# Introduction to QCD

FS 10, Series 9

Due date: 05.05.2010, 1 pm

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**Exercise 1** The plus distributions may be defined as

$$\left[\frac{f(z)}{1-z}\right]_+ = \frac{f(z)}{1-z} - \delta(1-z)\int_0^z dy \frac{f(y)}{1-y}$$

where $f$ is some smooth function of $z$ on the integration domain $0 < z < 1$. This definition is only valid when the plus distribution is being integrated over $0 < z < 1$. Show that

$$\int_x^1 dz g(z)\left[\frac{f(z)}{1-z}\right]_+ = \int_x^1 dz f(z)\left(\frac{g(z)-g(1)}{1-z}\right) - g(1)\int_0^x dz \frac{f(z)}{1-z}.$$

*Hint:* Use a suitable $\theta$-function to remap the integrand from 0 to 1.

**Exercise 2** The hadronic tensor $W^{\mu\nu}$ can be factorised into a perturbative part $\hat{W}$ convoluted with a nonperturbative part $f_i(x, Q^2)$ (the parton distribution function):

$$W^{\mu\nu} = \sum_i \int_0^1 \frac{d\xi}{\xi} f_i(\xi)\hat{W}_i^{\mu\nu}(\xi p, q_\gamma).$$

The *real* correction to the process $q\gamma^* \to q'$ was given as

$$\hat{W} = 4e_q^2\alpha_s C_F \int d\Phi_2 \left[\frac{g.q'}{g.q} + \frac{g.q}{g.q'} + \frac{Q^2 q.q'}{g.q g.q'}\right]$$

where $d\Phi_2$ denotes the $2 \to 2$ particle phase space measure.

a) Show that in the center of mass frame of the incoming quark $q$ and the virtual photon $\gamma^*$ (of virtuality $-Q^2$ ) the momenta may be parameterised as

$$\begin{aligned}
q^\mu &= \frac{s+Q^2}{2\sqrt{s}}(1,0,0,1) \\
{q_\gamma}^\mu &= (\frac{s-Q^2}{2\sqrt{s}}, 0, 0, -\frac{s+Q^2}{2\sqrt{s}}) \\
q'^\mu &= \frac{\sqrt{s}}{2}(1, -\sin\theta, 0, -\cos\theta) \\
g^\mu &= \frac{\sqrt{s}}{2}(1, \sin\theta, 0, \cos\theta)
\end{aligned} \tag{1}$$

where $s = (q_\gamma + q)^2$ shall denote the center of mass energy of the system. Assuming that $q$ is collinear to the proton ($q = \xi p$) show that

$$s = \frac{Q^2(1-z)}{z}$$

where we define $x = Q^2/2p.q_\gamma$ and $z = \xi/x$.

b) Show that the (remaining) Lorentz invariants then take the following form

$$\begin{aligned} 2g.q &= \frac{Q^2}{2z}(1-\cos\theta) \\ 2q.q' &= \frac{Q^2}{2z}(1+\cos\theta). \end{aligned} \tag{2}$$

c) Hence show that

$$\hat{W} = \frac{e_q^2 \alpha_s C_F}{4\pi} \int_{-1}^{1} d\cos\theta \left[ \frac{2(1-z)}{1-\cos\theta} + \frac{1-\cos\theta}{2(1-z)} + \frac{2z(1+\cos\theta)}{(1-z)(1-\cos\theta)} \right]$$

(where we used $d\Phi_2 = d\cos\theta/16\pi$ see **Series 1**).

d) We are interested in the collinear singularity. Expand around $\theta = 0$ ( to $O(\theta^0)$) to isolate the collinear pole

$$\hat{W} = e_q^2 C_F \frac{\alpha_s}{2\pi} \int_0^\pi \frac{d\theta}{\theta} \left[ \frac{1+z^2}{1-z} \right] + O(\theta^0).$$

e) We see that apart from the collinear singularity there is also a soft singularity associated with $z \to 1$. Argue that if we had added the virtual correction, which lives at $z = 1$, this soft singularity should have canceled. Use the plus description to accomplish this. Further more argue that the virtual correction could add another term proportional to $\delta(1-z)$. And that we should thus have arrived at

$$\hat{W}^{(real+virtual)} = e_q^2 C_F \frac{\alpha_s}{2\pi} \int_0^\pi \frac{d\theta}{\theta} P_{qq}(z) + O(\theta^0)$$

where

$$P_{qq}(z) = \frac{1+z^2}{(1-z)_+} + C_{qq}\delta(1-z)$$

is the $q \to q$ Altarelli-Parisi splitting function.

f) Use the DGLAP equation for $q^{NS}$ in Mellin space and the fact that $\int_0^1 dx q^{NS}(x) = constant$ to prove that

$$\gamma_{qq}(1) = \int_0^1 dz P_{qq}(z) = 0.$$

Derive $C_{qq} = 3/2$.