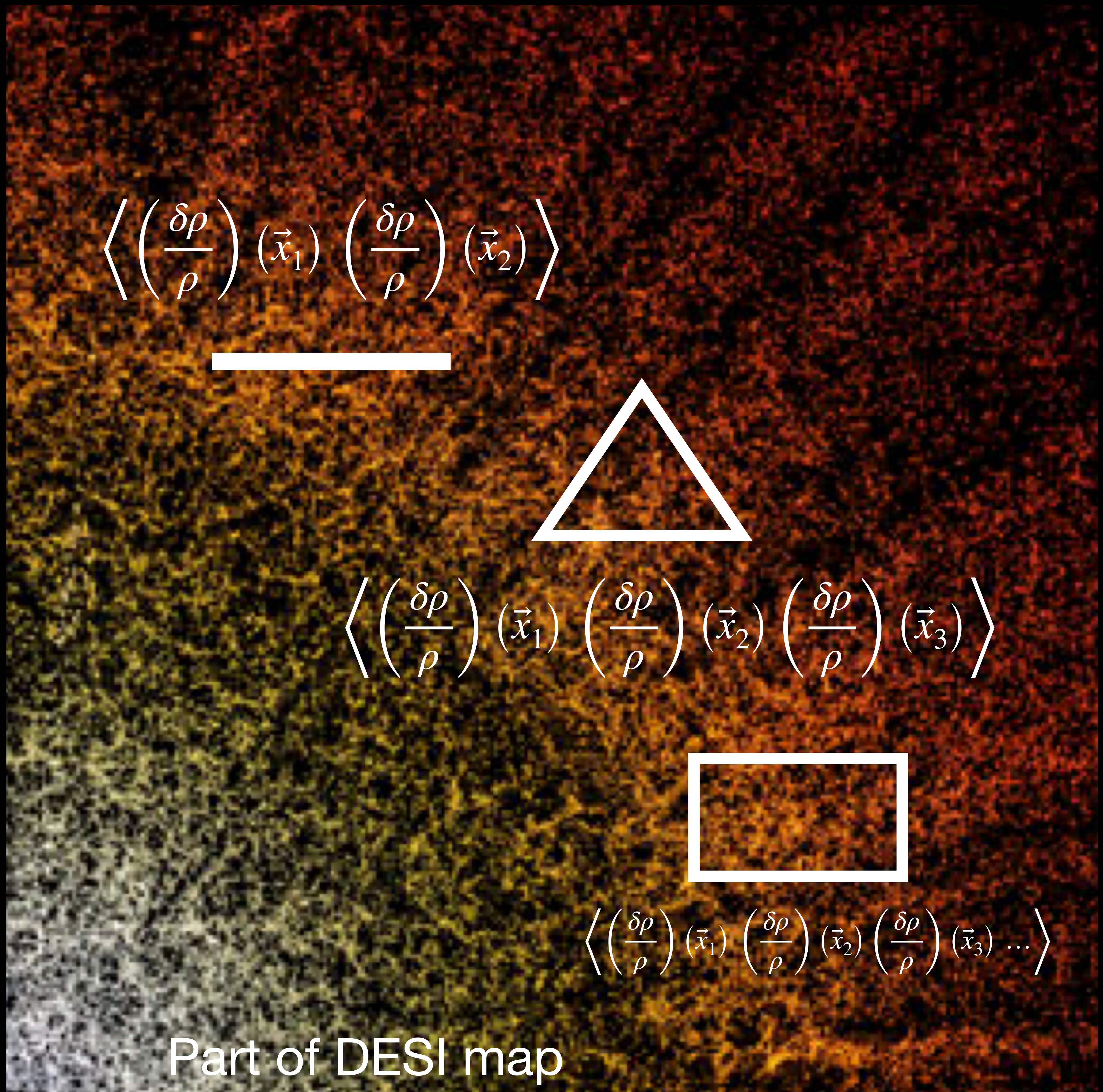


- Bispectrum



- Higher point correlators

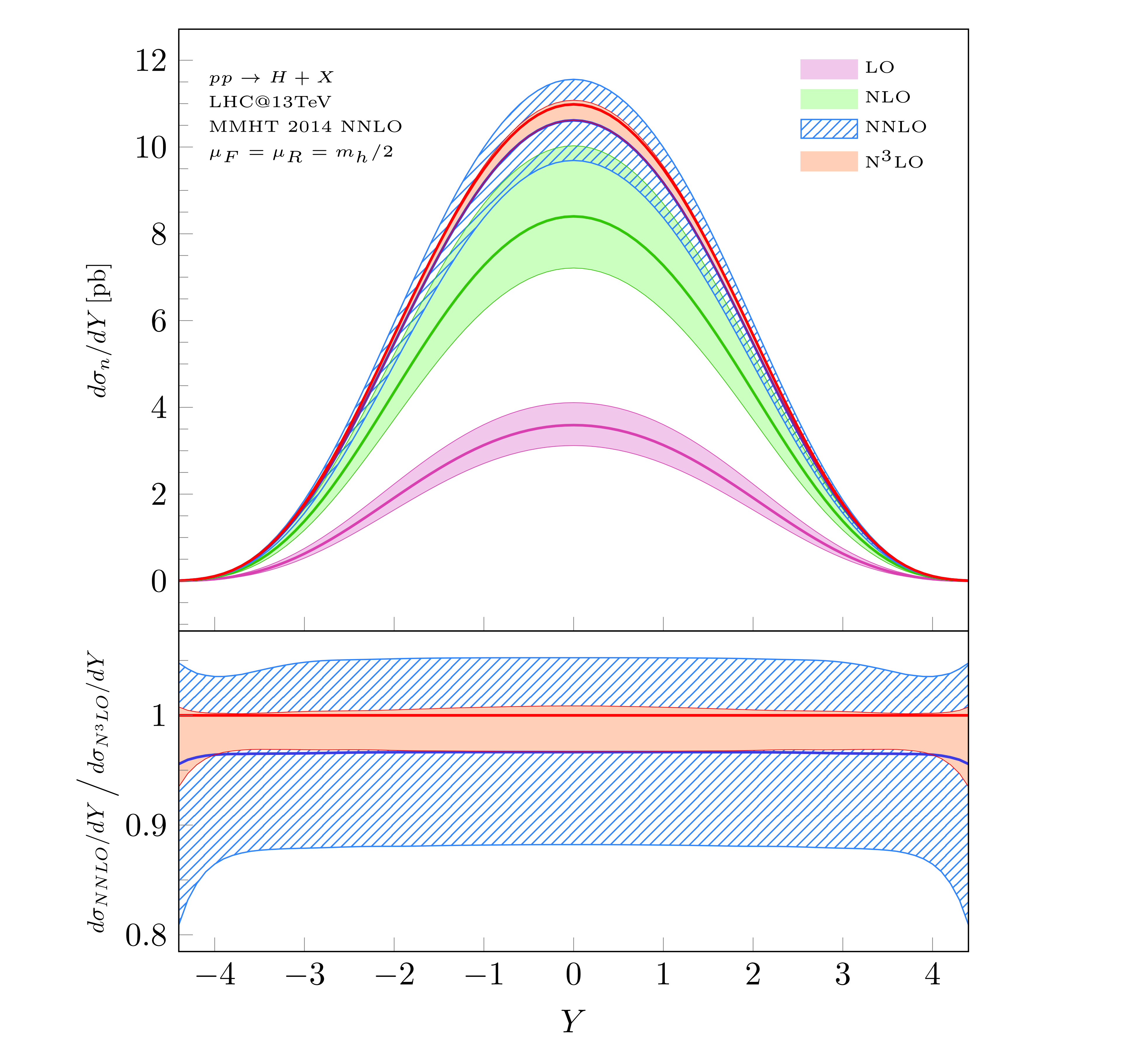




High precision theory predictions in perturbative QCD

e.g. Higgs Rapidity

- Innovative perturbative methods (mostly analytic)
- High precision theoretical prediction (~2%) at N3LO.
- Awaiting data from the LHC at the high luminosity phase.



Dulat, Mistlberger, Pelloni, [1810.09462]

Numerical methods for perturbation theory

- Fantastic progress with analytic approaches, so far.
- Scattering amplitudes for many massive and massless particles depend on many variables. Hard to solve analytically.
- Cosmological correlators depend non-trivially on all parameters of the cosmological model, as well as kinematic variables.
- Recent progress is fast towards semi-analytic/semin-numerical methods.
- We would like to develop universal, process dependent numerical methods. Letting the computer to do the hard work…

Effective Field Theory of the Large Scale Structure

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}$$

Integrate our short modes

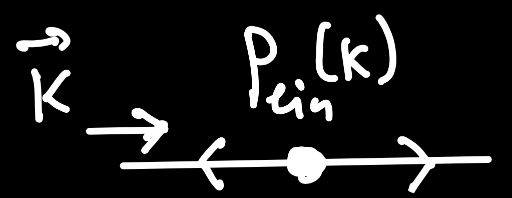
$$\frac{\delta\rho}{\rho} = \delta + \int \delta_{\rm short}$$

$$\delta \sim \delta^{(1)} + \delta^{(2)} + \dots$$

Baumann, Nicolis, Senatore, Zaldarriaga [1004.2488] Carrasco, Hertzberg,, Senatore [1206.2926] Porto, Senatore, Zaldarriaga [1311.2168] Senatore, Zaldarriaga [1404.5954]

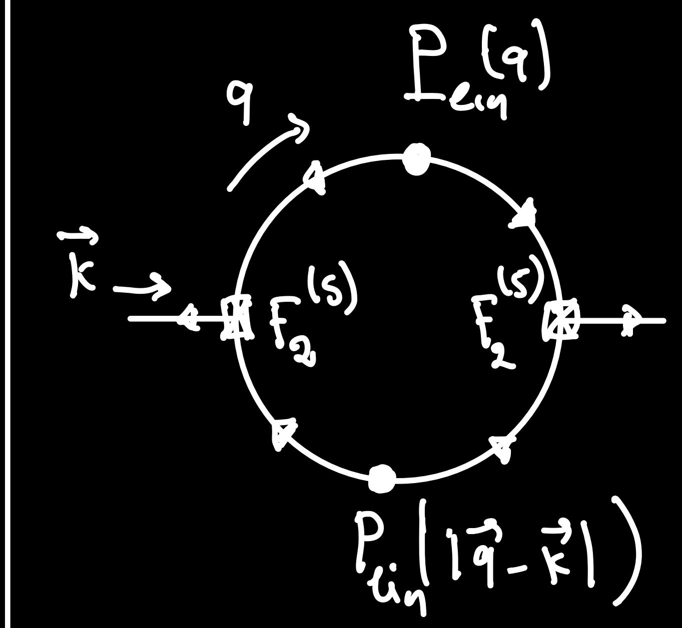
...

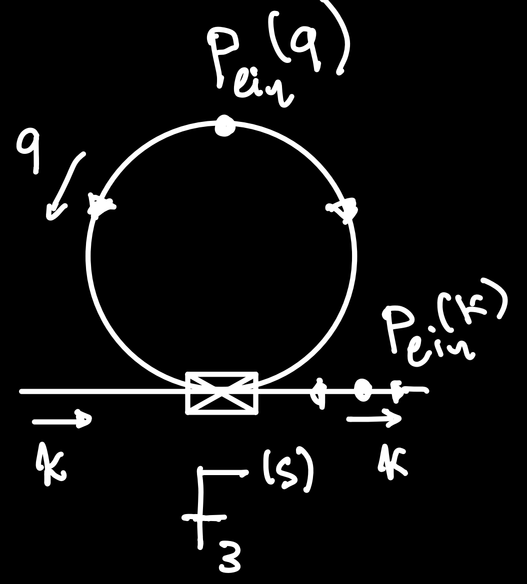
Tree

$\mathscr{P}(k) =$ 

$+$

One loop



$+$ 

$+ \text{CTs}$ $+$ Two-loop, ...

Solution of linearised equations. Depends on cosmological parameters, e.g. Hubble constant

Weak non-linearities as loop corrections. Cosmological parameters enter implicitly, through the propagator lines.

Loops in EFT of Large Scale Structure

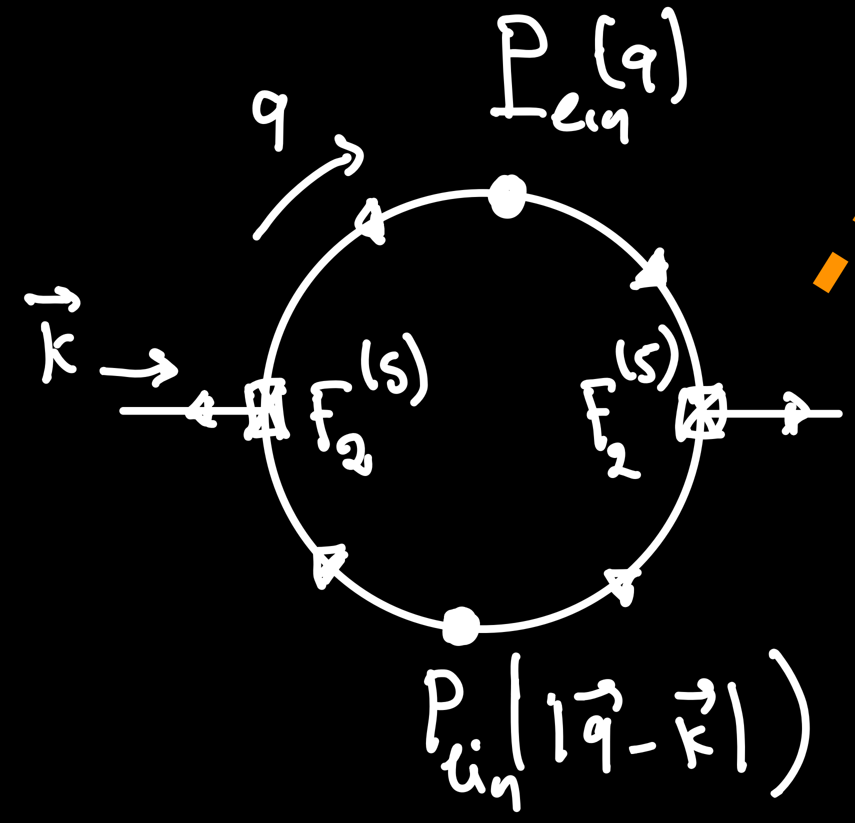
Euclidean loop integration $\displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3}$

Infinite range (UV singularities)

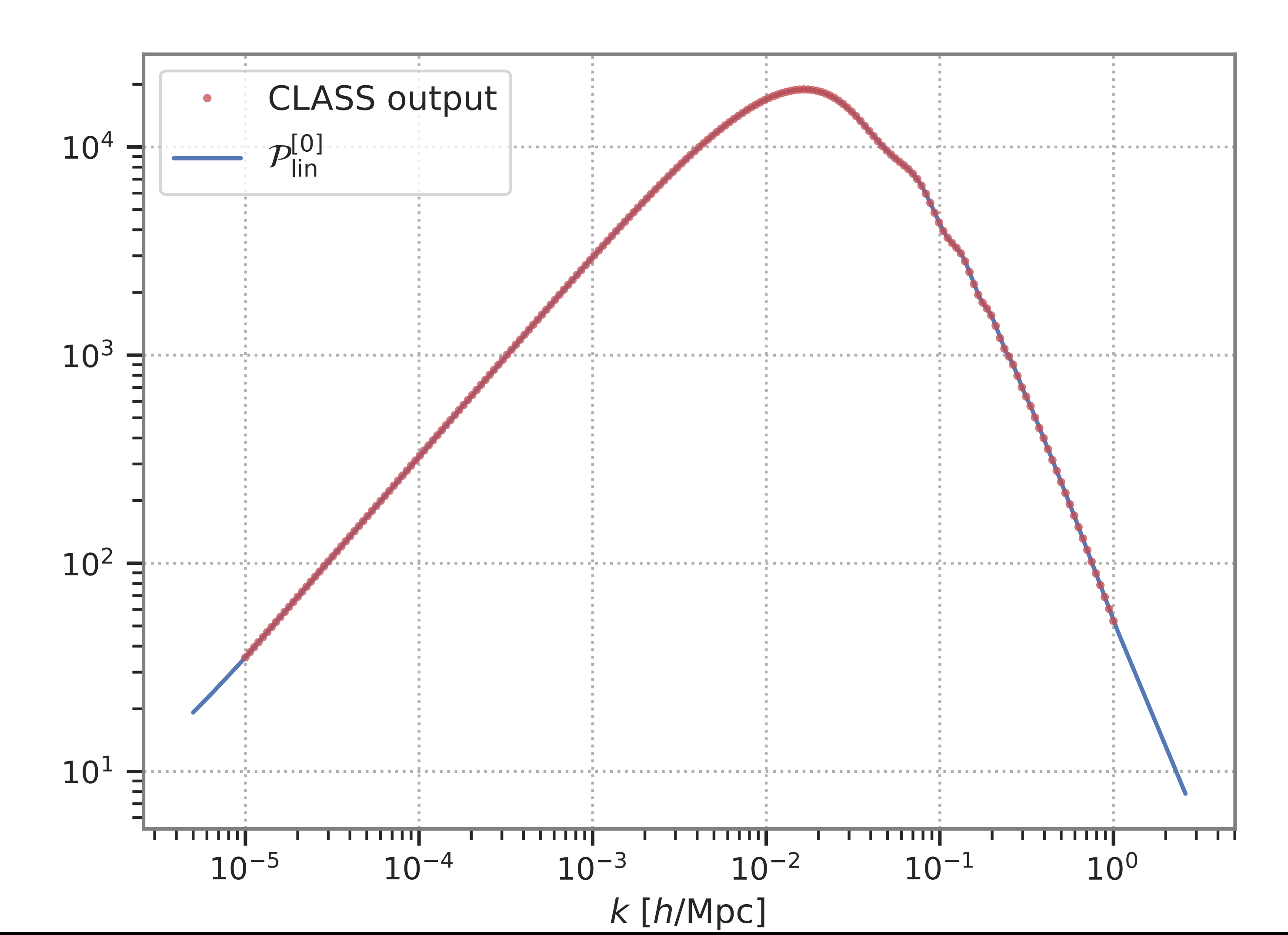
VERTICES

$$28\,F_2 = 10 + 3\,\frac{k^2}{q^2} + 3\frac{k^2}{(q-k)^2} - 5\frac{k^2}{(q-k)^2} - 5\frac{(q-k)^2}{k^2} + 2\frac{(k^2)^2}{q^2\,(q-k)^2}$$

Massless QCD-type propagator denominators (IR singularities)



PROPAGATORS

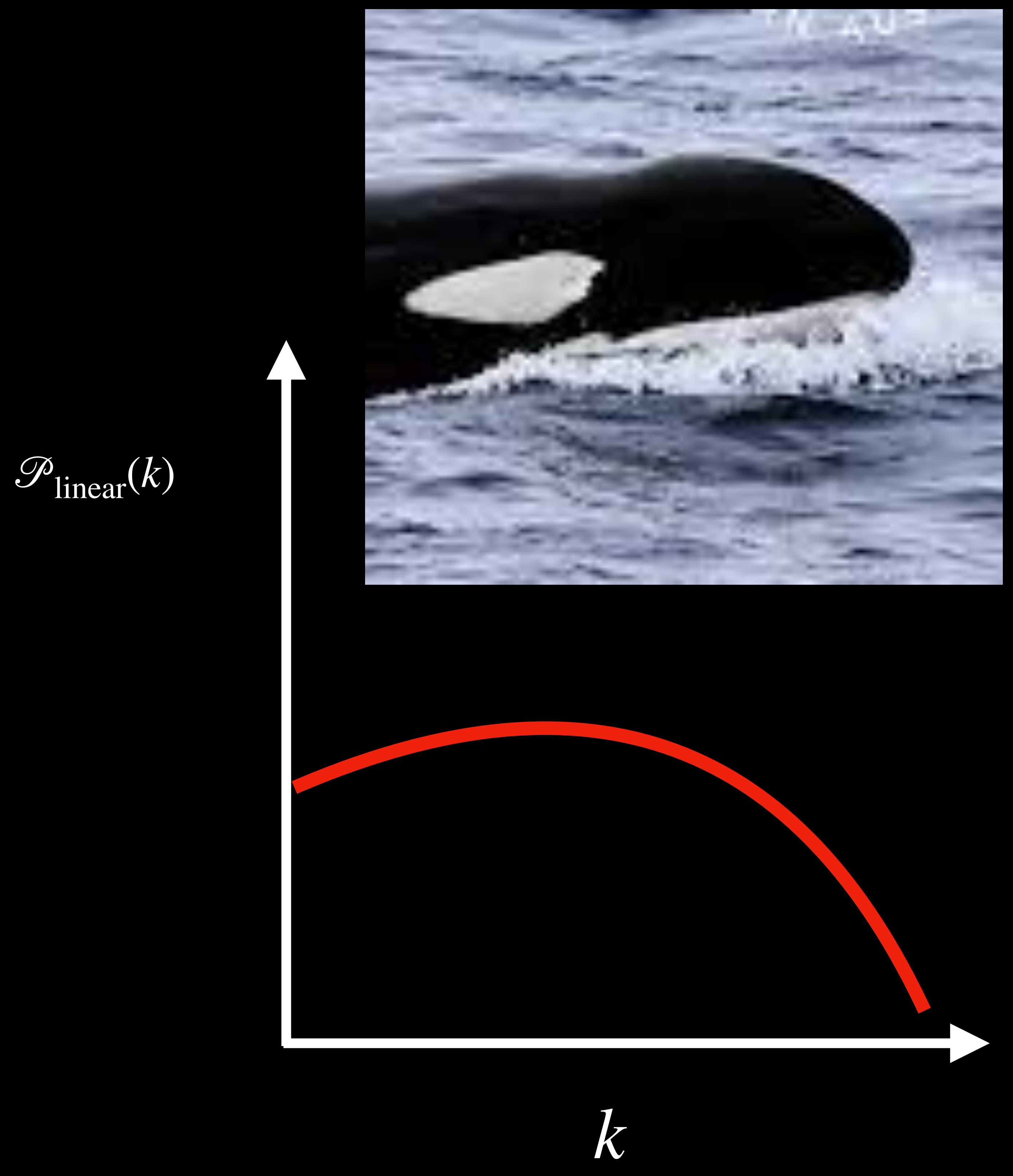


A function of wavelength (loop momentum) and cosmological parameters

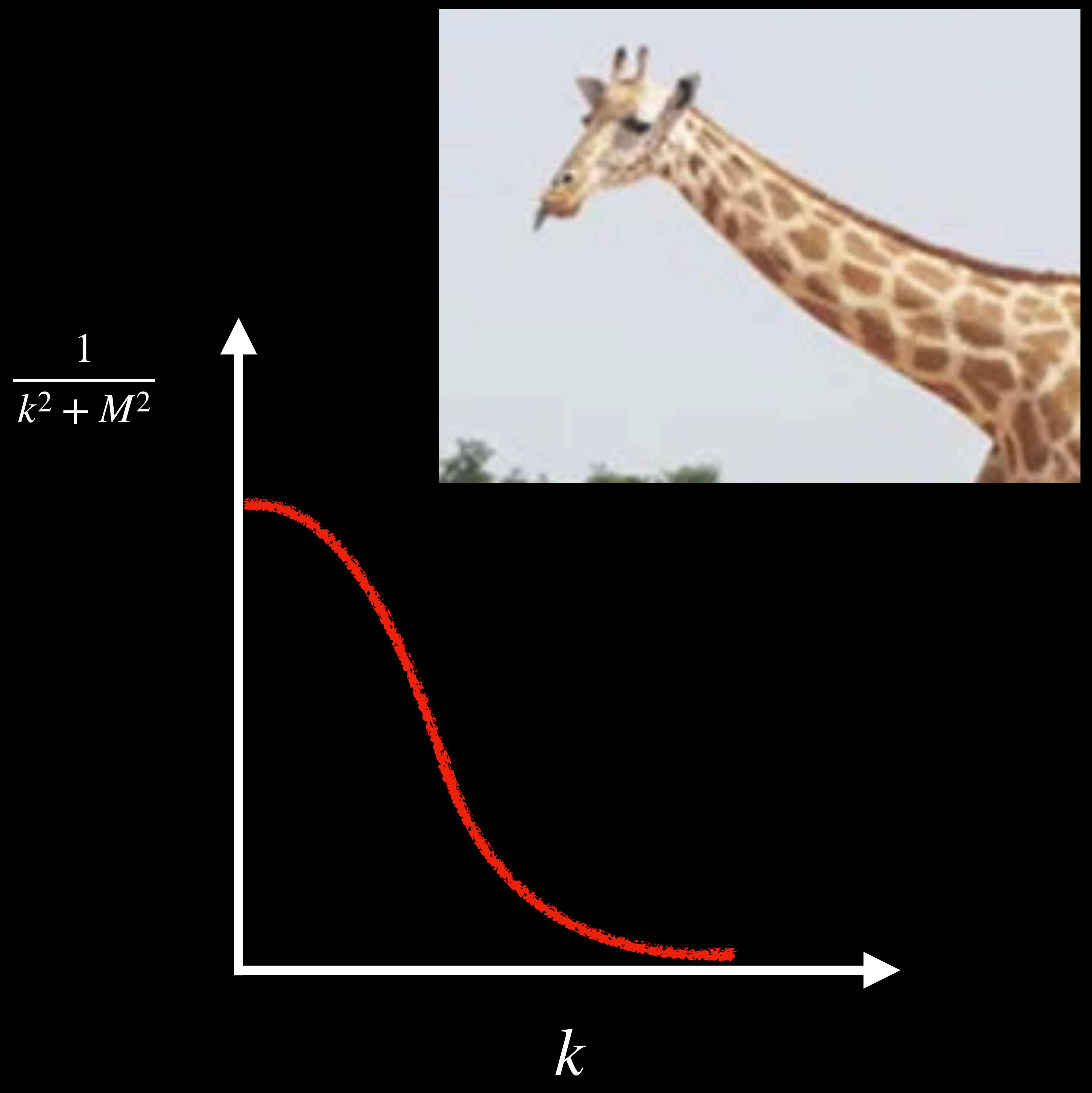
IR and UV singularities motivate analytic methods for the evaluation of loop corrections

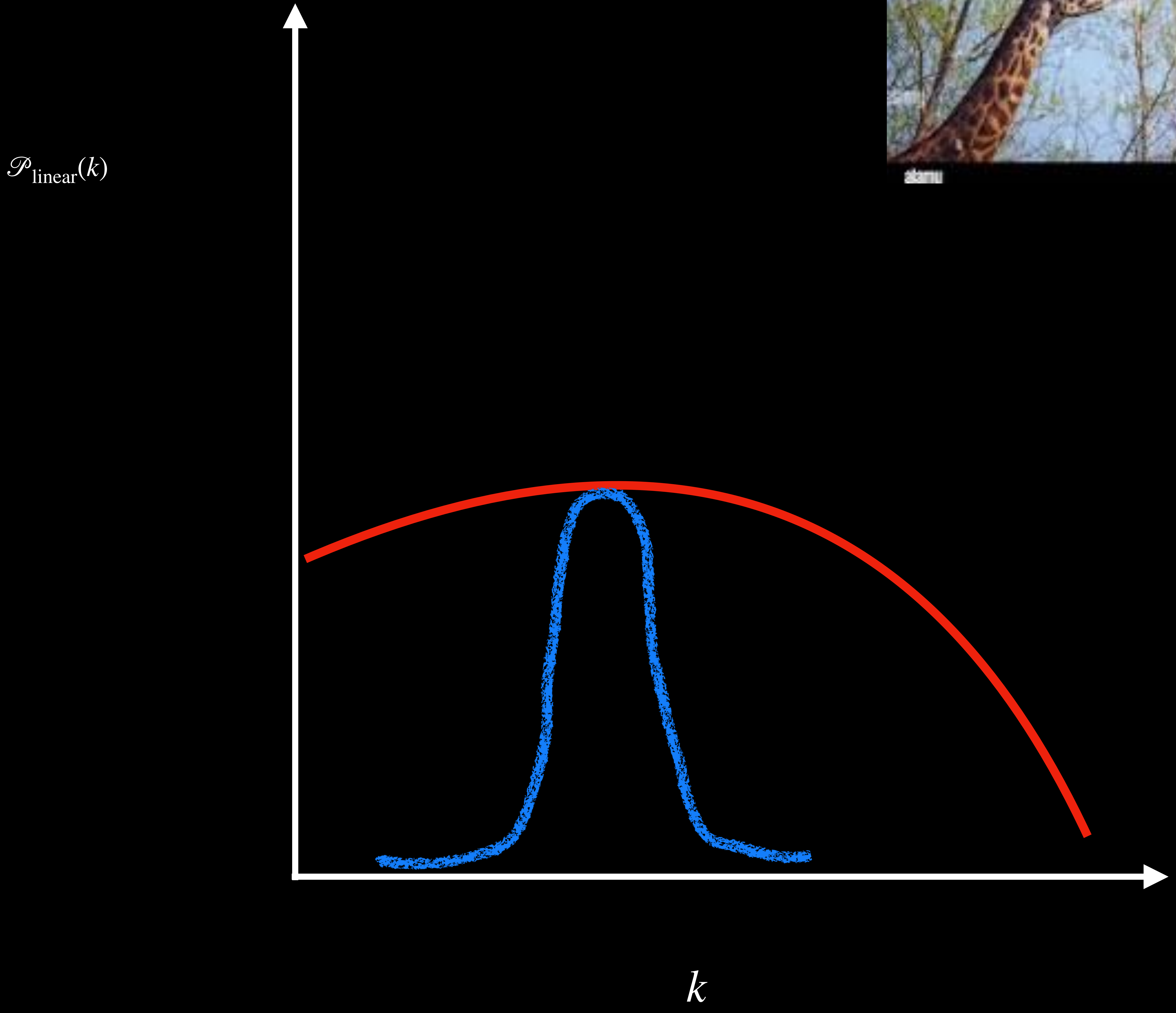
Motivates numerical methods for the evaluation of loop corrections

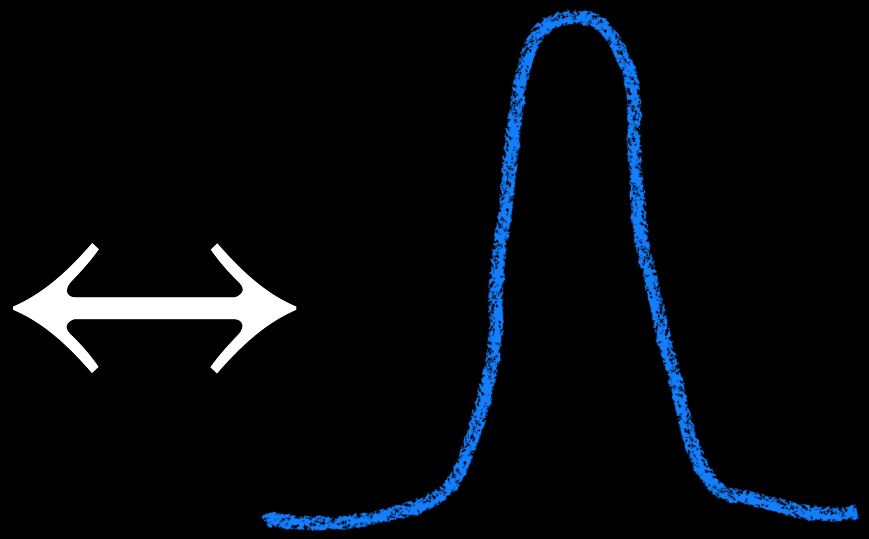
Tree level propagators



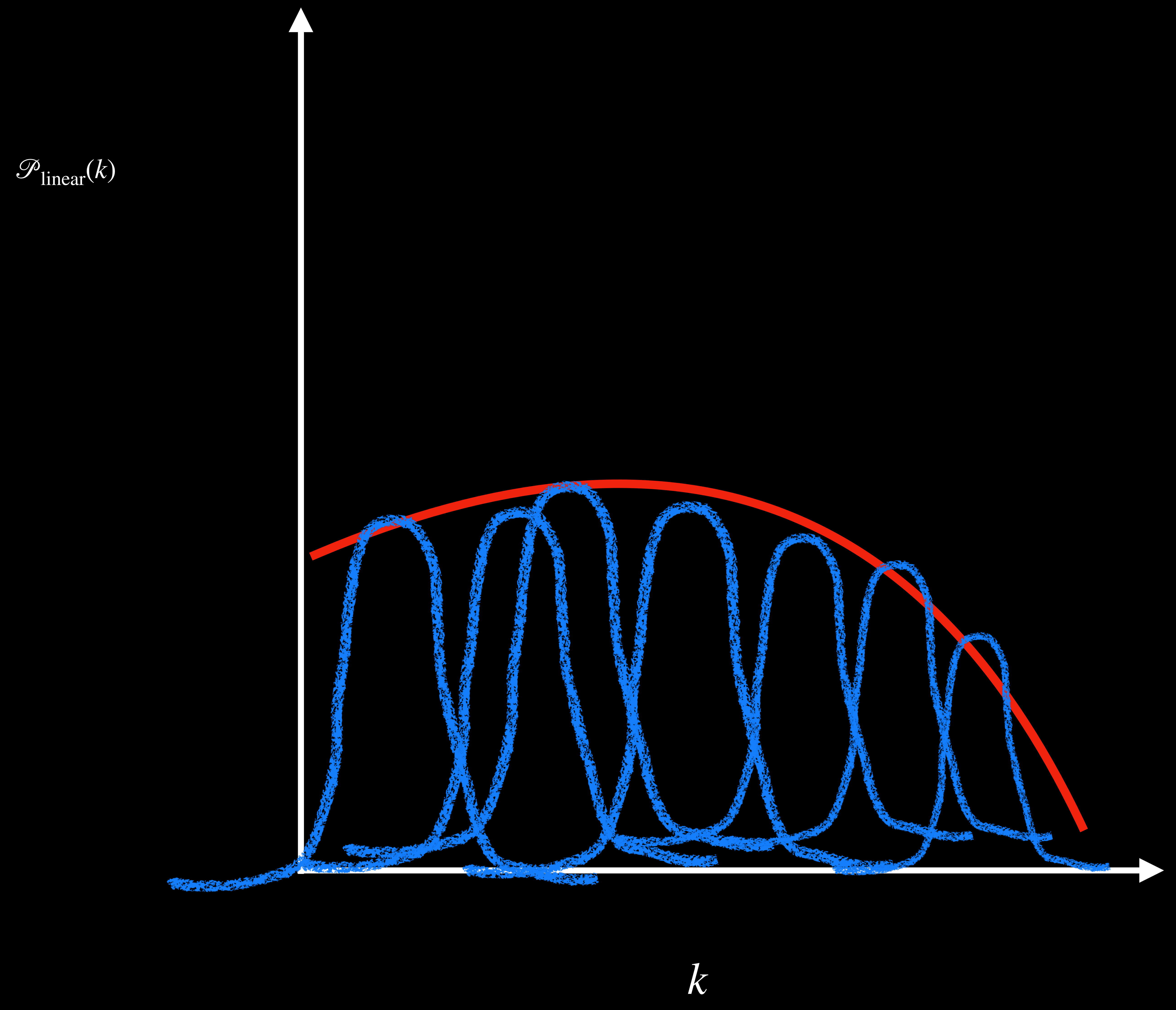
Versus





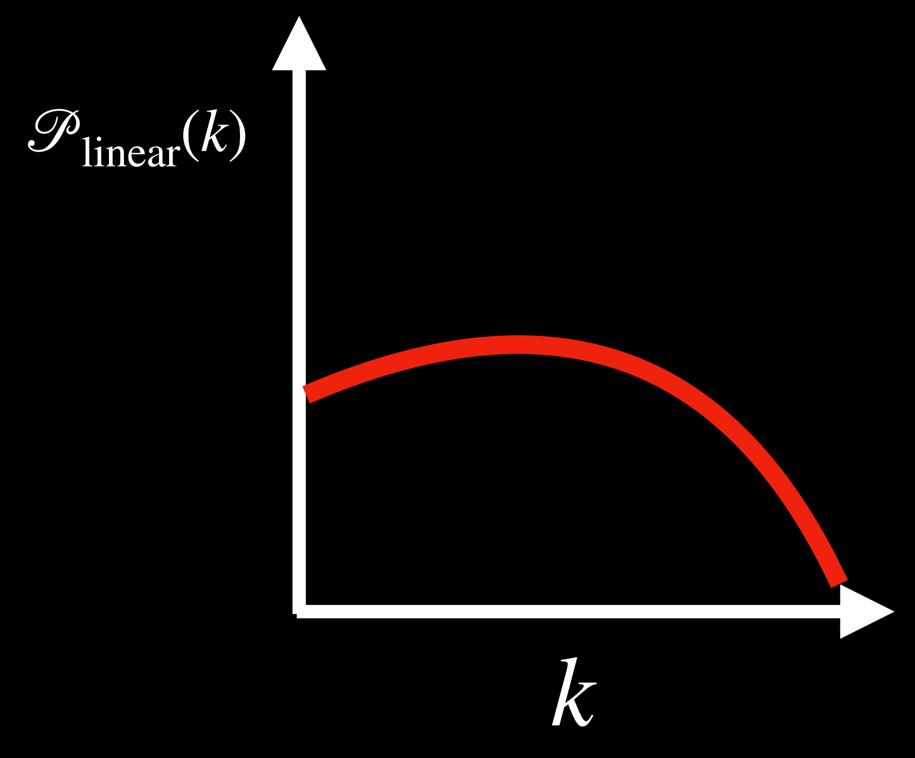


Brett Wigner


$\mathscr{P}_{\text{linear}}(k)$
k

Analytic method at one-loop

CA, Braganca, Senatore, Zheng
[2212.07421]



$$= \sum_n C_{nm}(H, \Omega, \ldots) \frac{1}{\left(k^2 + M_n + i\Gamma_n\right)^{\nu_m}}$$

EFT of LSS propagator

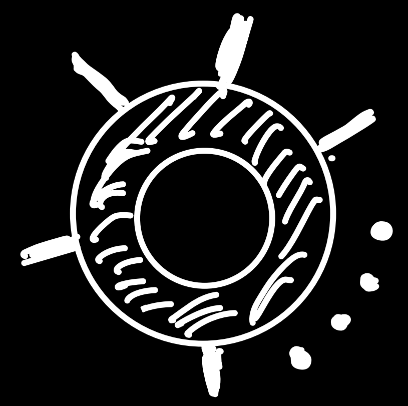
Massive QFT propagators

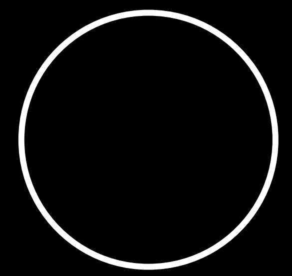
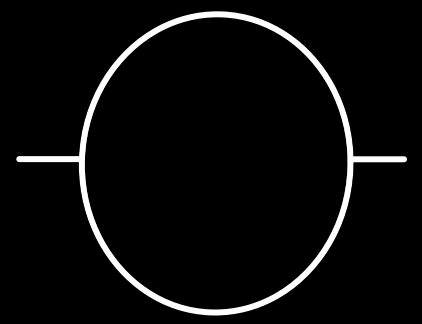
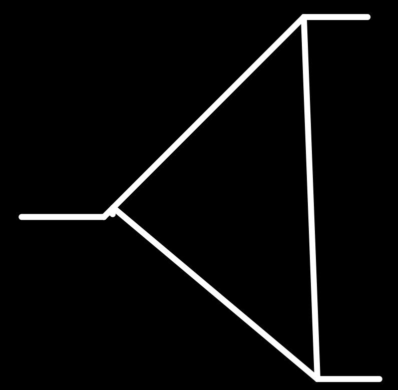
Raised to integer powers

- Enables powerful methods to reduce number of integrals (master integrals)
- Enables master integral evaluation methods with analytic control over UV and IR singularities.

Generic one-loop N-point correlators in EFT of LSS.

CA, Braganca, Senatore, Zheng [2212.07421]

 $= \sum_n C^{(n)}_{H,\Omega,\ldots} \mathcal{I}^{(N_p)}_n$

$\mathcal{I}^{(N_p)}_n = \quad d^{tadp.}_n$  $+ \quad d^{bubble}_n$  $+ \quad d^{trian.}_n$ 

Integration by parts identities can be brought to a diagonal form for arbitrary number of external legs

$$\nu_b \, \hat{\mathbf{b}}^+ = \sum_{j=1}^{N} \tilde{d}_{bj} \left[\left(\nu_j + \sum_{i=1}^{N} \nu_i - D \right) \hat{\mathbf{0}} + \sum_{a \neq j} \nu_a \mathbf{a}^+ \mathbf{j}^- \right] .$$

No box, pentagon, hexagon,… master integrals in three dimensions

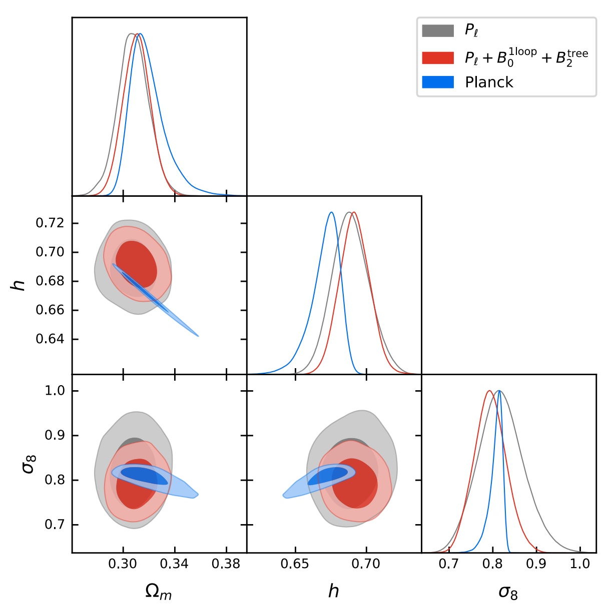
Van Neerven, Vermaseren [Phys.Lett.B 137 (1984) 241-244]

$$\left[-2\rho_4 - \frac{1}{2} \sum_{i,j=1}^{3} (\rho_i - \rho_4)\, \Pi_{ij} \, (\rho_j - \rho_4) \right] \int d^D q \, \frac{1}{\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4} =$$

$$- \frac{1}{2} \int d^D q \, \frac{2\mathcal{A}_4 + \sum_{i,j=1}^{3} (\rho_i - \rho_4)\, \Pi_{ij} \, (\mathcal{A}_j - \mathcal{A}_4)}{\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4} + \mathcal{O}(\epsilon) .$$

One-loop bispectrum in EFT of LSS in phenomenology

Loop corrections enhance the range of applicability of perturbation theory to include shorter scales (weakly non-linear regime).



best-fit mean ± σ	Ω_m	h	σ_8	ω_{cdm}	$\ln(10^{10}A_s)$	S_8
P_ℓ	0.2984 0.308 ± 0.012	0.6763 $0.689^{+0.012}_{-0.014}$	0.8305 $0.819^{+0.049}_{-0.055}$	0.1143 0.1232 ± 0.0075	3.123 3.02 ± 0.15	0.8283 $0.830^{+0.051}_{-0.060}$
$P_\ell + B_0^{\rm tree}$	0.3101 0.309 ± 0.011	0.6907 0.691 ± 0.012	0.8063 0.804 ± 0.049	0.1248 0.1246 ± 0.0058	2.98 2.97 ± 0.13	0.8197 $0.816^{+0.050}_{-0.057}$
$P_\ell + B_0^{\rm 1loop}$	0.3210 0.314 ± 0.011	0.6956 0.693 ± 0.011	0.7882 $0.790^{+0.033}_{-0.037}$	0.1331 0.1278 ± 0.0061	2.82 2.90 ± 0.11	0.8153 $0.807^{+0.037}_{-0.043}$
$P_\ell + B_0^{\rm 1loop} + B_2^{\rm tree}$	0.3082 0.311 ± 0.010	0.6928 0.692 ± 0.011	0.7856 0.794 ± 0.037	0.1258 0.1255 ± 0.0057	2.88 2.94 ± 0.11	0.7962 0.808 ± 0.041
Planck	$0.3191^{+0.0085}_{-0.016}$	$0.671^{+0.012}_{-0.0067}$	$0.807^{+0.018}_{-0.0079}$	0.1201 ± 0.0013	3.046 ± 0.015	0.832 ± 0.013

Figure 1: Triangle plots, best-fit values, and relative 68%-credible intervals of base cosmological parameters measured from the analysis of BOSS power spectrum multipoles P_ℓ, $\ell = 0, 2$, at one-loop, bispectrum monopole B_0 at tree or one-loop level, and bispectrum quadrupole B_2 at tree-level. Planck $\nu\Lambda$CDM results are shown for comparison.

BOSS

D' Amico, Donath, Lewandowski, Senatore, Zheng [2206.08327]

$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	n_s	Ω_k
P_ℓ	0.014	0.2	0.012	0.066	0.027
P_ℓ+B_0	0.01	0.13	0.009	0.037	0.017

$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	$\log m_\nu^{\rm tot.}(^{\sigma^+}_{\sigma^-})$
P_ℓ	0.011	0.14	0.011	$2.3\ (^{+0.84}_{-0.089})$
P_ℓ+B_0	0.0092	0.095	0.008	$1.6\ (^{+0.41}_{-0.080})$

$\sigma(\cdot)$	$f_{\rm NL}^{\rm loc.}$	$f_{\rm NL}^{\rm eq.}$	$f_{\rm NL}^{\rm orth.}$
P_ℓ+B_0	25	266	94

BOSS

$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	n_s	Ω_k
P	0.0061	0.12	0.0045	0.027	0.051
P+B	0.0042	0.035	0.0023	0.011	0.013

$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	$\log m_\nu^{\rm tot.}(^{\sigma^+}_{\sigma^-})$
P	0.0065	0.051	0.0072	$1.3\ (^{+0.26}_{-0.072})$
P+B	0.0042	0.025	0.0034	$0.63\ (^{+0.087}_{-0.047})$

$\sigma(\cdot)$	$f_{\rm NL}^{\rm loc.}$	$f_{\rm NL}^{\rm eq.}$	$f_{\rm NL}^{\rm orth.}$
P+B	3.5	114	30

DESI

$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	n_s	Ω_k
P	0.0036	0.021	0.0012	0.006	0.0076
$P+B$	0.0021	0.0052	0.0003	0.002	0.0015

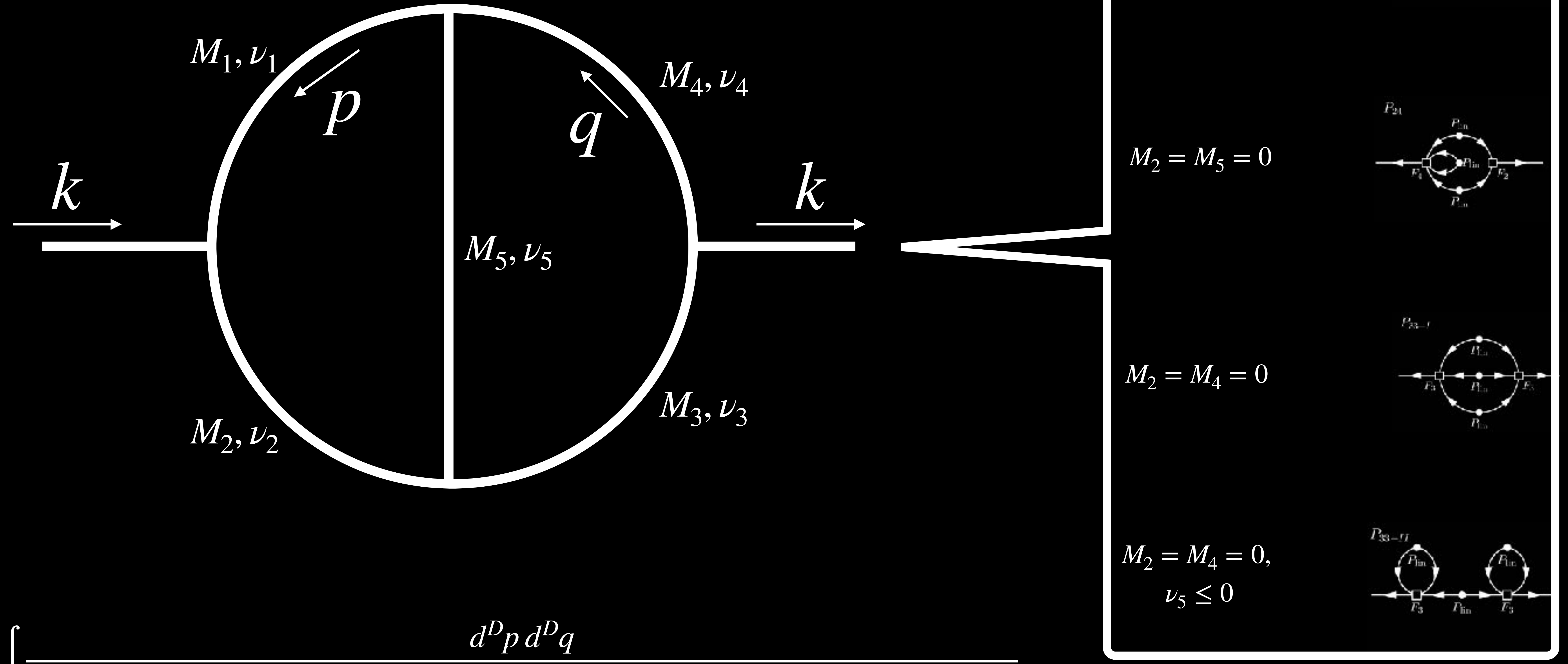
$\sigma(\cdot)$	h	$\ln(10^{10}A_s)$	Ω_m	$\log m_\nu^{\rm tot.}(^{\sigma^+}_{\sigma^-})$
P	0.0023	0.022	0.0015	$0.38\ (^{+0.046}_{-0.031})$
P+B	0.0010	0.0039	0.0004	$0.083\ (^{+0.009}_{-0.008})$

$\sigma(\cdot)$	$f_{\rm NL}^{\rm loc.}$	$f_{\rm NL}^{\rm eq.}$	$f_{\rm NL}^{\rm orth.}$
P+B	0.27	17.7	4.6

MegaMapper

Braganca, Donath, Senatore, Zheng [2307.04992]

Two-loop QFT integrals for the power spectrum: the SunCut topology



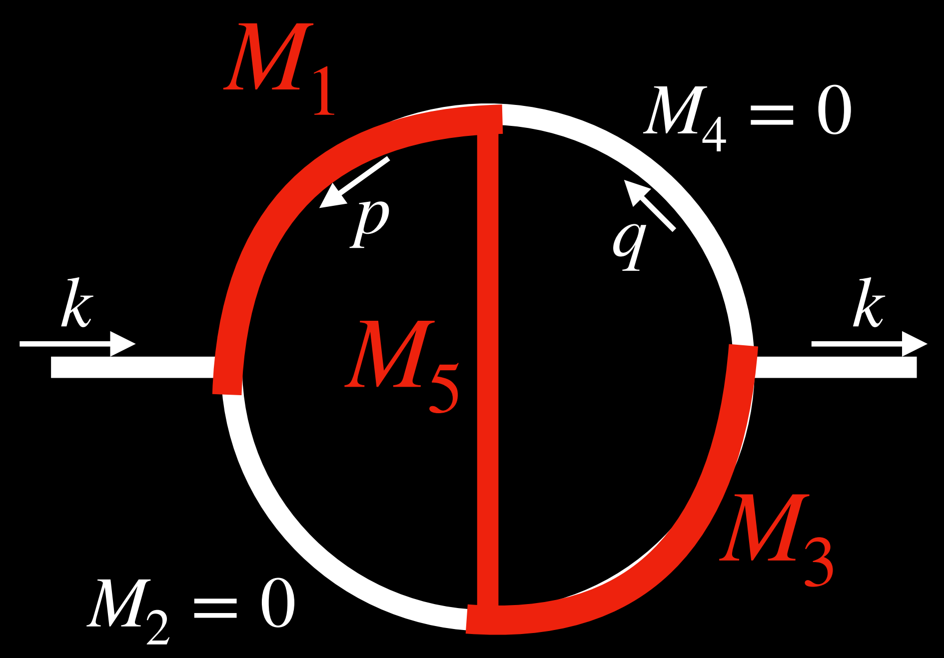
$$\int \frac{d^D p \, d^D q}{\left[p^2 + M_1\right]^{\nu_1} \left[(k+p)^2 + M_2\right]^{\nu_2} \left[(k+q)^2 + M_3\right]^{\nu_3} \left[q^2 + M_4\right]^{\nu_4} \left[(q-p)^2 + M_5\right]^{\nu_5}}$$

$$D = 3 - 2\epsilon$$

Differences with integrals from one-loop cosmological correlators

- At one-loop, in $D = 3 - 2\epsilon$, **ALL** integrals are free of $1/\epsilon$ poles (finite or dim. Reg. zeros).
- At one loop, IBP identities can be easily diagonalised, symbolically (no Laporta algorithm needed).
- Reduction to master integrals numerically (setting D=3 exactly) and masses/invariants to their values.
- All (three) master integrals computed analytically in closed form (logs).

- UV/IR divergent integrals, with $1/\epsilon$ poles.
- IBP reduction identities are coupled.
- Master integrals are neither known nor easy to compute analytically.



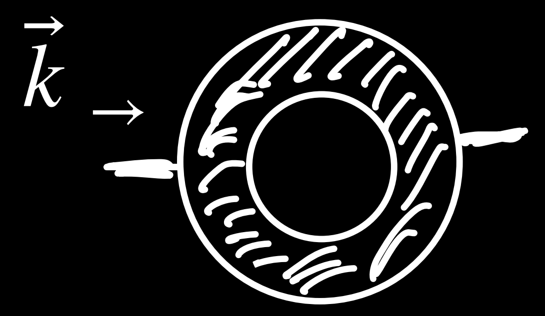
Novel Elliptic Polylogarithms

Gireaux, Pokraka, Porkert, Sohnle [2401.14307]

Ingredients for a numerical evaluation of loop corrections

- Computations purposed for cosmological parameter inference. Need to decouple cosmological parameter dependence from integrations.
- No ultraviolet singularities or enhancements (UV counterterms of EFT of LSS)
- No infrared singularities (general property of EFT of LSS)
- We need smooth integrands, not just integrals, free of UV+IR singularities so that a numerical integration is possible and efficient.

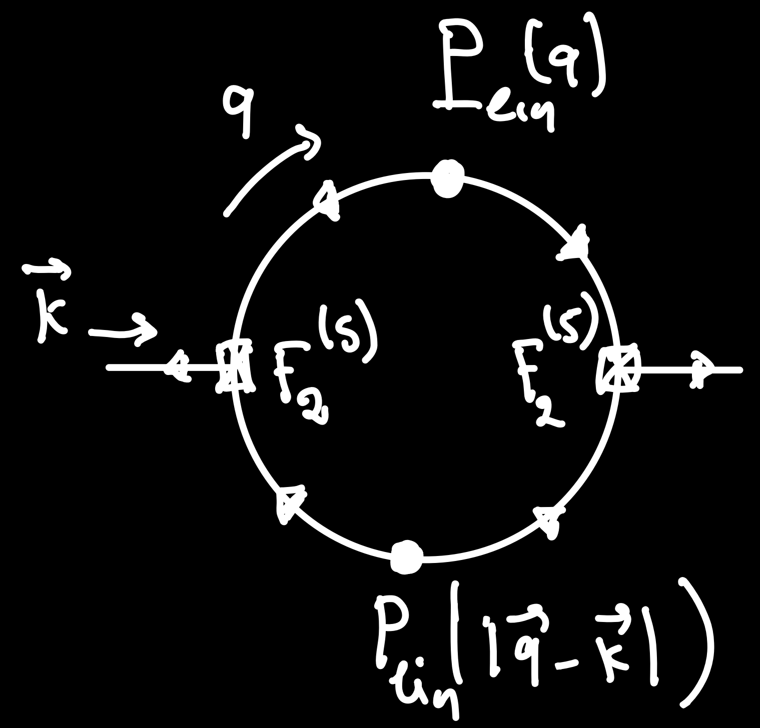
Cosmology dependence



$$= C_n\left(H,\Omega_m,\sum m_\nu,\ldots\right)\int [dl_i]\,\mathscr{I}_n\left(l_i,k\right)$$

Free of UV+IR singularities

The one-loop corrections



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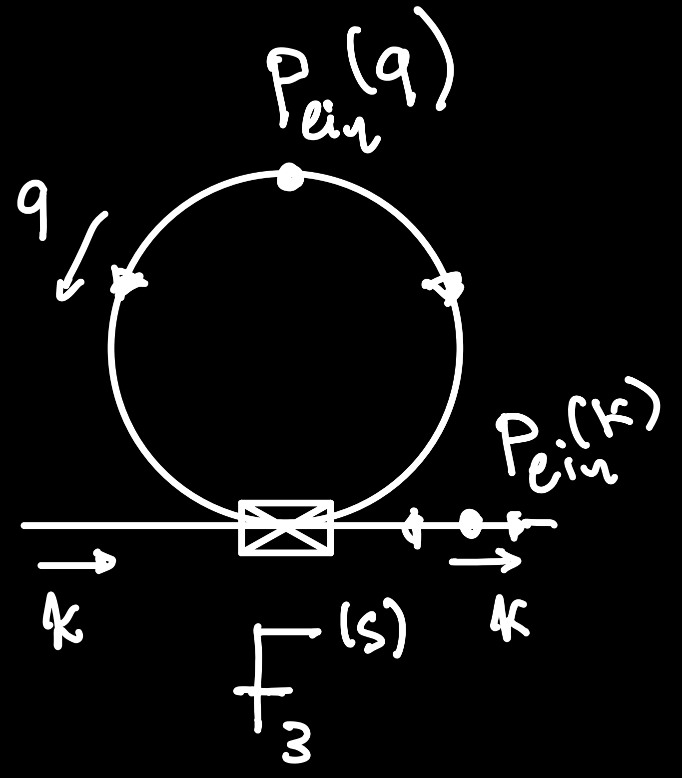
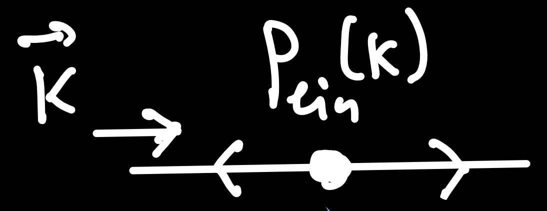


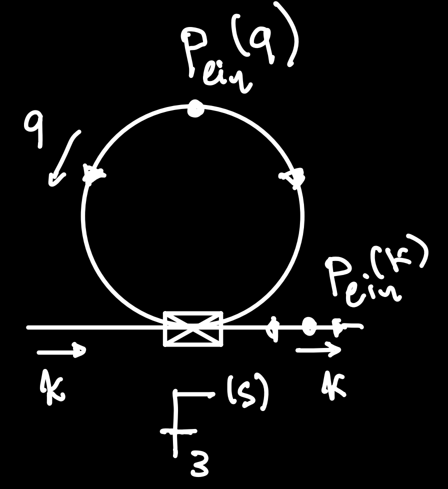
+ $C_0 \frac{k^2}{k_{UV}^2}$



EFT counterterm

Removing leading ultraviolet behaviour

- Power-counting to determine UV limit of the integrand
- Write an approximation of the integrand with the same limiting UV behaviour.
- The UV approximation should not introduce new undesired singularities.
- To restrict the counterterm in the UV region only, we need to introduce an IR ``cutpoff” (a renormalization mass scale)
- Define a UV-fintie integral with a subtraction.
- Absorb UV behaviour into the UV counterterms of the EFT of LSS.



$$= \int d^3\vec{q}\, p_{13}\left(\vec{k},\vec{q}\right)$$

$$p_{13}\left(\vec{k},\vec{q}\right) = 6\mathscr{P}_{lin}(k)\,\mathscr{P}_{lin}(q)\,F_3^{(s)}(\vec{q},-\vec{q},\vec{k})$$

Power counting

$$\mathscr{P}_{lin}\left(\frac{q}{\delta}\right) = \frac{C_{lin}}{q^\nu}\delta^\nu + \mathcal{O}(\delta^{\nu+1})$$

$$F_3^{(s)}\left(\frac{\vec{q}}{\delta},-\frac{\vec{q}}{\delta},\vec{k}\right) = \delta^2\,\mathscr{T}_{q\to\infty}\left[F_3^{(s)}(\vec{q},-\vec{q},\vec{k})\right] + \mathcal{O}(\delta^3)$$

$$\mathscr{T}_{q\to\infty}\left[F_3^{(s)}(\vec{q},-\vec{q},\vec{k})\right] = \frac{1}{126\,k^2}\,\frac{28(\vec{k}\cdot\vec{q})^4 - 59k^2(\vec{k}\cdot\vec{q})^2q^2 + 10(k^2)^2(q^2)^2}{(q^2)^3}$$

$$d^3\vec{q}\,p_{13}(\vec{q},\vec{k}) \sim \delta^{\nu-1}$$

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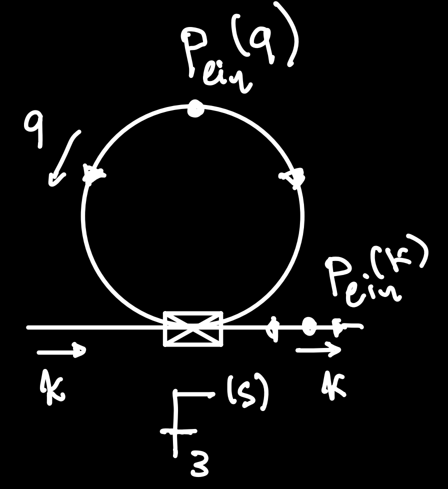


$$= \int d^3\vec{q}\, p_{13}\left(\vec{k},\vec{q}\right)$$

$$p_{13}\left(\vec{k},\vec{q}\right) = 6\mathscr{P}_{lin}(k)\,\mathscr{P}_{lin}(q)\,F_3^{(s)}(\vec{q},-\vec{q},\vec{k})$$

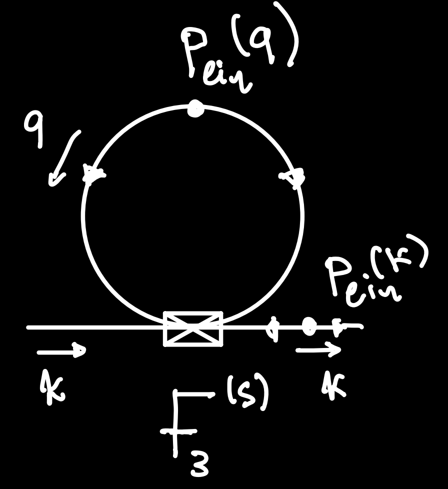
UV approximation

$$p_{13}^{UV}\left(\vec{k},\vec{q}\right) = 6\mathscr{P}_{lin}(k)\,\mathscr{P}_{lin}(q)\,\mathscr{T}_{q\to\infty}\left[F_3^{(s)}(\vec{q},-\vec{q},\vec{k})\right]\frac{q^2}{q^2+M^2}$$

$$\mathscr{T}_{q\to\infty}\left[F_3^{(s)}(\vec{q},-\vec{q},\vec{k})\right] = \frac{1}{126\,k^2}\,\frac{28(\vec{k}\cdot\vec{q})^4 - 59k^2(\vec{k}\cdot\vec{q})^2q^2 + 10(k^2)^2(q^2)^2}{(q^2)^3}$$

Removing leading ultraviolet behaviour

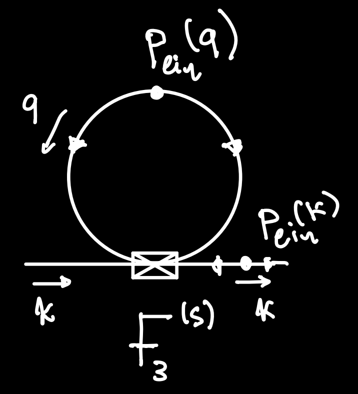
- Power-counting to determine UV limit of the integrand
- Write an approximation of the integrand with the same limiting UV behaviour.
- The UV approximation should not introduce new undesired singularities.
- To restrict the counterterm in the UV region only, we need to introduce an IR ``cutpoff” (a renormalization mass scale)
- Define a UV-fintie integral with a subtraction.
- Absorb UV behaviour into the UV counterterms of the EFT of LSS.



$$= \int d^3\vec{q}\, p_{13}\left(\vec{k},\vec{q}\right)$$

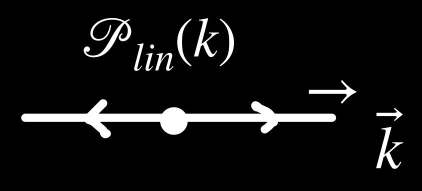
UV approximation

$$p_{13}^{reg.}\left(\vec{k},\vec{q}\right) = p_{13}\left(\vec{k},\vec{q}\right) - p_{13}^{UV}\left(\vec{k},\vec{q}\right)$$



$$= \int d^3\vec{q}\, p_{13}^{reg}\left(\vec{k},\vec{q}\right) + \int d^3\vec{q}\, p_{13}^{UV}\left(\vec{k},\vec{q}\right)$$

$$= \int d^3\vec{q}\, p_{13}^{reg}\left(\vec{k},\vec{q}\right) - \left[\frac{61}{315}\int_q \frac{\mathscr{P}_{lin}(q)}{q^2+M^2}\right] k^2$$



Numeric integration

Degenerate to EFT counterterm

Removing leading ultraviolet behaviour

- Power-counting to determine UV limit of the integrand
- Write an approximation of the integrand with the same limiting UV behaviour.
- The UV approximation should not introduce new undesired singularities.
- To restrict the counterterm in the UV region only, we need to introduce an IR ``cutpoff” (a renormalization mass scale)
- Define a UV-fintie integral with a subtraction.
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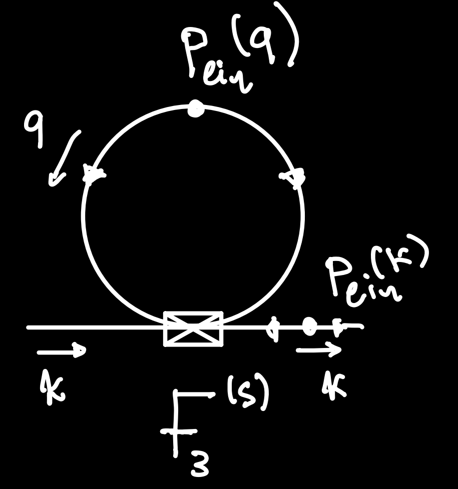
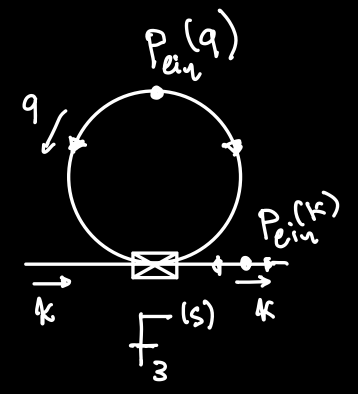


$$= \int d^3\vec{q}\, p_{13}\left(\vec{k},\vec{q}\right)$$

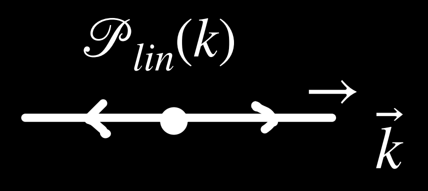
UV approximation

$$p_{13}^{reg.}\left(\vec{k},\vec{q}\right) = p_{13}\left(\vec{k},\vec{q}\right) - p_{13}^{UV}\left(\vec{k},\vec{q}\right)$$



$$= \int d^3\vec{q}\, p_{13}^{reg}\left(\vec{k},\vec{q}\right) + \int d^3\vec{q}\, p_{13}^{UV}\left(\vec{k},\vec{q}\right)$$

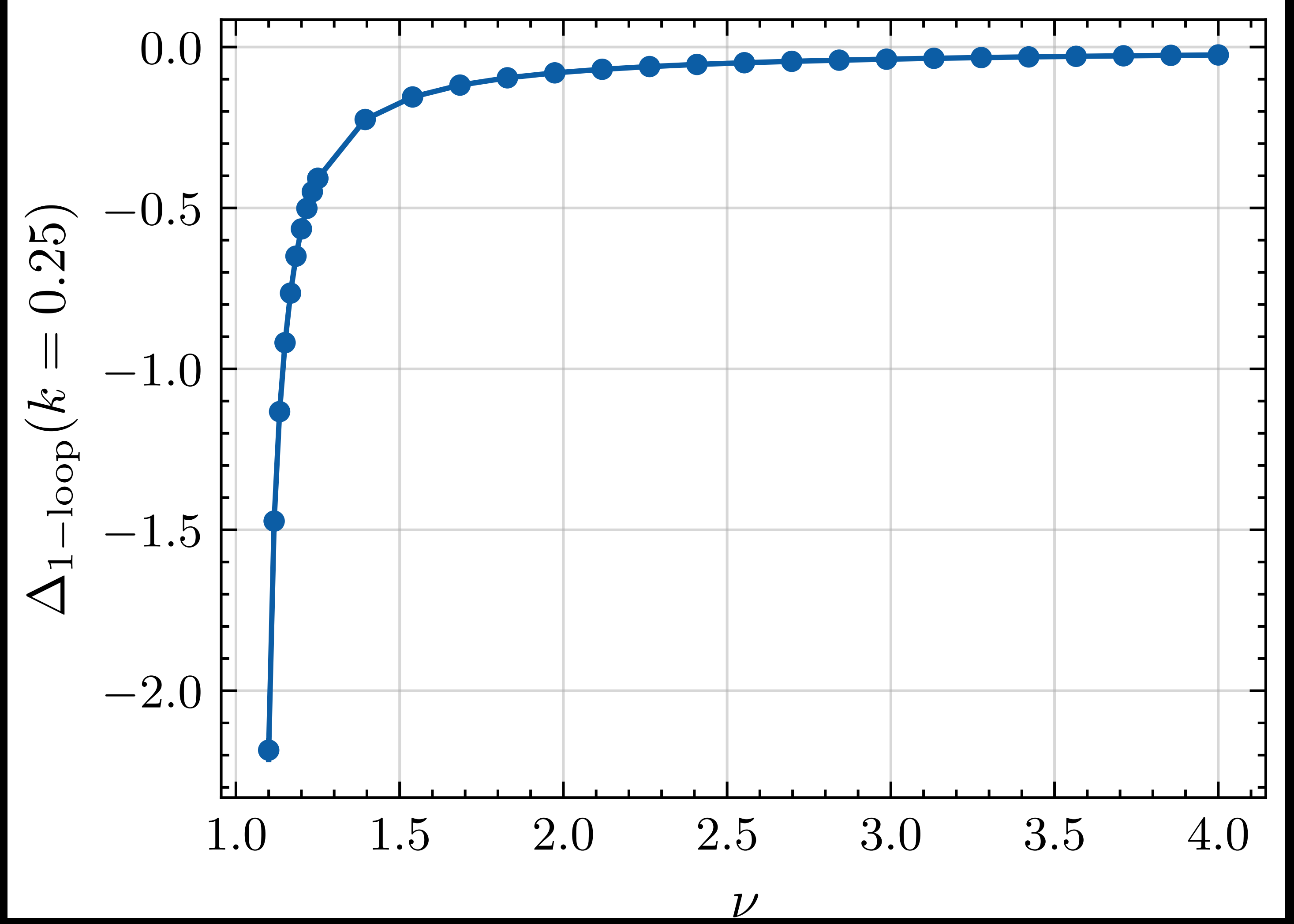
$$= \int d^3\vec{q}\, p_{13}^{reg}\left(\vec{k},\vec{q}\right) - [\text{ A CONSTANT }]\, k^2$$



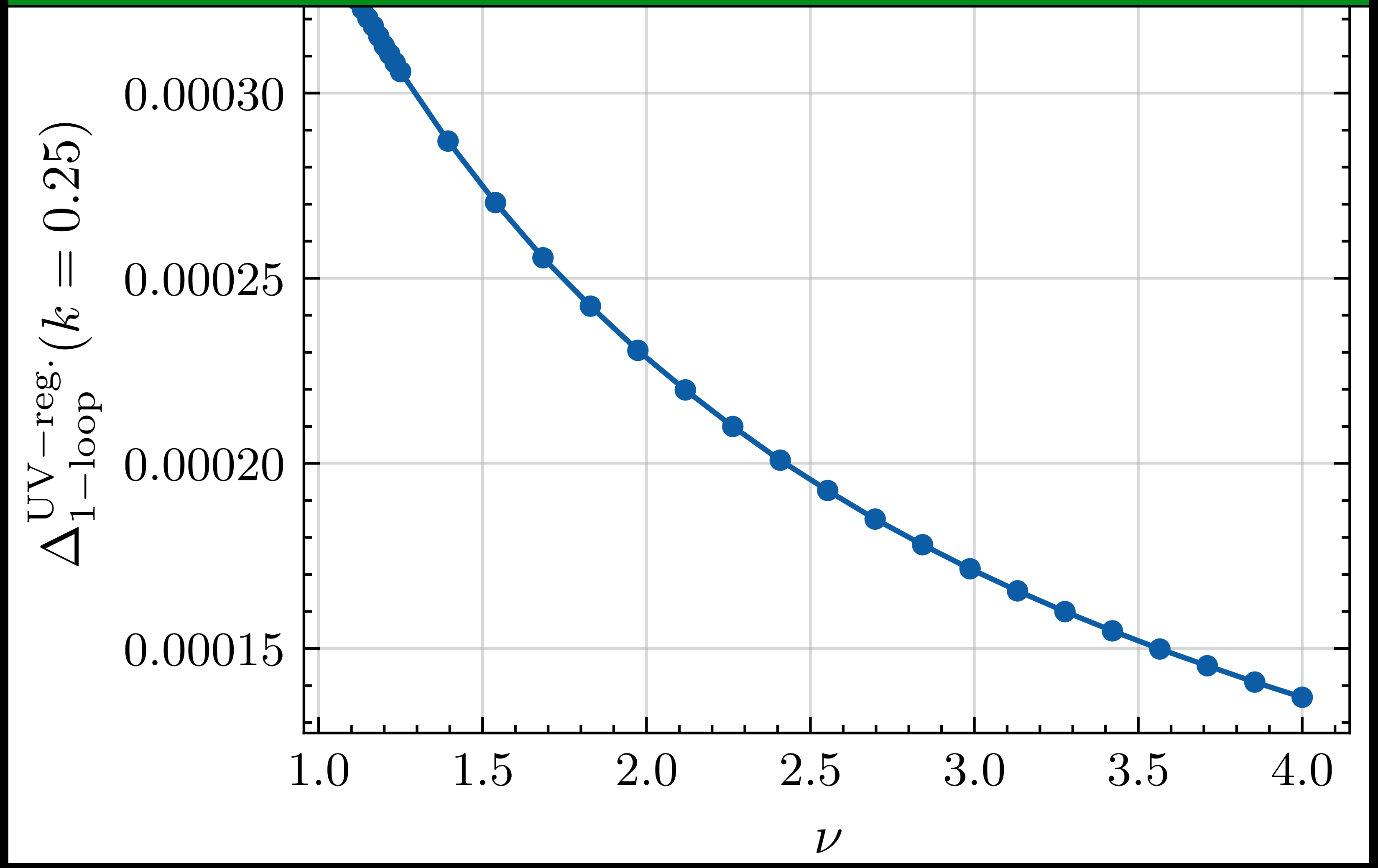
Numeric integration

Degenerate to EFT counterterm

Effect of UV subtraction at one-loop

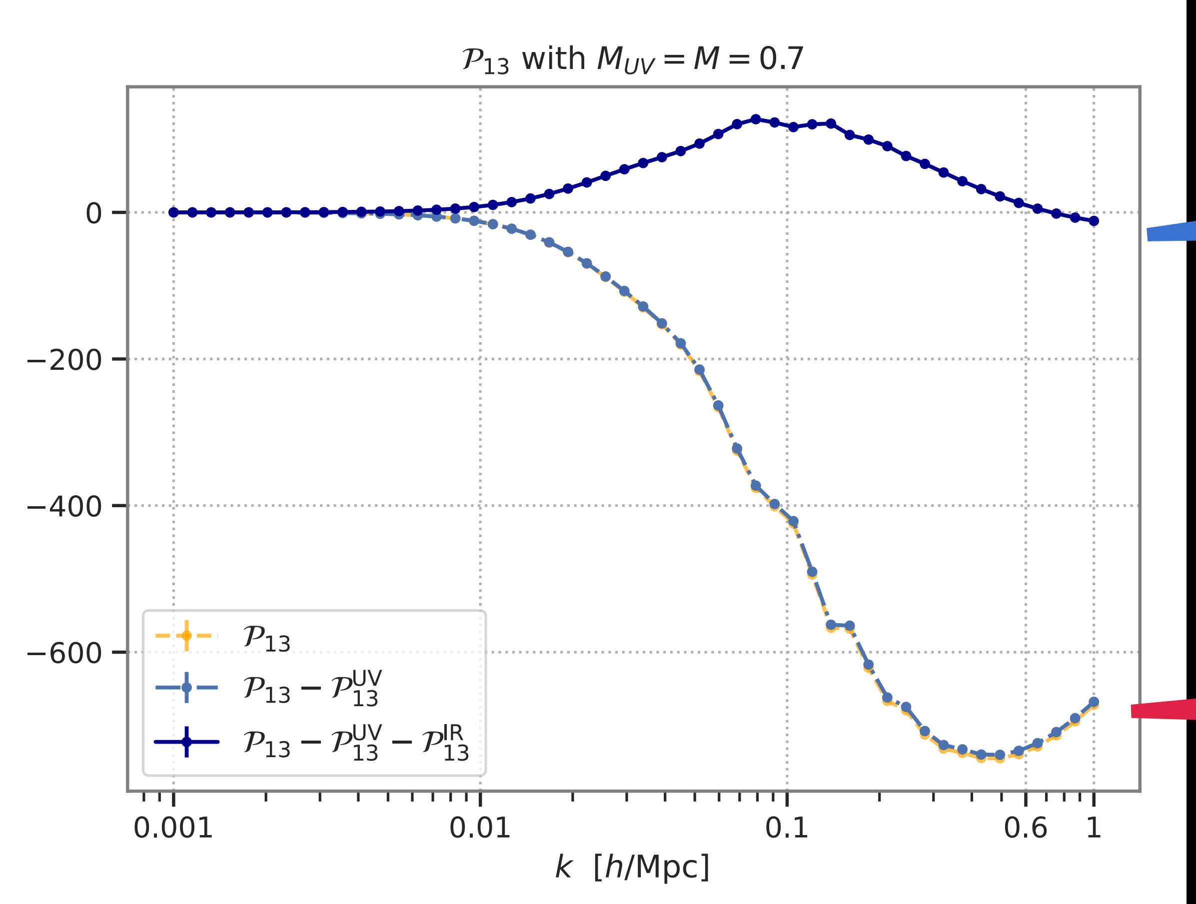


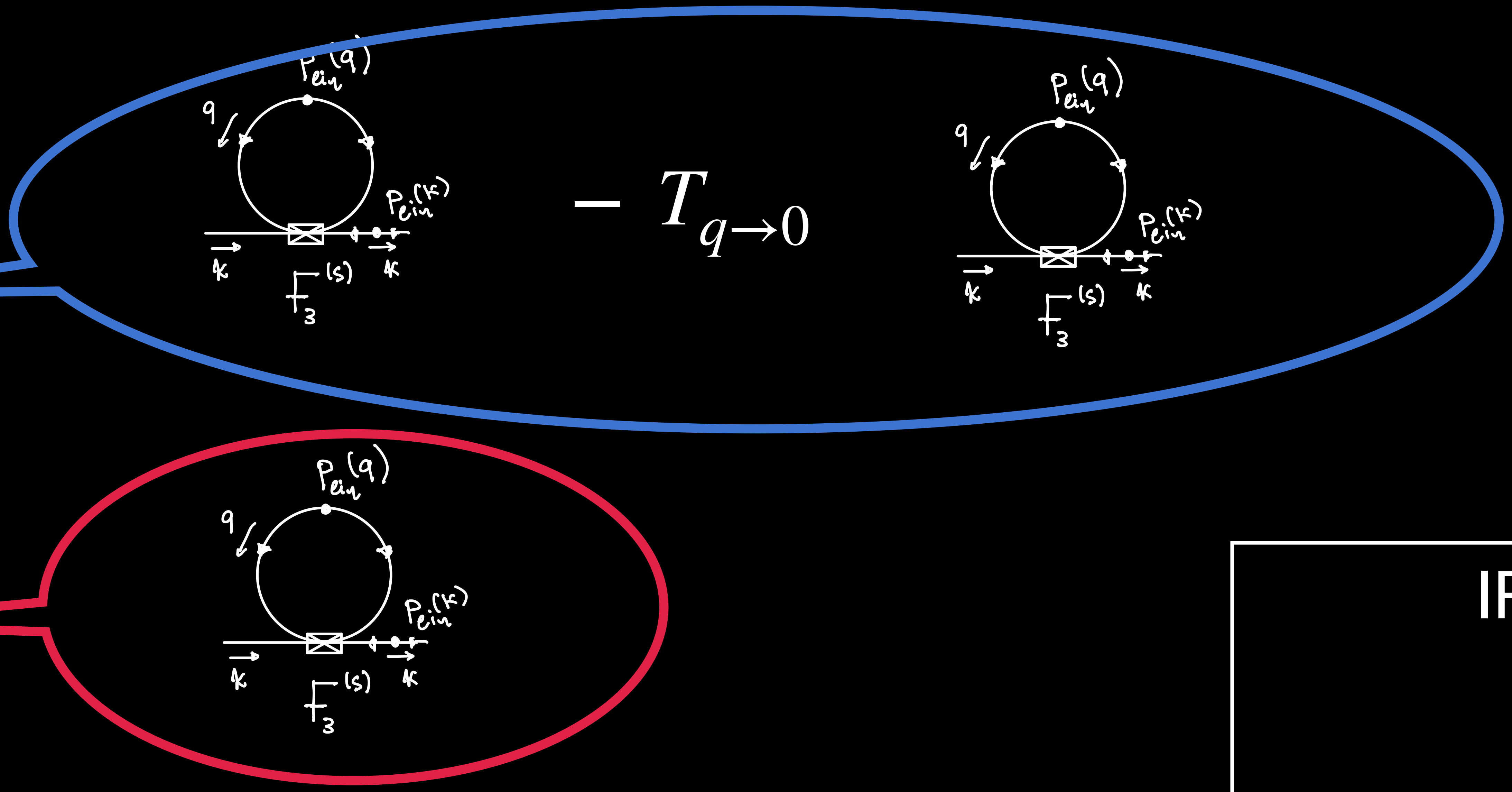
UV subtracted diagrams receive very small contributions from non-linear region



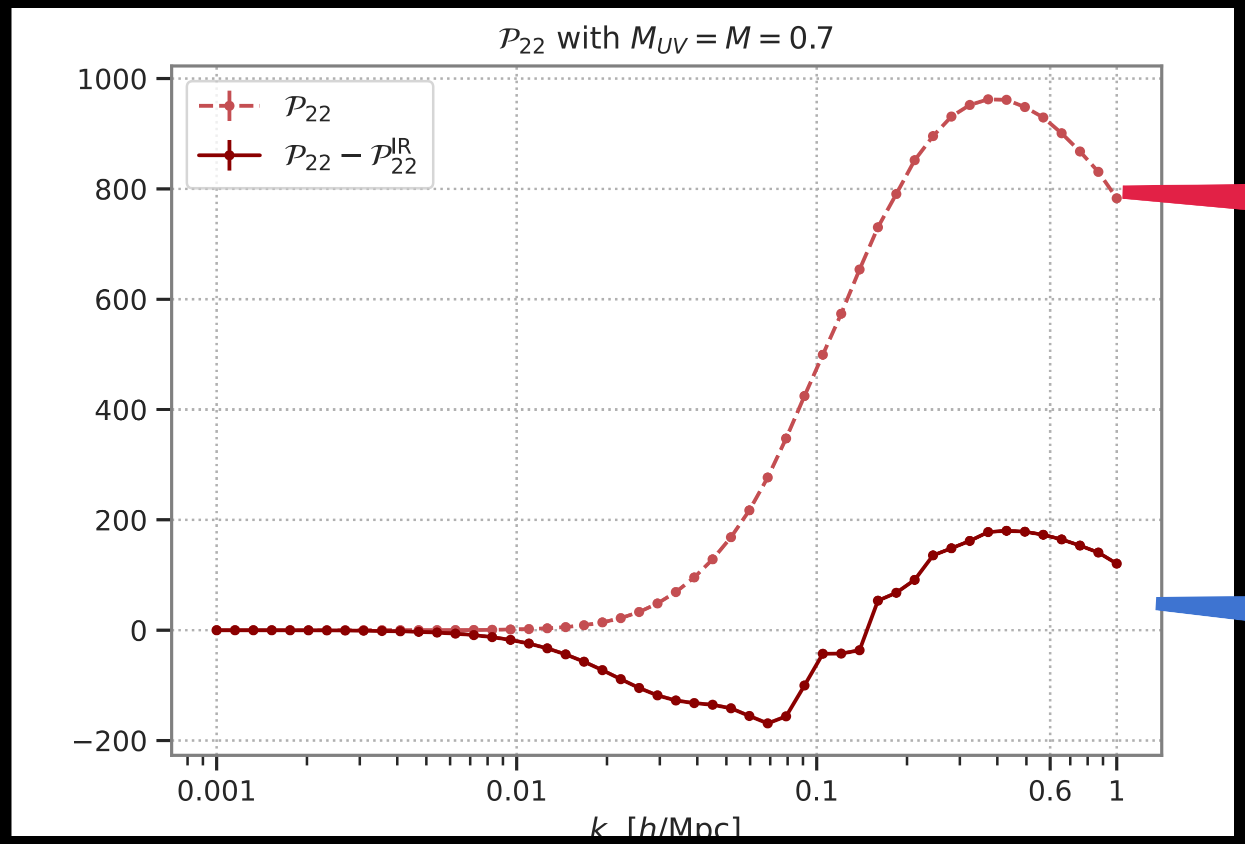
Fraction of SPT one-loop diagrams above $1\ h/Mpc$

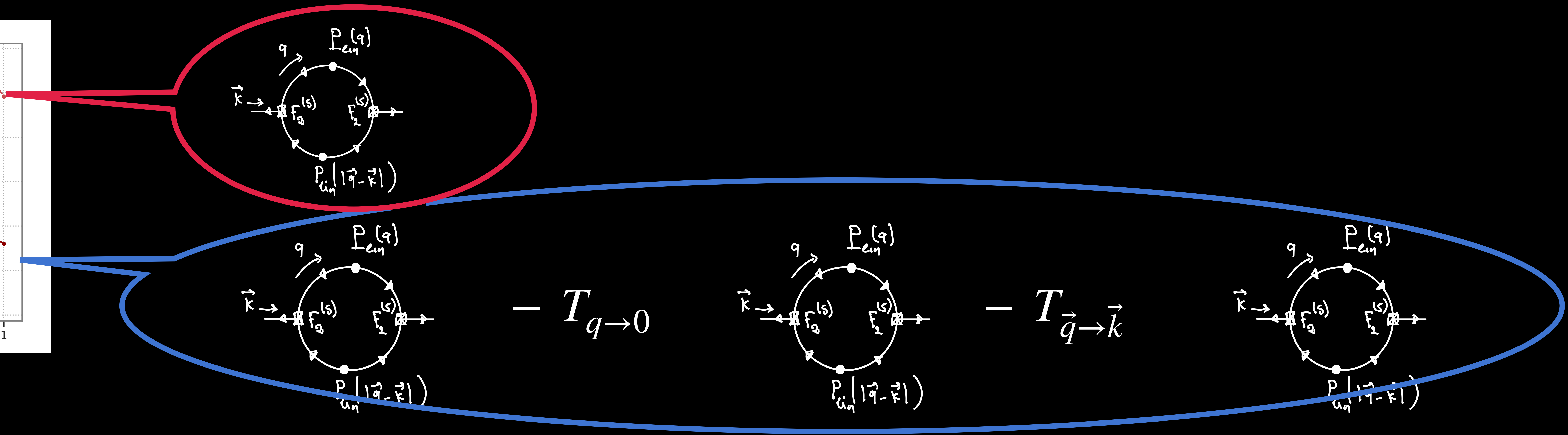
Large IR cancellations among diagrams



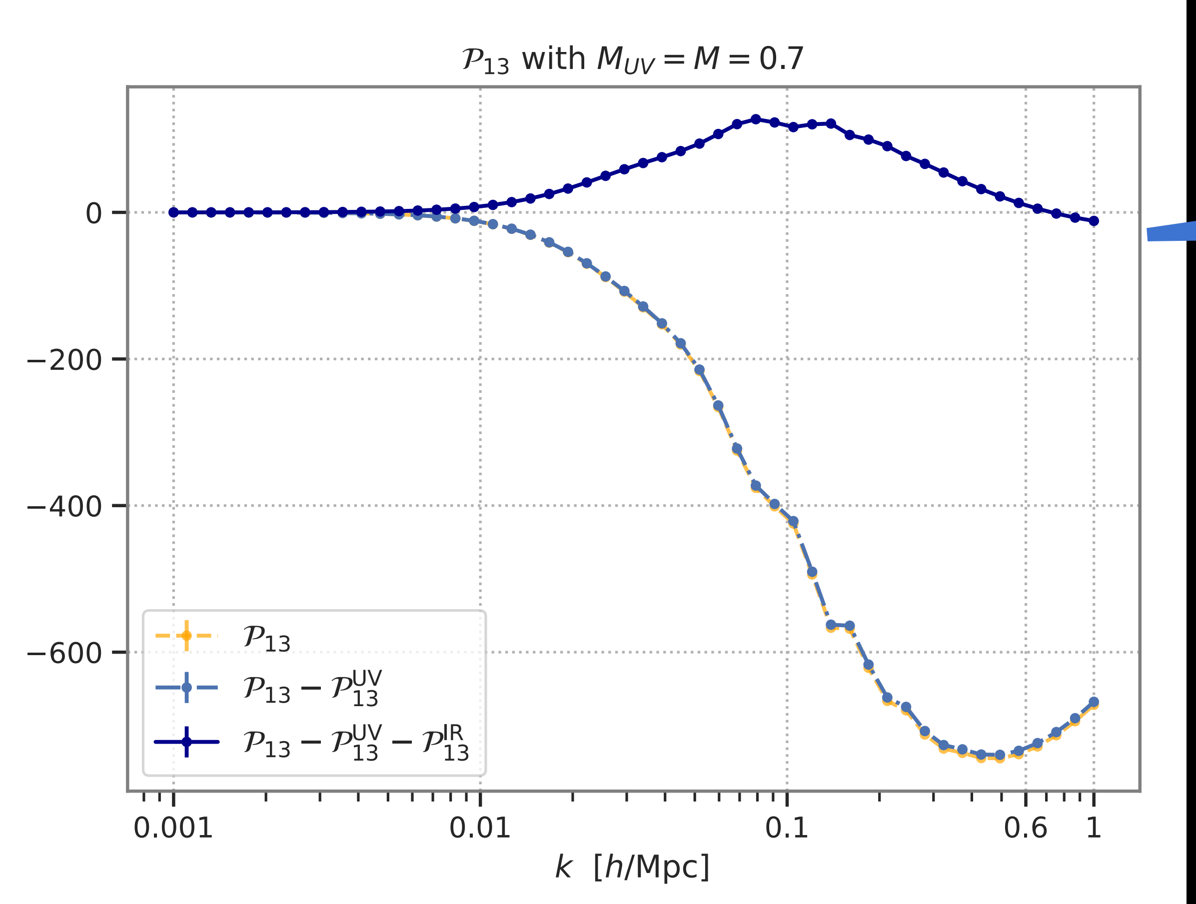


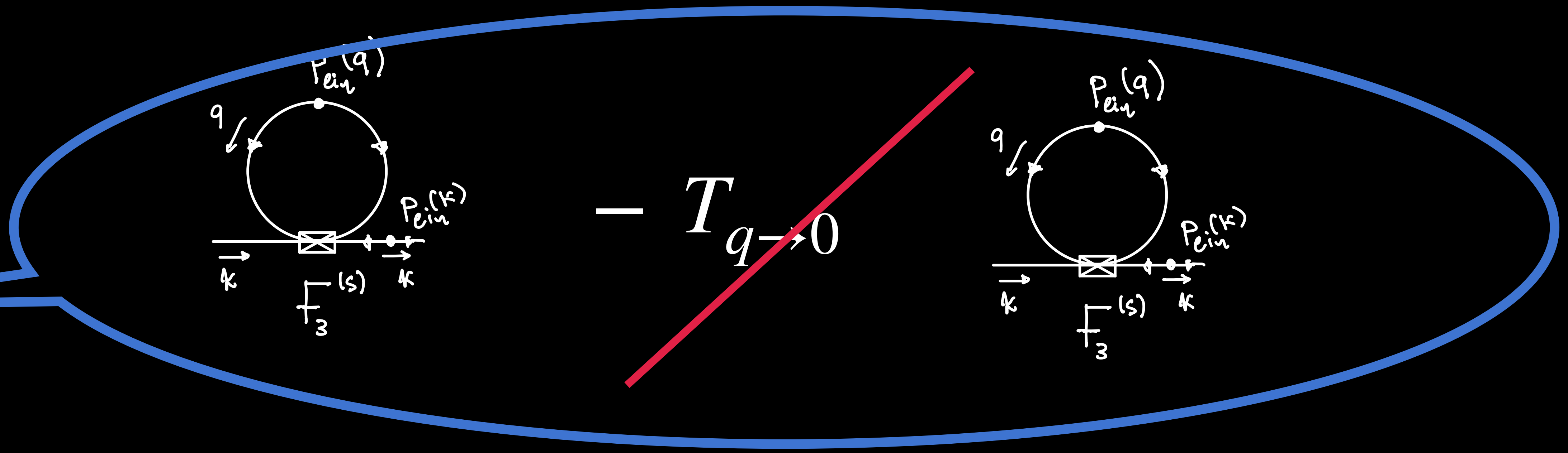
IR subtracted diagrams do not exhibit large cancellations



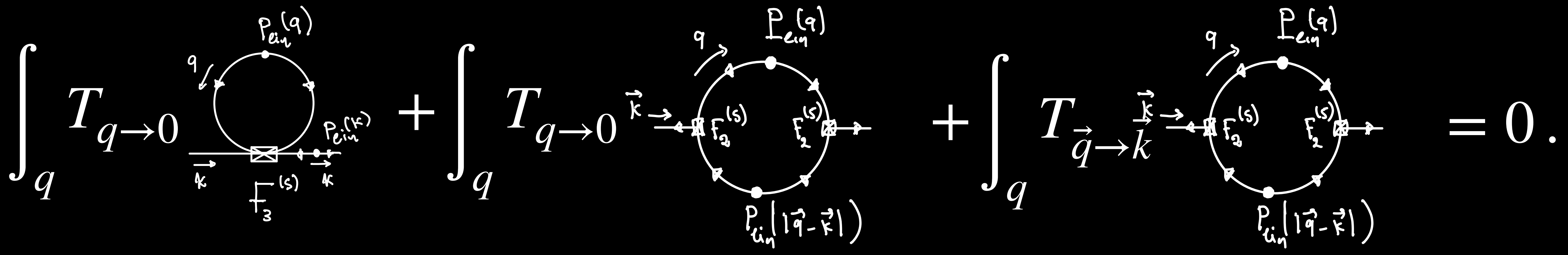


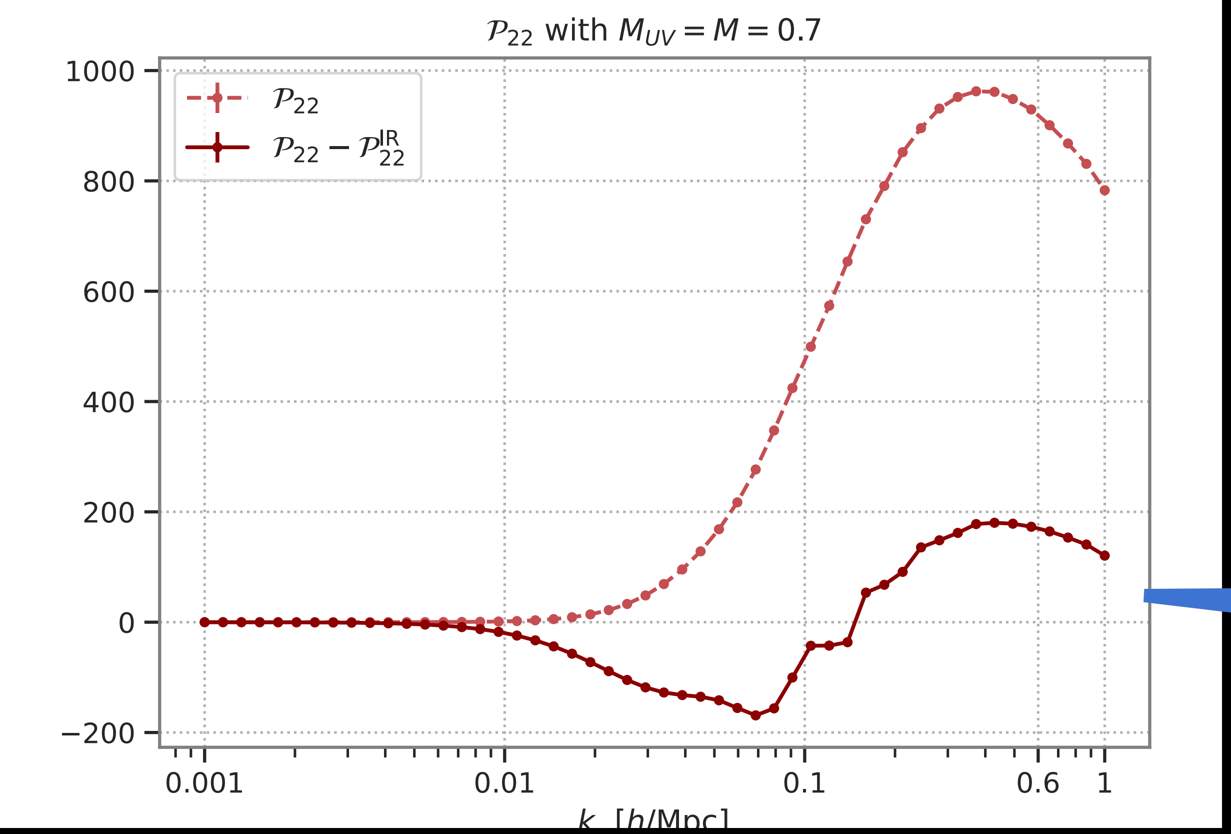
Theory free of IR singularities

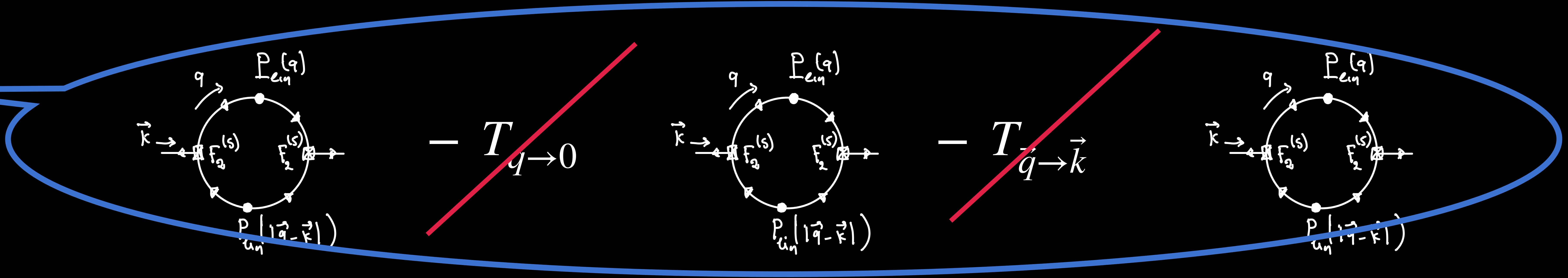




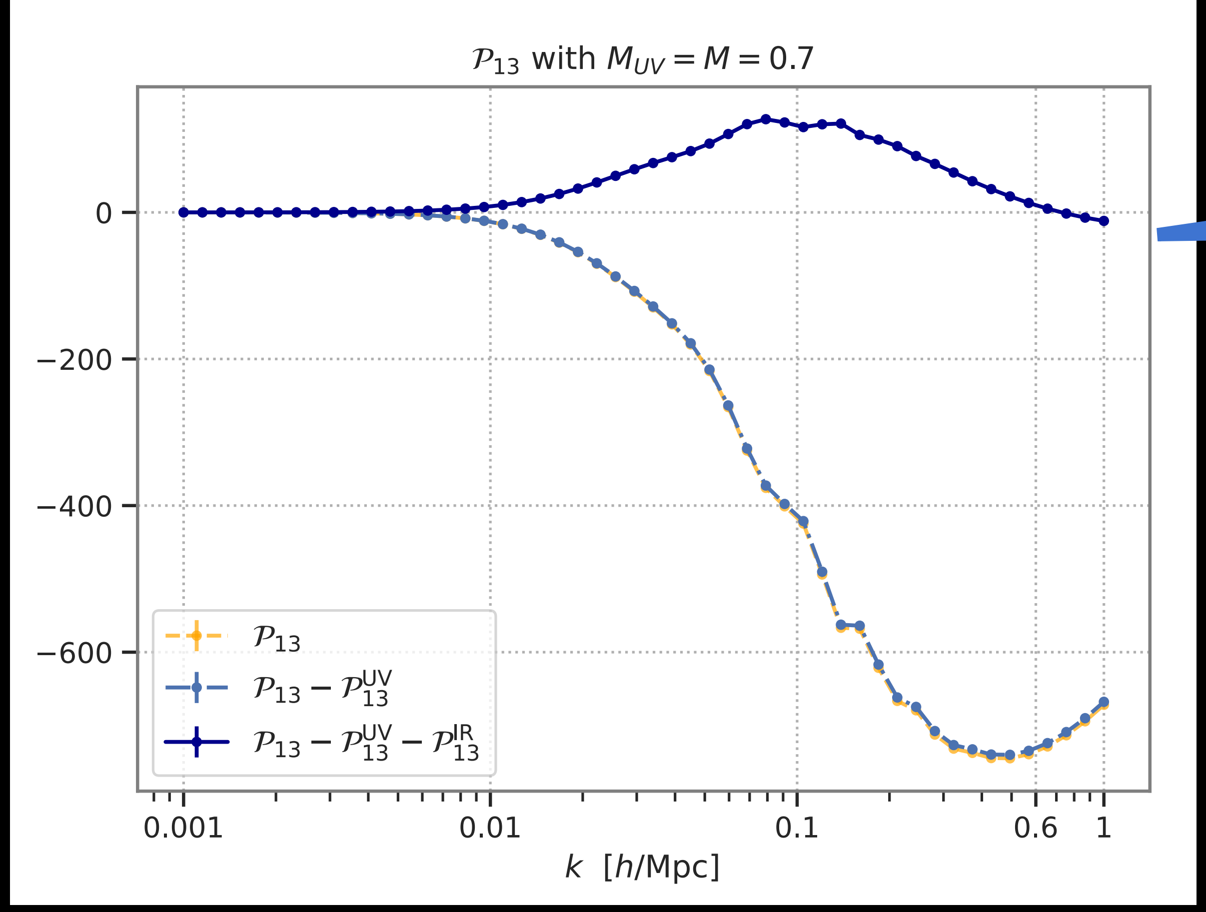
In the sum of diagrams all IR divergences dancel

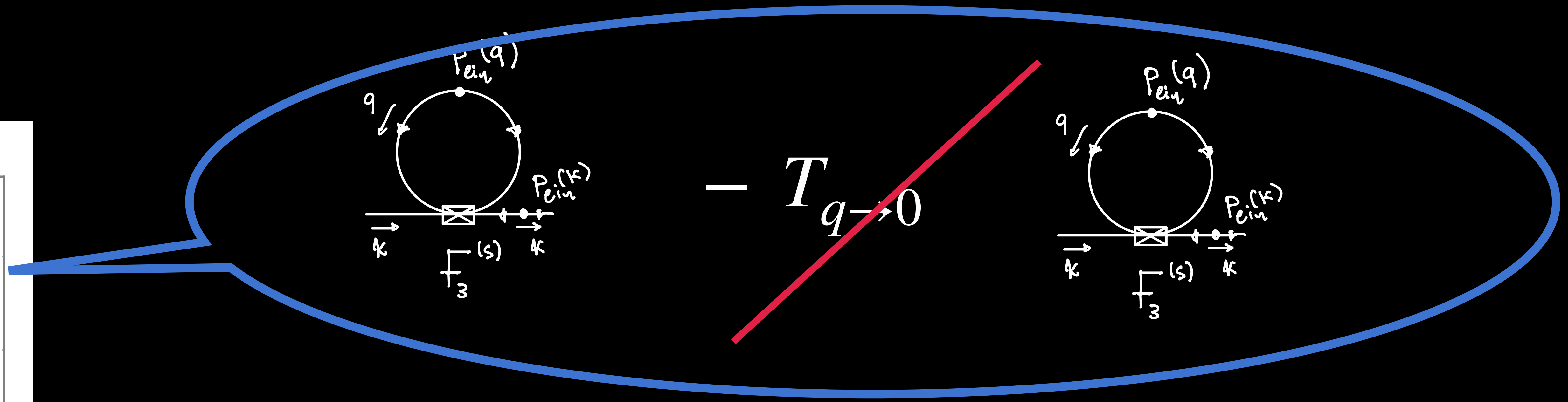






Theory free of IR singularities





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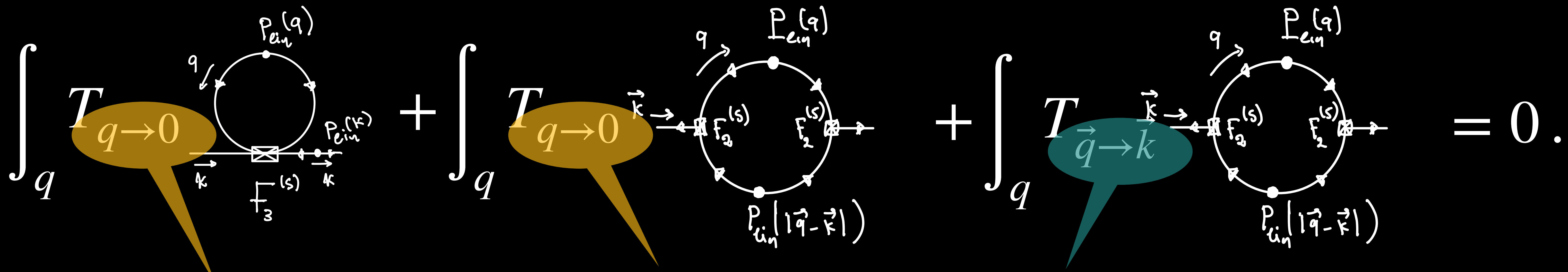
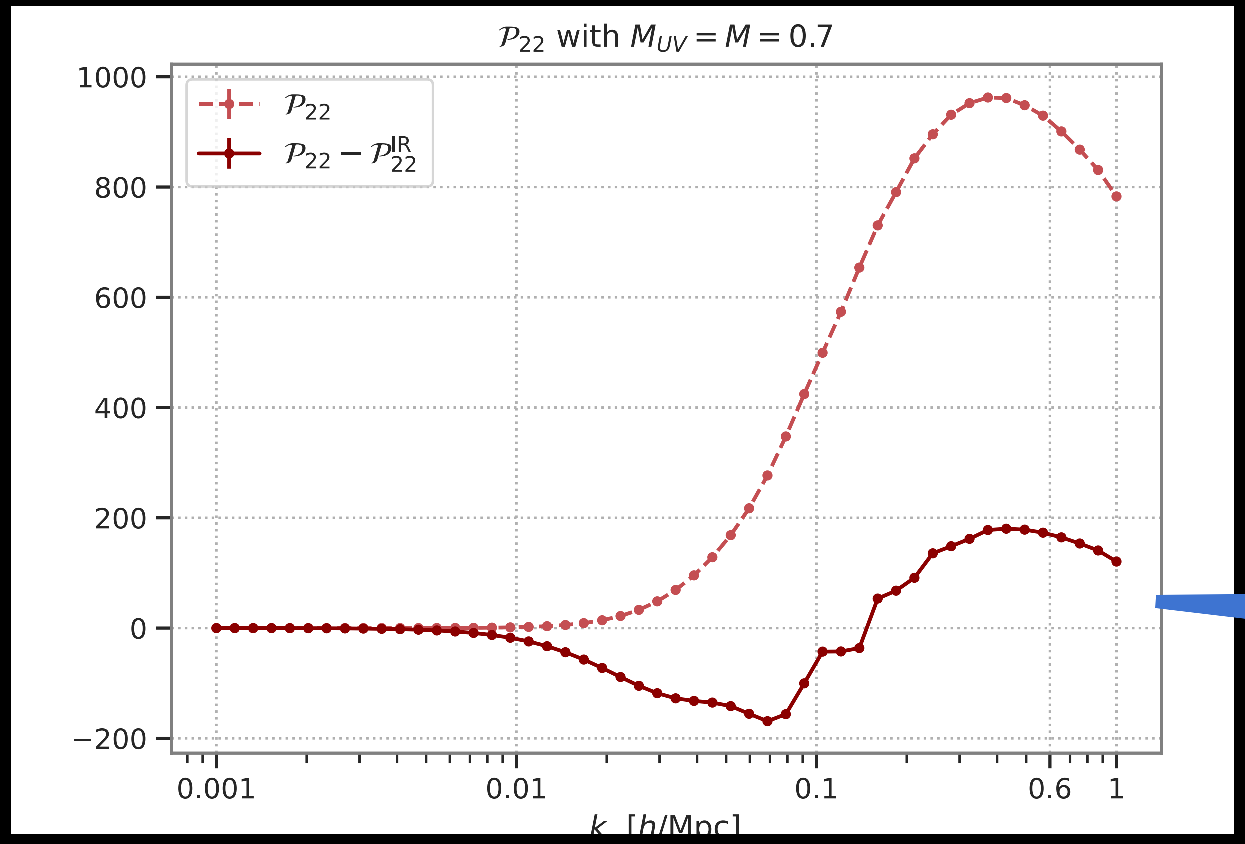
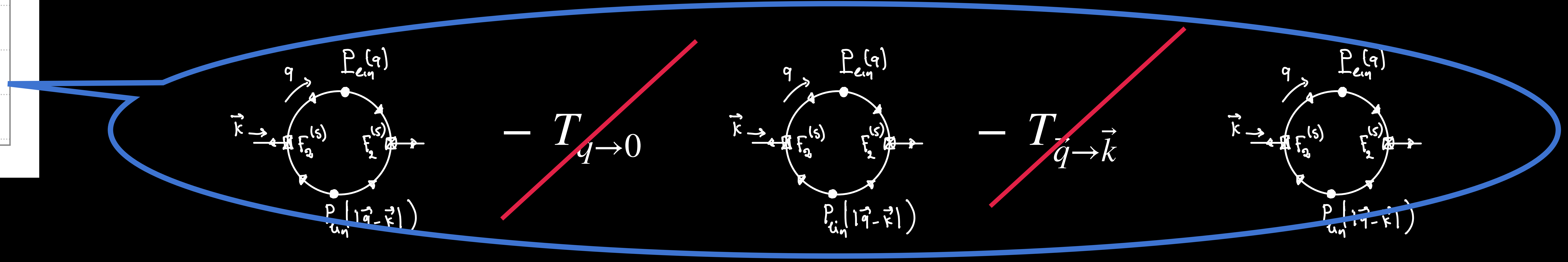
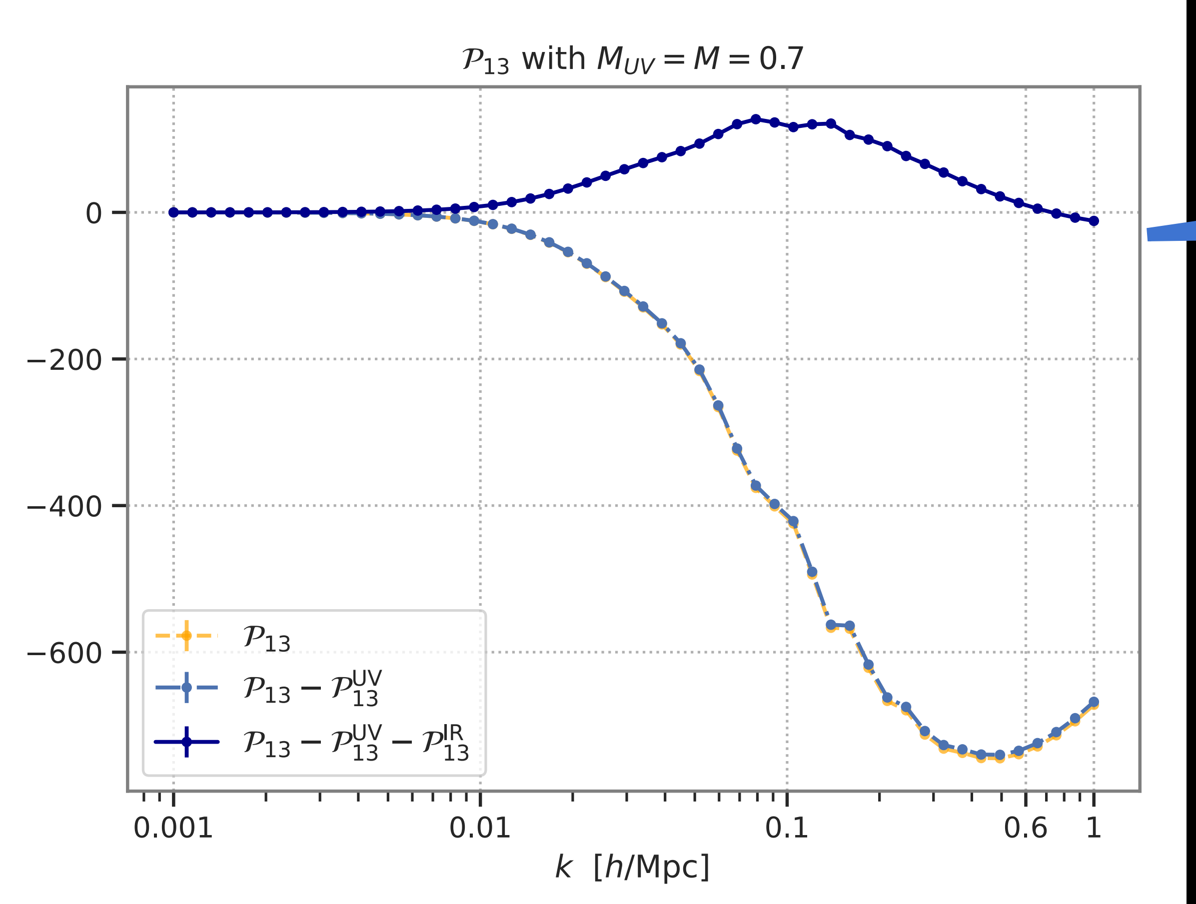


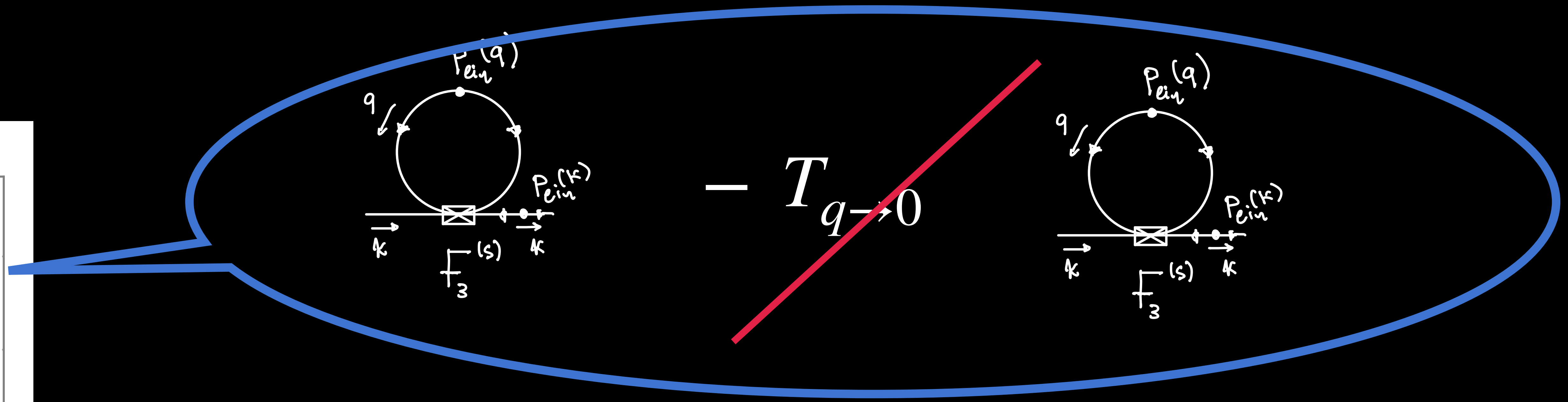
Cancellations are not local





Theory free of IR singularities





In the sum of diagrams all IR divergences dancel

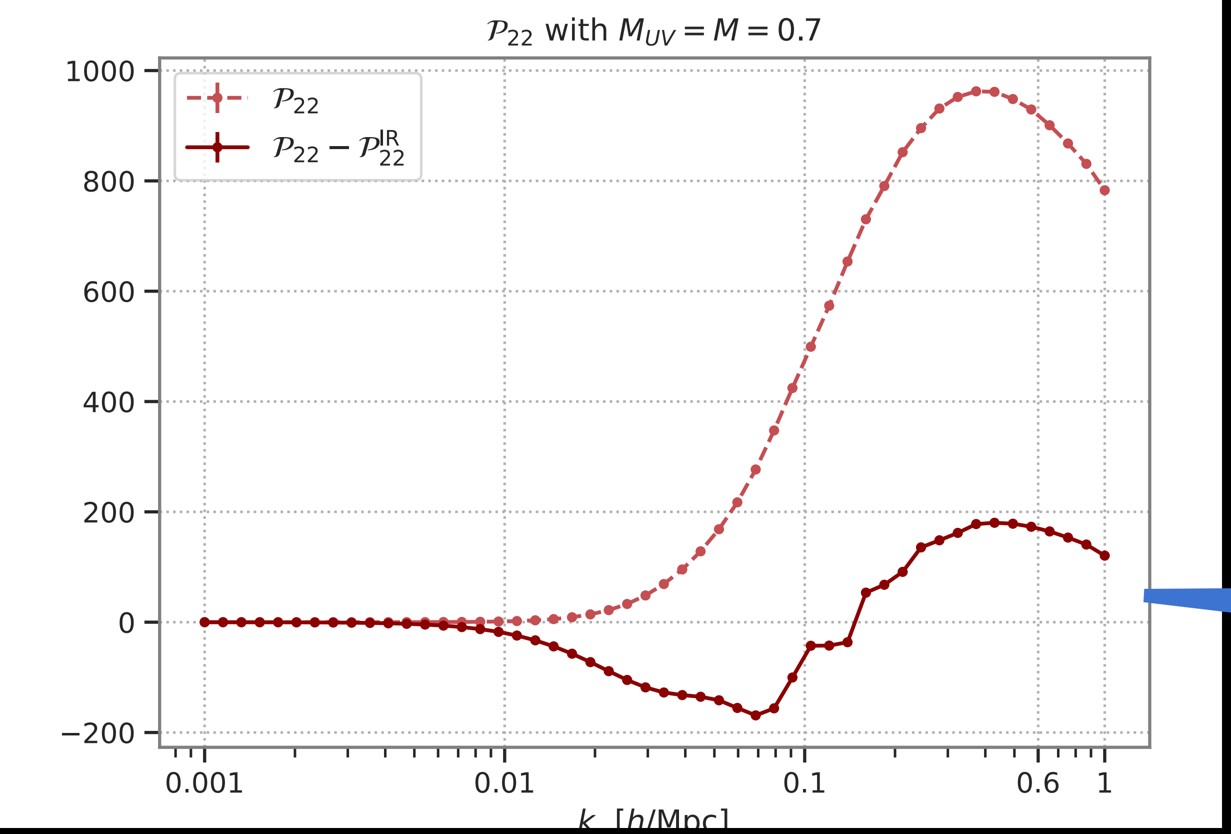
For an efficient numerical method, we want to implement these cancellations before integration!

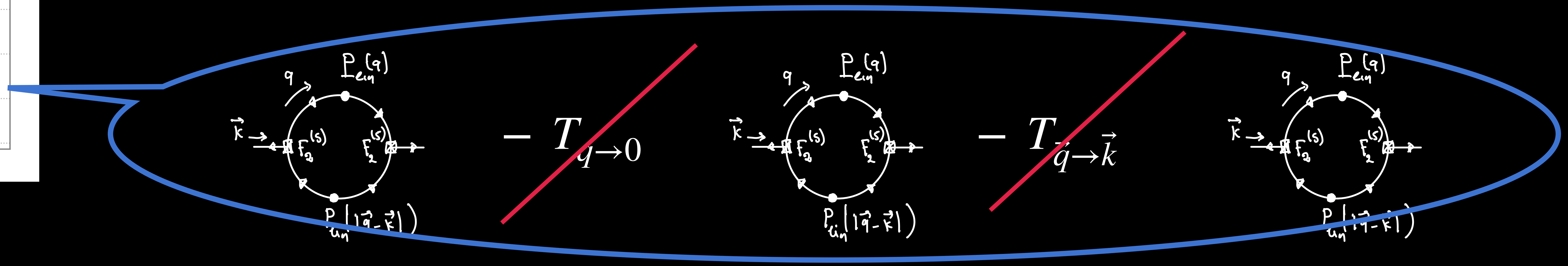
Method 1: Local IR subtractions per diagram (as in this slide)

e.g. C.A.,Sterman [1812.03753]

Method 2: Construct a single integrand for the sum of all diagrams with local IR cancellations.-

Carrasco, Foreman, Green, Senatore [1304.4946]

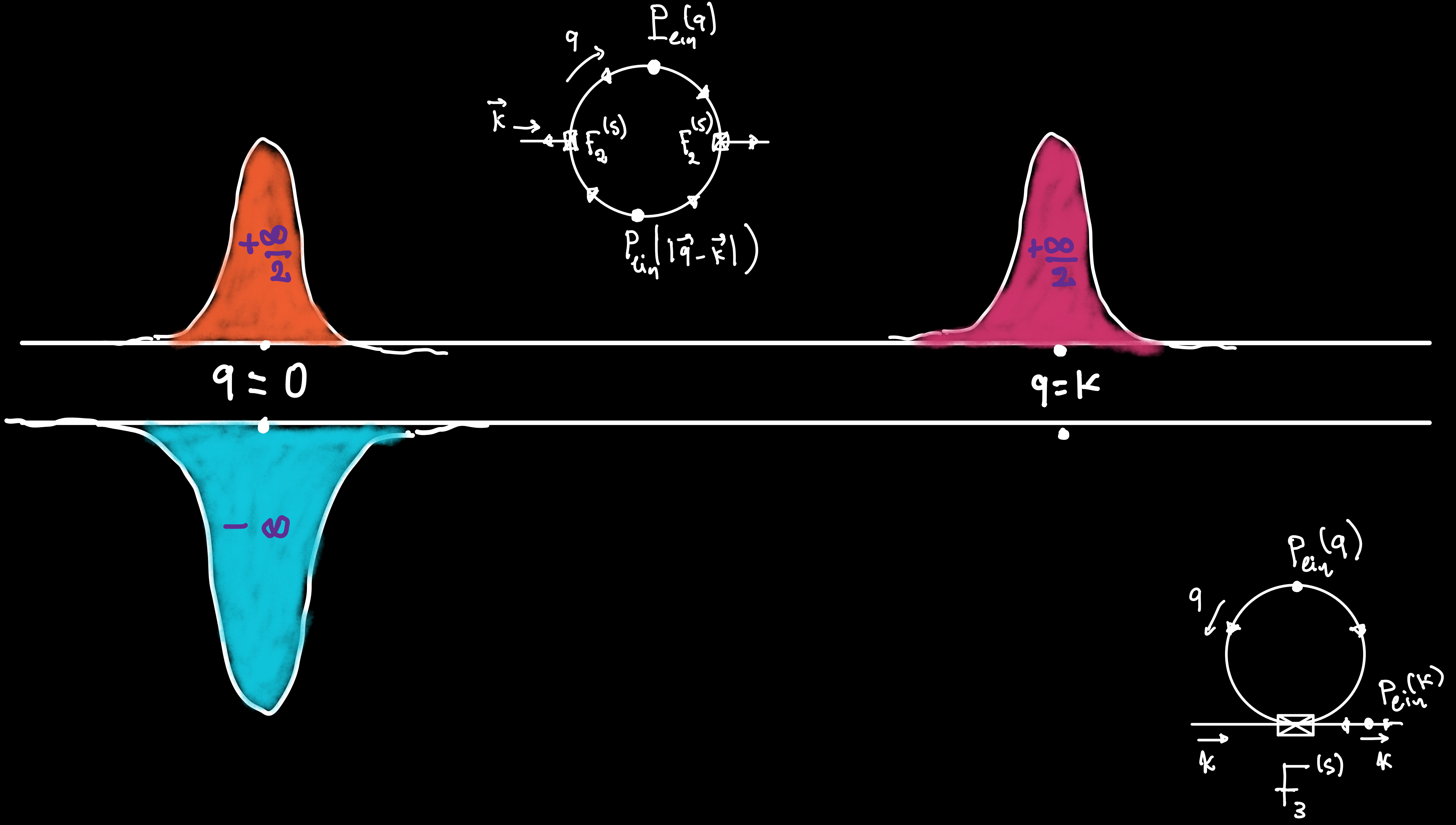




IR safe one-loop integrand

In the spirit of Carrasco, Foreman, Green, Senatore [1304.4946]

- Map all infrared singularities from all diagrams to a single point (the origin $\vec{q} = 0$) in the integration domain.
- Singularities are then destined to cancel locally.
- Partition of unity method to disentancle singularities.



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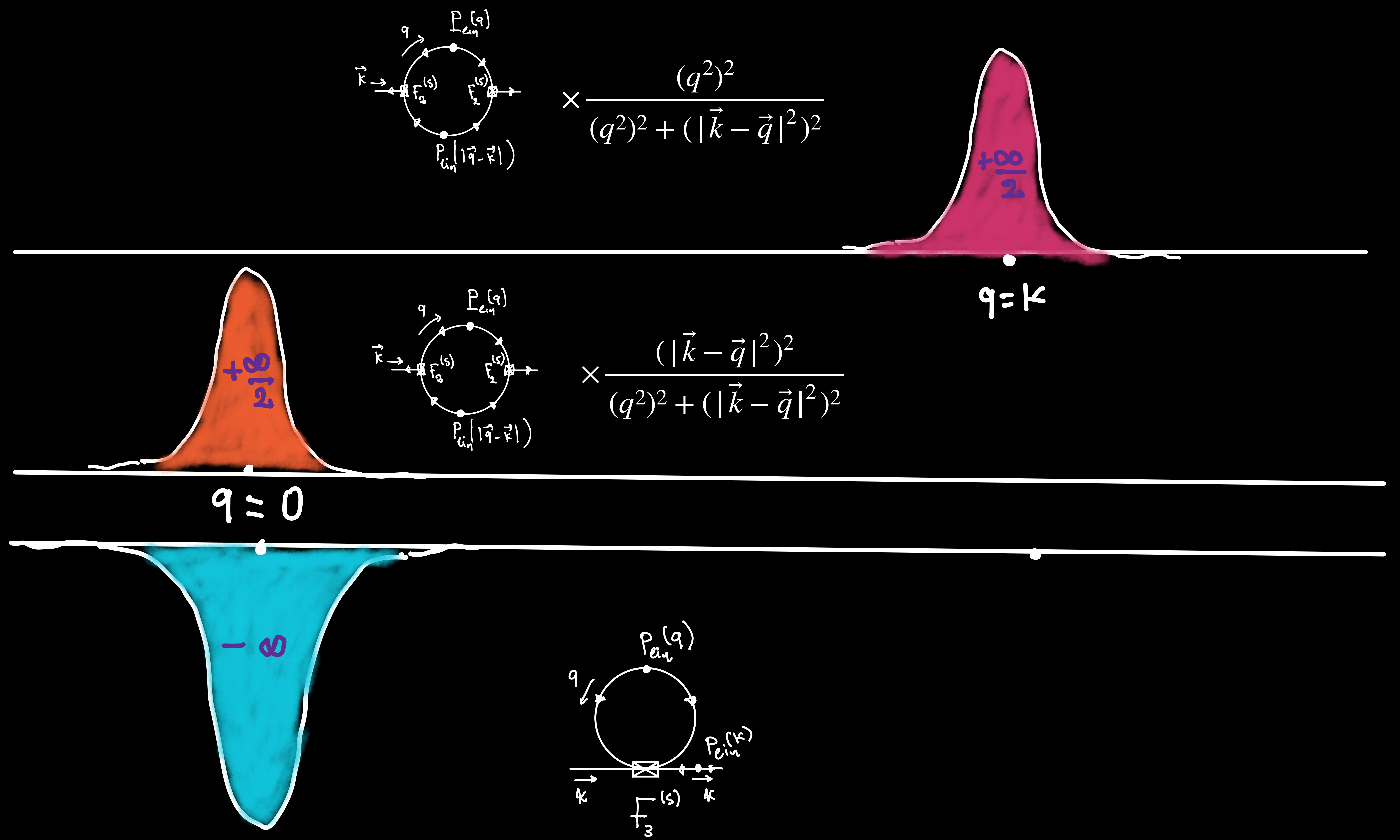

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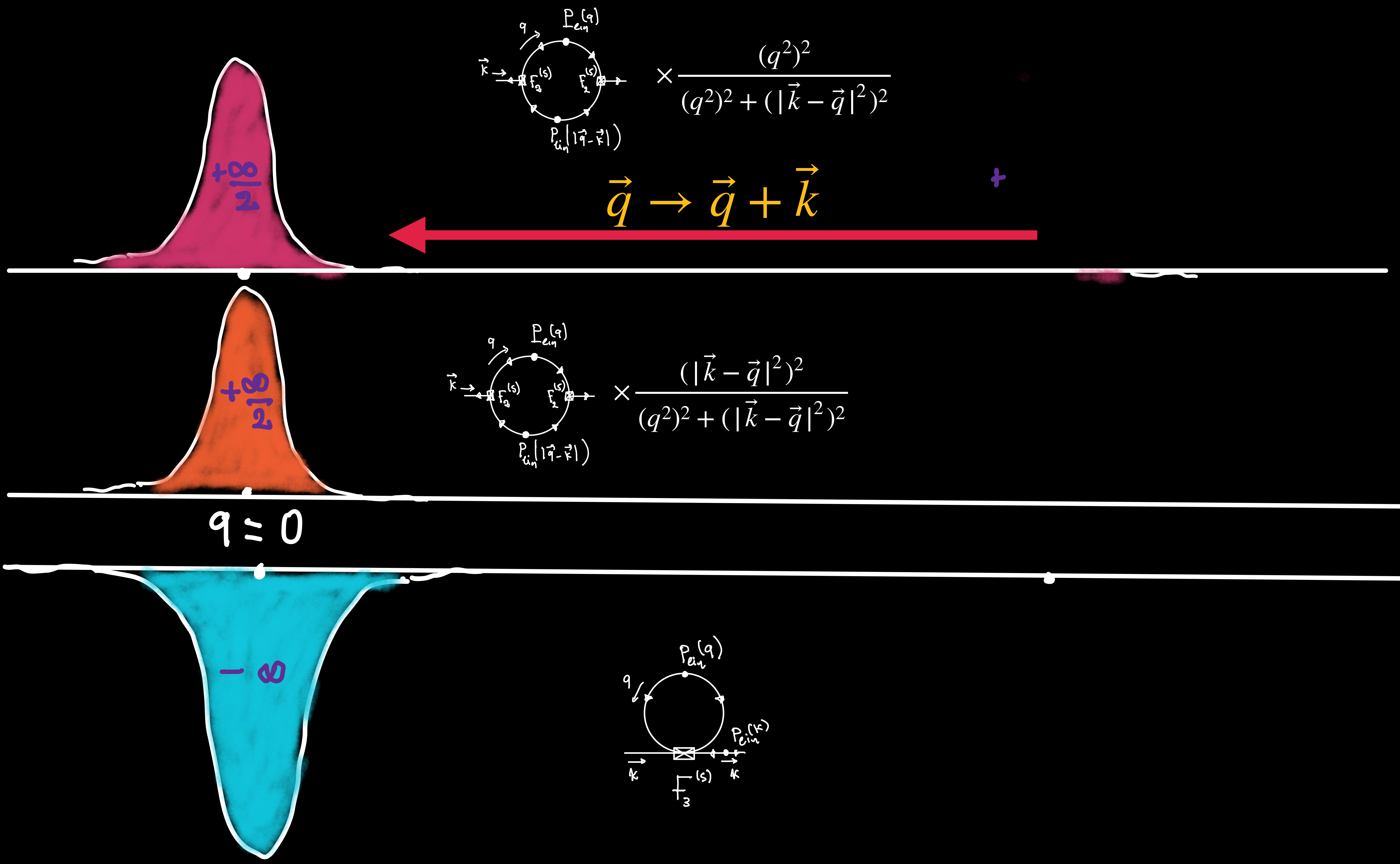
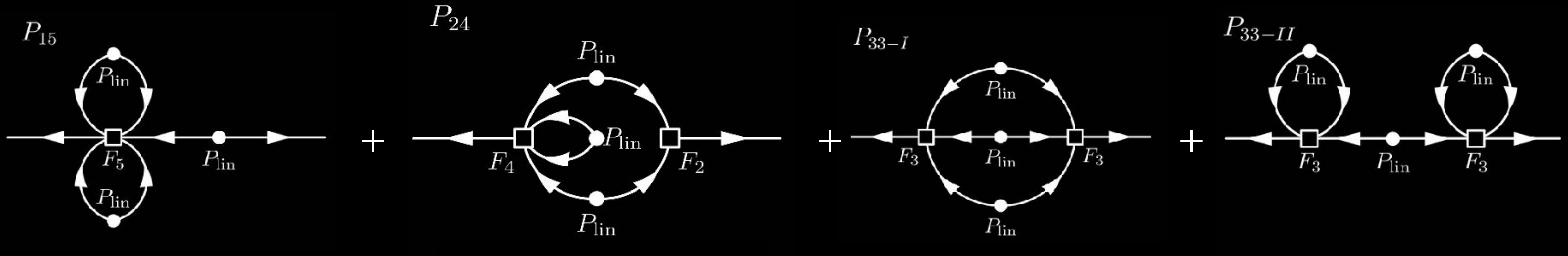


The two-loop corrections



+ EFT counterterms

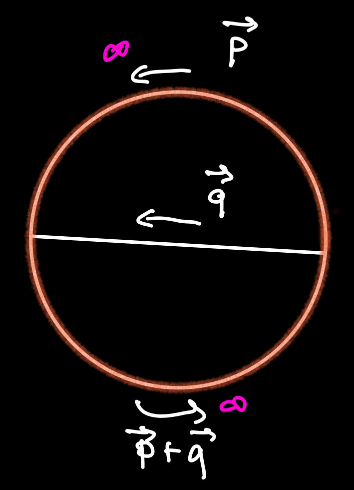
Two-loop UV subtraction

$$P_{ij}(\vec{k}) = \int d^3\vec{p}\,d^3\vec{p}\; p_{ij}\left(\vec{p}, \vec{q}, \vec{k}\right)$$

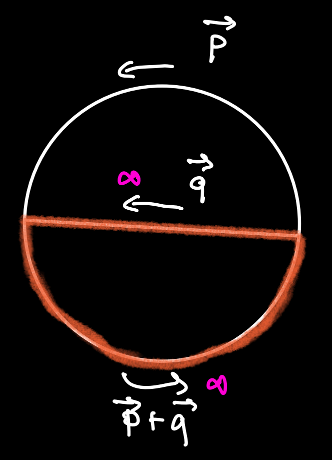
- "Single" and "double" UV singularities
- A nested set of subtractions (BPHZ-like)
- In a first step, subtract first single UV singularities.
- Subtract the double UV singularity from the outcome of the first step.

Double UV: $\vec{p} \to \infty$ and $\vec{q} \to \infty$

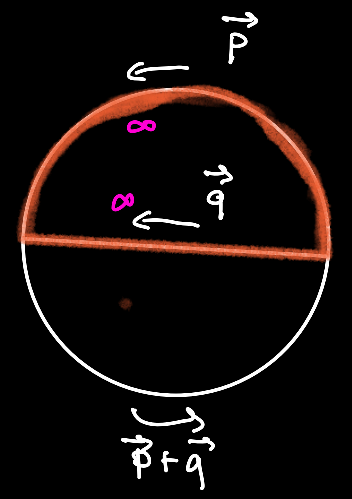
Single UV: $\vec{p} \to \infty$ and $\vec{q}$ fixed



Or $\vec{q} \to \infty$ and $\vec{p}$ fixed



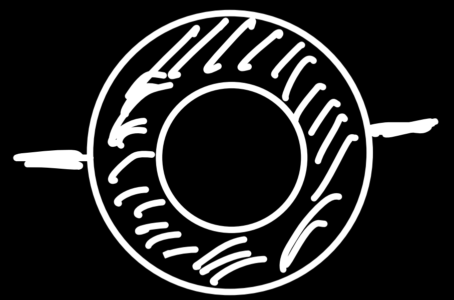
Or $\vec{q} \to \infty$ and $\vec{p} + \vec{q}$ fixed

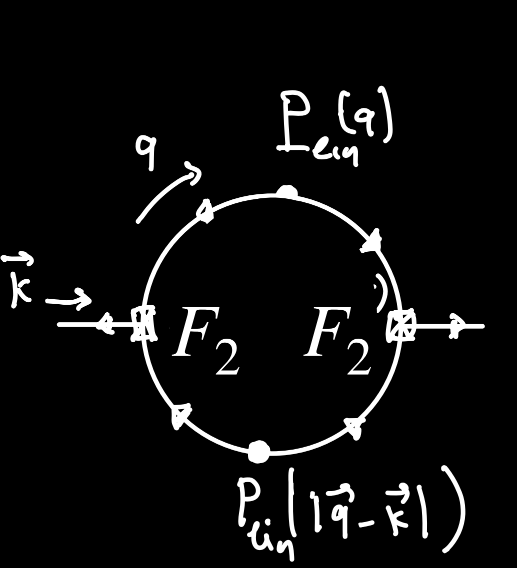


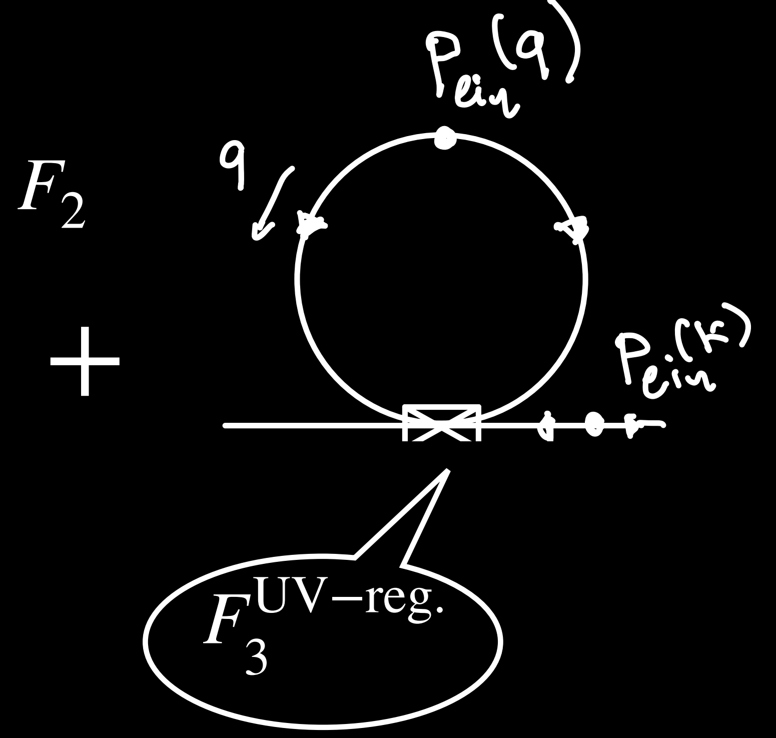
$$p_{ij}^{\mathrm{UV-reg.}} = \left(1 - R_{double-UV}\right)\left(1 - R_{single-UV}\right) p_{ij}$$

ALL UV subtractions can be made to act on the momentum-dependent vertices.

UV-safe power spectrum through two-loops

$$\mathscr{P}(k) \equiv$$  $$= \mathscr{P}^{\mathrm{UV-reg.}}_{1-\mathrm{loop}} + \mathscr{P}^{\mathrm{UV-reg.}}_{2-\mathrm{loop}} + (\mathrm{EFT\ counterterms})'$$

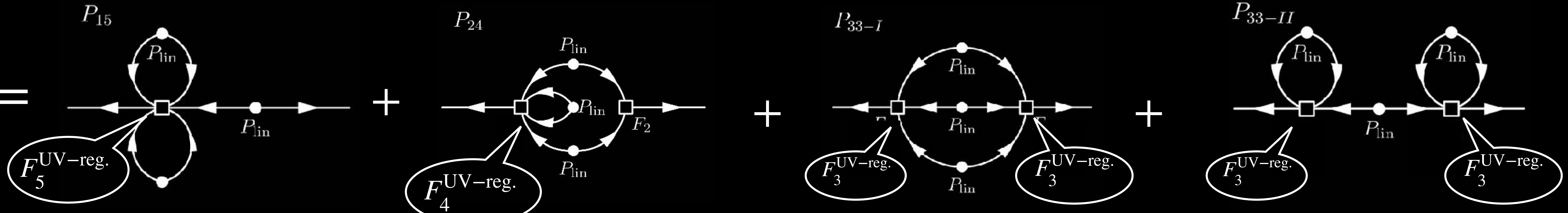
$$\mathscr{P}^{\mathrm{UV-reg.}}_{1-\mathrm{loop}} =$$ 

$+$ 

$$F_3^{\mathrm{UV-reg.}} = \left(1 - \mathscr{R}_{q\to\infty}\right) F_3^{(\mathrm{s})} \qquad F_4^{\mathrm{UV-reg.}} = \left(1 - \mathscr{R}_{p\to\infty}\right) F_4^{(\mathrm{s})}$$

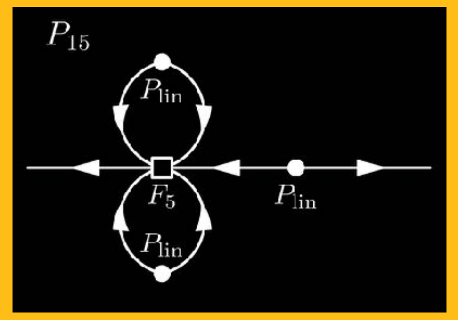
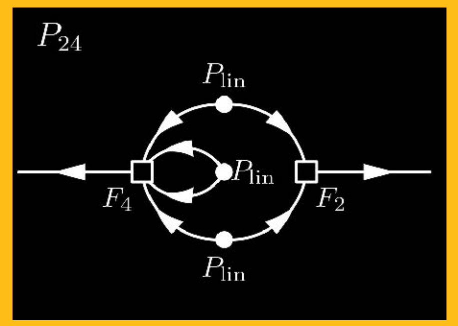
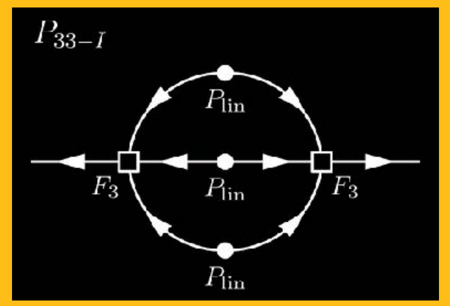
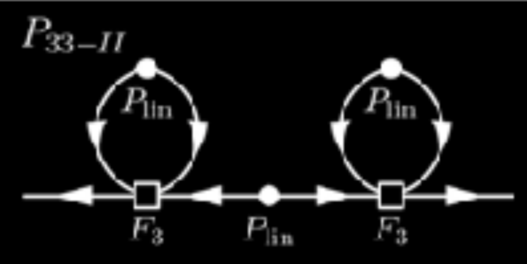
$$F_5^{\mathrm{UV-reg.}} = \left(1 - \mathscr{R}^{(0)}_{p,q\to\infty} - \mathscr{R}^{(2)}_{p,q\to\infty}\right)\left(1 - \mathscr{R}_{q\to\infty} - \mathscr{R}_{p\to\infty}\right) F_5^{(\mathrm{s})}$$

Modified vertices

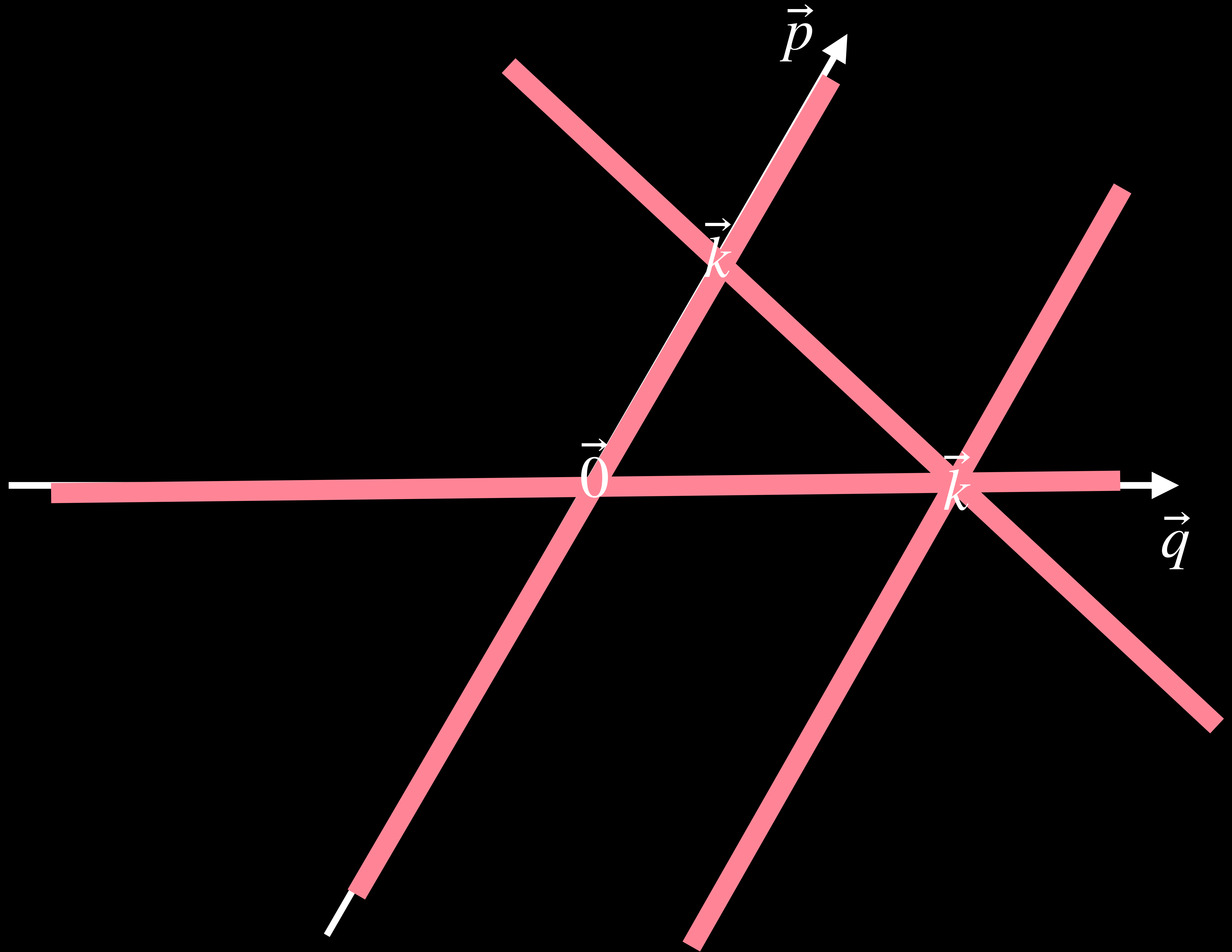
$$\mathscr{P}^{\mathrm{UV-reg.}}_{2-\mathrm{loop}} =$$ 

$F_4^{\mathrm{UV-reg.}}$

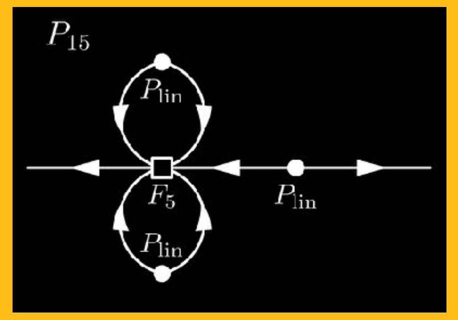
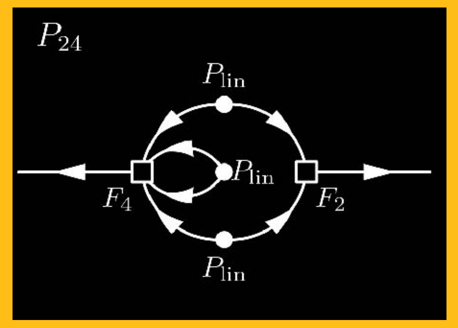
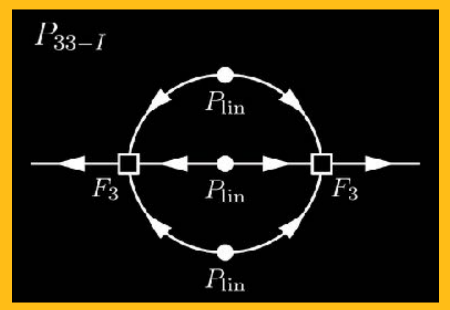
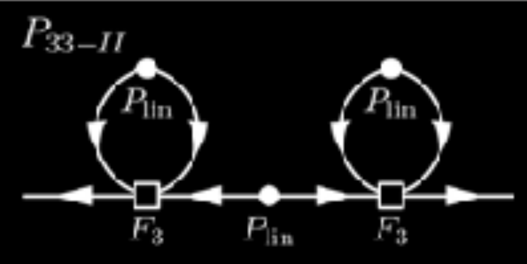
IR-safe two-loop power spectrum

DIAGRAM	LOCATION OF *SINGLE* IR SINGULARITIES
	$\vec{p} = 0$ or $\vec{q} = 0$
	$\vec{p} = 0$, or $\vec{q} = 0$, or $\vec{q} = \vec{k}$
	$\vec{p} = 0$, or $\vec{q} = 0$, or $\vec{p} + \vec{q} = \vec{k}$
	$\vec{p} = 0$ or $\vec{q} = 0$

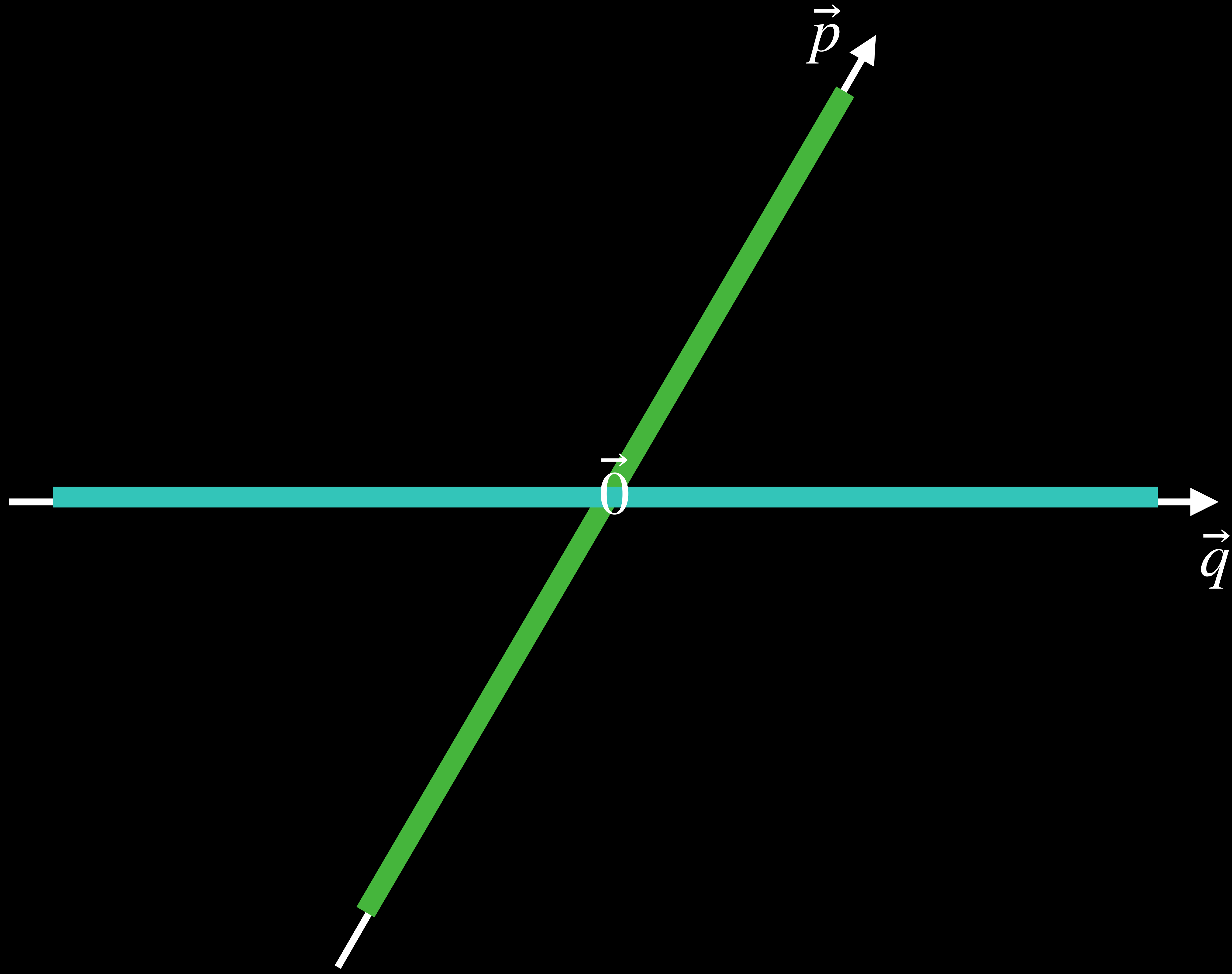
Web of leading IR singularities



IR-safe two-loop power spectrum

DIAGRAM	LOCATION OF *SINGLE* IR SINGULARITIES
	$\vec{p}=0$ or $\vec{q}=0$
	$\vec{p}=0$, or $\vec{q}=0$, or $\vec{q}=\vec{k}$
	$\vec{p}=0$, or $\vec{q}=0$, or $\vec{p}+\vec{q}=\vec{k}$
	$\vec{p}=0$ or $\vec{q}=0$

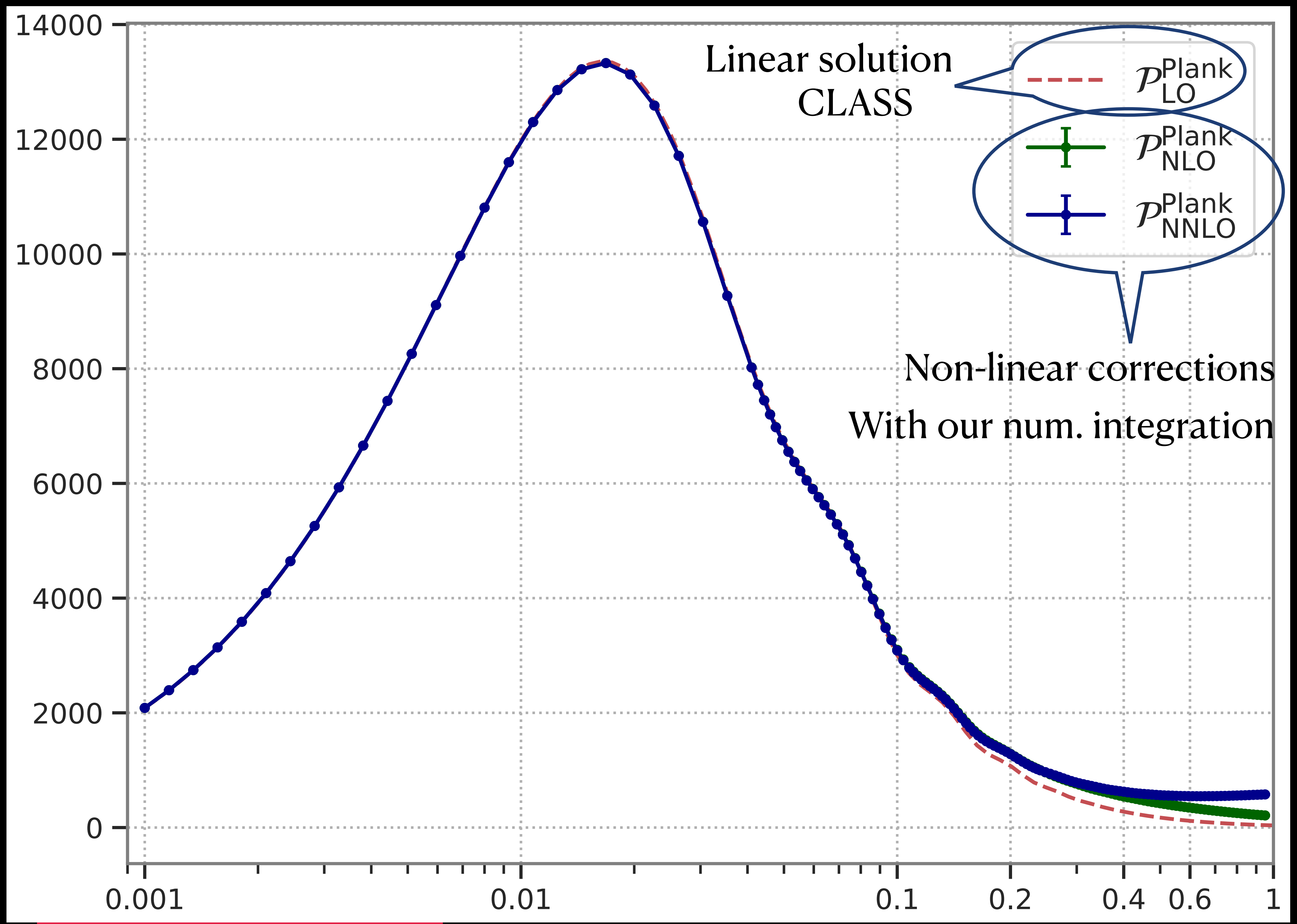
After partition of unity and shifts leading IR singularities cancel locally

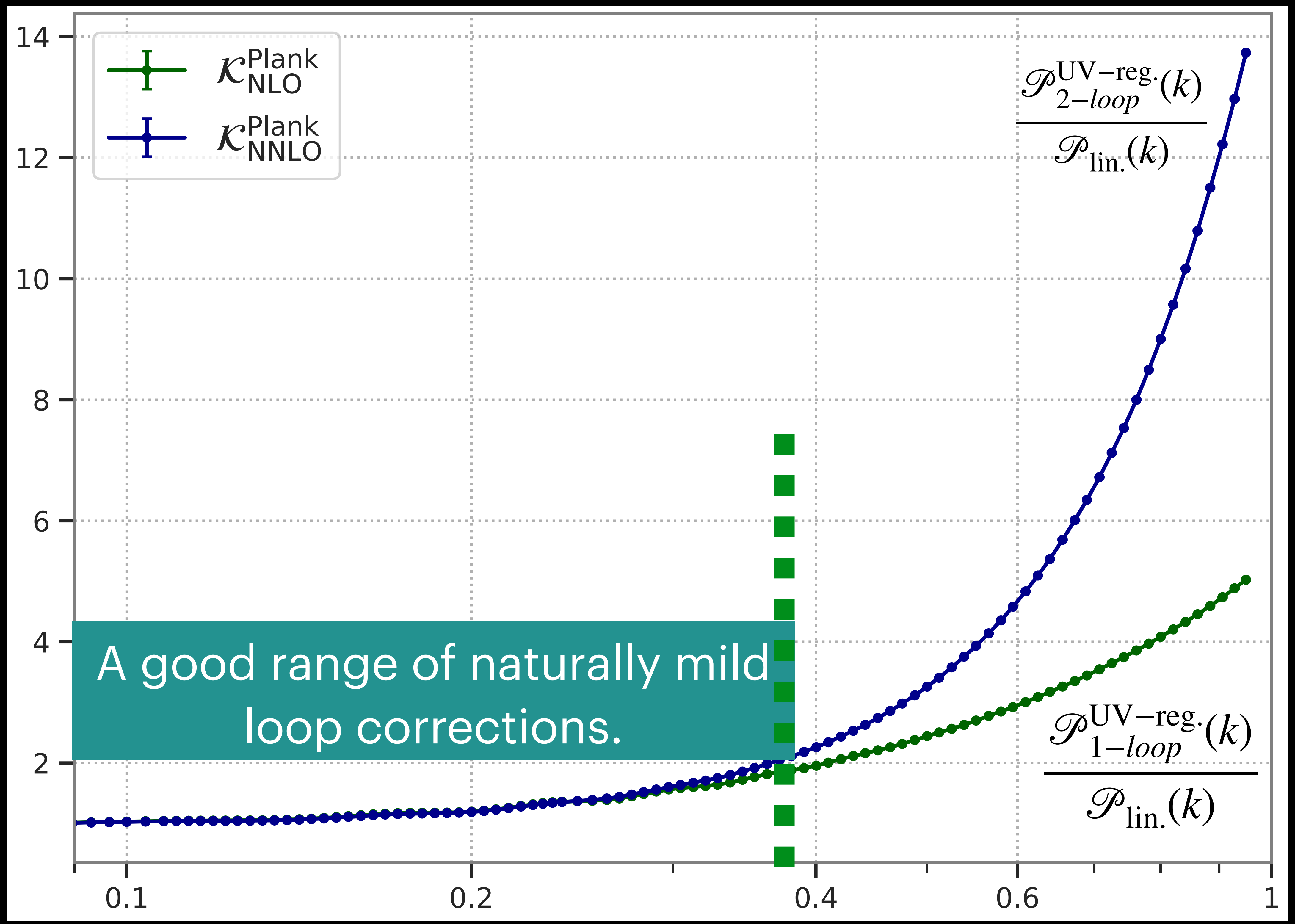


Numerical impact of loop corrections

$$\mathscr{P}_{\mathrm{NLO}}(k) = \mathscr{P}_{\mathrm{lin}}(k) + \mathscr{P}^{\mathrm{UV-reg.}}_{1-\mathrm{loop}}(k, M) + \mathrm{CTs}^{(1)}(k, M)$$

$$\mathscr{P}_{\mathrm{NNLO}}(k) = \mathscr{P}_{\mathrm{lin}}(k) + \mathscr{P}^{\mathrm{UV-reg.}}_{1-\mathrm{loop}}(k, M) + \mathscr{P}^{\mathrm{UV-reg.}}_{2-\mathrm{loop}}(k, M) + \mathrm{CTs}^{(2)}(k, M)$$





ΛCDM model with parameters fitting CMB data of Planck

$$\omega_b^{(\mathrm{Planck})} = 0.02237$$
$$\omega_{\mathrm{cdm}}^{(\mathrm{Planck})} = 0.1203$$
$$\ln 10^{10} A_s^{(\mathrm{Planck})} = 3.044$$
$$n_s^{(\mathrm{Planck})} = 0.965$$
$$\sum m_{\nu_i}^{(\mathrm{Planck})} = 0.06$$

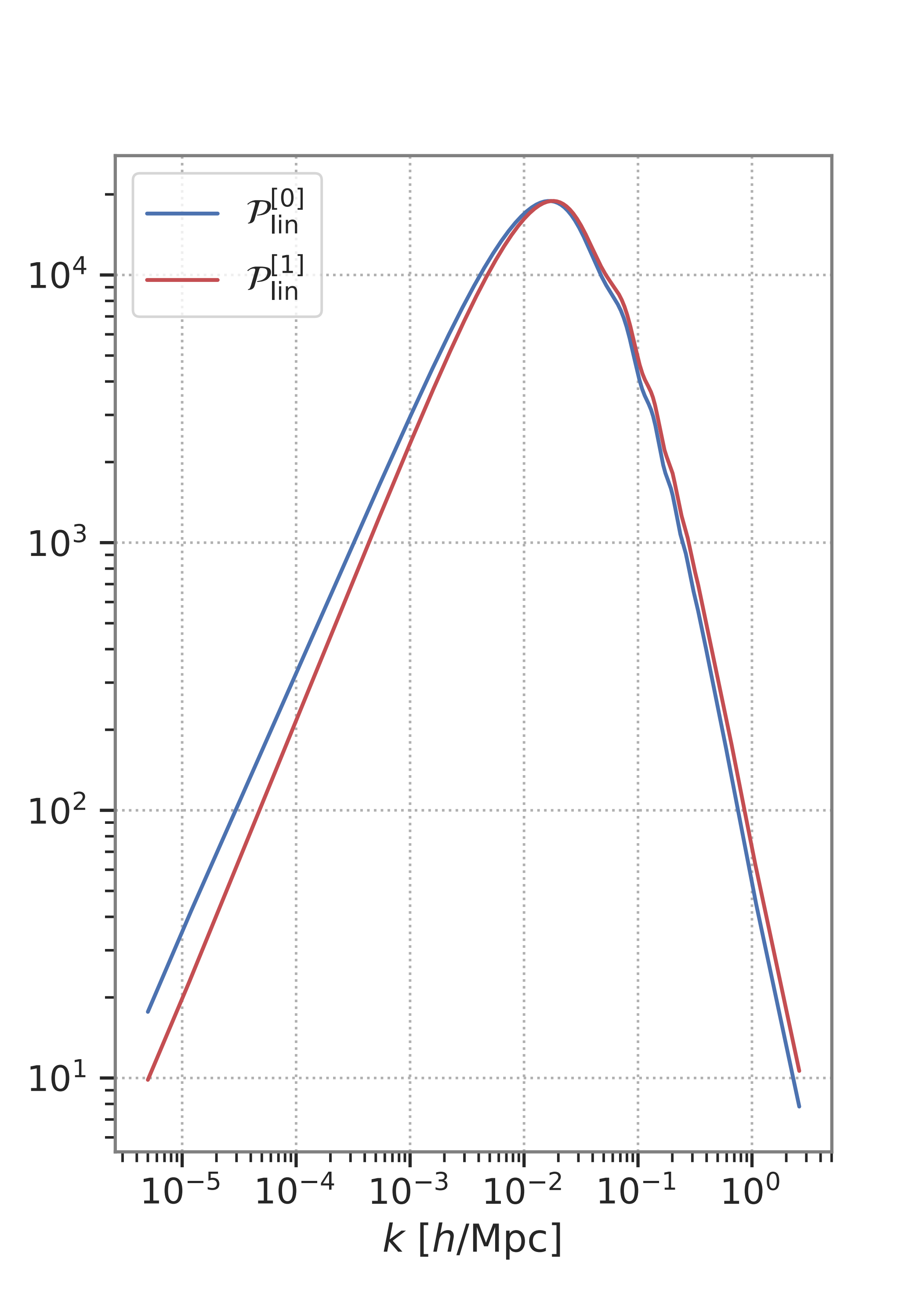
EFT parameters

$$M = 0.7h/\mathrm{Mpc}$$
$$\mathrm{CTs}^{(1)} = \mathrm{CTs}^{(2)} = 0.$$

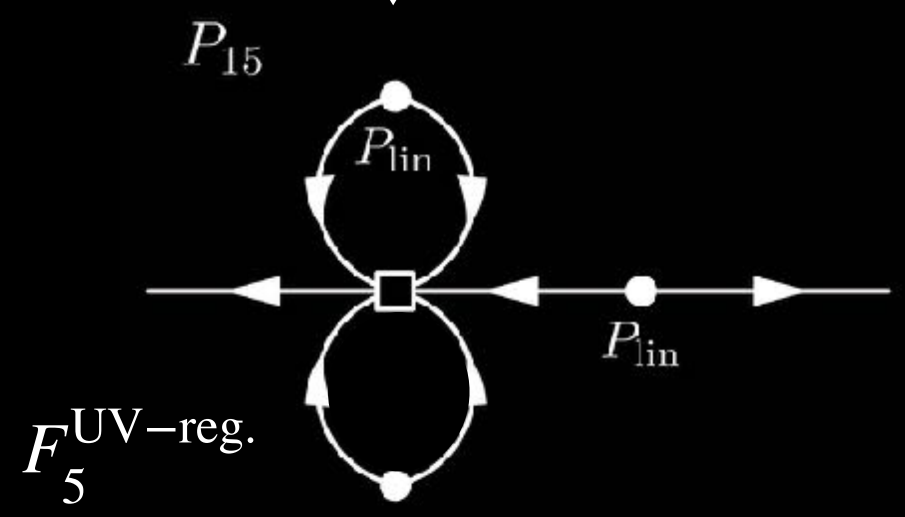
Avoiding new numerical integrations in inferring cosmological model parameters

- I presented numerical results for one cosmological model and one set of values for its parameters.
- Very fast (few laptop minutes per wavenumber point) numerical integration thanks to having eliminated locally, at the integrand ultraviolet and infrared singularities.
- This may be NOT fast enough for practically scanning the full model parameter space and parameter inference from data.

$$h = h^{(\mathrm{Planck})} + \Delta h$$
$$\omega_b = \omega_b^{(\mathrm{Planck})} + \Delta\omega_b$$
$$\omega_{\mathrm{cdm}} = \omega_{\mathrm{cdm}}^{(\mathrm{Planck})} + \Delta\omega_{\mathrm{cdm}}$$
$$A_s = A_s^{(\mathrm{Planck})} + \Delta A_s$$
$$n_s = n_s^{(\mathrm{Planck})} + \Delta n_s$$
$$\sum m_\nu = \sum m_\nu^{(\mathrm{Planck})} + \Delta \sum m_\nu$$

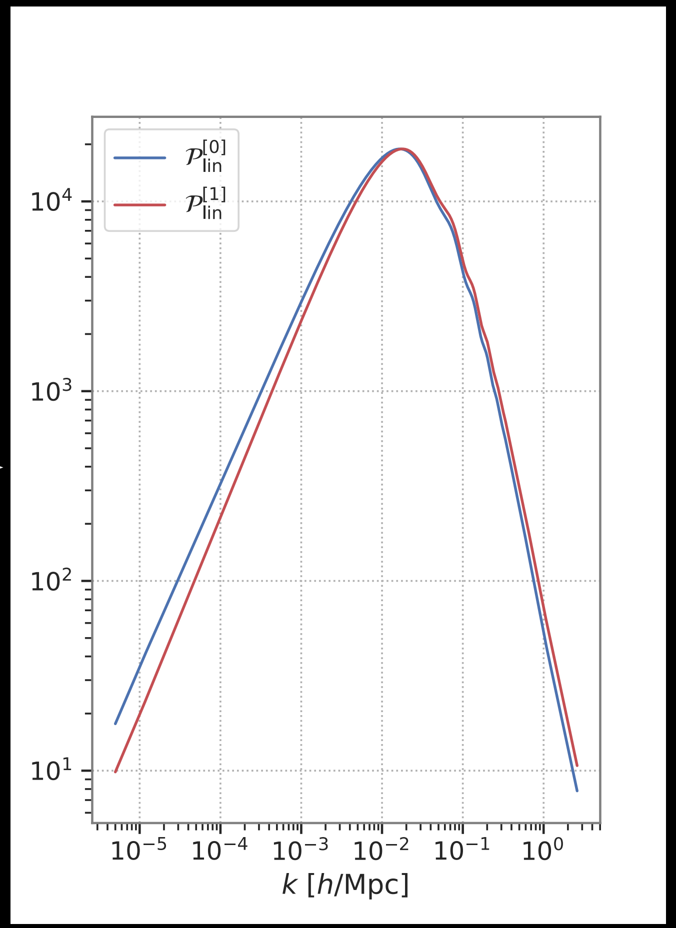


$$\mathscr{P}_{\mathrm{lin.}}(q) = N\,\mathscr{P}^{\mathrm{Planck}}_{\mathrm{lin.}}(q) + \Delta\mathscr{P}_{\mathrm{lin.}}(q)$$

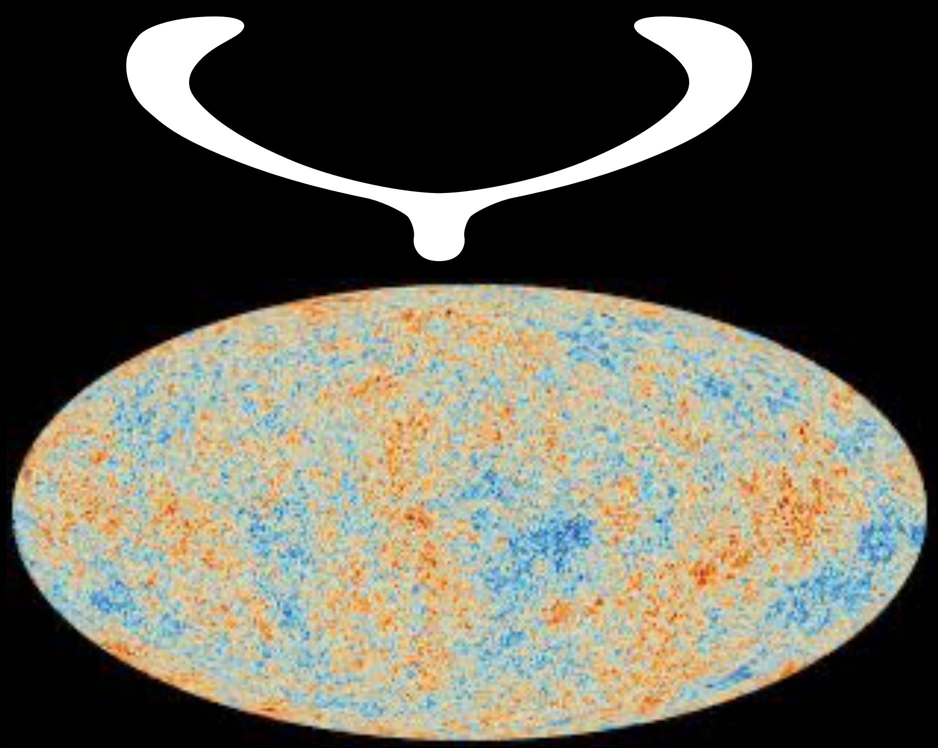


Perurbative expansion in cosmological model parameter space

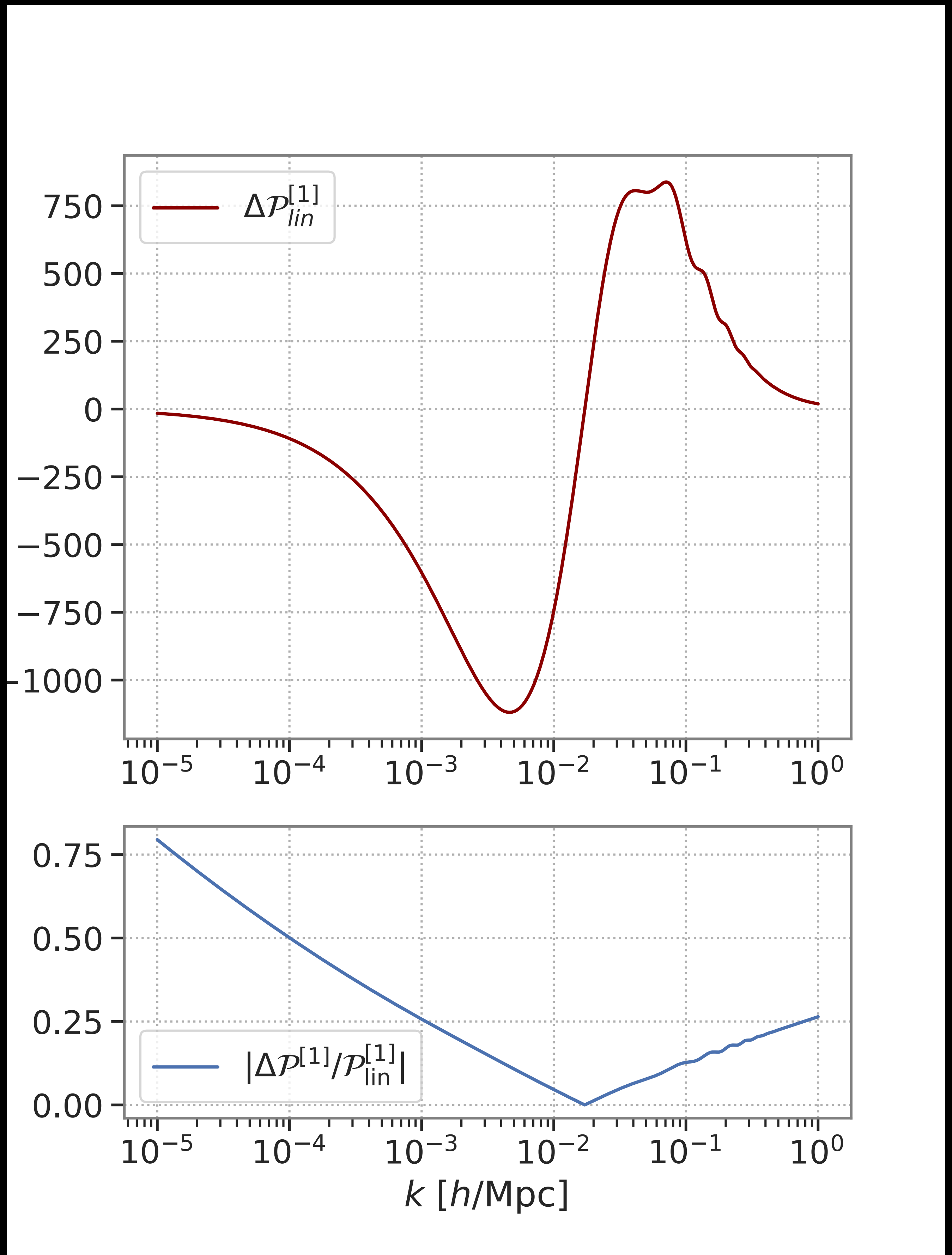
$$h = h^{(\text{Planck})} + \Delta h$$
$$\omega_b = \omega_b^{(\text{Planck})} + \Delta\omega_b$$
$$\omega_{\text{cdm}} = \omega_{\text{cdm}}^{(\text{Planck})} + \Delta\omega_{\text{cdm}}$$
$$A_s = A_s^{(\text{Planck})} + \Delta A_s$$
$$n_s = n_s^{(\text{Planck})} + \Delta n_s$$
$$\sum m_\nu = \sum m_\nu^{(\text{Planck})} + \Delta \sum m_\nu$$



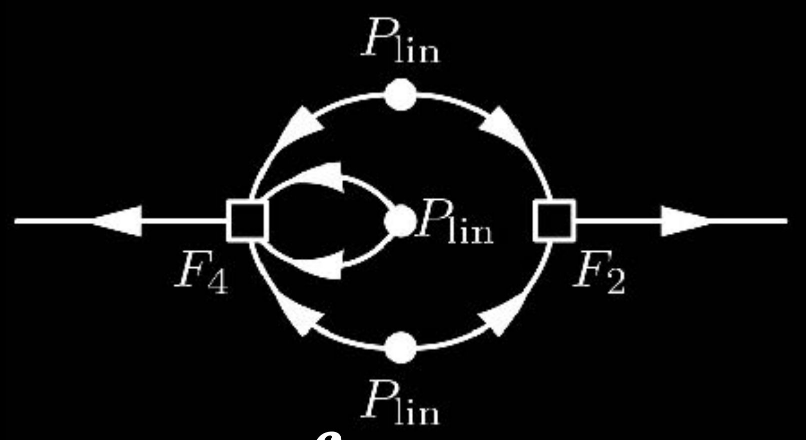
$$\mathcal{P}_{\text{lin.}}(q) = N\,\mathcal{P}_{\text{lin.}}^{\text{Planck}}(q) + \Delta\mathcal{P}_{\text{lin.}}(q)$$



Small



We are scanning over viable parameter space of viable models. The linear power spectra of scanned "cosmologies" should not differ drastically.

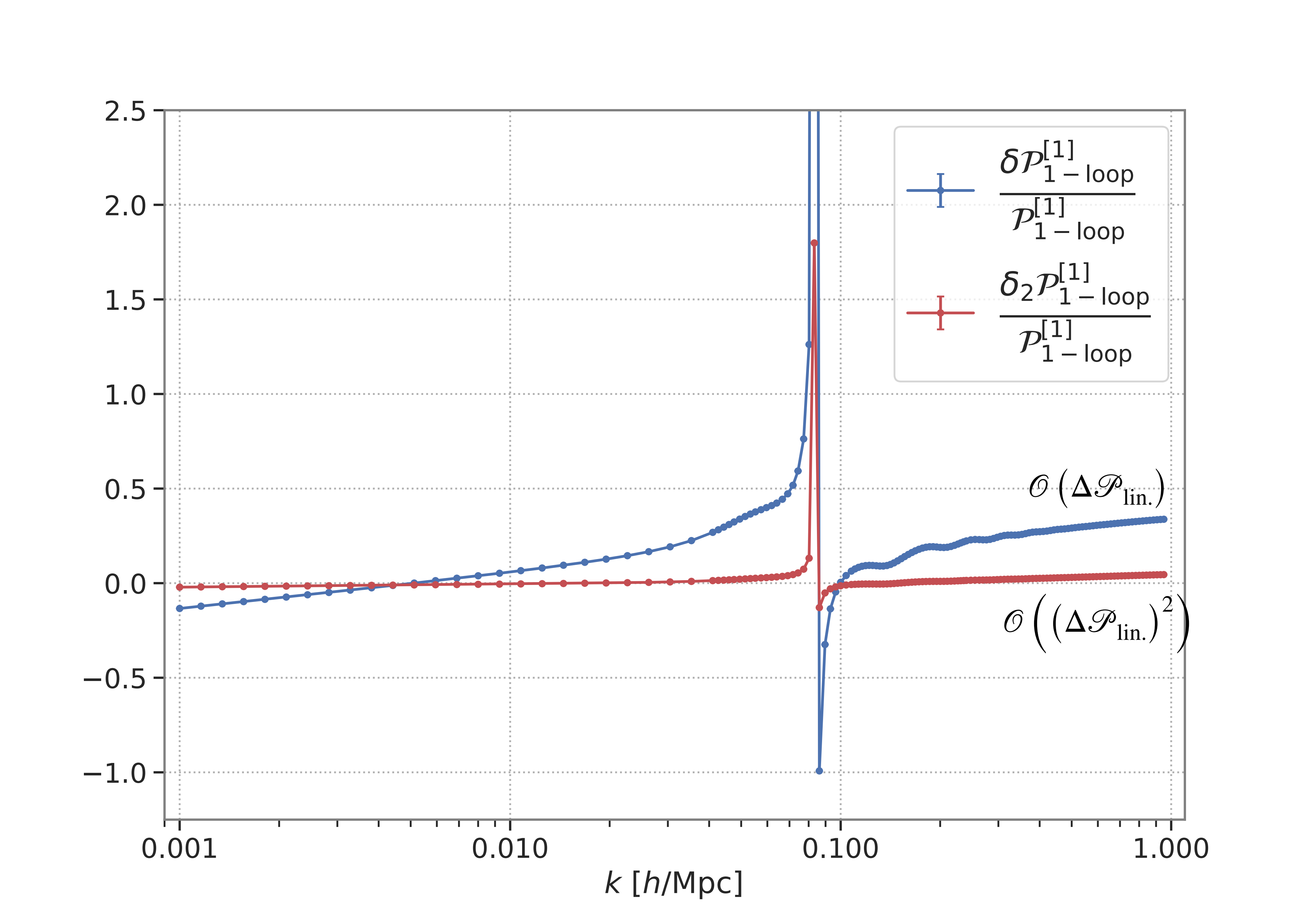


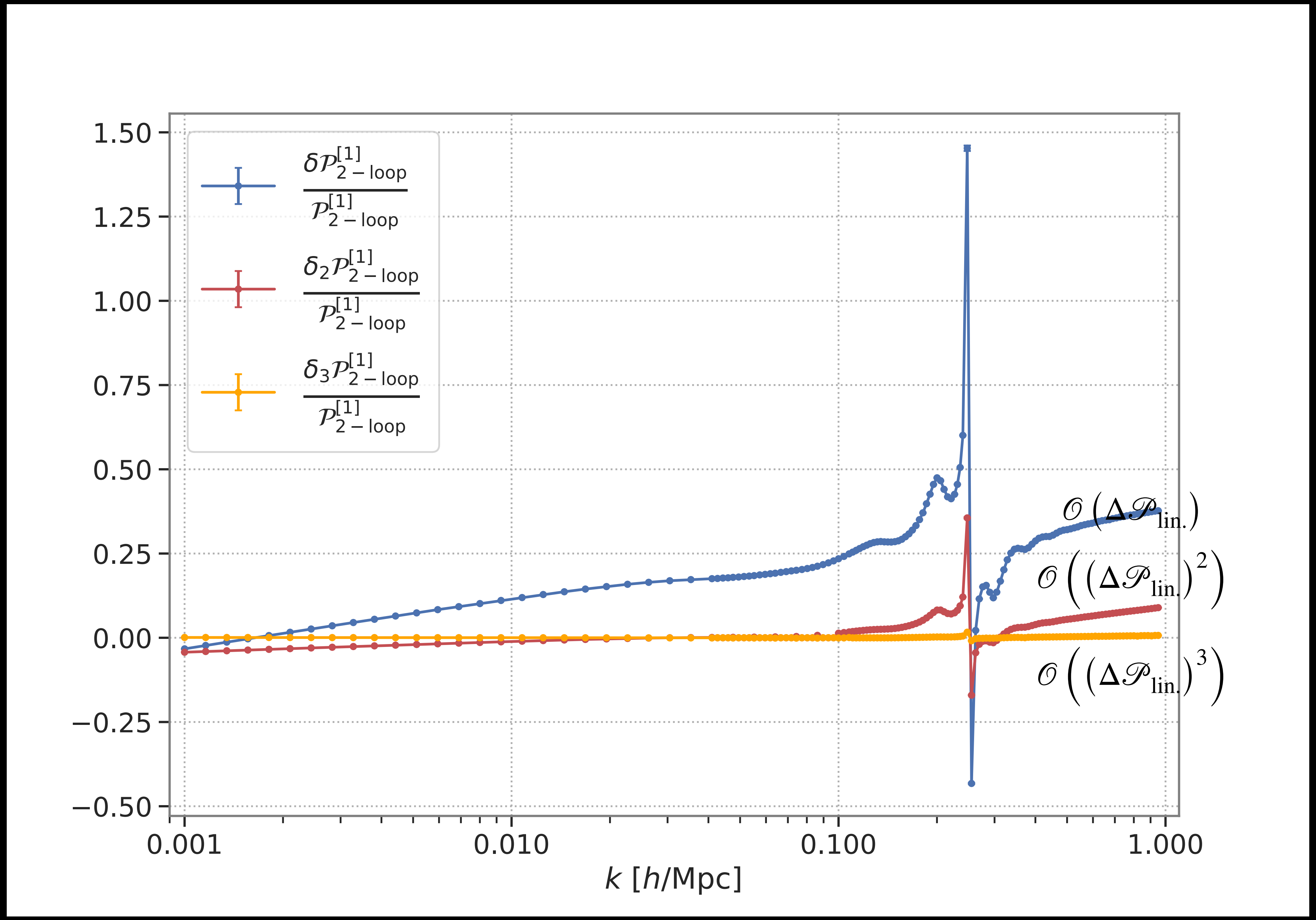
$$\sim \int F^2\,\mathcal{P}_{\text{lin.}}^3 = \int F^2\left[\mathcal{P}_{\text{lin.}}^{\text{Planck}} + \Delta P\right]^3$$

$$\sim \int F^2\left[\mathcal{P}_{\text{lin.}}^{\text{Planck}}\right]^3 + \int F^2\left[\mathcal{P}_{\text{lin.}}^{\text{Planck}}\right]^2\Delta P + \int F^2\,\mathcal{P}_{\text{lin.}}^{\text{Planck}}\left[\Delta\mathcal{P}\right]^2 + \int F^2\left[\Delta\mathcal{P}\right]^3$$

Big | Small | Smaller | Tiny

Perturbative expansion in cosmological models



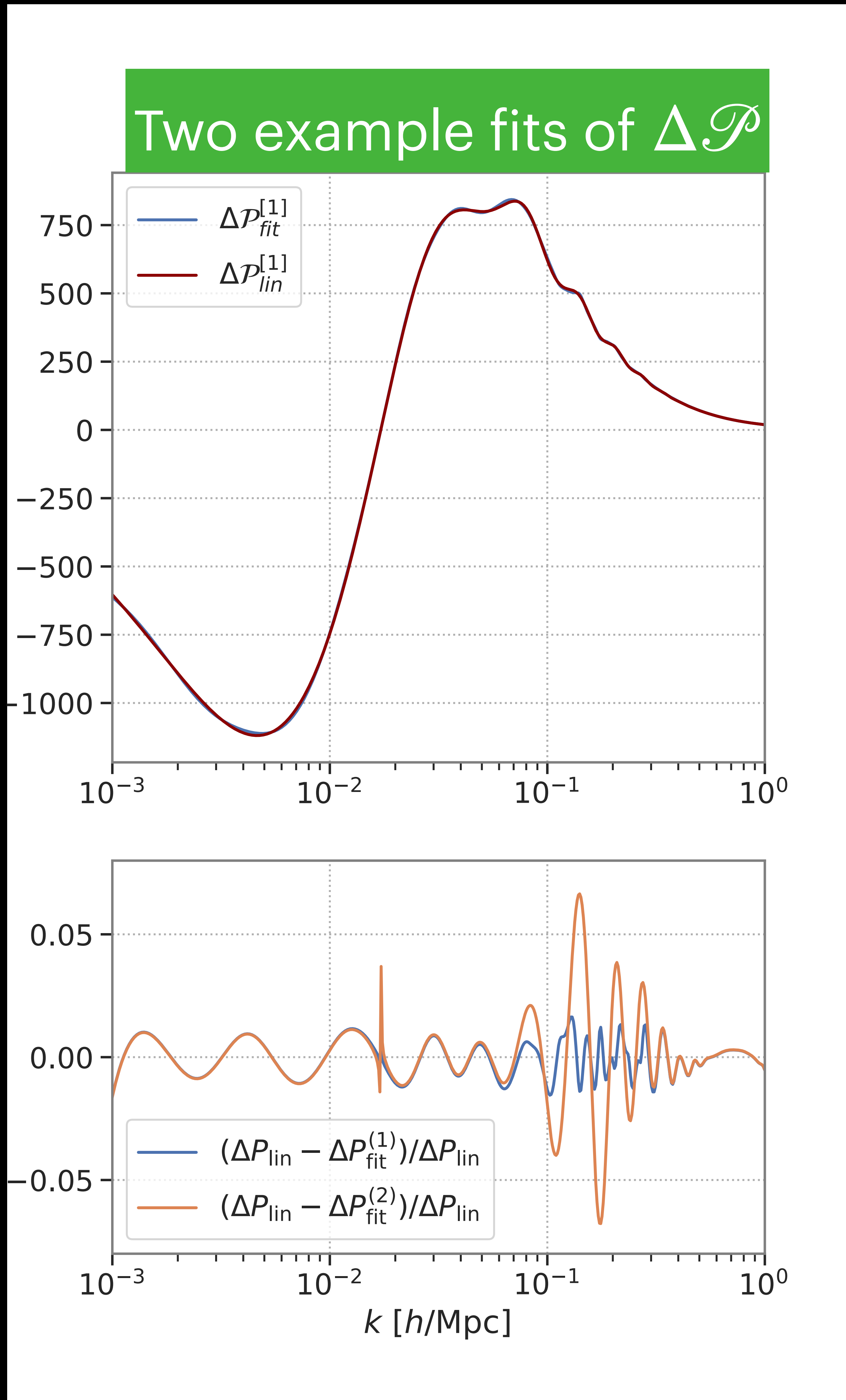


Decoupling cosmological model and integrations.

$$\mathscr{P}_{\text{lin.}}(q) = N\,\mathscr{P}^{\text{Planck}}_{\text{lin.}}(q) + \Delta\mathscr{P}_{\text{lin.}}(q)$$

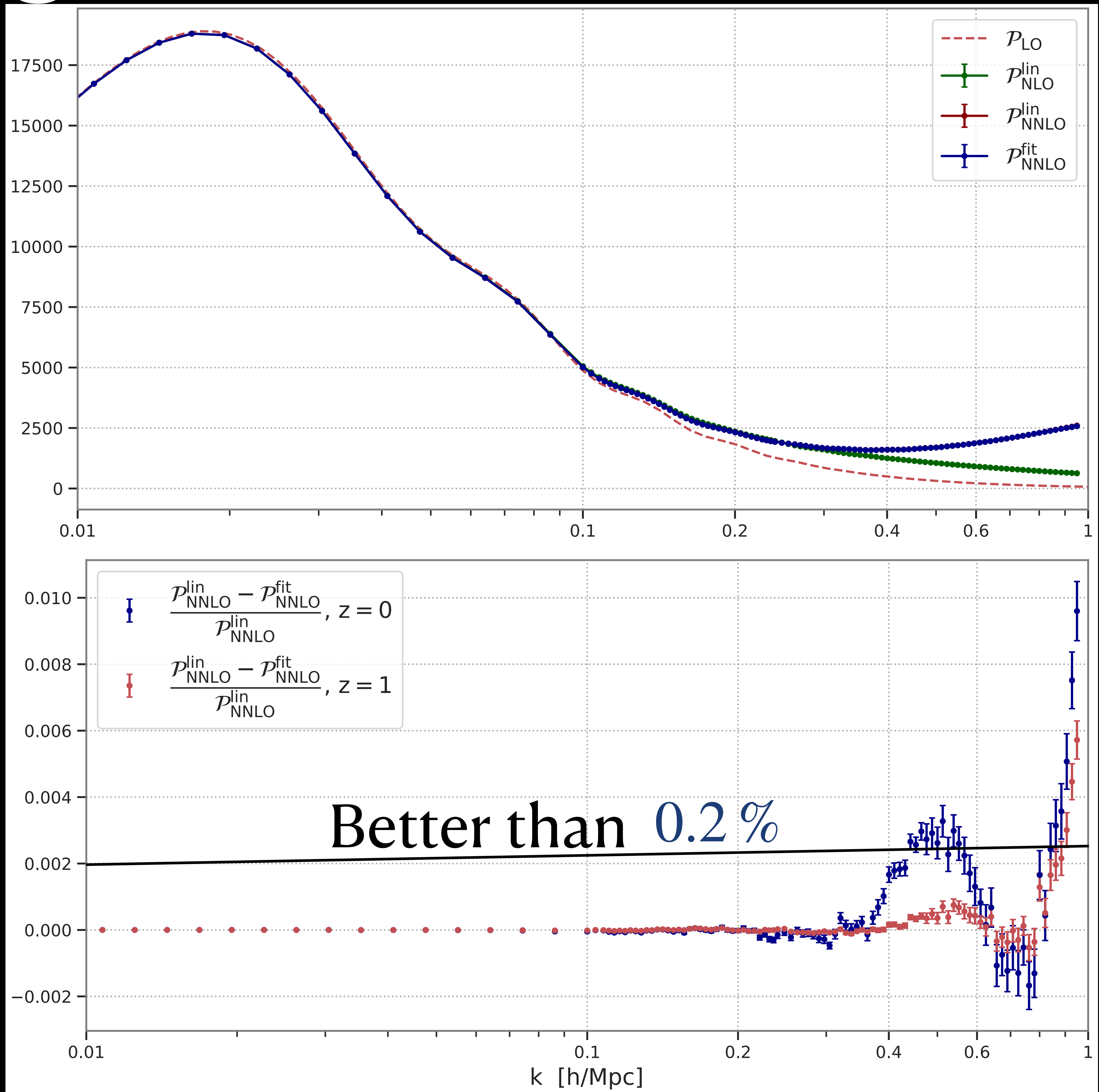
$$\Delta\mathscr{P}_{\text{lin.}}(k) = \sum_{n,m} C_{nm}(H,\Omega,\ldots)\,\frac{1}{\left(k^2 + M_n + i\Gamma_n\right)^{\nu_m}}$$

Quality of fit depends on the number of our basis functions.

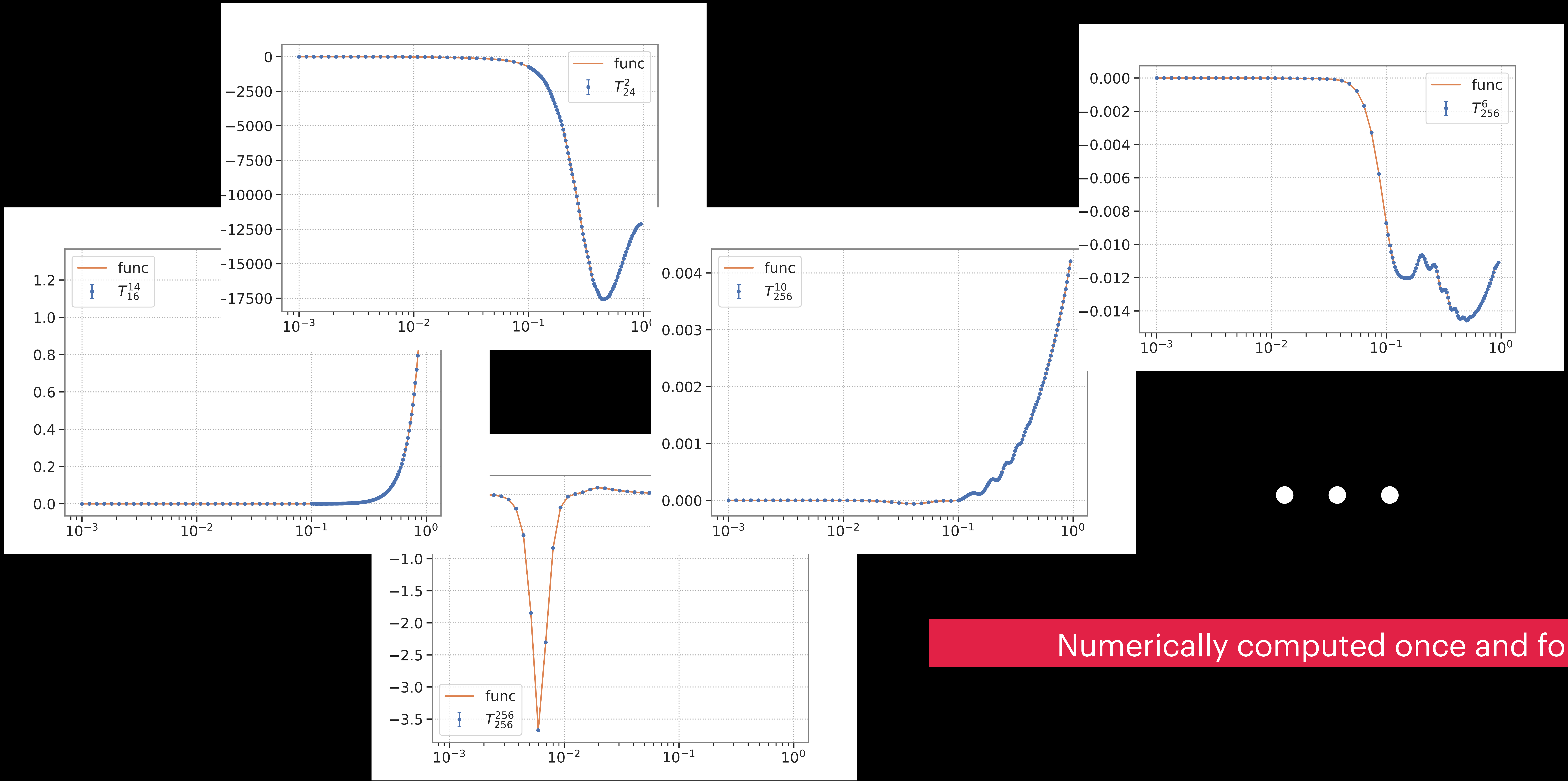


Decoupling cosmological model and integrations.

$$
\begin{aligned}
&\mathscr{P}_{2-loop}\left[\mathscr{P}_{\mathrm{lin}}(h,\Omega,\dots)\right](k)\\
=\,&\mathscr{P}_{2-loop}\left[\mathscr{P}_{\mathrm{lin}}(h^{\mathrm{Planck}},\Omega^{\mathrm{Planck}},\dots)\right](k)\\
&+\sum_{a}\mathrm{Coeff}_a\left(h,\Omega\right)\mathscr{I}^{a}\left(k,h^{\mathrm{Planck}},\Omega^{\mathrm{Planck}},\dots\right)\\
&+\sum_{a,b}\mathrm{Coeff}_a\left(h,\Omega\right)\mathrm{Coeff}_b\left(h,\Omega\right)\mathscr{I}^{ab}\left(k,h^{\mathrm{Planck}},\Omega^{\mathrm{Planck}},\dots\right)\\
&+\mathscr{O}\left(\Delta\mathscr{P}^{3}\right)
\end{aligned}
$$



Required $\mathcal{O}(300)$ basis integrals



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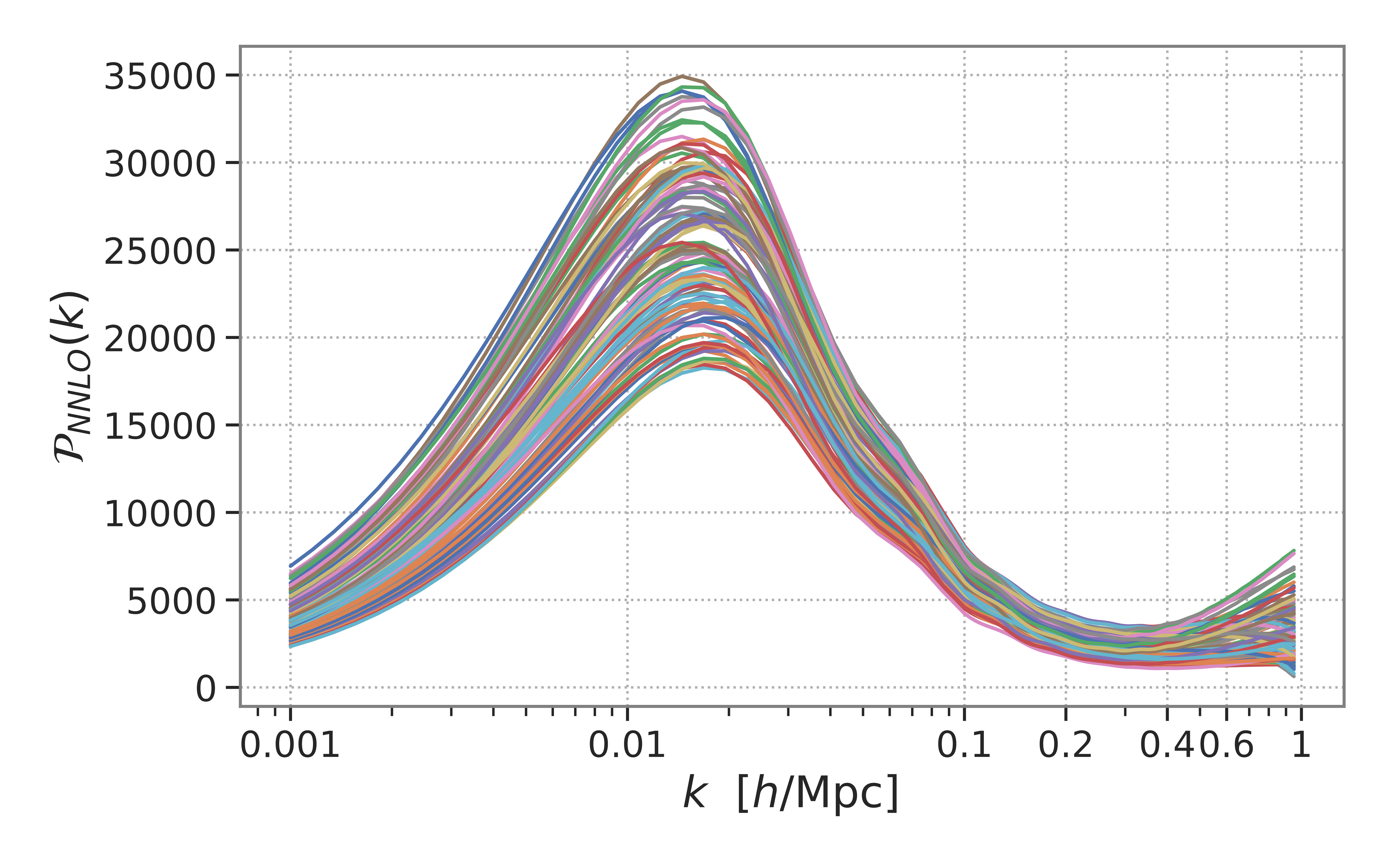
Numerically computed once and for all

The two-loop power-spectra for 100 sets of cosmological parameters



Developing a universal numerical method for QCD amplitudes

Two loop gauge theory amplitudes with direct integration

- Two-loop amplitudes with direct integration over loop momenta?
- Number of integrals is SIX.
- … for all two-loop amplitudes and kinematic configurations.
- Understand fully the singular structure of QCD amplitudes at two loops.

$$A_2\left(\left\{p_{ext_i}\right\},\{M_i\}\right)$$

$$= \int d^dk \int d^dl\, \mathcal{A}_2\left(k, l, \left\{p_{ext_i}\right\},\{M_i\}\right)$$

Monte-Carlo Integration?

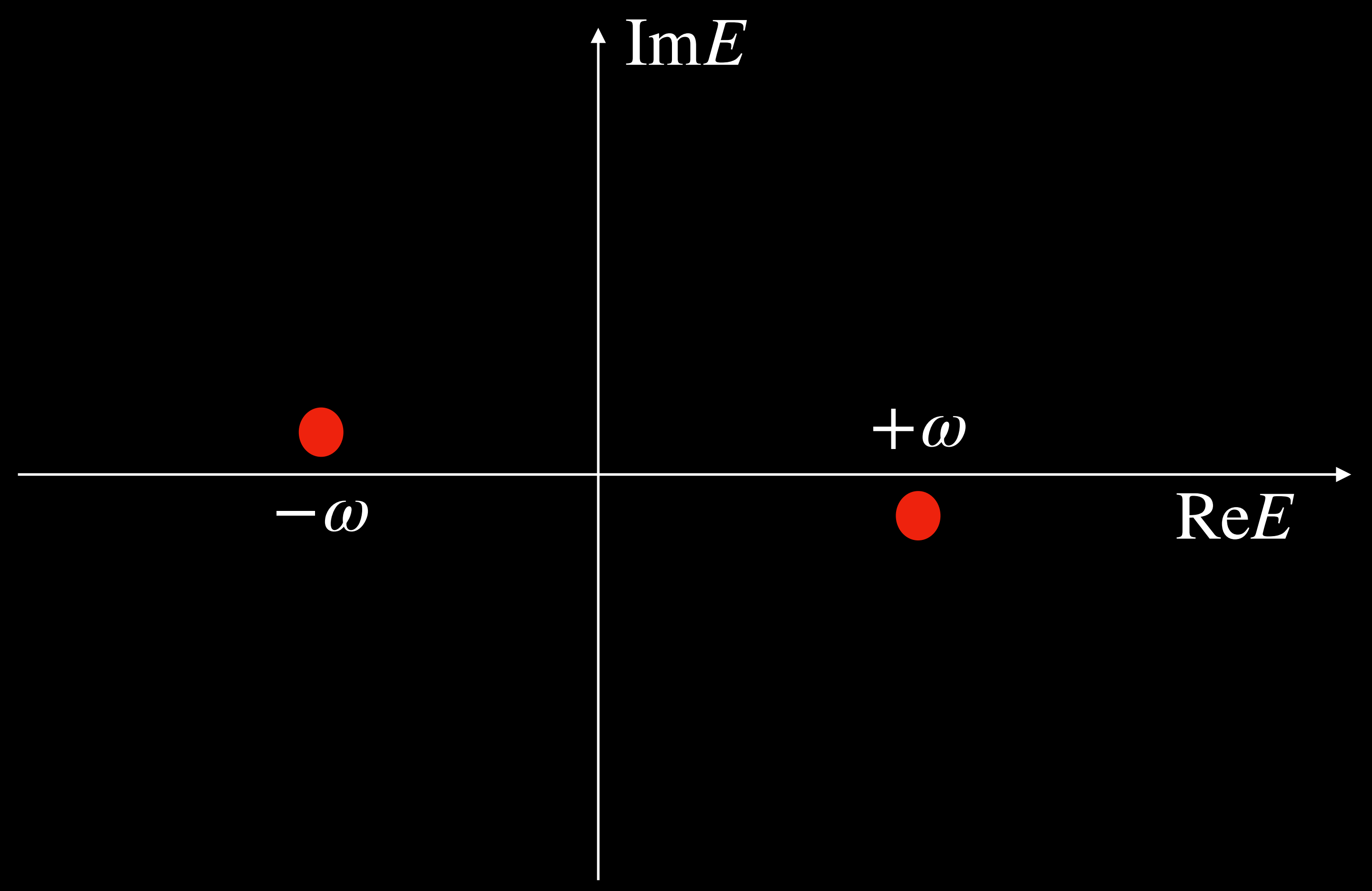
$$d \longrightarrow 4?$$

Singularities

Singularities of Feynman diagrams and scattering amplitudes

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega^2 + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.

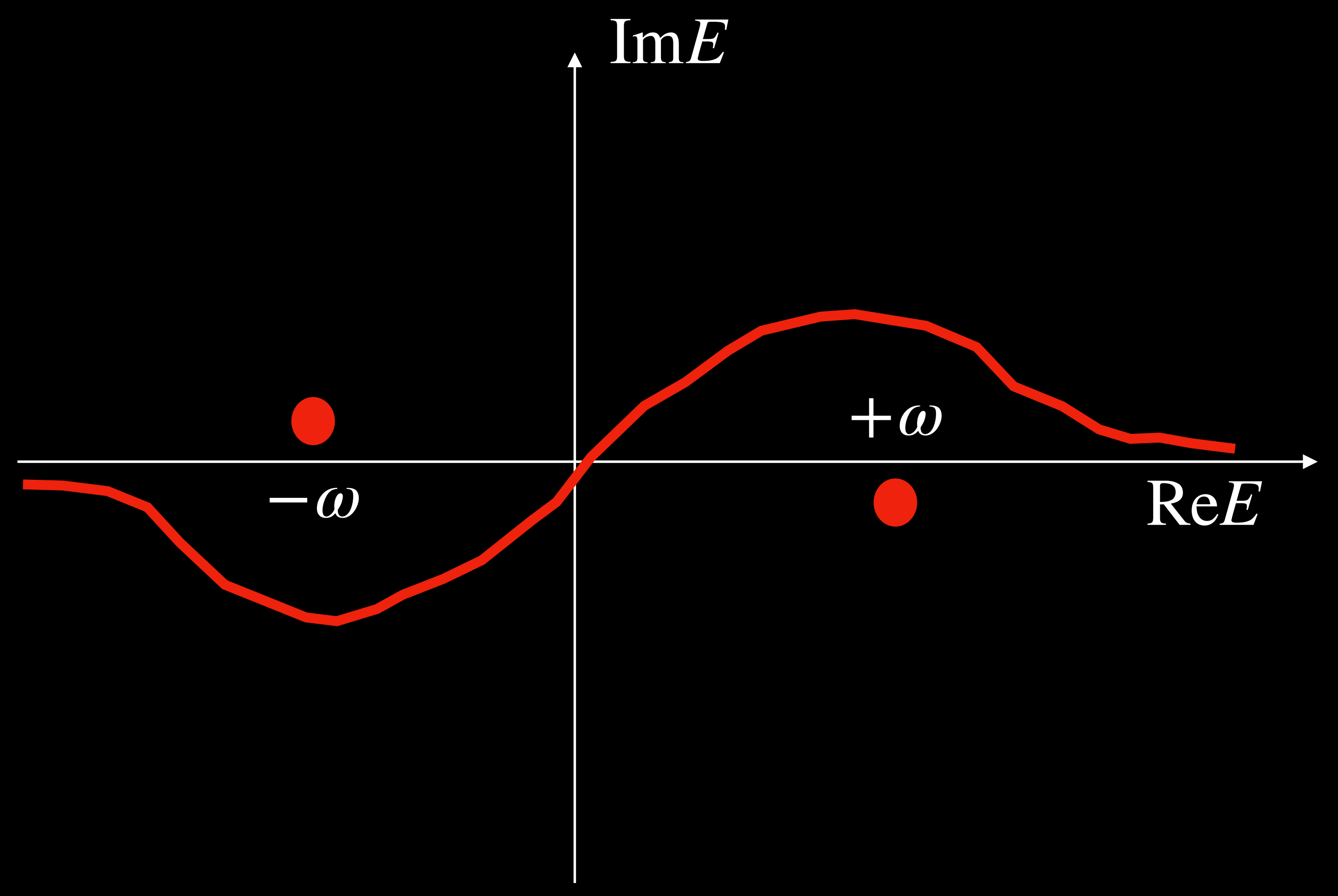


$\omega \to \omega - i\delta$ with $\delta \to 0$

Integrable Singularities

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{E^2 - \omega^2 + i\delta} = \int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{\omega} \left(\frac{1}{E - \omega + i\delta} - \frac{1}{E + \omega - i\delta} \right)$$

- The poles can lie inside the domain of integration.
- If we can deform the path of integration away from the poles, then they lead to no singularities



$\omega \to \omega - i\delta$ with $\delta \to 0$

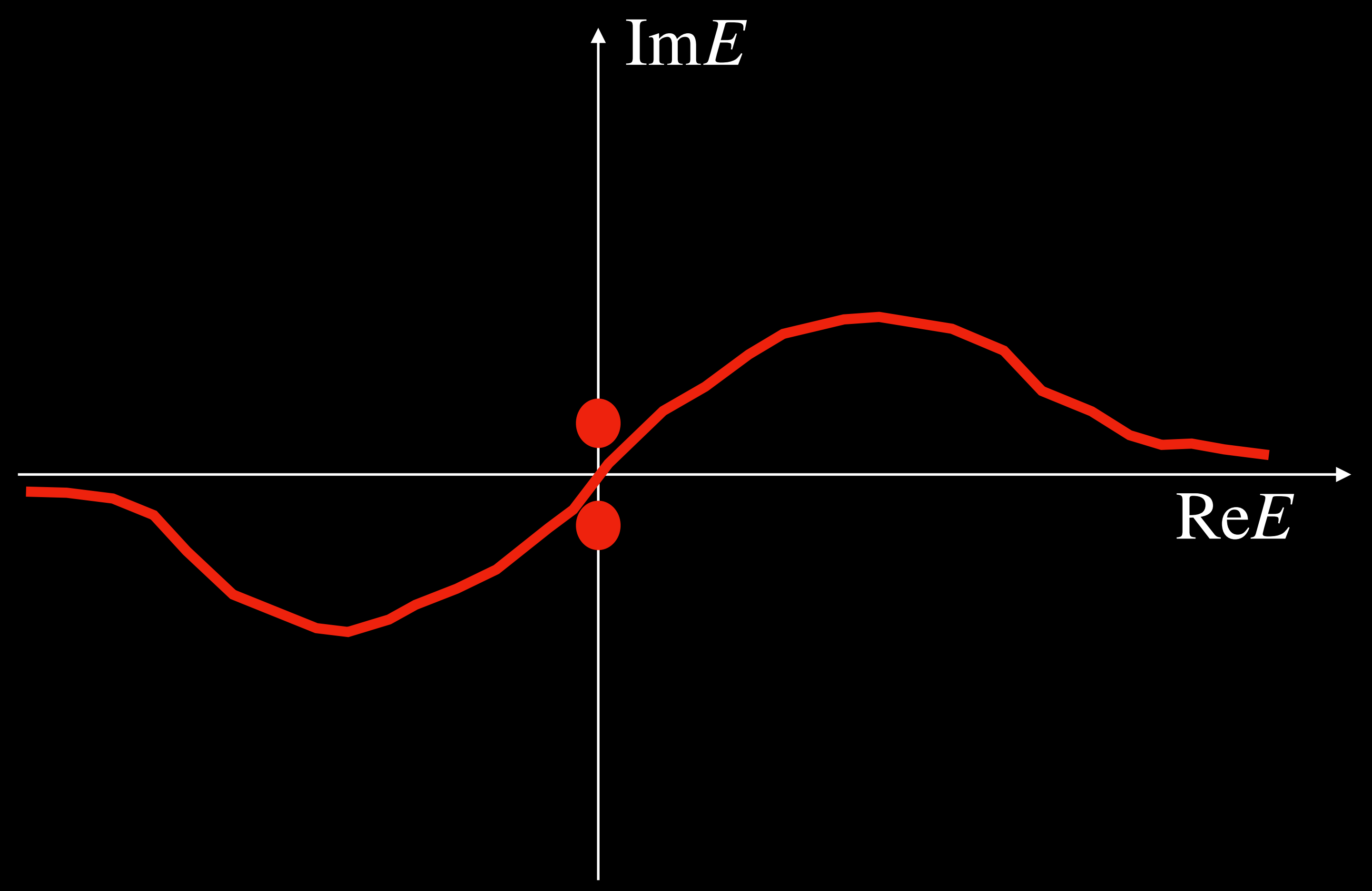
Soft massless particles

$$\int_{-\infty}^{\infty} dE \ldots \frac{\cdots}{(E + i\delta)\,(E - i\delta)}$$

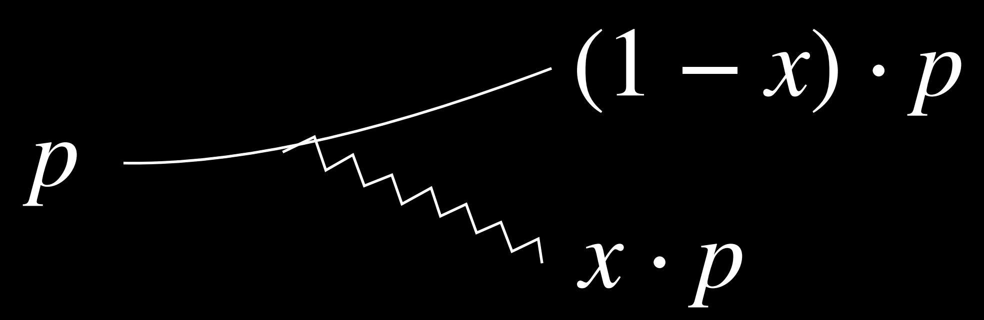
- Poles due to soft massless particles.

- These singularities pinch the integration path from both sides.

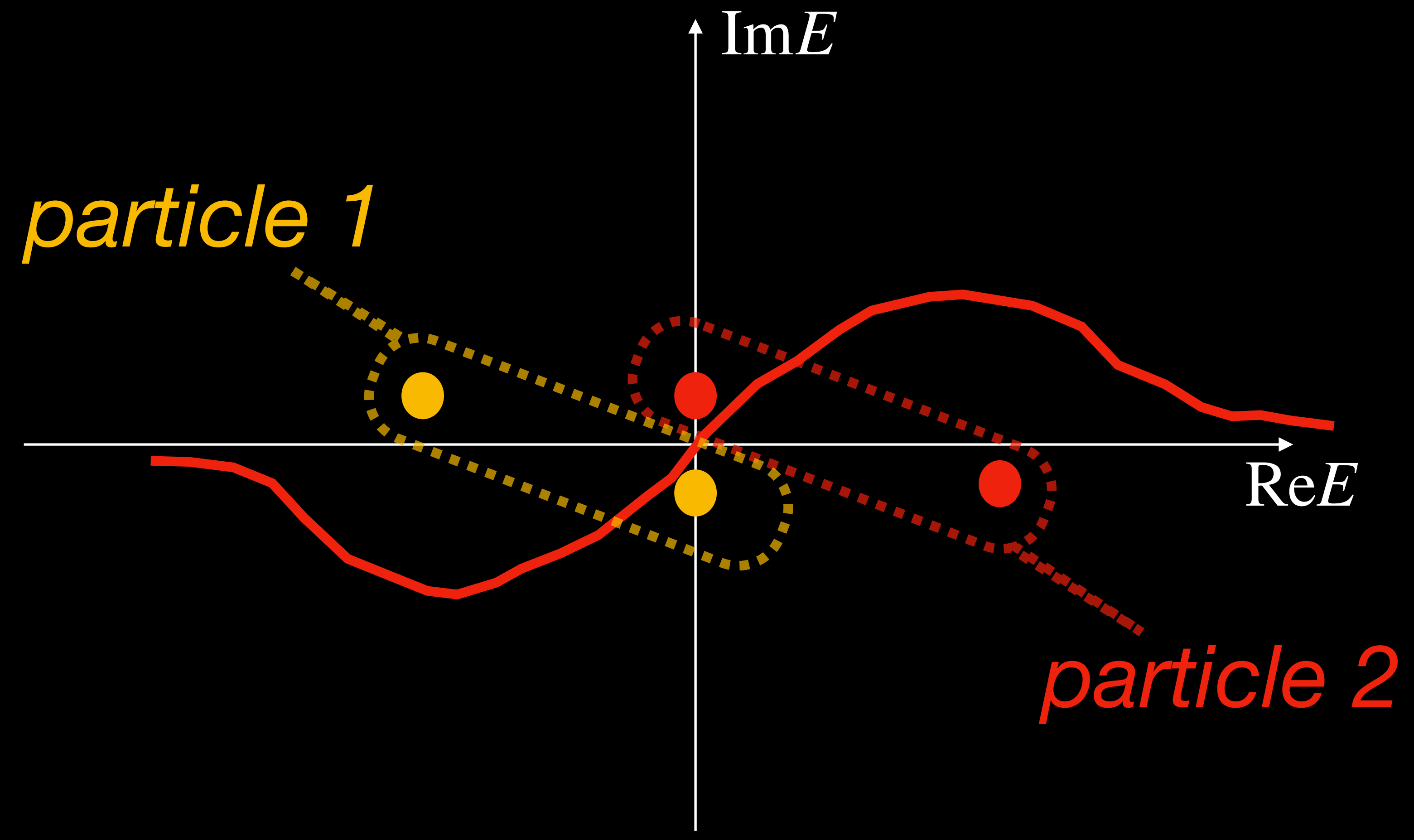
- Condition for a TRUE INFINITY

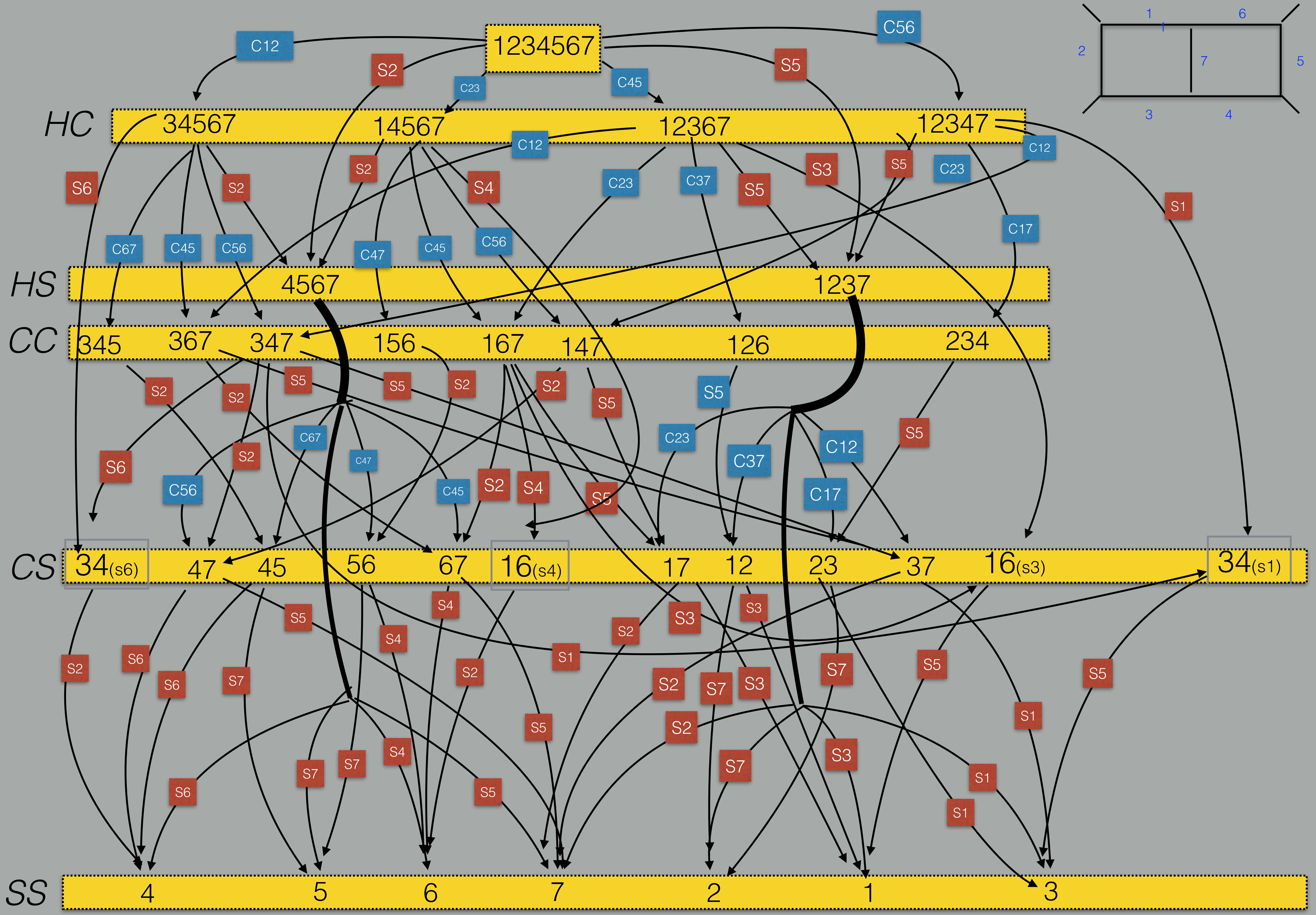


Collinear massless particles



- A second source of infinities due to massless collinear particles.
- A singularity of one particle in the lower half-plane lines up with the singularity of a collinear particle in the higher half-pane.
- The singularities pinch the integration path from both sides.
- We cannot deform the path, a condition for a TRUE INFINITY!



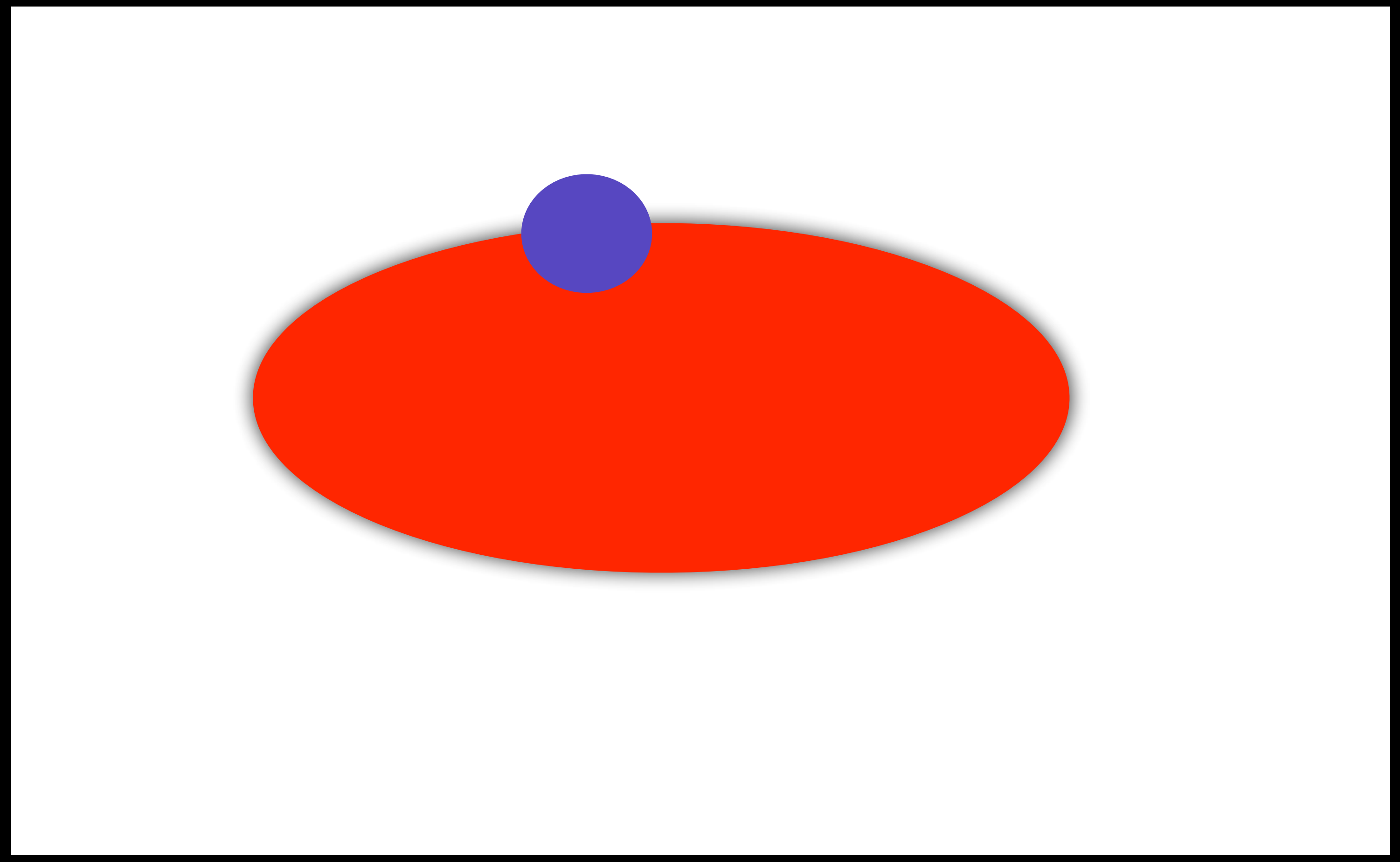


A complicated structure of overlapping singularities.

Nested subtractions

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
- Order the singular regions by their “volume”



Nested subtractions

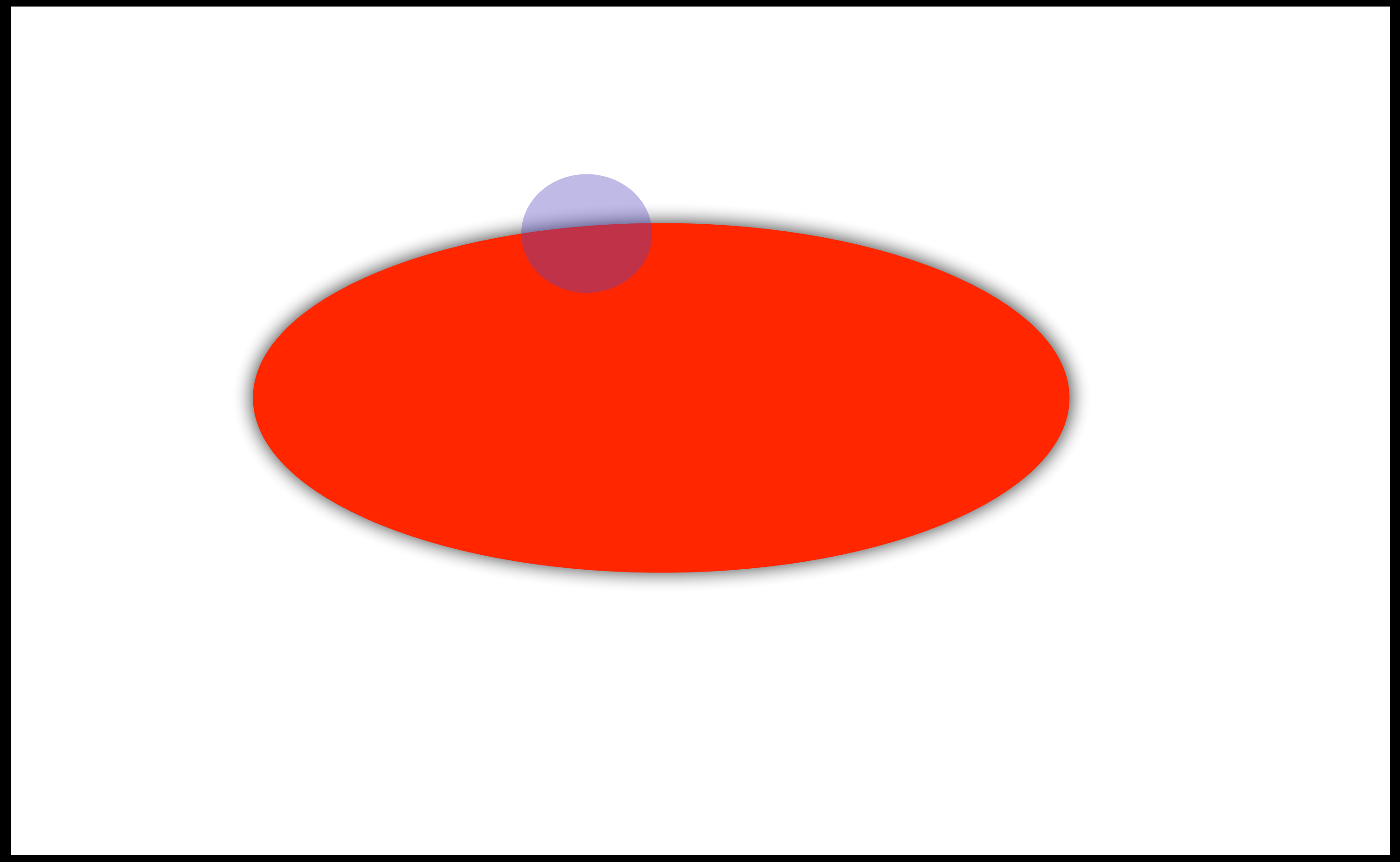
Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?

- Order the singular regions by their “volume”

- Subtract an approximation of the integrand in the smallest volume

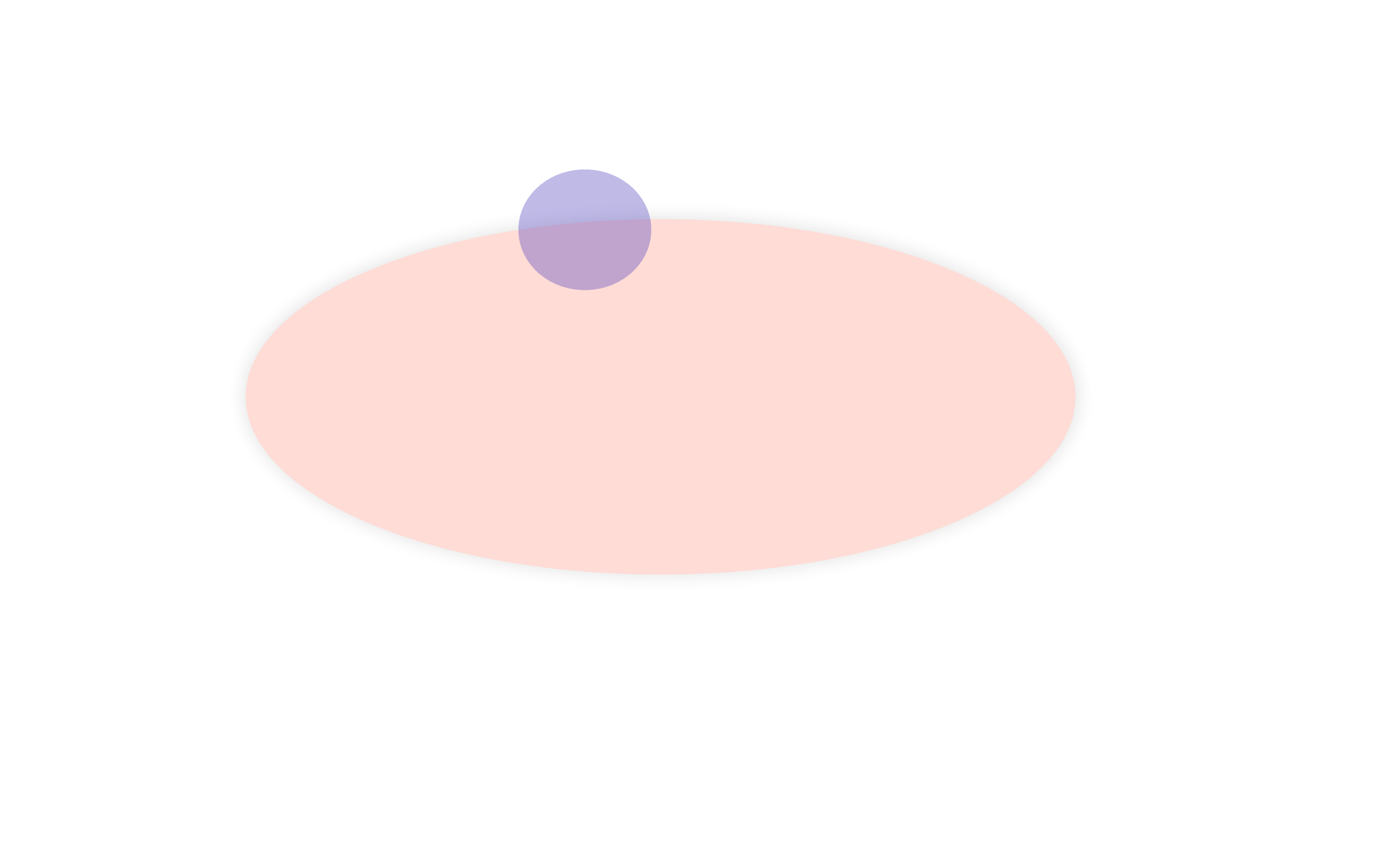
-



Nested subtractions

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

- Singular regions are interconnected. How can we create systematically an approximation of the loop integrals in all singular regions?
- Order the singular regions by their "volume"
- Subtract an approximation of the integrand in the smallest volume
- Then, proceed to the next volume and repeat until there are no more singularities to remove.



Nested subtractions

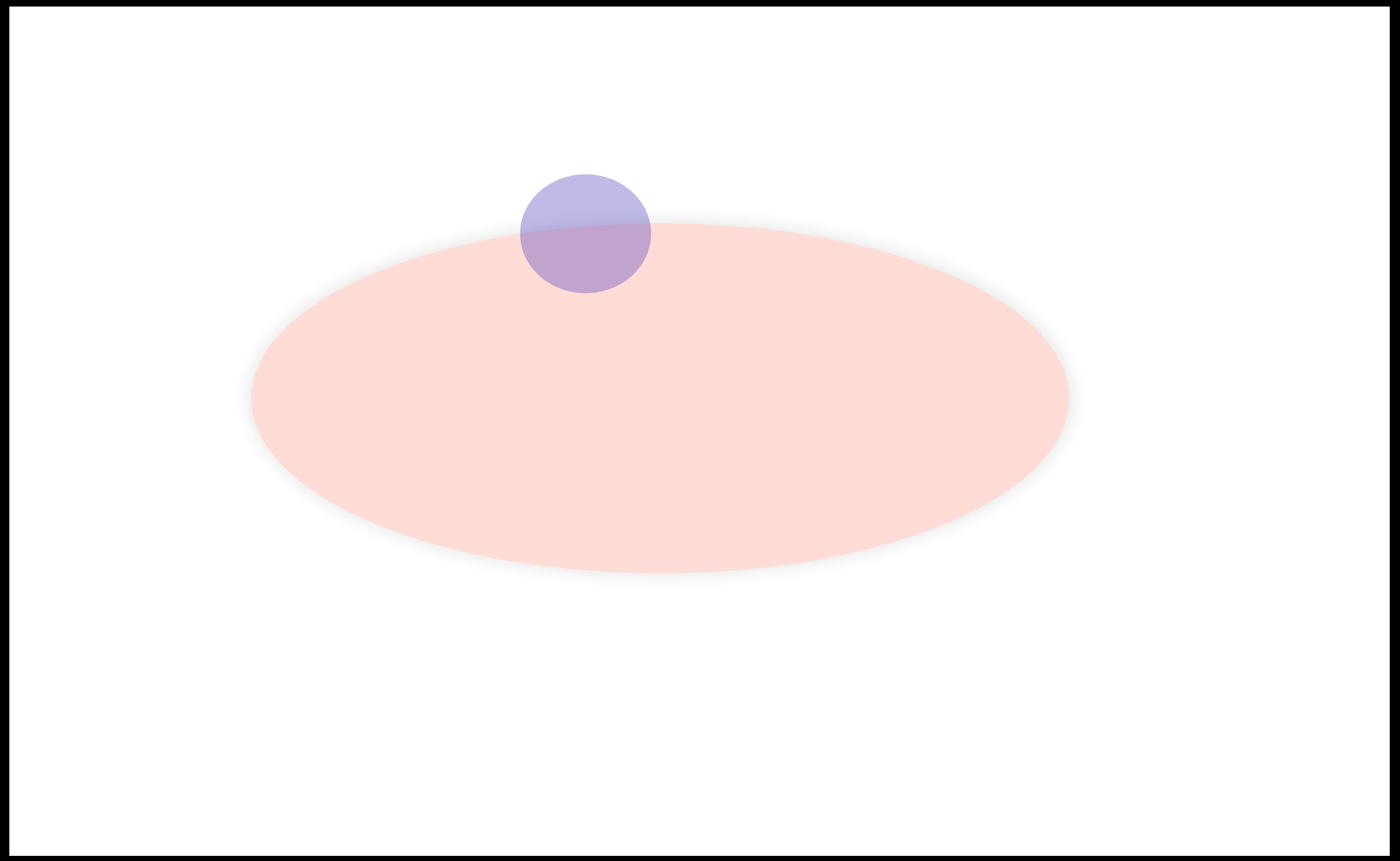
- The procedure of nested subtractions has a solution for the finite remainder at any loop order as a Forest formula (similarly to BPHZ of UV renormalzation)

- It is valid term by term in an amplitude or a Feynman diagram.

- This forest formula structure combined with gauge symmetry, gives rise to the factorization of gauge theory amplitudes in terms of Jets, Soft and Hard fucntions.

Ma; Erdogan, Sterman; Collins; Collins, Soper, Sterman

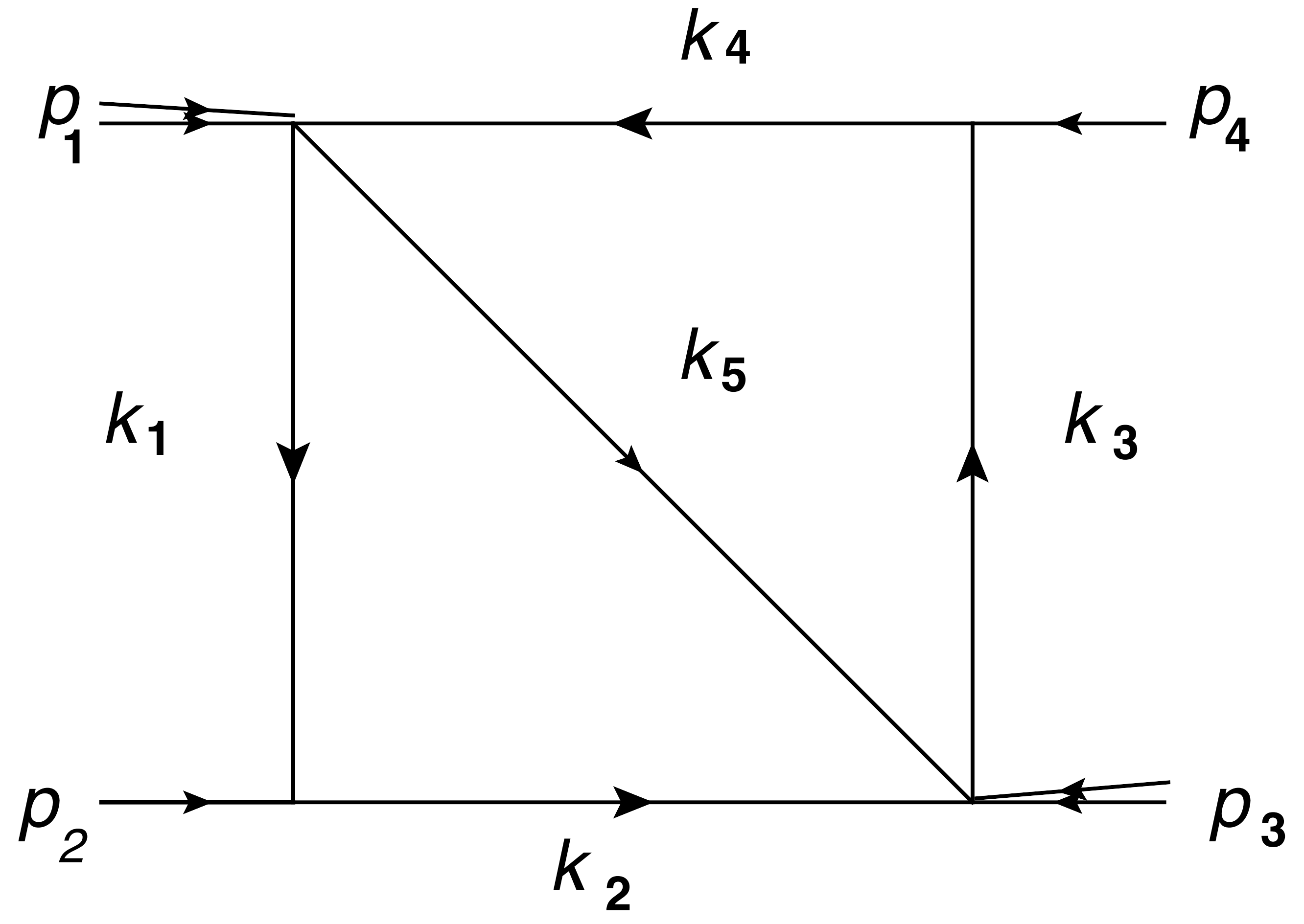
$$R^{(n)}\,\gamma^{(n)} = \gamma^{(n)} + \sum_{N\in\mathcal{N}[\gamma^{(n)}]} \prod_{\rho\in N} (-t_\rho)\,\gamma^{(n)},$$



Constructing finite two-loop integrals

- The method of nested subtractions guarantees that we can always remove the infrared singularities
- … of ANY integral at ANY order in perturbation theory.
- Subtractions can be made to take a simple form.
- Method demonstrated with examples at two-loops

CA, G. Sterman *JHEP 07 (2019) 056*



$$
\begin{aligned}
\mathrm{D_{box}}\big|_{\mathrm{fin}} = \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_4}{i\pi^{\frac{d}{2}}} \Bigg\{ & \frac{1}{A_1 A_2 A_3 A_4 A_5} \\
& - \left[\frac{1}{A_1 A_2} - \frac{1}{(A_1 - \mu^2)(A_2 - \mu^2)}\right] \left[\frac{1}{A_3 A_4 A_5}\right]_{k_1 = -x_2 p_2} \\
& - \left[\frac{1}{A_3 A_4} - \frac{1}{(A_4 - \mu^2)(A_3 - \mu^2)}\right] \left[\frac{1}{A_1 A_2 A_5}\right]_{k_4 = x_4 p_4} \\
+ \left[\frac{1}{A_1 A_2} - \frac{1}{(A_1 - \mu^2)(A_2 - \mu^2)}\right] & \left[\frac{1}{A_5}\right]_{\substack{k_4 = x_4 p_4, \\ k_1 = -x_2 p_2}} \left[\frac{1}{A_3 A_4} - \frac{1}{(A_3 - \mu^2)(A_4 - \mu^2)}\right] \Bigg\}
\end{aligned}
$$

Constructing finite two-loop integrals

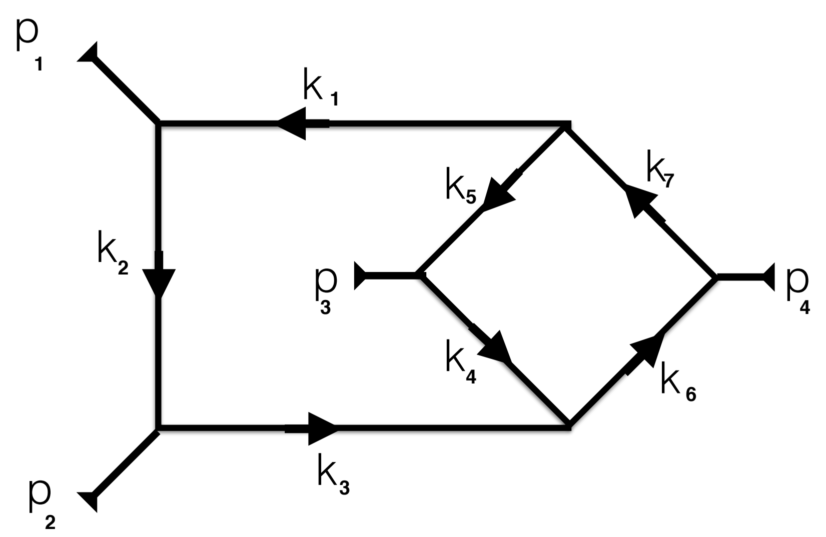
- There is a lot of value in constructing finite integrands for Feynman integrals.

- And simplifications or elegance.

- Subtractions can be made to take a simple form.

- For example, the full two-loop crossed box has a mixed transcendentally. But the integration over our finite remainder gives uniform weight four…

CA, G. Sterman *JHEP 07 (2019) 056*



$$s^3 {\mathrm{X}_{\mathrm{box}}}^{\mathrm{fin}} = \frac{f_{\mathrm{X}_{\mathrm{box}}}(y)}{y} + \frac{f_{\mathrm{X}_{\mathrm{box}}}(1-y)}{1-y},$$

$$f_{\mathrm{X}_{\mathrm{box}}}(y) = [G_R(y) + i\pi G_I(y)] \, \log\left(\frac{\mu^2}{s}\right) + E_R(y) + i\pi E_I(y)$$

$$\begin{aligned}
E_R(y) = & -8\,\pi^2\,\mathrm{Li}_2(y) + 8\,\mathrm{Li}_2(y)\,\log(1-y)^2 - 28\,\log(y)\,\mathrm{Li}_2(y)\,\log(1-y) - 18\,\mathrm{Li}_2(y)\,\log(y)^2 \\
& +44\,\mathrm{Li}_3(y)\,\log(1-y) + 96\,\mathrm{Li}_3(y)\,\log(y) - 188\,\mathrm{Li}_4(y) + \frac{17}{36}\,\pi^4 + \frac{1}{12}\,\log(1-y)^4 \\
& +7\,\log(y)\,\log(1-y)\,\pi^2 - \frac{25}{6}\,\pi^2\,\log(1-y)^2 - \frac{3}{2}\,\log(y)^2\,\pi^2 + \log(y)\,\log(1-y)^3 \\
& +44\,S_{12}(y)\,\log(1-y) - 52\,S_{12}(y)\,\log(y) + 84\,S_{13}(y) + 88\,S_{22}(y) - 44\,\zeta_3\,\log(1-y) \\
& -4\,\log(y)\,\zeta_3 - \frac{1}{4}\,\log(y)^4 + \log(y)^3\,\log(1-y) - \frac{9}{2}\,\log(y)^2\,\log(1-y)^2,
\end{aligned}$$

QCD Amplitudes are SIMPLER than Feynman integrals

- In singular IR regions of loop momenta, virtual gauge bosons acquire (unphysical) longitudinal polarizations.
- The propagation of unphysical degrees of freedom is "prohibited" by Ward identities.
- Cancellations of IR singularities among Feynman diagrams.

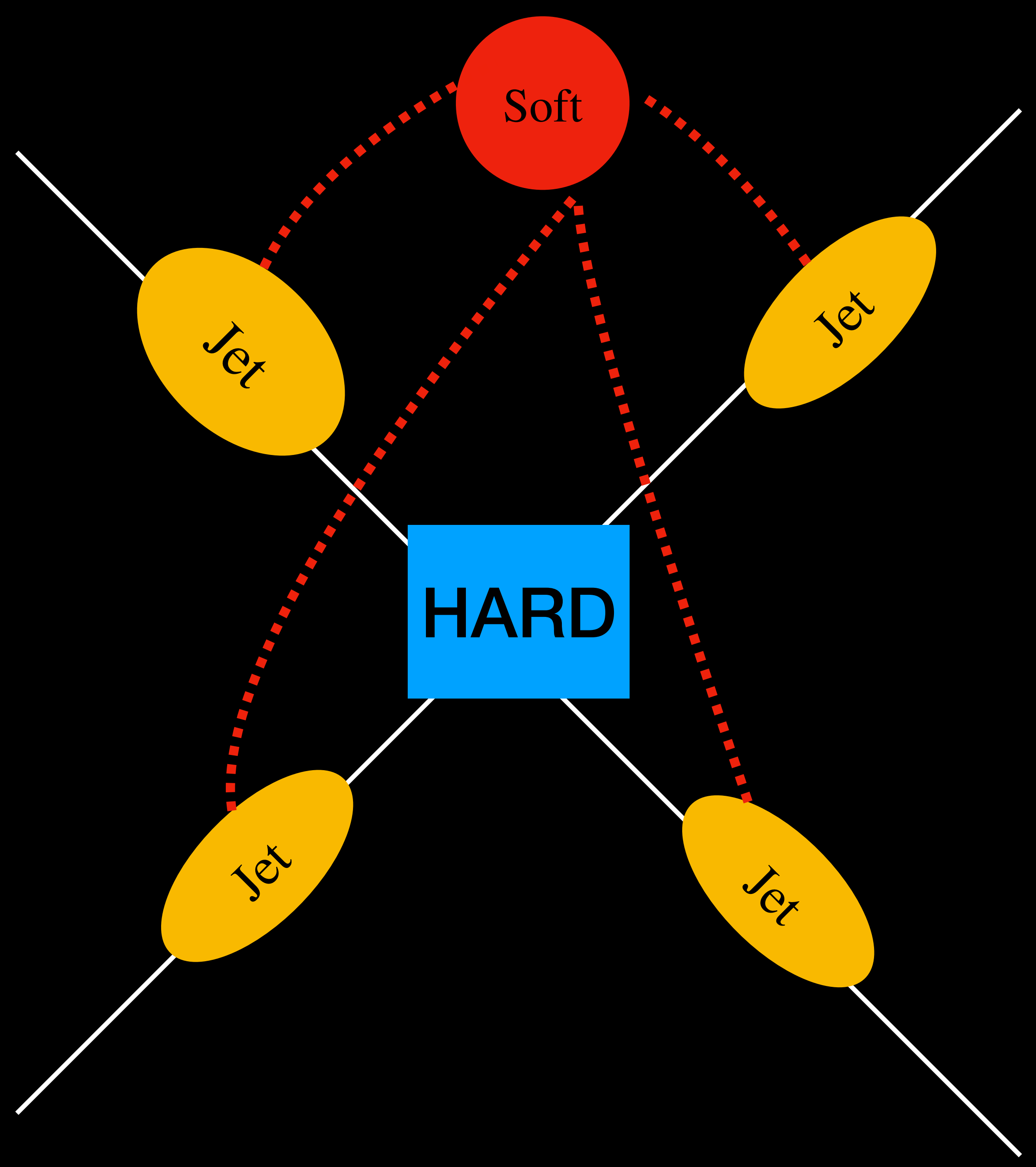
Infrared amplitude factorization

- UV Renormalized scattering amplitudes for well-separated final-states take a simple factorized form

$$Amplitude = \mathrm{hard} \cdot \mathrm{soft} \cdot \prod_i \mathrm{Jet}_i \,.$$

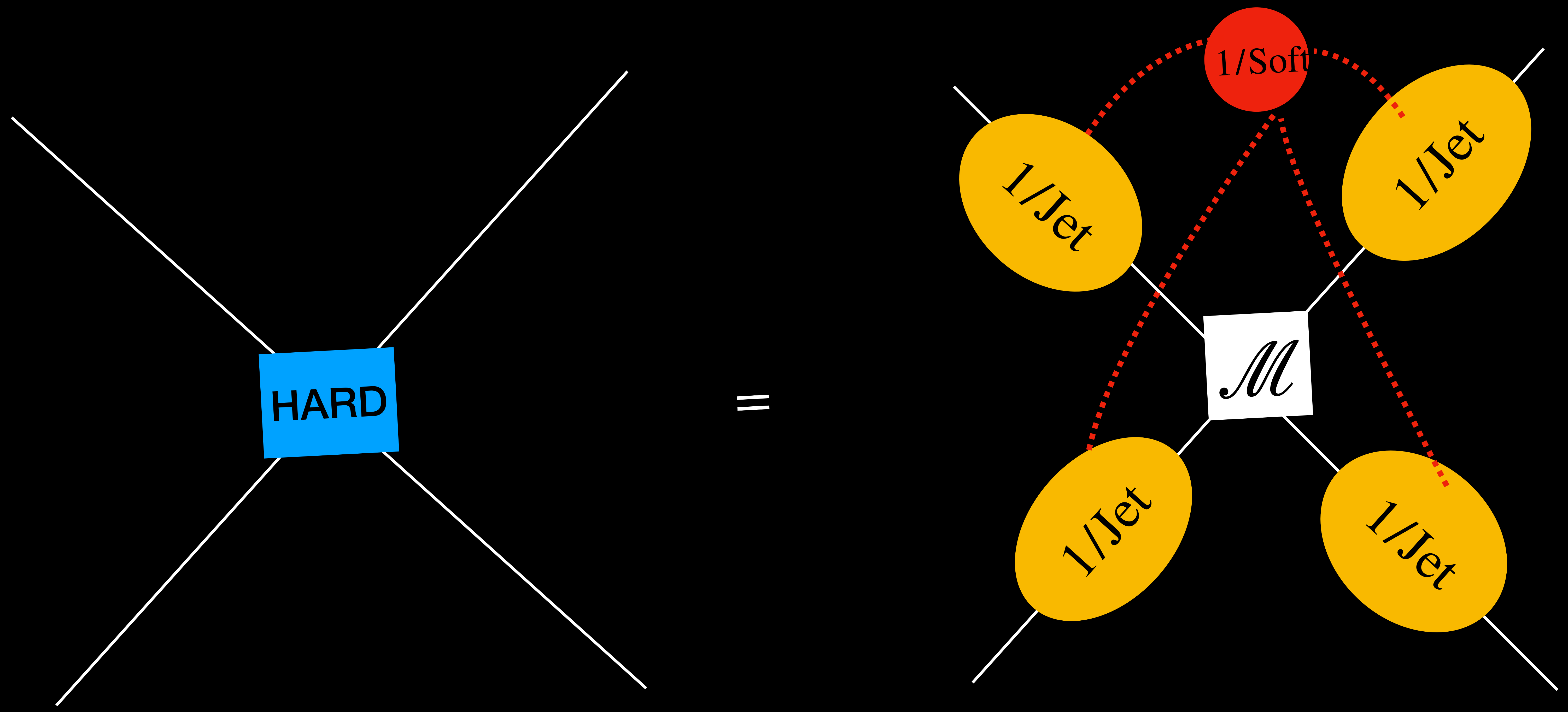
  - "soft" and "jet" functions contain all divergences.

- These are universal functions. For any new process we should need to compute only the "hard" function.
- So far, we do not have a way to compute the "hard" function directly.
- But, what if we did?



Ma;
Erdogan, Sterman; Feige, Schwartz; Collins

How could we imagine using factorisation?



An inverted factorization theorem

How could we imagine using factorization?

$$A = \int [dk]\, \mathcal{A}(k) = \int \mathcal{S} \prod_i \mathcal{J}_i \cdot \int [dk]\, \mathcal{A}(k) \cdot \mathcal{S}^{-1}(k) \cdot \prod_i \mathcal{J}_i^{-1}(k)$$

soft/collinear *Hard*

Divergent | Finite

Analytic Integration in $D = 4 - 2\epsilon$, known to at least three-loops

Numerical integration in exactly $D = 4$.

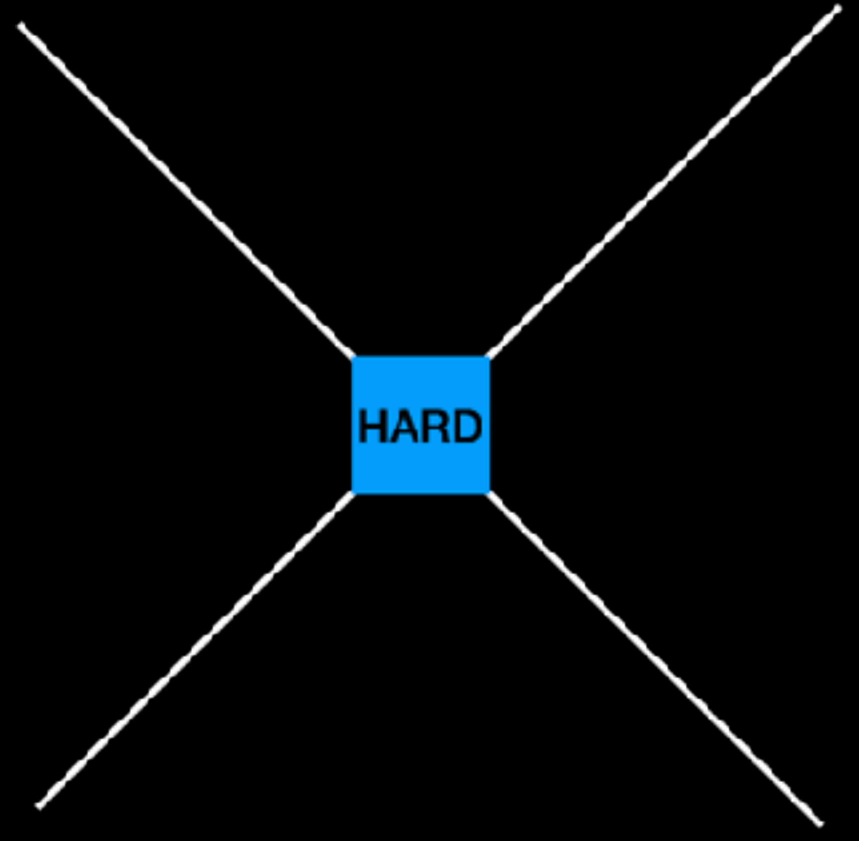
Universal | Process dependent

This procedure is universal…could be applied to any process, irrespectively of the complexity of its final state.

From factorisation we could identify, remove and integrate separately the singular parts of amplitudes order by order in perturbation theory:

$$\mathcal{H}^{(0)} = \mathcal{A}^{(0)} \qquad \mathcal{H}^{(1)} = \mathcal{A}^{(1)} - \mathcal{J}^{(1)}\mathcal{H}^{(0)} - \mathcal{S}^{(1)}\mathcal{H}^{(0)} \qquad \mathcal{H}^{(2)} = \mathcal{A}^{(2)} - \mathcal{J}^{(1)}\mathcal{H}^{(1)} - \mathcal{S}^{(1)}\mathcal{H}^{(1)} - \mathcal{J}^{(2)}\mathcal{H}^{(0)} - \mathcal{S}^{(2)}\mathcal{H}^{(0)} + \mathcal{J}^{(1)}\mathcal{S}^{(1)}\mathcal{H}^{(0)} \quad \ldots$$

Factorisation and integrands



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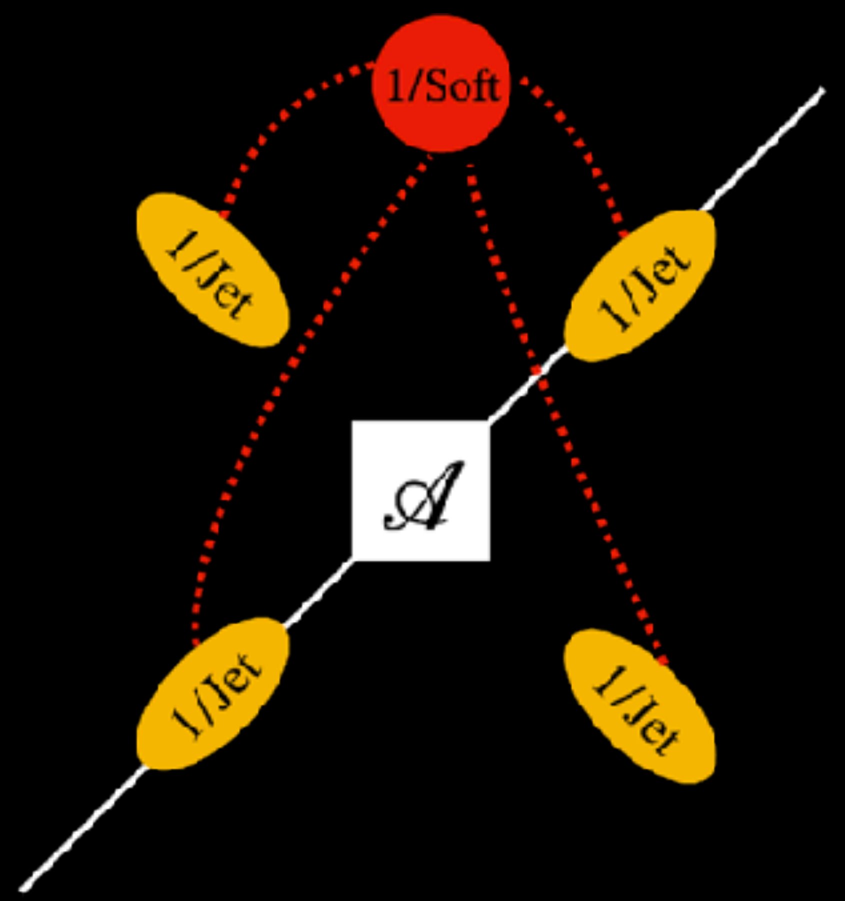
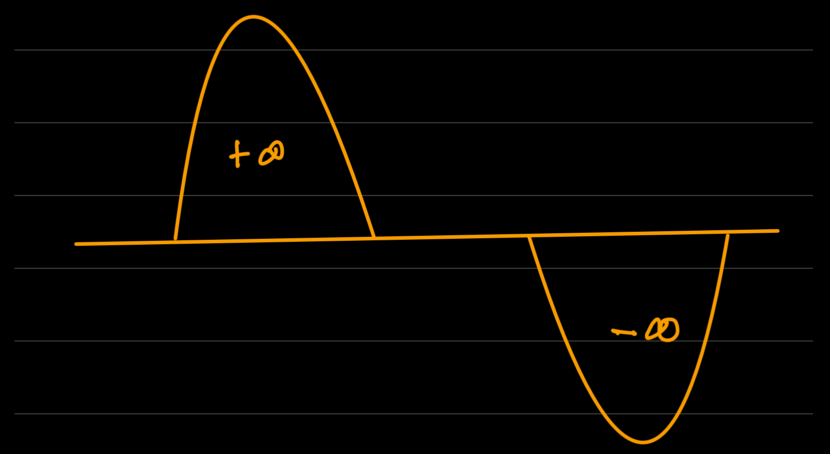
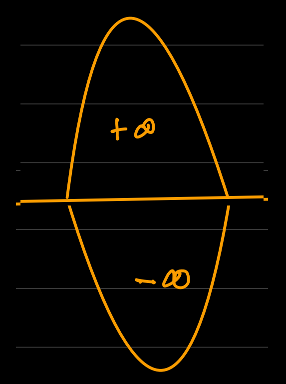




Non-local cancellations

$$\int_0^{10} dx \left[\frac{1}{x-3+i0^+} - \frac{1}{x-7+i0^+}\right]$$



Local cancellations
Numerically integrable

$$\int_0^{10} dx \left[\frac{e^{-\frac{1}{(x-3)^2}}}{x-3+i0^+} - \frac{\cos(x-3)}{x-3+i0^+}\right]$$

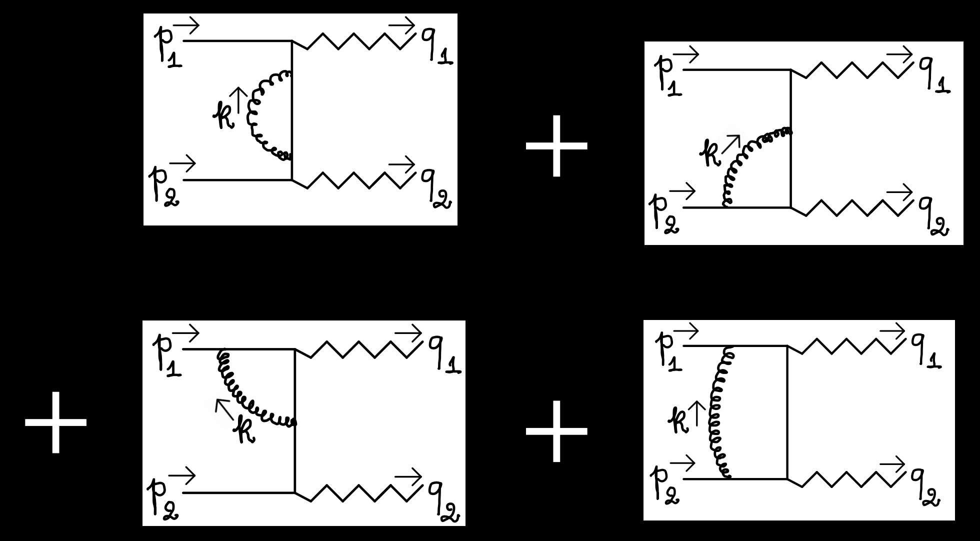
- In the integral expression of the process dependent “HARD” function, we need singularities to be cancelled locally, AT THE INTEGRAND.
- A naive construction leads to non-local cancellations.
- Integrands with non-local cancellations cannot be integrated numerically.
- To enable Monte-Carlo integration methods, can we ensure that ALL soft, collinear and ultraviolet singularities cancel point by point in the integrand?
- A challenge!

Locally finite integrands for electroweak production in quark annihilation

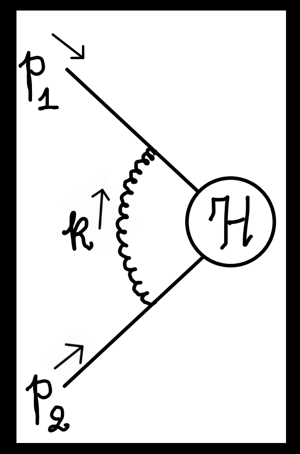
$$q + \bar{q} \to V_1 + V_2 + \dots V_n \,, \quad V_i = W, Z, \gamma^*$$

$$\mathcal{H}^{(1),R}_{q\bar{q}\to ew}(k) = \mathcal{M}^{(1),R}_{q\bar{q}\to ew}(k) - \mathcal{F}^{(1),R}_{q\bar{q}}(k) \left[\mathbf{P}_1 \widetilde{\mathcal{M}}^{(0)}_{q\bar{q}\to ew} \mathbf{P}_1 \right]$$

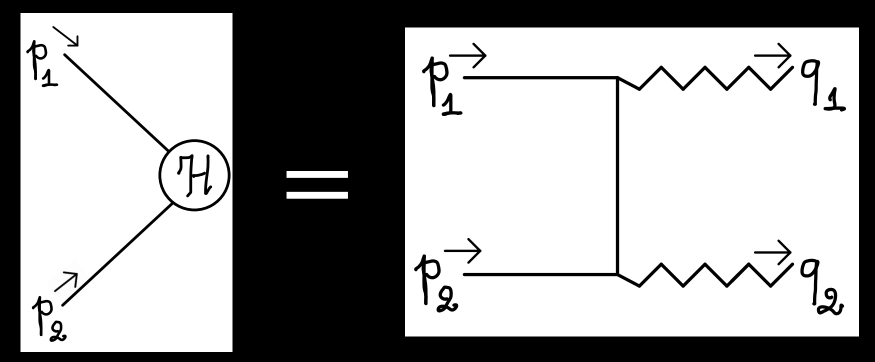
- Finite one-loop amplitude integrand
- Free of all IR and UV singularities locally
- Integrable in D=4 exactly



- One-loop amplitude integrand
- As derived with Feynman rules
- Momentum flow assignment

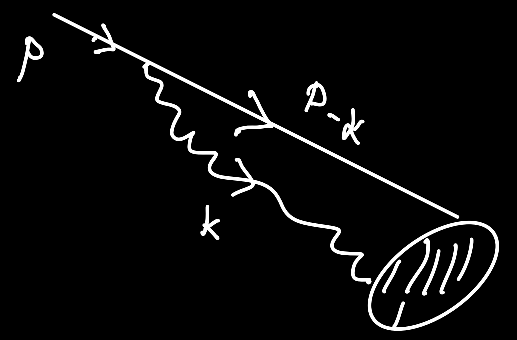


- One-loop amplitude integrand for simplest $2 \to 1$ process



- The external current is the full tree-level amplitude
- I.e. the finite integrand of the previous perturbative order

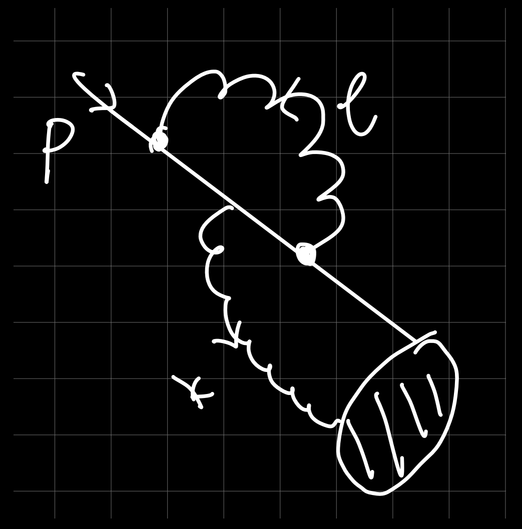
Non-local cancellations?



$$\xrightarrow{k \to x p} \quad \ldots u(p) \frac{2(1-x)}{x} \frac{k^\mu}{k^2 (k+p)^2}$$

Longitudinal polarization

- Collinear gluons off one-loop vertices acquire random polarisations.

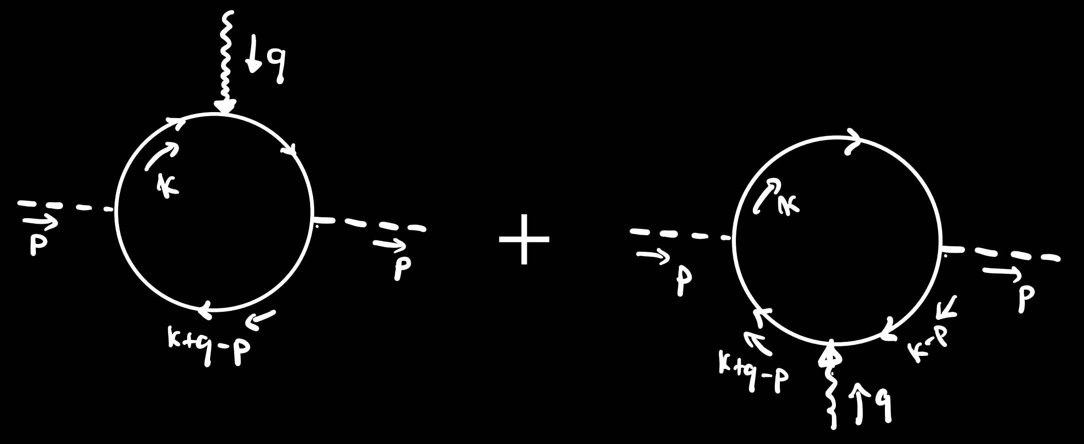


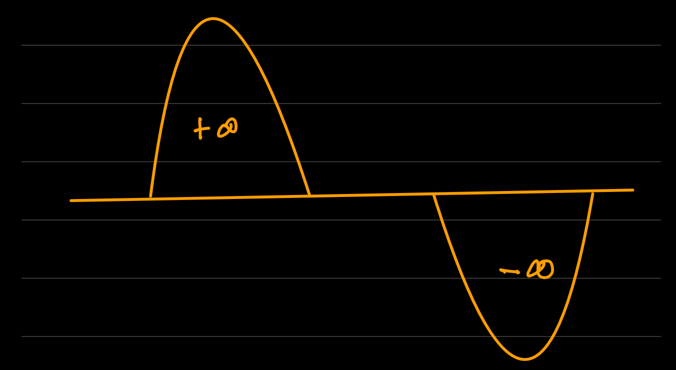
$$\xrightarrow[l \sim hard]{k \to x p} \quad \frac{\ldots k^\mu + \ldots l^\mu}{k^2 (k+p)^2 l^2 (l+p)^2 (k+l+p)^2}$$

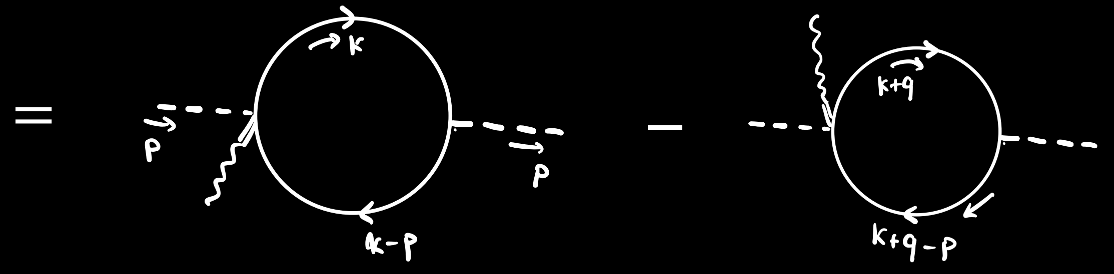
Longitudinal polarization

Loop polarization

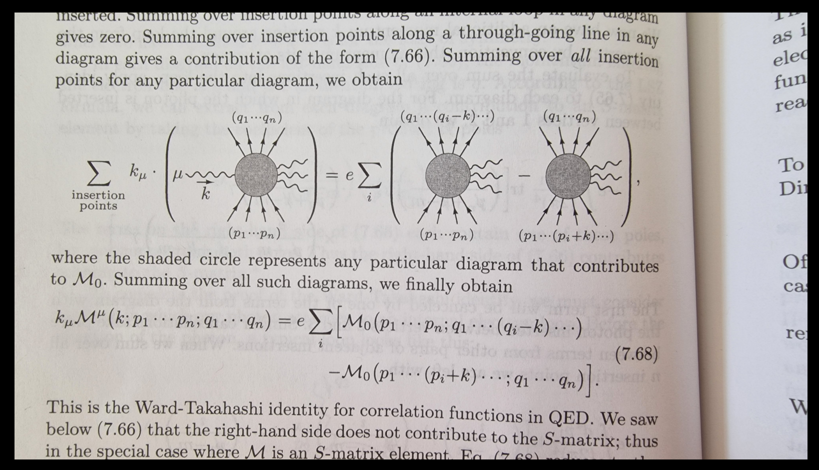
- Ward identities generate non-local zeros.







$$\stackrel{\int}{=} 0 .$$



inserted. Summing over insertion points along an internal loop in any diagram gives zero. Summing over insertion points along a through-going line in any diagram gives a contribution of the form (7.66). Summing over *all* insertion points for any particular diagram, we obtain

$$\sum_{\substack{\text{insertion}\\ \text{points}}} k_\mu \cdot \left(\ldots \right) = e \sum_i \left(\ldots - \ldots \right),$$

where the shaded circle represents any particular diagram that contributes to $\mathcal{M}_0$. Summing over all such diagrams, we finally obtain

$$k_\mu \mathcal{M}^\mu(k; p_1 \cdots p_n; q_1 \cdots q_n) = e \sum_i \Big[\mathcal{M}_0(p_1 \cdots p_n; q_1 \cdots (q_i - k) \cdots) - \mathcal{M}_0(p_1 \cdots (p_i + k) \cdots; q_1 \cdots q_n) \Big]. \quad (7.68)$$

This is the Ward-Takahashi identity for correlation functions in QED. We saw below (7.66) that the right-hand side does not contribute to the S-matrix; thus in the special case where $\mathcal{M}$ is an S-matrix element, Eq. (7.68) ...

...from Peskin and Schroeder

Locally finite integrands for electroweak production in quark annihilation

$$q + \bar{q} \to V_1 + V_2 + \dots V_n \,, \quad V_i = W, Z, \gamma^*$$

- In singular IR regions of loop momenta, virtual gauge bosons acquire (unphysical) longitudinal polarizations.

$$\mathbf{M}^{(2),R}_{q\bar{q}\to ew}(k,l) = \mathscr{M}^{(2),R}_{q\bar{q}\to ew}(k,l) - \Delta\mathscr{M}^{(2),R}_{q\bar{q}\to ew}(k,l)$$

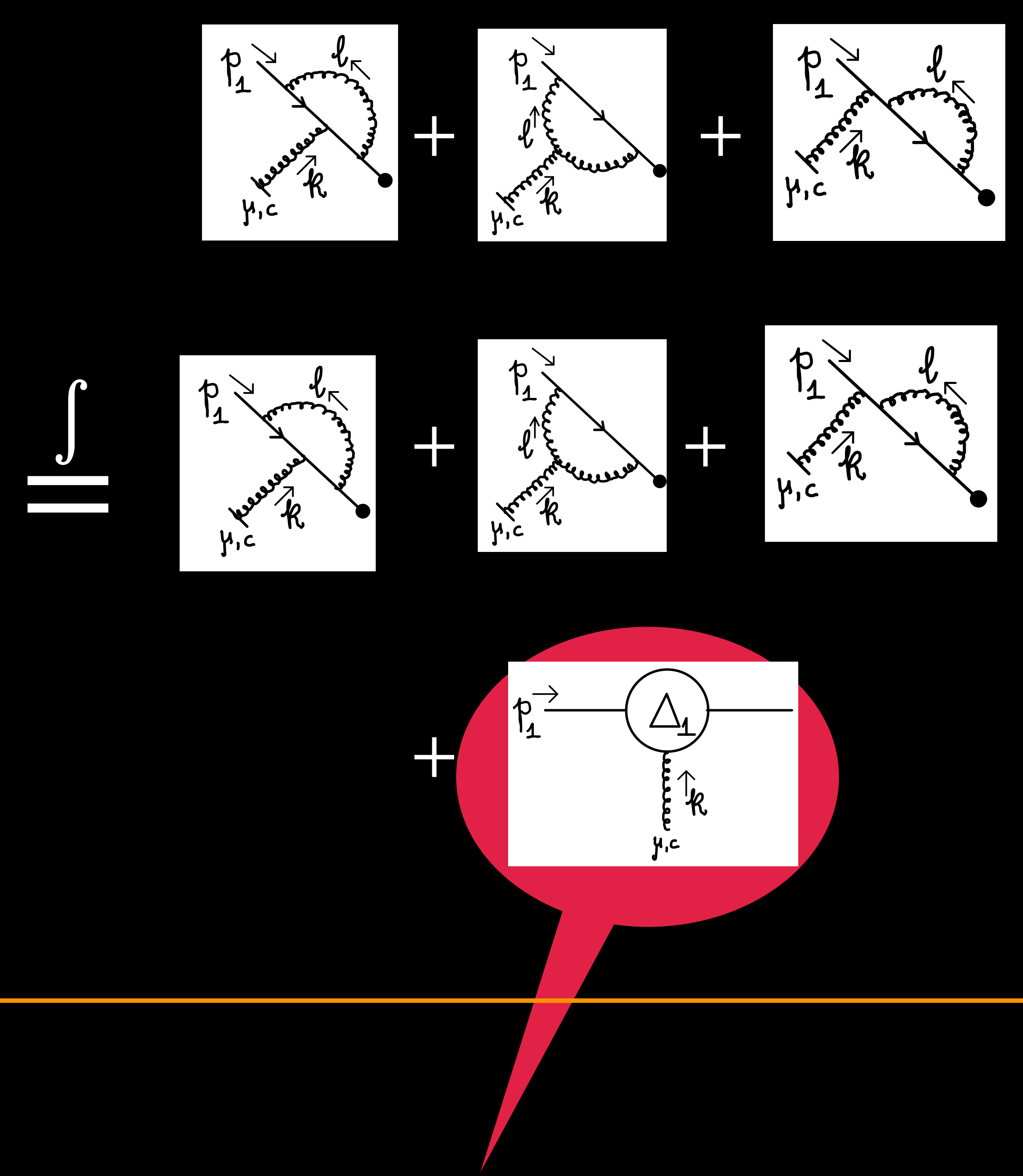
- The two-loop amplitude integrand as derived from Feynman diagrams
- “Shift” Counterterms

- Same scattering amplitude. “Shift” counterterms integrate to zero.

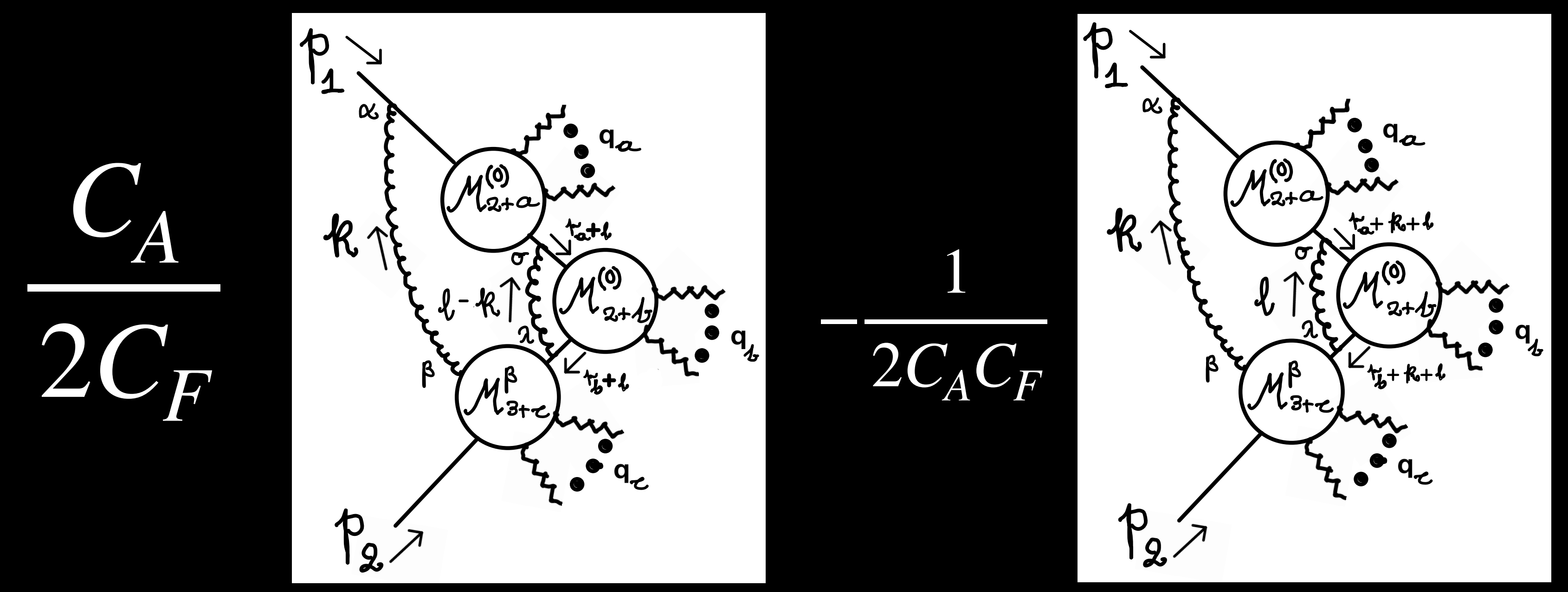
$$\int d^D k \int d^D l \, \Delta\mathscr{M}^{(2),R}_{q\bar{q}\to ew}(k,l) = 0$$

- All Ward identity cancellations are made local. All collinear gluons have longitudinal polarizations

What cures non-local cancellations?



- Integrates to zero
- Eliminates loop polarizations



- Attribute different momentum routing to the various colour components of Feynman diagrams

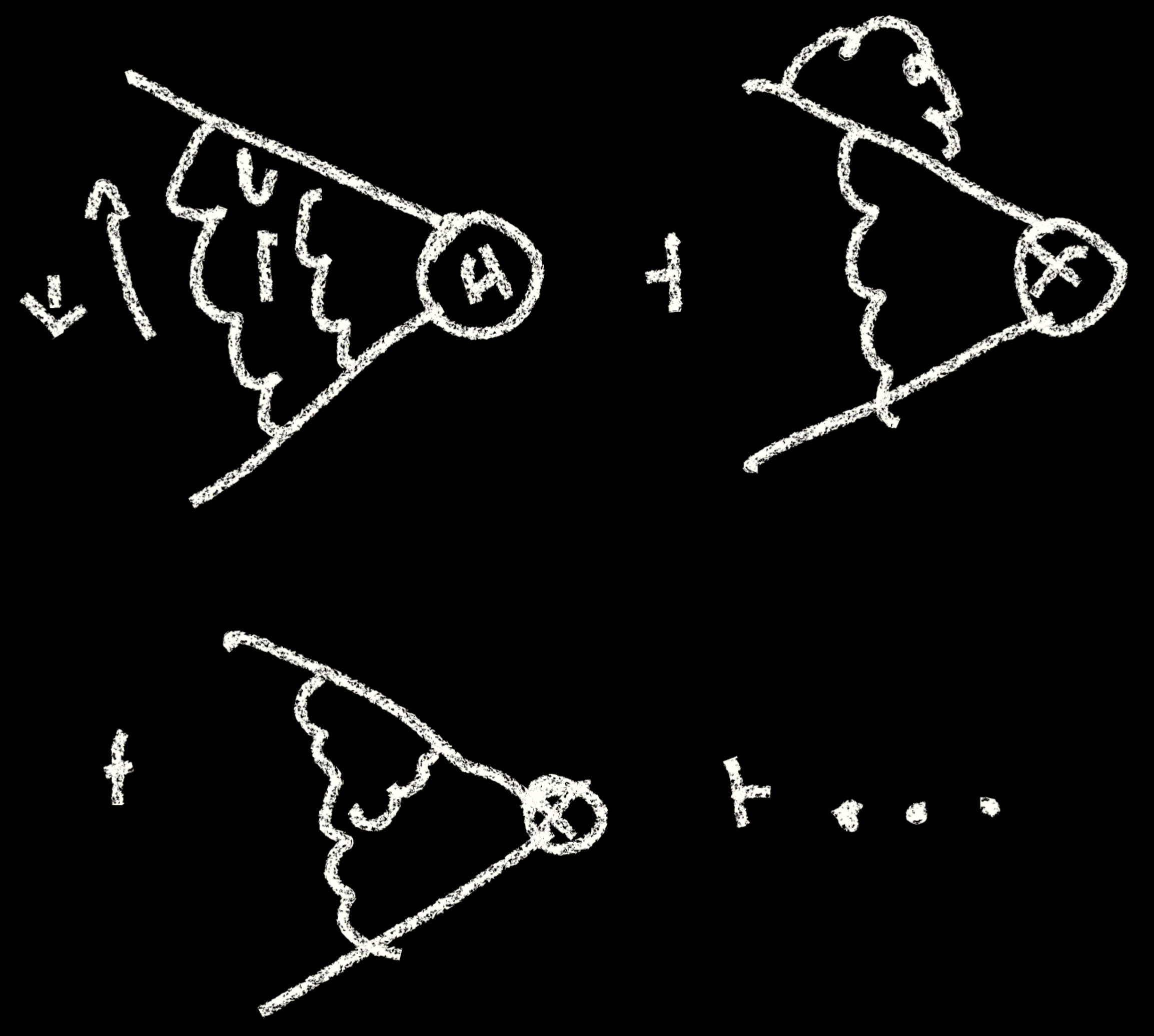
Locally finite integrands for electroweak production in quark annihilation

$$q + \bar{q} \to V_1 + V_2 + \ldots V_n\,, \quad V_i = W, Z, \gamma^*$$

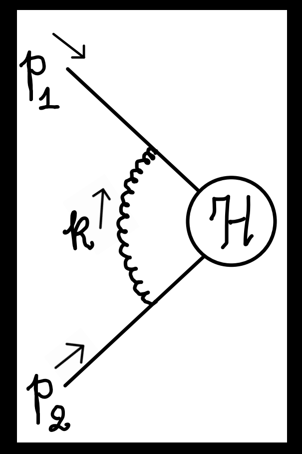
$$\mathcal{H}^{(2),R}_{q\bar{q}\to ew}(k,l) = \mathbf{M}^{(2),R}_{q\bar{q}\to ew}(k,l) - \mathcal{F}^{(2),R}_{q\bar{q}}(k,l)\left[\mathbf{P}_1 \widetilde{\mathcal{M}}^{(0)}_{q\bar{q}\to ew}\mathbf{P}_1\right] - \mathcal{F}^{(1),R}_{q\bar{q}}(k)\left[\mathbf{P}_1 \widetilde{\mathcal{H}}^{(1),R}_{q\bar{q}\to ew}(l)\mathbf{P}_1\right]$$

- Finite two-loop amplitude integrand
- Free of all IR and UV singularities locally
- Integrable in D=4 exactly

- Two-loop amplitude integrand
- As derived with Feynman rules
- AND “Shift” counterterms
- Momentum flow assignment



- Two-loop amplitude integrand for simplest $2 \to 1$ process



- One-loop amplitude integrand for simplest $2 \to 1$ process

- The external current is the finite one-loop amplitude intend
- I.e. the finite integrand of the previous perturbative order

Locally finite integrands for a class of two-loop QCD amplitudes (gluon fusion)

$$g + g \to V_1 + V_2 + \ldots V_n\,, \quad V_i = Higgs, W, Z, \gamma^*$$ ✔

$$\mathscr{H}^{(2),R}_{gg\to\text{colorless}}(k,l) = \mathbf{M}^{(2),R}_{gg\to\text{colorless}}(k,l) - \mathscr{F}^{(1),R}_{\text{scalar}}(k)\,\mathscr{M}^{(1),R}_{gg\to\text{colorless}}(l)$$

$$\mathbf{M}^{(2),R}_{gg\to\text{colorless}}(k,l) = \mathscr{M}^{(2),R}_{gg\to\text{colorless}}(k,l) - \Delta\mathscr{M}^{(2),R}_{gg\to\text{colorless}}(k,l)$$

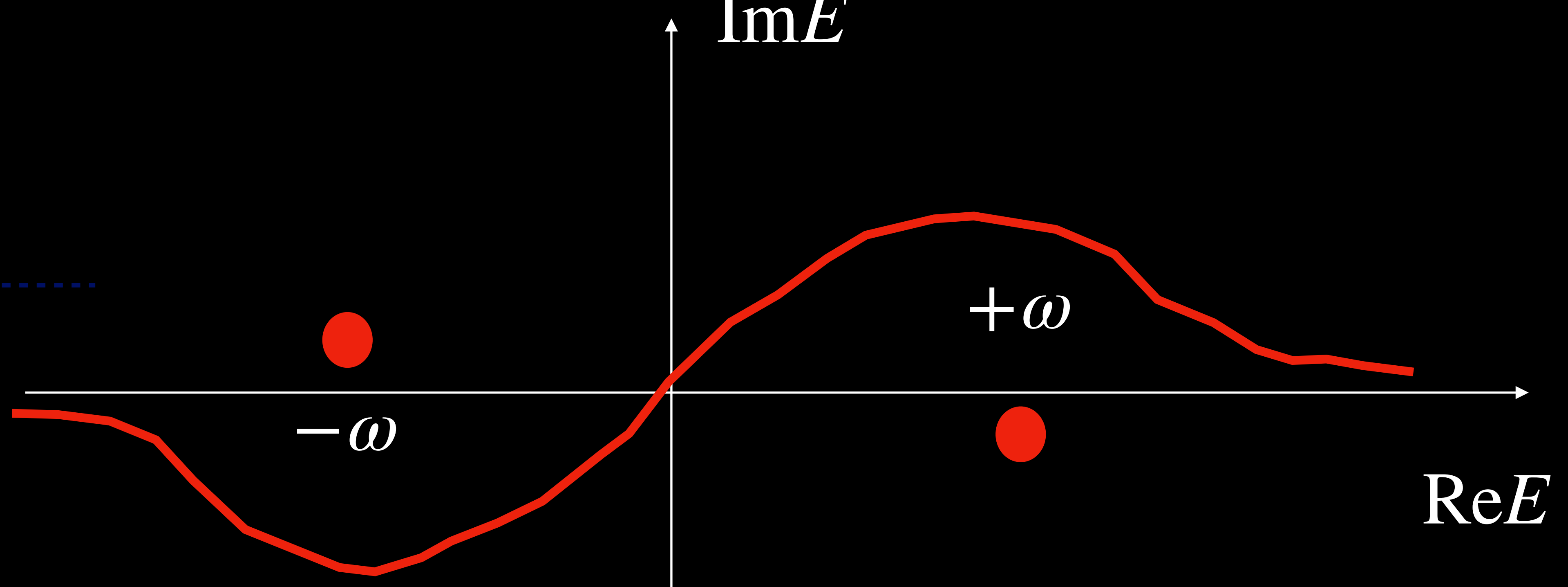
CA, Julia Karlen, George Sterman, Ani Venkata (JHEP 11 (2024) 043)

Numerical integration

- Can such IR subtractions be used for evaluating loop amplitudes numerically?
- They are an important ingredient! They remove “pinch” singularities.
- Other singularities which can be avoided with appropriate contour-deformations are equally important.
- Breakthroughs and excellent ideas.

A novel contour deformation method

Capatti, Hirschi, Kermaschah, Pelloni, Ruijl [1912.0929]



A novel “threshold subtraction” method

Kermaschah [2110.06869]

Kermaschah, Vicini [2407.21511,2407.18051]

Nf-virtual contribution to NNLO electroweak cross-sections
Kermanschah, Vicini arXiv:2407.18051

centre-of-mass energy	$E_{CM} \equiv \sqrt{s} = 13\,\text{TeV}$
phase-space cuts	$p_T^{\min} = \begin{cases} 50\,\text{GeV} & \text{for massless bosons} \\ 25\,\text{GeV} & \text{for massive bosons} \end{cases}$
PDFs	CT10 NLO PDF, lhapdf_id: 11000 [182] flavours $= \begin{cases} d, \bar{d} & \text{if } Z \text{ boson in final state} \\ u, \bar{u}, d, \bar{d}, c, \bar{c}, s, \bar{s}, b, \bar{b} & \text{otherwise} \end{cases}$
masses / scales	$M_Z = 91.1876\,\text{GeV}$, $M_{\gamma_1^*} = 20\,\text{GeV}$, $M_{\gamma_2^*} = 50\,\text{GeV}$ $\mu = \mu_F = M = M_{\gamma^*} = M_Z$
couplings at M_Z	$\alpha_s = 0.118001$, $\alpha = (132.5070)^{-1}$, $a_{Zd} = \frac{1}{4\cos\theta_W \sin\theta_W}$, $v_{Zd} = \frac{3-4\sin^2\theta_W}{12\cos\theta_W\sin\theta_W}$, $\sin^2\theta_W = 0.22225$
other constants	$N_c = 3$, $N_f = 5$, $T_F = \frac{1}{2}$, $C_F = \frac{4}{3}$

- No known analytic results, or master integrals.
- Seamless combined integration of loop and phase-space integrations.
- Process universality

Process	Order	Part	N_p [10^6]	t/p [μs]	Exp.	Result [pb]	Δ [%]
$pp \to \gamma\gamma\gamma$	LO	$\boldsymbol{\sigma_{\text{LO}}}$	10	0.3	10^{-3}	**2.5999 ± 0.0023**	**0.089**
		reference			10^{-3}	**2.5980 ± 0.0018**	**0.069**
	NLO	$\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V},(0)}$	10	1	10^{-4}	-1.0386 ± 0.0017	0.165
		$\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V},(1)}$	1420	6	10^{-4}	2.5260 ± 0.0139	0.549
		$\boldsymbol{\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V,CS}}}$			10^{-4}	**1.4874 ± 0.0140**	**0.940**
		reference			10^{-4}	**1.5090 ± 0.0010**	**0.066**
	NNLO, N_f	$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(0)}$	10	1	10^{-6}	1.4781 ± 0.0074	0.499
		$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(1)}$			10^{-6}	2.6355 ± 0.0145	0.549
		$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(2)}$	7703	12	10^{-5}	-2.9574 ± 0.0236	0.798
		$\boldsymbol{\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V,CS}}}$			10^{-5}	**-2.5460 ± 0.0237**	**0.929**
		reference	**177**		10^{-5}	**-2.5732 ± 0.0006**	**0.023**
$p_d p_d \to Z\gamma_1^*\gamma_2^*$	LO	$\boldsymbol{\sigma_{\text{LO}}}$	11	5	10^{-3}	**1.3831 ± 0.0010**	**0.072**
		reference			10^{-3}	**1.3830 ± 0.0009**	**0.066**
	NLO	$\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V},(0)}$	11	21	10^{-5}	-2.6582 ± 0.0077	0.289
		$\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V},(1)}$	108	1056	10^{-4}	2.7258 ± 0.0210	0.771
		$\boldsymbol{\left(\frac{\alpha_s}{4\pi}\right)\sigma_{\text{NLO}}^{\text{V,CS}}}$			10^{-4}	**2.4600 ± 0.0210**	**0.855**
		reference			10^{-4}	**2.4440 ± 0.0016**	**0.065**
	NNLO, N_f	$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(0)}$	19	20	10^{-7}	-2.1367 ± 0.0204	0.954
		$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(1)}$			10^{-6}	2.8440 ± 0.0219	0.771
		$\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V},(2)}$	927	2403	10^{-5}	-2.6932 ± 0.0228	0.845
		$\boldsymbol{\left(\frac{\alpha_s}{4\pi}\right)^2\sigma_{\text{NNLO},N_f}^{\text{V,CS}}}$			10^{-5}	**-2.4301 ± 0.0229**	**0.941**

Table 7: Virtual contributions to $2 \to 3$ cross sections.

Conclusions

- A thriving programme of precision studies in cosmology from surveys of the Large Scale Structure and at the LHC.

- Perturbative corrections can be tackled in common, with numerical methods.

- New results for the two-loop power-spectrum in EFT of LSS.

- New results for triple electroweak production at the LHC.

- An exciting time for precision phenomenology in particle physics and in cosmology.