Introduction to QCD

FS 10, Series 5

Due date: 12.04.2010, 6 pm

Exercise 1

The renormalization group equations for QCD are

$$\Big[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} - \gamma_m(g_R,\alpha_R)m_R\frac{\partial}{\partial m_R} + \delta(g_R,\alpha_R)\frac{\partial}{\partial \alpha_R}$$
$$-n_G\gamma_G(g_R,\alpha_R) - n_F\gamma_F(g_R,\alpha_R))\Big]F_{n_G,n_F}(\lambda p, g_R, m_R, \alpha_R, \mu) = 0, \tag{1}$$

where F_{n_G,n_F} is the truncated connected Green function (times $-i$) with n_G gluon and n_F quark legs in momentum space, and the renormalization group functions β, γ_m, δ, γ_G, and γ_F are defined by

$$\beta(g_R,\alpha_R) = \mu\frac{\partial g_R}{\partial\mu} \tag{2}$$
$$\gamma_m(g_R,\alpha_R) = -\frac{\mu}{m_R}\frac{\partial m_R}{\partial\mu} \tag{3}$$
$$\delta(g_R,\alpha_R) = \mu\frac{\partial\alpha_R}{\partial\mu} = -2\alpha_R\gamma_G(g_R,\alpha_R) \tag{4}$$
$$\gamma_G(g_R,\alpha_R) = \frac{\mu}{2Z_3}\frac{\partial Z_3}{\partial\mu} \tag{5}$$
$$\gamma_F(g_R,\alpha_R) = \frac{\mu}{2Z_2}\frac{\partial Z_2}{\partial\mu}. \tag{6}$$

(i) Using the fact that in the MS scheme the renormalization constant Z_m is given by

$$Z_m = 1 + \frac{Z^{(1)}g_R^2}{\epsilon} + \mathcal{O}(g_R^4) \tag{7}$$

show that γ_m does not depend on the gauge parameter.

(ii) Use the definition of γ_m and eq. (7) to show that $\gamma_{m0} = -Z_m^{(1)}$.

(iii) (a) Using λ as the renormalization scale instead of μ, show that

$$\frac{1}{m(t)}\frac{dm}{dt} = -1 + \gamma_m(g(t)), \tag{8}$$

where $t = -\ln\lambda$.

(b) Integrate the differential equation obtained in the previous item, and use

$$g(t)^2 = \frac{g^2}{1 + 2\beta_0 g^2 t} \tag{9}$$

to show that

$$m(t) = m\, t^{\frac{\gamma_{m0}}{2\beta_0}}\, e^{-t}. \tag{10}$$