Symmetries in Physics

Problem Sets

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The current version of this work can be found at:
http://people.phys.ethz.ch/~nbeisert/lectures/.

Contents

1.1. Defining a group

Consider the following minimalistic definition of a group:
A group is a set G together with a map

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G}, \qquad (g, h) \mapsto gh \tag{1.1}$$

satisfying the following axioms:

- *associativity:*
$$a(bc) = (ab)c \qquad \text{for all } a, b, c \in \mathrm{G}; \tag{1.2}$$
- *unit element:* there exists an $e \in \mathrm{G}$ such that
$$ea = a \qquad \text{for all } a \in \mathrm{G}; \tag{1.3}$$
- *inverse:* for all $a \in \mathrm{G}$ there exists an $a^{-1} \in \mathrm{G}$ such that
$$a^{-1}a = e. \tag{1.4}$$

a) Show, using only the axioms above, that $aa^{-1} = e$. Using this show that $ae = a$ and finally that the unit is unique.

b) Replace the unit axiom (1.3) above with

$$ae = a \qquad \text{for all } a \in \mathrm{G}. \tag{1.5}$$

Find a set that fulfils the modified axioms (1.2), (1.5), (1.4) and is not a group.
Hint: Think of a subset of 2×2 matrices.

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1.2. Familiarising with groups

Consider the following sets and establish whether they form a group or not.

a) The set of all non-zero real numbers with ordinary multiplication as the group operation. What changes if we include also the element zero?

b) The set of all real numbers (including zero) with ordinary addition as the group operation.

c) The set of permutations S_n acting on the set of n symbols $A_n := \{1, 2, \ldots, n\}$. What is the number of elements, i.e. the order, of S_n?

Consider in particular S_3. All its elements can be generated by the iterated products of two elements σ_1 and σ_2 satisfying the conditions

$$\sigma_1^2 = \sigma_2^2 = e, \tag{1.6}$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2. \tag{1.7}$$

d) Find all the elements of S_3, and argue that no further elements are generated.

e) Find a suitable action for σ_1, σ_2 on the set A_3.

f) Find a 3-dimensional representation for these elements.
Hint: Think of A_3 as a basis for the 3-dimensional representation space.

g) *optional:* Find a non-trivial 2-dimensional representation for these elements.

h) *optional:* Find a non-trivial 1-dimensional representation for these elements.

2.1. Maps between groups

For the whole exercise let G and H be groups and $g, g_1, g_2 \in \mathrm{G}$ and $h, h_1, h_2 \in \mathrm{H}$.

a) A *group homomorphism* is a map $\varphi : \mathrm{G} \to \mathrm{H}$ that respects the group structure:

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \qquad \text{for all } g_1, g_2 \in \mathrm{G}. \tag{2.1}$$

Show that $\varphi(e_\mathrm{G}) = e_\mathrm{H}$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$, where e_G and e_H denote the identity elements of G and H, respectively.

b) A bijective group homomorphism is called *group isomorphism*. Consider the set of *automorphisms* of a group G defined as

$$\mathrm{Aut(G)} := \{\varphi : \mathrm{G} \to \mathrm{G}; \varphi \text{ is a group isomorphism}\}. \tag{2.2}$$

Show that Aut(G) together with the composition of maps $\circ$ forms a group.

c) Show that the *direct product* $\mathrm{G} \times \mathrm{H}$ defined with multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 h_2) \tag{2.3}$$

is a group.

d) Let G act on H by a group homomorphism $\varphi : \mathrm{G} \to \mathrm{Aut(H)}$, in particular $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$ and $\varphi(g)(h_1 h_2) = \varphi(g)(h_1)\varphi(g)(h_2)$.
Show that the *semi-direct product* $\mathrm{H} \rtimes_\varphi \mathrm{G}$ defined with the multiplication

$$(h_1, g_1)(h_2, g_2) := \big(h_1 \varphi(g_1)(h_2), g_1 g_2\big) \tag{2.4}$$

is again a group.

2.2. On triangles and hexagons

a) What are the symmetries of an equilateral triangle? This group is commonly denoted as D_3. Write down its multiplication table.

b) Label the vertices of your triangle as $1, 2, 3$ and consider the action of the symmetric group S_3 on these vertices. Write down an explicit isomorphism from S_3 to D_3.

c) Write down the elements of the group D_6, the symmetries of a regular hexagon. Give an injective homomorphism from D_3 into D_6 and use this to show that its image is a subgroup of D_6 (i.e., a subset of D_6 which itself forms a group with the restricted multiplication map).

d) Give an injective homomorphism from D_6 into S_6. Stare out the window for a bit and convince yourself the image is again a subgroup.

Hint: It suffices to give the map for a set of generators and check the generator relations.

e) Can you give a surjective homomorphism from D_6 to D_3?

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2.3. Some small problems on permutation groups

Recall the cycle notation for permutations: a cycle $(a_1, \ldots, a_n)$ means that a_1 is sent to a_2, a_2 is sent to a_3, etc. In this way, any permutation can be written as a product of disjoint cycles. 1-Cycles, which correspond to fixed points, are usually left out for convenience sake. Conventionally, the lowest number occurring in a cycle is written first.

a) Write down the elements of S_{496} which send the number 20, 146, and 400 among themselves. Show that this is a subgroup of S_{496}.

b) Recall that S_n is generated by the 2-cycles. Prove that $\mathrm{A}_n \subset \mathrm{S}_n$, the subgroup of even permutations, is generated by the 3-cycles of S_n.

c) Let H be a subgroup of S_n which is not contained in A_n. Prove that exactly half of the elements of H are even permutations.

3.1. The group SO(3)

Consider the set of proper rotations in three dimensions

$$\mathrm{SO}(3) = \big\{R \in \mathrm{Aut}(\mathbb{R}^3); R^{\mathsf{T}}R = 1, \det R = 1\big\}. \tag{3.1}$$

a) Show that SO(3) together with the matrix multiplication forms a group.

b) Show that each element $A \in \mathrm{SO}(3)$ is a rotation about some axis, i.e. show that for all $A \in \mathrm{SO}(3)$ there exists a $\vec{v} \in \mathbb{R}^3$ such that $A\vec{v} = \vec{v}$.

Hint: When is $\ker(A - 1)$ non-trivial?

c) Fix a vector $\vec{v} \in \mathbb{R}^3 \backslash \{0\}$. The *stabiliser subgroup* of SO(3) with respect to $\vec{v}$ is defined as

$$\mathrm{SO}(3)_{\vec{v}} := \{A \in \mathrm{SO}(3); A\vec{v} = \vec{v}\}. \tag{3.2}$$

For a given $A \in \mathrm{SO}(3)$ show that

$$\mathrm{SO}(3)_{A\vec{v}} = A\,\mathrm{SO}(3)_{\vec{v}}\,A^{-1} := \{ARA^{-1}; R \in \mathrm{SO}(3)_{\vec{v}}\} \tag{3.3}$$

and furthermore that $\mathrm{SO}(3)_{\vec{v}} \equiv \mathrm{SO}(2)$.

d) Consider a cube centred at the origin. How many rotations are there that map the cube to itself?

3.2. Euler angles

Any rotation in three dimensions $R \in \mathrm{SO}(3)$ can be expressed as

$$R = R^z_\phi R^y_\theta R^z_\psi \tag{3.4}$$

in terms of the three Euler angles $0 \leq \psi, \phi < 2\pi$, $0 \leq \theta \leq \pi$ and the rotation matrices around the z-axis and the y-axis

$$R^y_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad R^z_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

a) Find the Euler angles for the rotation around the x-axis $0 \leq \alpha < 2\pi$

$$R^x_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.6}$$

b) Show that it is indeed sufficient to restrict to the angles $0 \leq \theta \leq \pi$. In other words, show that a rotation from the domain $\pi < \theta < 2\pi$ can be expressed through a rotation in the domain $0 \leq \theta \leq \pi$.

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3.3. Group representations and the group algebra

Let G be a group and let k be a field (you may take $k = \mathbb{C}$). Let $k[\mathrm{G}]$ denote the vector space over k with basis $\{e_g; g \in \mathrm{G}\}$. Define a multiplication on $k[\mathrm{G}]$ by

$$e_g \cdot e_h = e_{gh}. \tag{3.7}$$

This turns $k[\mathrm{G}]$ into an associative algebra with unit e_e, called the *group algebra of* G.

a) Let

$$\rho : \mathrm{G} \to \mathrm{Aut}(\mathbb{V}) \tag{3.8}$$

be a representation of G, where $\mathbb{V}$ is a vector space over k. Show that ρ induces a linear map

$$\tilde{\rho} : k[\mathrm{G}] \to \mathrm{End}(\mathbb{V}) \tag{3.9}$$

that satisfies

$$\begin{aligned} \tilde{\rho}(ab) &= \tilde{\rho}(a)\tilde{\rho}(b), \\ \tilde{\rho}(e_e) &= 1. \end{aligned} \tag{3.10}$$

b) Conversely, suppose we are given a map $\tilde{\rho} : k[\mathrm{G}] \to \mathrm{End}(\mathbb{V})$ as above. Show that we can recover a representation $\rho : G \to \mathrm{Aut}(\mathbb{V})$ of G from $\tilde{\rho}$.

4.1. From curves to Lie algebras

Let $\mathrm{G} \subset \mathrm{Aut}(\mathbb{V})$ be a continuous matrix group and $A(t) \subset \mathrm{G}$ a differentiable curve parametrised by t, such that $A(0) = 1$, where 1 is the identity element of G. Then the derivative

$$a := \frac{\mathrm{d}}{\mathrm{d}t} A(t)\bigg|_{t=0} \tag{4.1}$$

defines an element of the tangent space $\mathfrak{g}$ of the identity.

a) Show that the set of all derivatives of such curves defines a real vector space.
Hint: Take two curves $A_1(t)$, $A_2(t)$ and consider $A_3(t) = A_1(\lambda_1 t)A_2(\lambda_2 t)$. Show that $A_3(t)$ uniquely defines an element a_3 of $\mathfrak{g}$, and write it in terms of a_1, a_2.

b) Define the adjoint action of a group element $R \in \mathrm{G}$ on the Lie algebra as

$$\mathrm{Ad}(R)(a_1) := \frac{\mathrm{d}}{\mathrm{d}t}\left(RA_1(t)R^{-1}\right)\bigg|_{t=0} \in \mathfrak{g}, \qquad \text{where} \qquad a_1 = \frac{\mathrm{d}}{\mathrm{d}t}\left(A_1(t)\right)\bigg|_{t=0} \in \mathfrak{g}. \tag{4.2}$$

Show that $\mathrm{Ad}(R)(a_1)$ is indeed an element of the Lie algebra for any $R \in \mathrm{G}$.

c) *Optional:* To which space does the object $(\mathrm{d}/\mathrm{d}t)[RA_1(t)]|_{t=0}$ belong? How about $(\mathrm{d}/\mathrm{d}t)A_1(t)R^{-1}|_{t=0}$? Can you find a bijection from these spaces to the Lie algebra $\mathfrak{g}$?

d) Consider now $\mathrm{Ad}(A_2(s))(a_1) \subset \mathfrak{g}$, where $A_2(s) \subset \mathrm{G}$ is another curve in G such that $A_2(0) = 1$. This defines a curve on the Lie algebra $\mathfrak{g}$, and since derivatives of curves on a vector space are themselves elements of the vector space, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Ad}(A_2(s))(a_1)\right)\bigg|_{s=0} \in \mathfrak{g}. \tag{4.3}$$

Evaluate this derivative. What happens if we exchange A_1 and A_2?
Hint: Show that $(\mathrm{d}/\mathrm{d}t)[A(t)^{-1}]|_{t=0} = -a$.

e) Consider now the group $\mathrm{G} = \mathrm{SO}(3)$. Starting from the properties of the curves $R(t) \subset \mathrm{G}$, construct the corresponding Lie algebra $\mathfrak{so}(3)$, find an explicit basis J_i $i = 1, 2, 3$, and compute the Lie brackets of the basis elements $[\![\mathrm{J}_i, \mathrm{J}_j]\!]$.

f) Consider the curve $R(t) \subset \mathrm{SO}(3)$ given by

$$R(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \theta(0) = 0. \tag{4.4}$$

Determine the corresponding Lie algebra element, and expand it on the basis you found before.

g) Starting again from the notion of curves, construct the Lie algebra corresponding to the SU(2) group and find a basis for it.
Can you find a relationship between the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ algebras?

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4.2. Lorentz transformations

The Lorentz transformations are the coordinate transformations that leave the Minkowski metric tensor $\eta = \mathrm{diag}(-1, +1, +1, +1)$, and correspondingly the distance squared $x^\intercal \eta x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$, invariant, i.e.

$$\mathrm{SO}(3,1) := \big\{ A \in \mathrm{Aut}(\mathbb{R}^4);\, A^\intercal \eta A = \eta \big\}. \tag{4.5}$$

Find its Lie algebra, determine a set of basis generators, interpret them physically, and find their commutation relations.

5.1. The Lie algebra $\mathfrak{so}(4)$

a) Determine the generators and their Lie brackets for the Lie algebra $\mathfrak{so}(4)$. Here $\mathfrak{so}(4)$ is the Lie algebra associated to the group SO(4) consisting of real orthogonal 4×4 matrices with determinant 1.

Hint: Consider the combination $R_{ij} := -\mathring{\imath}E_{ij} + \mathring{\imath}E_{ji}$, where E_{ij} is a matrix with a 1 at position (i, j) and zeros everywhere else.

b) Show that, as a real Lie algebra, $\mathfrak{so}(4) \equiv \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined to be the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces with the requirement that $[\![\mathfrak{g}_1, \mathfrak{g}_2]\!] = 0$.

Hint: Consider linear combinations of R_{ij} and R_{kl} with distinct $i, j, k, l \in \{1, 2, 3, 4\}$.

5.2. BCH formula

Let $\mathfrak{g}$ be a Lie algebra and $A, B \in \mathfrak{g}$. The Baker–Campbell–Hausdorff (BCH) formula states that

$$\exp(A) \cdot \exp(B) = \exp(A \star B), \tag{5.1}$$

where $A \star B \in \mathfrak{g}$ equals

$$A \star B = A + B + \tfrac{1}{2}[\![A, B]\!] + \tfrac{1}{12}[\![A, [\![A, B]\!]]\!] + \tfrac{1}{12}[\![B, [\![B, A]\!]]\!] + \ldots\ . \tag{5.2}$$

Prove the BCH formula to this order assuming that A, B are matrices.

Hint: Replace A, B by tA, tB and expand both sides to the appropriate order in t.

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5.3. Universal enveloping algebra

For a general Lie algebra $\mathfrak{g}$, the Lie bracket $[\![\cdot,\cdot]\!] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ is merely an abstract binary operation that *a priori* has nothing to do with matrices, multiplication or commutators. In this exercise we will embed $\mathfrak{g}$ into an associative algebra $\mathcal{U}(\mathfrak{g})$ such that the Lie bracket is mapped to the commutator in $\mathcal{U}(\mathfrak{g})$, i.e. we would like that $[\![a, b]\!] = a{\cdot}b - b{\cdot}a$ for all $a, b \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$. To achieve this we need to construct the associative algebra $\mathcal{U}(\mathfrak{g})$ with this multiplication rule.

In the former few parts you will derive what is the underlying space $\mathcal{U}(\mathfrak{g})$, which are its properties, and whether they are coherent with our wishes. This will allow us to define $\mathcal{U}(\mathfrak{g})$ explicitly, and to study it in the latter few parts.

a) Show that the Jacobi identity is satisfied if multiplication in $\mathcal{U}(\mathfrak{g})$ is associative.

b) As a first attempt we consider the space $\mathbb{K}\oplus\mathfrak{g}\oplus(\mathfrak{g}\otimes\mathfrak{g})$ where $\mathbb{K}$ is the field (usually $\mathbb{R}$ or $\mathbb{C}$) over which $\mathfrak{g}$ is defined. The product $A{\cdot}B := A\otimes B$ is defined by the associative tensor product $\otimes$ over $\mathfrak{g}$; for example, for any $a, b \in \mathfrak{g}$ and $c, d \in \mathbb{K}$ the product is given by $a{\cdot}b = a\otimes b$ as well as $c{\cdot}a = a{\cdot}c = ca$ and $c{\cdot}d = cd$. This allows us to multiply Lie algebra elements as well as numbers, and to implement the Lie bracket as the commutator.

Why does this not define an associative algebra?

c) Extend $\mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g})$ until you obtain a space $\mathcal{T}(\mathfrak{g})$ that can be turned into an associative algebra with the above multiplication.

Now that we have seen that $\mathcal{T}(\mathfrak{g})$ has a consistent tensor product, we implement the Lie bracket relationship on $\mathcal{T}(\mathfrak{g})$ by declaring the following equivalence relation for any $a, b \in \mathfrak{g}$:

$$a \otimes b - b \otimes a \sim [\![a, b]\!]. \tag{5.3}$$

d) Show that this equivalence defines a left and right ideal $\mathcal{I}(\mathfrak{g}) \subset \mathcal{T}(\mathfrak{g})$, and describe its elements.

Define the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ as the quotient

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}(\mathfrak{g})/\mathcal{I}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\sim, \tag{5.4}$$

where the equivalence relation becomes an identity or where elements of the ideal $\mathcal{I}(\mathfrak{g})$ are identified with zero.

e) Let $\{x_i\}$ be a basis of $\mathfrak{g}$. Write the Lie algebra $\mathfrak{g}$ and its enveloping algebra $\mathcal{U}(\mathfrak{g})$ in terms of generators and generator relations.

f) Show that there is a one-to-one correspondence between representations of a Lie algebra $\mathfrak{g}$ and representations of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Hint: A representation of an associative algebra A is a map

$$\rho : A \to \mathrm{End}(\mathbb{V}) \tag{5.5}$$

to a vector space $\mathrm{End}(\mathbb{V})$ over a field $\mathbb{K}$ such that $\rho(a{\cdot}b) = \rho(a)\rho(b)$ and $\rho(1) = \mathrm{id}$.

g) *Optional:* Consider

$$C^{(m)} = \kappa^{i_1 i_2 \ldots i_m} x_{i_1}{\cdot}x_{i_2}{\cdot}\ldots{\cdot}x_{i_m} \in \mathcal{U}(\mathfrak{g}). \tag{5.6}$$

What property must the coefficients κ satisfy for $C^{(m)}$ to be a Casimir operator, i.e. to satisfy $[\![C^{(m)}, x_j]\!] = 0$ for all $x_j \in \mathfrak{g}$?

6.1. Orbit-stabiliser theorem

Given the action of a finite group G on a set X, we define the stabiliser of an element $x \in X$ as the subset of transformations that map x onto itself,

$$\mathrm{G}_x := \big\{ g \in \mathrm{G}; g\cdot x = x \big\}. \tag{6.1}$$

The orbit-stabiliser theorem states that the order $|\mathrm{G}|$ of the group G can be calculated as the product of the order of the stabiliser of x times the cardinality $|X_x|$ of the orbit $X_x = \mathrm{G}\cdot x := \{g\cdot x; g \in \mathrm{G}\}$

$$|X_x| \cdot |\mathrm{G}_x| = |\mathrm{G}|, \tag{6.2}$$

for any $x \in X$.

a) Consider the symmetry group O of the cube. Verify the orbit-stabiliser theorem by considering the action of O on

i. the set F of faces of a cube;
ii. the set V of vertices of a cube.

b) Prove the orbit-stabiliser theorem in the general case.

Hint: fix an element $x \in X$, then

i. show that the relation $g \sim h$ if $g\cdot x = h\cdot x$ is an equivalence relation on G;
ii. show that the number of elements of G in each equivalence class is equal and compute it;
iii. show that the number of equivalence classes into which G is partitioned via $\sim$ is equal to the cardinality of X_x.

6.2. Eigenvalues of representations of finite groups

Consider a finite-dimensional representation ρ of a finite group G. Show that the eigenvalues of $\rho(g)$ are roots of unity for any $g \in \mathrm{G}$.

7.1. Characters of irreducible representations

Given two representations ρ_A and ρ_B of a finite group G, we define the inner product of their characters as

$$\langle \chi_A, \chi_B \rangle := \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \chi_A(g^{-1}) \chi_B(g). \tag{7.1}$$

By Schur's lemma the characters χ_k of irreducible representations ρ_k are orthonormal, $\langle \chi_k, \chi_j \rangle = \delta_{kj}$. The following parts should be addressed using this relationship along with properties of scalar products.

a) Confirm the orthogonality relation for the irreducible representations of the group S_3.

b) Show that for a representation ρ_A and irreducible representation ρ_k, the inner product $\langle \chi_A, \chi_k \rangle$ gives the multiplicity with which ρ_k appears in ρ_A. In particular, confirm that for the group S_3 the three-dimensional representation ρ_3 decomposes as $\rho_3 = \rho_2 \oplus \rho_1$.

c) Show that a representation ρ_A is irreducible if and only if $\langle \chi_A, \chi_A \rangle = 1$.

d) Using these results, show that the tensor product of any irreducible representation with a one-dimensional representation is again an irreducible representation.

7.2. Irreps and characters of quaternions

Consider the group of elementary quaternions $\mathrm{H} = \{\pm 1, \pm \hat{\imath}, \pm \hat{\jmath}, \pm \hat{k}\}$.

a) Find the conjugacy classes of H. What does this tell you about the irreducible representations of H and their dimensions?

b) Let $Z(\mathrm{H}) := \{x \in \mathrm{H}; xy = yx \text{ for all } y \in \mathrm{H}\}$ denote the centre of H. Show that $\mathrm{H}/Z(\mathrm{H})$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

c) Show that representations of $\mathrm{H}/Z(\mathrm{H})$ induce representations of H and combine this with the above to find the four one-dimensional irreducible representations of H. What is the remaining two-dimensional representation?

d) Write down the character table of H.

7.3. Irreducible representations of abelian groups

Using Schur's lemma, show that the irreducible representations of a finite abelian group G are all one-dimensional.

8.1. Representations of the dihedral group

Recall that the dihedral group D_n is generated by two elements r and s satisfying the relations

$$r^n = s^2 = e, \qquad sr^k s = r^{-k} \quad \text{for all } k \in \mathbb{Z}. \tag{8.1}$$

a) Argue that a general element of this group can be written as r^a or sr^a with $a \in \mathbb{Z}_n$.

b) A simple dihedral group is $\mathrm{D}_3 = \mathrm{S}_3$, the symmetry group of an equilateral triangle. Identify the elements r and s with symmetries of the triangle and verify that the relations (8.1) are satisfied. Compute the multiplication table of D_3.

c) Write the result of the conjugation of $\{r^a, sr^a\}$ by $\{r^b, sr^b\}$ as a general element of the form above using the defining relations.

d) Using the following steps, find the conjugacy classes of D_n for the case of even $n = 2\ell$:

i. Consider elements of the form r^a. How many conjugacy classes do you find, and with how many elements?

ii. Consider elements of the form sr^a. How many conjugacy classes do you find, and with how many elements?

iii. Check that the conjugacy classes that you found contain all elements of D_n.

iv. Check that you have the right results by verifying the orbit-stabiliser theorem for the adjoint action of the group by considering a representative of each conjugacy class.

e) Repeat part d) for the case of odd $n = 2\ell + 1$.

Hint: Some details in steps i and ii are different.

f) Find all the irreducible representations of the dihedral group D_n for the case of even $n = 2\ell$ using the following steps:

i. Recall that there are as many irreducible representations as there are conjugacy classes. Recall also that

$$|\mathrm{D}_n| = \sum_{i=1}^{m} (\dim \mathbb{V}_i)^2, \tag{8.2}$$

where the sum runs over m inequivalent irreducible representations.
Knowing that the irreducible representations of D_n have either dimension 1 or 2, how many one-dimensional and two-dimensional representations do you expect to find?

ii. Find all the irreducible representations of D_n. Check that they are indeed inequivalent using their characters.
Hint: The two-dimensional representations of r and s are rotations and reflections in $\mathbb{R}^2$, respectively.

g) Repeat part f) for the case of odd $n = 2\ell + 1$.

h) Decompose the tensor products of all irreducible representations.

9.1. Three-dimensional Lie algebras

Consider the following three-dimensional complex Lie algebras defined in terms of generators x, y, z and the commutation relations

$$\begin{array}{lllll} \text{i.} & [\![x,y]\!]=0, & [\![x,z]\!]=0, & [\![y,z]\!]=0, & \text{(abelian algebra)} \\ \text{ii.} & [\![x,y]\!]=z, & [\![x,z]\!]=0, & [\![y,z]\!]=0, & \text{(Heisenberg algebra)} \\ \text{iii.} & [\![x,y]\!]=x, & [\![x,z]\!]=0, & [\![y,z]\!]=0, & \text{(direct product)} \\ \text{iv.} & [\![x,y]\!]=y, & [\![x,z]\!]=y+z, & [\![y,z]\!]=0, & \text{(Bianchi type IV)} \\ \text{v.} & [\![x,y]\!]=y, & [\![x,z]\!]=-z, & [\![y,z]\!]=x. & (\mathfrak{sl}(2)) \end{array} \tag{9.1}$$

Calculate the following for each Lie algebra:

a) Every Lie algebra acts on itself via the adjoint action $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined as $\mathrm{ad}(g)h := [\![g,h]\!]$. In particular, $\rho(g) := \mathrm{ad}(g)$ defines a representation, the *adjoint representation*.

Find the adjoint representation in matrix form.

b) Find the Killing form defined as $\kappa(g,h) := \mathrm{tr}(\mathrm{ad}(g)\,\mathrm{ad}(h))$ in the x,y,z-basis.

c) Find the derived algebra $\mathfrak{g}_1 := [\![\mathfrak{g},\mathfrak{g}]\!] = \{[g,h]; g,h \in \mathfrak{g}\}$, the second derived algebra $\mathfrak{g}_2 := [\![\mathfrak{g},\mathfrak{g}_1]\!] = \{[g,h]; g \in \mathfrak{g}, h \in \mathfrak{g}_1\}$ and the derived algebra $\mathfrak{g}_{1,1} := [\![\mathfrak{g}_1,\mathfrak{g}_1]\!]$ of $\mathfrak{g}_1$.

d) Which of these algebras is simple, i.e. has no non-trivial ideal?

An *ideal* is a subalgebra $\mathfrak{i} \subset \mathfrak{g}$ such that $[\![\mathfrak{g},\mathfrak{i}]\!] \subset \mathfrak{i}$.

9.2. Simple Lie groups and simple Lie algebras

A simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups (a subgroup H of a group G is called *normal* if for all $h \in \mathrm{H}$ and $g \in \mathrm{G}$, $ghg^{-1} \in \mathrm{H}$). We want to understand what this condition means in terms of the Lie algebra of the group.

Note: In this exercise, we will always consider compact connected Lie groups, for which the exponential map is onto, i.e. each $g \in \mathrm{G}$ can be written as $g = \mathrm{e}^A$ for some $A \in \mathfrak{g}$, and you may assume that the BCH formula converges.

a) A *subalgebra* is a subspace of an algebra which is closed under multiplication. In the case of a Lie algebra this means that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ obeys

$$\mathfrak{h} \subset \mathfrak{g}, \qquad [\![\mathfrak{h},\mathfrak{h}]\!] \subset \mathfrak{h}. \tag{9.2}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.

b) An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that

$$[\![\mathfrak{g},\mathfrak{h}]\!] \subset \mathfrak{h}. \tag{9.3}$$

A proper ideal of $\mathfrak{g}$ is an ideal which is neither trivial nor all of $\mathfrak{g}$. Show, using the exponential map, that a Lie group is simple if and only if its Lie algebra contains no proper ideals.

10.1. The Killing form

Consider a Lie algebra $\mathfrak{g}$ together with its Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{C}$, which is given by $\kappa(X, Y) = \mathrm{tr}[\mathrm{ad}(X)\,\mathrm{ad}(Y)]$ and satisfies

$$\kappa\big(\mathrm{ad}(X)Y, Z\big) + \kappa\big(Y, \mathrm{ad}(X)Z\big) = 0, \tag{10.1}$$

for any $X, Y, Z \in \mathfrak{g}$.

a) A Lie algebra $\mathfrak{g}$ is called semi-simple if the only abelian ideal subalgebra of $\mathfrak{g}$ is $\{0\}$. Show that if κ is non-degenerate, i.e. $\ker(\kappa) := \{Y \in \mathfrak{g}; \kappa(X, Y) = 0 \text{ for all } X \in \mathfrak{g}\} = \{0\}$, then $\mathfrak{g}$ is semi-simple.

Hint: Note that for $Y \in \mathfrak{g}$ and fixed $X \in \mathfrak{g}$, $A \in \mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{g}$ is an abelian ideal of $\mathfrak{g}$, the map $n(Y) := \mathrm{ad}(X)\,\mathrm{ad}(A)Y$ maps $\mathfrak{g}$ to $\mathfrak{a}$. Then show that n is nilpotent.

b) Show that if κ is non-degenerate then the centre of $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{0\}$.

c) Suppose $\mathfrak{a} \subset \mathfrak{g}$ is an ideal subalgebra of $\mathfrak{g}$. Show that

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g}; \kappa(X, Y) = 0 \text{ for all } Y \in \mathfrak{a}\} \subset \mathfrak{g} \tag{10.2}$$

is an ideal subalgebra of $\mathfrak{g}$ as well.

10.2. Root systems

Consider a semi-simple Lie algebra $\mathfrak{g}$, which can be decomposed as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha, \tag{10.3}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e. a maximal abelian subalgebra of $\mathfrak{g}$. Recall that the rank r of $\mathfrak{g}$ is given by $r = \dim(\mathfrak{h})$.

a) Knowing that $\dim(\mathfrak{g}_\alpha) = 1$, show that $|\Delta| = \dim(\mathfrak{g}) - r$.

b) Show that if $\alpha \in \Delta$, then $-\alpha \in \Delta$.

Hint: Prove first that if $\alpha + \beta \neq 0$ then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$.

c) Consider $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$. Show that $[\![E_\alpha, F_\alpha]\!] = \kappa(E_\alpha, F_\alpha)T_\alpha$, where $T_\alpha \in \mathfrak{h}$ is defined such that $\kappa(T_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$.

Show that $\{T_\alpha, E_\alpha, F_\alpha\}$ forms a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

d) In class you have defined the simple roots α_i, $i = 1, \ldots, r$, as the set of positive roots which cannot be expressed as combinations of other positive roots. Show that if α_i and α_j are simple roots, then $\alpha_i - \alpha_j$ is not a root.

e) The Cartan matrix is an $r \times r$ matrix defined as

$$A_{jk} := \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_j, \alpha_j \rangle} = \alpha_k(\mathrm{H}_j). \tag{10.4}$$

What are the diagonal elements of the Cartan matrix? Knowing that $\langle \alpha_i, \alpha_j \rangle < 0$ for $i \neq j$, show that $A_{ij}A_{ji} < 4$ for $i \neq j$.

Hint: Use the Cauchy–Schwarz inequality.

f) Using the results above, find all semi-simple Lie algebras with rank $r = 2$.

11.1. Representation of $\mathfrak{su}(3)$

Construct explicitly the finite-dimensional irreducible representation $\rho : \mathfrak{su}(3) \to \mathrm{End}(\mathbb{V})$ that is generated from the highest-weight state $|\mu\rangle$ satisfying

$$\rho(L_{12})|\mu\rangle = \rho(L_{13})|\mu\rangle = \rho(L_{23})|\mu\rangle = 0, \tag{11.1}$$

with the weight given by ($H_{jk} := L_{jj} - L_{kk}$)

$$\rho(H_{12})|\mu\rangle = 2|\mu\rangle, \qquad \rho(H_{23})|\mu\rangle = 0. \tag{11.2}$$

Determine, in particular, the dimension of ρ and the eigenvalues (with multiplicities) of $\rho(H_{12})$ and $\rho(H_{23})$. Proceed as follows:

a) Show that $\mathbb{V}$ is spanned by the vectors $W|\mu\rangle$, where W is any word in $\rho(L_{21})$ and $\rho(L_{32})$.

b) Acting on $|\mu\rangle$ with $\rho(L_{21})$ and $\rho(L_{32})$, construct a basis of $\mathbb{V}$. For any new vector, compute its eigenvalues under $\rho(H_{12})$ and $\rho(H_{23})$, and verify that you can go back to the vectors previously constructed (and thus, by recursion, to $|\mu\rangle$) by acting with $\rho(L_{12})$ and $\rho(L_{23})$. If this is not possible the vector must be 0, since by assumption the representation is irreducible.

11.2. Representation theory of $\mathfrak{sl}(5, \mathbb{C})$

Develop the representation theory of the complexification $\mathfrak{sl}(5, \mathbb{C})$ of the Lie algebra $\mathfrak{su}(5)$.

a) Identify the Cartan subalgebra $\mathfrak{h}$ and define a suitable basis for the dual space $\mathfrak{h}^*$. Find the roots of the algebra and describe them in terms of this basis $\mathfrak{h}^*$.

b) Identify subalgebras $\mathfrak{sl}(2, \mathbb{C})$ inside $\mathfrak{sl}(5, \mathbb{C})$, and deduce the structure of the possible weights of any finite-dimensional representation of $\mathfrak{sl}(5, \mathbb{C})$.

c) Choose a linear functional on the dual Cartan subalgebra, and partition the roots into positive and negative roots. Then identify the possible highest weights of any finite dimensional representation. Show that these highest weights can be labelled by four non-negative integers. Describe the defining and conjugate defining representation by suitable numbers.

12.1. Tensor Product Representations

Let $\rho_1 : \mathfrak{g} \to \mathbb{V}_1$ and $\rho_2 : \mathfrak{g} \to \mathbb{V}_2$ be two irreducible representations of the Lie algebra $\mathfrak{g}$. We define the tensor product representation $\rho : \mathfrak{g} \to \mathbb{V}_1 \otimes \mathbb{V}_2$ such that for any $X \in \mathfrak{g}$ we have

$$\rho(X) := \rho_1(X) \otimes 1 + 1 \otimes \rho_2(X), \tag{12.1}$$

where 1 denotes the identity map in both vector spaces $\mathbb{V}_1$ and $\mathbb{V}_2$.

a) Show that if two states $|\lambda_1\rangle \in \mathbb{V}_1$ and $|\lambda_2\rangle \in \mathbb{V}_2$ have definite weights λ_1 and λ_2 then the state $|\lambda_1\lambda_2\rangle := |\lambda_1\rangle \otimes |\lambda_2\rangle \in \mathbb{V}_1 \otimes \mathbb{V}_2$ has the weight $\lambda_1 + \lambda_2$.

b) Show that if $|\mu_1\rangle \in \mathbb{V}_1$ and $|\mu_2\rangle \in \mathbb{V}_2$ are the highest-weight states of the representations ρ_1 and ρ_2, respectively, then so is the state $|\mu\rangle := |\mu_1\rangle \otimes |\mu_2\rangle$ of the tensor product representation ρ.

Hint: Consider the action of the positive root generators.

Consider the defining representation $\mathbf{3}$ of the Lie algebra $\mathfrak{su}(3)$. Denote the basis of the representation space $\mathbb{V}_{\mathbf{3}} = \mathbb{C}^3$ by the three states $|u\rangle$, $|d\rangle$ and $|s\rangle$ with the corresponding weights $(1, 0)$, $(0, -1)$ and $(-1, 1)$ in some basis of $\mathfrak{h}^*$, and assume that $|u\rangle$ is the highest-weight state.

c) List all the states $|ab\rangle := |a\rangle \otimes |b\rangle$ in the tensor product representation $\mathbf{3} \otimes \mathbf{3}$. Find their weights and corresponding multiplicities.

d) Construct the irreducible representation based on the highest-weight state defined in part b) by acting on it with negative root generators. This representation does not span the entire space $\mathbb{V}_{\mathbf{3}} \otimes \mathbb{V}_{\mathbf{3}}$. Find the states that span the remaining part of the space, and identify the highest-weight state of the corresponding irrep.

e) Repeat part c) and part d) for the tensor product representation $\mathbf{3} \otimes \mathbf{3}^*$ involving the conjugate defining representation $\mathbf{3}^*$.

13.1. Algebra isomorphism

Consider the algebra $\mathfrak{g}^{(M)}$ defined by

$$\mathfrak{g}^{(M)} := \big\{A \in \mathrm{End}(\mathbb{C}^n); A^\intercal M = -MA\big\}, \tag{13.1}$$

where M is an invertible $n \times n$ matrix. We want to show that $\mathfrak{g}^{(M)} \equiv \mathfrak{so}(n, \mathbb{C})$ if M is symmetric.

a) Prove that $\mathfrak{g}^{(M)}$ is a Lie algebra.

b) Consider a matrix $T = P^\intercal M P$ for some invertible matrices P and M. Show that the Lie algebras $\mathfrak{g}^{(M)}$ and $\mathfrak{g}^{(T)}$ are isomorphic.

c) Prove that $\mathfrak{g}^{(M)} \equiv \mathfrak{so}(n, \mathbb{C})$ for a symmetric M.

d) *optional:* Show that $[M^{-1}M^\intercal, A] = 0$ for all $A \in \mathfrak{g}^{(M)}$ and argue why one should choose M to be either symmetric or anti-symmetric.

13.2. Dynkin diagram of $\mathfrak{so}(2r + 1, \mathbb{C})$

Consider the algebra $\mathfrak{g}^{(S)}$ with $n = 2r + 1$ as defined in problem 13.1, where S is the invertible symmetric matrix in $(r, r, 1)$ block form

$$S := \begin{pmatrix} 0 & \mathrm{id}_r & 0 \\ \mathrm{id}_r & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.2}$$

a) Write the elements of $\mathfrak{g}^{(S)}$ as block matrices adapted to the blocks of S.

b) Let $\mathfrak{h}$ be the Cartan subalgebra, which is spanned by diagonal matrices of the form

$$H_i := L_{i,i} - L_{i+r,i+r}, \tag{13.3}$$

where the $L_{i,j}$ are matrices with 1 in row i and column j and 0 everywhere else.

Find the generators L_α corresponding to the roots α as well as the corresponding Cartan generators defined by $H_\alpha := [\![L_\alpha, L_{-\alpha}]\!]$. Check that $[\![H_\alpha, L_\alpha]\!] \neq 0$ for all roots.

Hint: It may be convenient to introduce the generator $H[h] := \sum_{i=1}^{r} h_i H_i$ such that the coefficients h_i serve as coordinates on $\mathfrak{h}$.

c) A basis for the dual Cartan subalgebra is then given by the simple roots

$$\begin{aligned} \beta_i &:= (H_i - H_{i+1})^*, \\ \beta_r &:= (H_r)^*. \end{aligned} \tag{13.4}$$

Identify the simple-root generators, then determine the Cartan matrix and the corresponding Dynkin diagram.