Introduction to QCD

FS 10, Series 12

Due date: 31.05.2010, 6 pm

Exercise 1: one-loop thrust distribution

1. Recall the definition of the thrust in e^+e^- annihilation:

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{q}_i \cdot \vec{n}|}{\sum_i |\vec{q}_i|} \,. \tag{1}$$

The axis $\vec{n}_T$ that maximises the sum eq. (1) is called the *thrust axis*.

   (a) Show that for two (back-to-back) massless particles, the thrust axis is along the direction of any of the two. Show also that in this case $T = 1$.

   (b) Optional: show that for three massless particles $\{q_1, q_2, q_3\}$ the thrust axis is along the most energetic particle q_1. Using also $q = q_1 + q_2 + q_3$ with $q^2 = Q^2$ show that

$$T = \frac{2(q_1 q)}{Q^2} = 1 - \frac{(q_2 + q_3)^2}{Q^2} \,. \tag{2}$$

   Hint: use the fact that the thrust axis divides the event in two hemispheres (L and R), such that

$$\sum_{i \in R} \vec{q}_i \times \vec{n}_T = \sum_{i \in L} \vec{q}_i \times \vec{n}_T = 0 \,. \tag{3}$$

2. Consider the one-loop thrust distribution in e^+e^- annihilation:

$$\Sigma(\tau) = 1 + 2C_F g^2 \int [dk] \, \frac{p\bar{p}}{(pk)(k\bar{p})} \left[\Theta\left(\tau - \frac{\omega}{Q}(1 - |\cos\theta|) \right) - 1 \right] \tag{4}$$

where momenta, neglecting recoil, are parametrised as follows

$$p = \frac{Q}{2}\,(1,0,0,1) \,, \qquad \bar{p} = \frac{Q}{2}\,(1,0,0,-1) \,, \qquad k = \omega(1, 0, \sin\theta, \cos\theta) \,. \tag{5}$$

After having verified that

$$[dk] = \frac{\omega d\omega}{8\pi^2} d\cos\theta \frac{d\phi}{2\pi} \,, \qquad \frac{p\bar{p}}{(pk)(k\bar{p})} = \frac{2}{\omega^2(1 - \cos^2\theta)} \,, \tag{6}$$

using $g^2 = 4\pi\alpha_s$, work out the integration in eq. (4) and show that

$$\Sigma(\tau) = 1 - C_F \frac{\alpha_s}{\pi} \left[\ln^2 \frac{1}{\tau} + \mathcal{O}(1) \right] \,. \tag{7}$$

Exercise 2: two-jet rate

A clustering algorithm in e^+e^- annihilation is a recursive procedure that works as follows. Fix a minimum required number of jets $n_{\rm jets}$ and let n be the number of particles.

1. If $n = n_{\rm jets}$ stop: the number of particles n is equal to the required number of jets $n_{\rm jets}$. If $n > n_{min}$, for any pair of particles q_i,q_j, construct an IRC safe distance d_{ij}, for instance
$$d_{ij} = \begin{cases} 2E_iE_j(1-\cos\theta_{ij}) & \text{(JADE)} \\ 2\min\{E_i^2, E_j^2\}(1-\cos\theta_{ij}) & \text{(Durham)} \end{cases} \tag{8}$$

2. Merge the pair q_i, q_j for which d_{ij} is minimum and replace them with a pseudoparticle q_{ij}, for instance $q_{ij} = q_i + q_j$ (E-scheme). Define also $d_n \equiv d_{ij}$

3. Let $n \to n-1$ and go back to step 1.

Let us define the n-jet resolution $y_{n-1,n} \equiv d_n/Q^2$. The n-jet rate $R_n(y_{\rm cut})$ is defined as the fraction of events for which $y_{n,n+1} < y_{\rm cut} < y_{n-1,n}$, with $y_{\rm cut}$ a given resolution parameter. It is evident that $R_2(y_{\rm cut})$ is the fraction of events for which $y_{23} < y_{cut}$.

For $n \leq 3$ show that

$$R_2(y_{\rm cut}) = 1 - R_3(y_{\rm cut}) \simeq 1 - 2C_F g^2 \int [dk] \frac{p\bar{p}}{(pk)(k\bar{p})} \Theta\left(d_3(k) - y_{\rm cut}Q^2\right) , \tag{9}$$

where the last approximation holds in the soft-collinear limit.

Using the kinematics of eq. (5), work out the integral in eq. (9) for both the JADE and the Durham algorithm and show that

$$R_2(y_{\rm cut}) \simeq \begin{cases} 1 - C_F \dfrac{\alpha_s}{\pi} \ln^2 \dfrac{1}{y_{\rm cut}} & \text{(JADE)} \\ 1 - C_F \dfrac{\alpha_s}{2\pi} \ln^2 \dfrac{1}{y_{\rm cut}} & \text{(Durham)} \end{cases} \tag{10}$$

where in the above expression we have neglected terms that do not contain logarithms of the jet-resolution parameter $y_{\rm cut}$.