Quantum Field Theory II

Problem Sets

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Contents

1.1. Transition amplitude for the harmonic oscillator

A standard exercise regarding path integrals in quantum mechanics is the computation of the transition amplitude for a harmonic oscillator. The harmonic oscillator is specified by the Lagrange function

$$L(q,\dot q) = \tfrac{1}{2}m\dot q^2 - \tfrac{1}{2}m\omega^2 q^2, \tag{1.1}$$

where q and $\dot q$ are the position and velocity variables.

Compute the transition amplitude for the harmonic oscillator

$$U(q_\mathrm{f}, q_\mathrm{i}, t) := \langle q_\mathrm{f}, t_\mathrm{f} | q_\mathrm{i}, t_\mathrm{i}\rangle = \int \mathrm{D}q \exp\left[\frac{\mathring{\imath}}{\hbar}\int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, L(q,\dot q)\right], \tag{1.2}$$

where $t := t_\mathrm{f} - t_\mathrm{i}$. Show that it equals

$$U(q_\mathrm{f}, q_\mathrm{i}, t) = \left[\frac{m\omega}{2\pi\hbar\mathring{\imath}\sin(\omega t)}\right]^{1/2} \exp\left[\frac{\mathring{\imath} m\omega}{\hbar}\,\frac{\frac{1}{2}(q_\mathrm{i}^2 + q_\mathrm{f}^2)\cos(\omega t) - q_\mathrm{i}q_\mathrm{f}}{\sin(\omega t)}\right]. \tag{1.3}$$

Walkthrough:

a) Split up the path $q(t)$ in n intermediate steps q_k, $k = 1, \ldots, n-1$,

$$U(q_\mathrm{f}, q_\mathrm{i}, t) = \lim_{n\to\infty} \int \left[\prod_{j=1}^{n-1} \mathrm{d}q_j\right] \left[\frac{nm}{2\pi\hbar\mathring{\imath}t}\right]^{n/2} \exp\left[\frac{\mathring{\imath}}{\hbar}\sum_{k=1}^{n}\frac{t}{n}\,L_k\right], \tag{1.4}$$

where $q_0 := q_\mathrm{i}$, $q_n := q_\mathrm{f}$ and $L_k = \langle q_k | L(\hat q, \hat{\dot q}) | q_{k-1}\rangle$.

Write the Lagrangian expectation value L_k in terms of the q_k. Choose a suitable definition of "ordering" of the operators with

$$\langle q_k | \hat{\dot q} | q_{k-1}\rangle = \frac{q_k - q_{k-1}}{t/n}\,; \tag{1.5}$$

make some choice for evaluating $V(\hat q)$.

b) Reexpress the exponential in (1.4) as a Gaußian function of the form

$$\exp\left[\frac{\mathring{\imath}}{\hbar}\sum_{k=1}^{n}\frac{t}{n}\,L_k\right] = \exp\left[\frac{\mathring{\imath} nm}{\hbar t}\left(\tfrac{1}{2}\vec q^{\,\mathsf{T}} M \vec q + \vec B^{\mathsf{T}}\vec q + C\right)\right], \tag{1.6}$$

where $\vec q = (q_1, q_2, \ldots, q_{n-1})$ is an $(n-1)$-dimensional vector, M is an $(n-1)\times(n-1)$ matrix, $\vec B$ is an $(n-1)$-dimensional vector and C a constant (the latter two depending on $q_0 = q_\mathrm{i}$, $q_n = q_\mathrm{f}$). With this form, you should be able to compute the integral (rather) easily.

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c) Let M_j denote the minor of M of size $j \times j$ with the last $n-1-j$ rows and columns eliminated. Show that the determinant of $D_j := \det M_j$ satisfies the equality

$$\frac{D_{j+1} - 2D_j + D_{j-1}}{(t/n)^2} = -a^2 D_j. \tag{1.7}$$

Determine the constant a, then solve the equation.

Hint: This difference equation is the discretised version of a well-known differential equation. In order to solve it, you can solve the continuum equations for $D(\tau)$, re-express the solution in discretised time and match the coefficients to some small values of j. You should find

$$D_j \simeq \frac{n}{\omega t} \sin(\omega t) + (j - n + 1) \cos(\omega t) + \mathcal{O}(1/n) \quad \text{for } j \approx n \to \infty. \tag{1.8}$$

d) Now you are left with the computation of the coefficients of q_i^2, q_f^2 and $q_\mathrm{i} q_\mathrm{f}$ in the exponent. You can compute them by computing the appropriate minors of M and get the correct result. Enjoy!

2.1. Generating functionals

Consider a Lagrangian for scalars with a quartic interaction

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \qquad \mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2, \quad \mathcal{L}_{\text{int}} = -\tfrac{1}{24}\lambda\phi^4. \tag{2.1}$$

From the lecture you know that the interacting generating functional is then

$$Z[j] = \exp\left[i\int \mathrm{d}x^4\,\mathcal{L}_{\text{int}}\left(\frac{-i\delta}{\delta j(x)}\right)\right] Z_0[j], \tag{2.2}$$

expressed in terms of the free generating functional

$$Z_0[j] = \int \mathrm{D}\phi\,\exp\left[i\int \mathrm{d}x^4\,\big(\mathcal{L}_0 + j(x)\phi(x)\big)\right]. \tag{2.3}$$

a) Show that

$$Z_0[j] = Z_0[0]\exp\left[\frac{i}{2}\int \mathrm{d}x^4\,\mathrm{d}y^4\,j(x)G_{\mathrm{F}}(x,y)j(y)\right], \tag{2.4}$$

where $G_{\mathrm{F}}(x,y)$ is exactly the *Feynman* propagator (and not just any Green function).

b) Compute the vacuum contributions to $Z[j]$ to order λ^2,

$$\begin{aligned}\frac{Z[0]}{Z_0[0]} &= \frac{1}{Z_0[0]}\exp\left[i\int \mathrm{d}x^4\,\mathcal{L}_{\text{int}}\left(\frac{-i\delta}{\delta j(x)}\right)\right] Z_0[j]\bigg|_{j=0}\\ &= 1 + \lambda\int \mathrm{d}x^4\,C_1 G_{\mathrm{F}}(x,x)^2\\ &\quad + \lambda^2\int \mathrm{d}x^4\,\mathrm{d}y^4\,\Big[C_{2,1}\,G_{\mathrm{F}}(x,x)^2 G_{\mathrm{F}}(y,y)^2\\ &\qquad + C_{2,2}G_{\mathrm{F}}(x,x)G_{\mathrm{F}}(y,y)G_{\mathrm{F}}(x,y)^2 + C_{2,3}\,G_{\mathrm{F}}(x,y)^4\Big] + \mathcal{O}(\lambda^3),\end{aligned} \tag{2.5}$$

being particularly careful in the computation of the combinatorial factors C_1, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$. These factors can be computed either from the functional derivative expression or as symmetry factors of the related diagrams.

Describe graphically each term and identify connected and disconnected terms.

c) Show that the functional

$$W[j] := -i\log\left[\frac{Z[j]}{Z_0[0]}\right] \tag{2.6}$$

at $j = 0$ generates only the connected contributions to the vacuum amplitude.

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2.2. Propagator of a free Klein–Gordon field

In this problem we will derive the Feynman propagator of a free scalar field using the path integral formalism. To that end, we will frequently need some generic Gaußian integrals.

a) Compute the Gaußian integrals

$$\int \mathrm{d}x^n \exp\bigl(-\tfrac{1}{2}x^{\mathsf{T}}Mx\bigr) \quad \text{and} \quad \int \mathrm{d}x^n\, x_j x_k \exp\bigl(-\tfrac{1}{2}x^{\mathsf{T}}Mx\bigr), \tag{2.7}$$

where x is an n-dimensional vector and M is a symmetric $n \times n$ matrix.

Hint: For the first integral, diagonalise M by a change of variables. Express the prefactor of the second integral using a derivative w.r.t. a suitable element of M.

To compute the relevant path integral

$$\begin{aligned}\langle 0|\,\mathrm{T}[\phi(x_1)\phi(x_2)]|0\rangle &= \lim_{T\to\infty(1-\mathring{\imath}\epsilon)} \frac{\int \mathrm{D}\phi\, \phi(x_1)\phi(x_2) \exp\bigl(\mathring{\imath}\int_{-T}^{+T} \mathrm{d}x^4\, \mathcal{L}\bigr)}{\int \mathrm{D}\phi \exp\bigl(\mathring{\imath}\int_{-T}^{+T} \mathrm{d}x^4\, \mathcal{L}\bigr)} \\ &= \int \frac{\mathrm{d}k^4}{(2\pi)^4} \frac{-\mathring{\imath}\, \mathrm{e}^{-\mathring{\imath}k(x_1-x_2)}}{k^2+m^2-\mathring{\imath}\epsilon}\,.\end{aligned} \tag{2.8}$$

we discretise spacetime by considering a four-dimensional lattice with N lattice points per side of length L. In the continuum limit, the lattice spacing L/N goes to zero, and the lattice size L goes to infinity. We perform a discrete mode expansion of the free scalar field

$$\phi(x) = \frac{1}{L^4}\sum_n \mathrm{e}^{-\mathring{\imath}k_n x}\, \phi(k_n), \tag{2.9}$$

where $k_n^\mu = 2\pi n^\mu/L$ is the discretised momentum, with $-N/2 \le n^\mu \le N/2$ an integer. The individual Fourier coefficients $\phi(k_n)$ are complex but the field $\phi(x)$ is real, so that $\phi(k_n)^* = \phi(-k_n)$. However, we can treat the real and imaginary part $\phi_\mathrm{r}(k), \phi_\mathrm{i}(k)$ of $\phi(k)$ as independent variables if we restrict ourselves to modes with $k_n^0 > 0$. Finally, the integrals are replaced by (the range '$n,+$' indicates the restriction to modes with $k_n^0 > 0$)

$$\int \mathrm{D}\phi \to \int \prod_{n,+}\bigl(\mathrm{d}\phi_\mathrm{r}(k_n)\, \mathrm{d}\phi_\mathrm{i}(k_n)\bigr), \qquad \int \frac{\mathrm{d}k^4}{(2\pi)^4} \to \frac{1}{L^4}\sum_n\,. \tag{2.10}$$

b) Find the discretised equivalent to the action of the Klein–Gordon field

$$S = \int \mathrm{d}x^4 \bigl(-\tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi(x)^2\bigr). \tag{2.11}$$

Hint: Use reality of $\phi(x)$ to express the result quadratically in $\phi_\mathrm{r}(k_n)$ and $\phi_\mathrm{i}(k_n)$.

c) Making use of the results obtained in part a), compute the discrete equivalent of

$$\int \mathrm{D}\phi\, \mathrm{e}^{\mathring{\imath}S}\,. \tag{2.12}$$

d) Now compute the discretised version of (2.8) by inserting the mode expansion for $\phi(x_1)\phi(x_2)$. Make use of the symmetry of the integrand (even or odd in ϕ) to maintain only non-vanishing terms. Can you relate your expressions to what you found in part a)? Finally, take the continuum limit and recover the Feynman propagator.

Quantum Field Theory II — Problem Set 3

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3.1. Classical and quantum field configurations

Consider the action for ϕ^4-theory

$$S[\phi] = \int \mathrm{d}x^D \left[-\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\,\phi^4 \right], \tag{3.1}$$

and its generating functional $Z[j]$ in the path integral

$$Z[j] = \int \mathrm{D}\phi \exp \mathring{\imath} \left[S[\phi] + \int \mathrm{d}x^D\, \phi(x) j(x) \right]. \tag{3.2}$$

$Z[j]$ may be approximated via the method of stationary phase. Namely, if $\phi_{\mathrm{C}}[j]$ is the field configuration where the functional derivative of the exponent's argument vanishes,

$$\frac{\delta S}{\delta\phi}\,[\phi_{\mathrm{C}}[j]] := \left.\frac{\delta S[\phi]}{\delta\phi}\right|_{\phi=\phi_{\mathrm{C}}[j]} = -j, \tag{3.3}$$

the path integral is approximated by evaluating the exponent at this point (up to an irrelevant factor independent of j)

$$Z[j] \simeq \exp \mathring{\imath} \left[S[\phi_{\mathrm{C}}[j]] + \int \mathrm{d}x^D\, \phi_{\mathrm{C}}[j](x)\, j(x) \right]. \tag{3.4}$$

This implies for the connected functional

$$W[j] = -\mathring{\imath} \log Z[j] \simeq S[\phi_{\mathrm{C}}[j]] + \int \mathrm{d}x^D\, \phi_{\mathrm{C}}[j](x)\, j(x) =: T[j], \tag{3.5}$$

where the last equality defines $T[j]$.

In the following two tasks use algebraic equations to obtain your answers, then give a graphical representation of results.

a) Compute the *classical field configuration* $\phi_{\mathrm{C}}[j]$ that solves the *classical equation of motion* (3.3), in position space, using perturbation theory up to and including order λ^2.

b) Compute the two-point correlation function that stems from $\mathring{\imath}T[j]$ by taking functional derivatives. What can be stated about higher-order perturbative corrections to this result?

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The *quantum effective action* $G[\phi]$ is defined as the Legendre transform of the connected functional $W[j]$:

$$\phi = \frac{\delta W}{\delta j}\,[j_{\mathrm{Q}}[\phi]], \qquad G[\phi] = W[j_{\mathrm{Q}}[\phi]] - \int \mathrm{d}x^D\, j_{\mathrm{Q}}[\phi](x)\,\phi(x). \tag{3.6}$$

c) The generating functional constructed using the quantum effective action then reads

$$Z_G[j] := \int \mathrm{D}\phi \exp \mathring{i}\left[G[\phi] + \int \mathrm{d}x^D\, \phi(x) j(x)\right]. \tag{3.7}$$

In the spirit of evaluating this path integral with the method of stationary phase, in $Z_G[j]$'s "classical limit" the only field configuration that contributes is the solution ϕ_{Q} of the *quantum equation of motion*

$$\frac{\delta G}{\delta \phi}\,[\phi_{\mathrm{Q}}[j]] = -j. \tag{3.8}$$

Prove that the *quantum field configuration* ϕ_{Q} is given by

$$\phi_{\mathrm{Q}}[j] = \frac{\delta W[j]}{\delta j} = \langle \phi \rangle_j. \tag{3.9}$$

d) Compute $\phi_{\mathrm{Q}}[j]$ in position space up to and including order λ using functional manipulations. After having drawn the diagrams that correspond to the terms you found, extend your answer to order λ^2 using graphical methods.

e) Recursively invert the functional $\phi_{\mathrm{Q}}[j]$ to obtain the functional $j_{\mathrm{Q}}[\phi]$ at order λ^2. In other words, solve the definition of the functional $j_{\mathrm{Q}}[\phi]$ in (3.6) at order λ^2.

f) Use the result for $j_{\mathrm{Q}}[\phi]$ to reconstruct the quantum effective action $G[\phi]$, as an expansion in λ at order λ^2, from its functional derivative (3.8)

$$\frac{\delta G[\phi]}{\delta \phi} = -j_{\mathrm{Q}}[\phi]. \tag{3.10}$$

4.1. BCH formula

The Baker–Campbell–Hausdorff (BCH) formula states that

$$\exp(A)\exp(B) = \exp(A + B + A \star B), \tag{4.1}$$

where $A \star B$ is an element of the Lie algebra generated by A and B given by

$$A \star B = \tfrac{1}{2}[\![A, B]\!] + \tfrac{1}{12}[\![A, [\![A, B]\!]]\!] + \tfrac{1}{12}[\![B, [\![B, A]\!]]\!] + \ldots . \tag{4.2}$$

Prove the BCH formula to this order assuming that A and B are matrices.

Hint: Replace the exponents X by ϵX and expand both sides to cubic order in ϵ.

4.2. Simple Lie groups and simple Lie algebras

A simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups (a subgroup H of a group G is called *normal* if $ghg^{-1} \in \mathrm{H}$ for all $h \in \mathrm{H}$, $g \in \mathrm{G}$; a subgroup of a group G is called *proper* if it is neither trivial nor the whole of G). We want to understand what this condition means in terms of the Lie algebra $\mathfrak{g}$ of the group.

In this exercise, we will only consider compact connected Lie groups, for which the exponential map is onto, i.e. each $g \in \mathrm{G}$ can be written as $g = \mathrm{e}^A$ for some $A \in \mathfrak{g}$.

a) A *subalgebra* is a subspace of an algebra which is closed under multiplication. In the case of a Lie algebra this means that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ obeys

$$\mathfrak{h} \subset \mathfrak{g}, \qquad [\![\mathfrak{h}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{4.3}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.

Hint: recall the Baker–Campbell–Hausdorff formula discussed in Problem 4.1.

b) An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that

$$[\![\mathfrak{g}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{4.4}$$

A *proper* subalgebra of $\mathfrak{g}$ is a subalgebra which is neither trivial nor all of $\mathfrak{g}$.

Show, using the exponential map, that a Lie group is simple if and only if its Lie algebra contains no proper ideals.

Hint: it is sufficient to work with infinitesimal elements of the group, i.e. $\exp(\epsilon A) = 1 + \epsilon A + \mathcal{O}(\epsilon^2)$.

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4.3. Tensor product representations

Representations of a Lie algebra $\mathfrak{g}$ can be combined to give bigger representations via direct sums and tensor products

$$R_{1\oplus 2}(a) = \begin{pmatrix} R_1(a) & 0 \\ 0 & R_2(a) \end{pmatrix}, \qquad R_{1\otimes 2}(a) = R_1(a) \otimes \mathrm{id}_2 + \mathrm{id}_1 \otimes R_2(a). \tag{4.5}$$

Here R_1, R_2 are representations of $\mathfrak{g}$ on the spaces $\mathbb{V}_1, \mathbb{V}_2$ and $a \in \mathfrak{g}$.

a) Verify that the tensor product of two representations is again a representation.

b) Consider the tensor product $R^{\otimes 2}$ of two identical representations R. Show that it can be decomposed as the direct sum $R^{\otimes 2} = R^+ \oplus R^-$ of representations $R^\pm$ acting on the symmetric and anti-symmetric subspaces $\mathbb{V}^\pm \subset \mathbb{V} \otimes \mathbb{V}$

$$R^\pm := P^\pm R^{\otimes 2}, \qquad P^\pm := \tfrac{1}{2}(\mathrm{id} \pm P), \tag{4.6}$$

where $P^\pm$ are projectors onto the subspaces $\mathbb{V}^\pm$ and P is the permutation on $\mathbb{V} \otimes \mathbb{V}$ acting as $P(v \otimes w) = w \otimes v$ for any two vectors $v, w \in \mathbb{V}$.
Hint: It is useful to show that $P^\pm R^{\otimes 2} = R^{\otimes 2} P^\pm$.

4.4. Dual and complex conjugate representations

The dual of a representation R of a Lie algebra $\mathfrak{g}$ on the vector space $\mathbb{V}$ is defined as the representation R^* on the dual vector space $\mathbb{V}^*$ by $R^*(a) = -R(a)^\intercal$ for all $a \in \mathfrak{g}$.

a) Verify that R^* is a representation.

b) For a unitary representation R of a real Lie algebra, show that R^* is the complex conjugate representation $\bar{R}$.

c) Show that the tensor product $R \otimes R^*$ contains the trivial representation.
Hint: Consider the representation acting on the state $v^i \otimes v^*_i$, where v^i is a basis of $\mathbb{V}$ and v^*_i is the dual basis of $\mathbb{V}^*$ with $v^*_i(v^k) = \delta^k_i$.

4.5. Killing form and Casimir invariant

The Killing bilinear form for a Lie algebra $\mathfrak{g}$ is defined as

$$K(a,b) := \mathrm{Tr}\big(R^{\mathrm{ad}}(a) R^{\mathrm{ad}}(b)\big) \qquad \text{for } a, b \in \mathfrak{g}. \tag{4.7}$$

It can be expanded in a basis T_a of $\mathfrak{g}$ as the matrix $k_{ab} := K(T_a, T_b)$. For semi-simple $\mathfrak{g}$ this matrix is invertible and its inverse shall be denoted by k^{ab}. Define the representation R of the quadratic Casimir invariant $C_2 = K^{-1}$ as

$$R(C_2) := k^{ab} R(T_a) R(T_b). \tag{4.8}$$

a) For generic $a, b, c \in \mathfrak{g}$ show that

$$K(a,b) = K(b,a), \qquad K(\llbracket c,a \rrbracket, b) + K(a, \llbracket c,b \rrbracket) = 0. \tag{4.9}$$

Express these relationships in terms of k_{ab} and $f_{abc} := f_{ab}{}^d k_{dc}$.

b) Show that $R(C_2)$ is invariant, i.e. $[R(a), R(C_2)] = 0$ for all $a \in \mathfrak{g}$.

c) Argue that $R(C_2) = C_2^R \, \mathrm{id}^R$ for an irreducible representation R.

5.1. Completeness relation and Casimirs for $\mathfrak{su}(N)$

The special unitary algebra $\mathfrak{su}(N)$ is defined as the commutator algebra on the space of anti-hermitian traceless matrices. We can use an imaginary basis T_a^{def}, $a = 1, \ldots N^2 - 1$, so that T_a^{def} is hermitian $(T_a^{\text{def}})^\dagger = T_a^{\text{def}}$. The Killing metric k_{ab} is

$$\operatorname{tr}[T_a^{\text{def}} T_b^{\text{def}}] = B^{\text{def}} k_{ab}. \tag{5.1}$$

a) Let X be a generic $N \times N$ complex matrix. Prove the completeness relation

$$k^{ab} \operatorname{tr}[T_a^{\text{def}} X]\, T_b^{\text{def}} = B^{\text{def}}\big(X - N^{-1} \operatorname{tr}[X]\, \text{id}\big). \tag{5.2}$$

Hint: consider the space of $N \times N$ complex matrices as an N^2-dimensional vector space over $\mathbb{C}$ and find a suitable basis by extending the basis of $\mathfrak{su}(N)$.

b) Knowing the previous identity (5.2), prove the completeness relation

$$k^{ab} T_a^{\text{def}} X T_b^{\text{def}} = B^{\text{def}}\big(\operatorname{tr}[X]\, \text{id} - N^{-1} X\big). \tag{5.3}$$

c) Consider the symmetric structure constants $d_{ab}{}^c$ as defined by

$$\{T_a^{\text{def}}, T_b^{\text{def}}\} = d_{ab}{}^c T_c^{\text{def}} + q\, k_{ab}\, \text{id}\,. \tag{5.4}$$

By choosing the coefficient q appropriately, show that $d_{ab}{}^c$ is traceless in the first two indices,

$$k^{ab} d_{ab}{}^c = 0. \tag{5.5}$$

d) Show that the quadratic and cubic Casimir invariants for the defining representation, $k^{ab} T_a^{\text{def}} T_b^{\text{def}} = C_2^{\text{def}}\, \text{id}$ and $d^{abc} T_a^{\text{def}} T_b^{\text{def}} T_c^{\text{def}} = C_3^{\text{def}}\, \text{id}$, are given by

$$C_2^{\text{def}} = \frac{N^2 - 1}{N}\, B^{\text{def}}, \qquad C_3^{\text{def}} = \frac{(N^2 - 4)(N^2 - 1)}{N^2}\, (B^{\text{def}})^2. \tag{5.6}$$

5.2. Chromodynamics

The chromodynamics Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{2g_{\text{YM}}^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}), \tag{5.7}$$

where $F_{\mu\nu}$ is the field strength tensor of a SU(3) gauge field A_μ and ψ is 3-vector of Dirac fields transforming in the defining representation of SU(3).

a) Write down the equations of motion for the fields.

b) Show that the fermionic current $(J^\mu)^\alpha{}_\beta = \mathring{\imath}\bar{\psi}_\beta \gamma^\mu \psi^\alpha$ is covariantly conserved

$$[D_\mu, J^\mu] = 0. \tag{5.8}$$

c) Expand the terms in the Lagrangian in terms of the gauge field $A_\mu = g T_a^{\text{def}} A_\mu^a$ and interpret pictorially the individual terms.

d) Can you write down a Lagrangian for a scalar field that is invariant under SU(3)?

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5.3. Wilson lines

Consider a group-valued smooth function $U(y,x)$ called comparator which allows to relate the gauge phase of fields at different points in spacetime. The comparator satisfies $U(x,x) = 1$, as well as $U(y,x)^{-1} = U(x,y)$. Under a gauge transformation $V(x)$ it transforms as

$$U'(y,x) = V(y)U(y,x)V^{-1}(x). \tag{5.9}$$

Expanding the comparator for a short distance in direction n^μ yields

$$U(x+\epsilon n, x) = 1 + \mathring{\imath}\epsilon n^\mu A_\mu(x) + \tfrac{1}{2}\epsilon^2\big[\mathring{\imath}n^\mu n^\nu \partial_\nu A_\mu(x) - (n^\mu A_\mu(x))^2\big] + \mathcal{O}(\epsilon^3). \tag{5.10}$$

a) Using the comparator, we can construct a quantity $W(x)$ which connects a spacetime point x to itself along a small quadrangle spanned by the vectors ϵm and ϵn

$$\begin{aligned} W(x) = U(x, x+\epsilon n)U(x+\epsilon n, x+\epsilon n+\epsilon m) \\ \cdot U(x+\epsilon n+\epsilon m, x+\epsilon m)U(x+\epsilon m, x). \end{aligned} \tag{5.11}$$

Show that this quantity is gauge covariant at x. Expand $W(x)$ to second order in ϵ and compare to the field strength $F_{\mu\nu} = -\mathring{\imath}[D_\mu, D_\nu]$.

Hint: You may restrict to the abelian case to get started.

The comparator is the infinitesimal version of an object called the Wilson line. Let us first consider the abelian case. Then the *Wilson line* for a path γ is defined by

$$U[\gamma] = \exp\left[\mathring{\imath}\int_\gamma A\right]. \tag{5.12}$$

Here $A(x) = \mathrm{d}x^\mu\, A_\mu(x)$ is the gauge field one-form. If the path γ is closed, $U[\gamma]$ is called a *Wilson loop*.

b) Using Stokes' theorem, rewrite an abelian Wilson loop $U[\gamma]$ in terms of the field strength $F = \frac{1}{2}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\, F_{\mu\nu}$. Conclude that the abelian Wilson loop is gauge invariant. Show that the Wilson loop $U[\gamma]$ reduces to $W(x)$ at leading non-trivial order for a small quadrangular path γ as described in part a).

c) Consider a fixed parametrised path $\gamma : \tau \mapsto y^\mu(\tau)$, and denote by $\gamma_{t,s}$ the sub-path on the interval $\tau \in [s,t]$.

Show that $U[\gamma_{t,s}]$ satisfies the differential equations ($\dot{y}^\mu := \mathrm{d}y^\mu/\mathrm{d}\tau$)

$$\partial_t U[\gamma_{t,s}] - \mathring{\imath}\dot{y}^\mu(t)A_\mu(y(t))U[\gamma_{t,s}] = \partial_s U[\gamma_{t,s}] + \mathring{\imath}\dot{y}^\mu(s)U[\gamma_{t,s}]A_\mu(y(s)) = 0. \tag{5.13}$$

Hint: Parametrise the integral of the Wilson line as

$$\int_{\gamma_{t,s}} A = \int_s^t \mathrm{d}\tau\, \dot{y}^\mu(\tau)A_\mu(y(\tau)). \tag{5.14}$$

Consider now a non-abelian gauge field for which the Wilson line is given by

$$U[\gamma_{t,s}] = \mathrm{P}\left\{\exp\left[\mathring{\imath}\int_s^t \mathrm{d}\tau\, \dot{y}^\mu(\tau)A_\mu(y(\tau))\right]\right\}. \tag{5.15}$$

Here P denotes path ordering which means that the terms in the expansion of the exponential are ordered in such a way that higher values of τ stand to the left.

d) Show that the non-abelian Wilson line satisfies the same differential equations as in the abelian case paying close attention to the particular ordering in (5.13).

e) Show that the differential equation (5.13) is gauge covariant. Note that $U[\gamma_{t,s}]$ transforms analogously to (5.9): $U'[\gamma_{t,s}] = V(y(t))U[\gamma_{t,s}]V(y(s))^{-1}$.

6.1. Gauge fixing in the path integral

a) Consider the action for pure electrodynamics

$$S_{\mathrm{ED}} = \int \mathrm{d}x^D \left[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right]. \tag{6.1}$$

Perform the gauge fixing via the Faddeev–Popov method, using the non-linear gauge condition

$$G[A, \mathit{\Omega}] = \partial_\mu A^\mu + \tfrac{1}{2}\zeta A_\mu A^\mu - \mathit{\Omega}. \tag{6.2}$$

Invert the kinetic operators that appear in the action to find the propagators for the photon field and for the ghost field.

Is the ghost field decoupled in this gauge? Show that in the limit $\zeta \to 0$ the modified Lorenz gauge with gauge-fixing parameter ξ is restored.

b) Consider the action for pure Yang–Mills theory

$$S_{\mathrm{YM}} = \int \mathrm{d}x^D \left[-\tfrac{1}{4} k_{ab} F^a_{\mu\nu} F^{b\,\mu\nu}\right]. \tag{6.3}$$

Perform the gauge fixing via the Faddeev–Popov method using the *axial* gauge condition along a fixed four-vector n^μ

$$G[A, \mathit{\Omega}]^a = n^\mu A^a_\mu - \mathit{\Omega}^a. \tag{6.4}$$

Invert the kinetic operators that appear in the action to find the propagators for the ghost field and for the gluon field.

Under which condition do ghosts decouple from gluon fields in this gauge? How can this be exploited in practice?

$\longrightarrow$

6.2. Construction of BRST transformations

Consider the Yang–Mills Lagrangian with fermions

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(\gamma^\mu D_\mu - m)\psi. \tag{6.5}$$

One possibility to construct BRST transformations for this theory is to proceed as follows: Start from the infinitesimal change of fields under a gauge transformation parametrised by $\alpha^a(x)$,

$$\delta A^a_\mu(x) = (D_\mu)^a{}_b \alpha^b(x), \qquad \delta\psi^i(x) = \mathring{\imath} g \alpha^a(x)(T_a)^i{}_j \psi^j(x), \tag{6.6}$$

where $(D_\mu)^a{}_b$ is the covariant derivative in the adjoint representation

$$(D_\mu)^a{}_b = \delta^a_b \partial_\mu - g f_{bc}{}^a A^c_\mu. \tag{6.7}$$

Then promote the parameter $\alpha^a(x)$ to a set of dynamical anti-commuting fields $C^a(x)$ and a constant anti-commuting variational parameter $\delta\epsilon$

$$\alpha^a(x) = \delta\epsilon\, C^a(x). \tag{6.8}$$

This defines the BRST transformation Q for matter and gauge fields via

$$\delta\phi \equiv \delta\epsilon\, Q\phi, \tag{6.9}$$

where ϕ is a generic field.

a) By requiring that the BRST transformation for a matter field is nilpotent, $Q^2\psi^i = 0$, determine the BRST variation QC^a of ghost fields.

b) Check explicitly that, using the same rule for the transformation of C^a, applying the Q operator two times on a gauge field A^a_μ or a ghost field C^a gives zero.

Because the BRST transformation acts exactly as a gauge transformation on matter fields, the Yang–Mills Lagrangian is obviously BRST-invariant. Moreover, since Q has been constructed in such a way that $Q^2 = 0$, any extra term

$$\mathcal{L}_{\rm BRST} = Q K_{\rm BRST}, \tag{6.10}$$

will make the total Lagrangian $\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm BRST}$ BRST-invariant.

c) Show that the choice

$$K_{\rm BRST} = k_{ab}\bar{C}^a(\tfrac{1}{2}\xi B^b - \partial^\mu A^b_\mu), \tag{6.11}$$

together with

$$Q\bar{C}^a = B^a, \qquad QB^a = 0, \tag{6.12}$$

still preserves trivially $Q^2 = 0$ and gives the gauge-fixing Lagrangian for modified Lorenz gauges with gauge-fixing parameter ξ, the ghost kinetic term and ghost interactions.

7.1. Passarino–Veltman reduction

We will study a method to reduce one-loop tensor integrals to linear combinations of scalar integrals. The integrals are defined in D dimensions with Wick-rotated momenta. Specifically, we define the integrals as

$$A_0(m^2) := \int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{1}{k^2 + m^2}, \tag{7.1}$$

$$B_{0,\mu,\mu\nu} := \int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{(1, k_\mu, k_\mu k_\nu)}{[k^2 + m_1^2][(k-p)^2 + m_2^2]}. \tag{7.2}$$

For simplicity, we shall suppress the arguments $(p; m_1^2, m_2^2)$ of the functions B.

The goal is to reduce the integrals $B_\mu, B_{\mu\nu}$ to a linear combination of integrals A_0, B_0 with suitable coefficient functions. The ansatz is to decompose the tensor integrals according to their Lorentz structure; the allowed structures for the two tensor integrals are

$$B_\mu = p_\mu B_1, \tag{7.3}$$

$$B_{\mu\nu} = p_\mu p_\nu B_{21} + \eta_{\mu\nu} B_{22}, \tag{7.4}$$

where B_1, B_{21}, B_{22} are linear combinations of A_0 and B_0 which are to be determined.

a) Express B_1 in (7.3) as a linear combination of $A_0(m_1^2)$, $A_0(m_2^2)$ and B_0.

Hint: Write B_μ as the integral of (7.2), contract both sides of (7.3) with p^μ and then rewrite the numerator of the integrand as a linear combination of the factors in the denominator of the integrand as well as m_1^2, m_2^2 and p^2. You should find

$$B_1 = -\frac{1}{2p^2} A_0(m_1^2) + \frac{1}{2p^2} A_0(m_2^2) + \frac{p^2 + m_2^2 - m_1^2}{2p^2} B_0. \tag{7.5}$$

b) For the tensor integral $B_{\mu\nu}$, we can obtain a 2×2 linear system of equations by contracting both sides of (7.4) with $\eta^{\mu\nu}$ and $p^\mu p^\nu$, respectively.

Repeat the steps for the manipulation of the numerator of the integrands. Show that the linear system expressing B_{21}, B_{22} in terms of the scalar integrals A_0, B_0 is

$$p^2 B_{21} + D B_{22} = X_1 A_0(m_2^2) + X_2 B_0, \tag{7.6}$$

$$p^2 B_{21} + B_{22} = Y_1 A_0(m_2^2) + Y_2 B_1, \tag{7.7}$$

where X_i, Y_i are some functions to be determined. Finally, solve for B_{21}, B_{22}.

Hint: You may recycle the results of part a); furthermore, recall the integral

$$\int \frac{\mathrm{d}k^D}{(2\pi)^D} \frac{k^\mu}{k^2 + m^2} = 0. \tag{7.8}$$

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7.2. Feynman's parametrisation of loop integrals

Prove the following identity, which is used to combine the denominators of integrands in loop integrals

$$\frac{1}{D_1^{a_1}\ldots D_n^{a_n}} = \frac{\Gamma(\sum_{j=1}^n a_j)}{\Gamma(a_1)\ldots\Gamma(a_n)} \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_n \, \frac{\delta(1-\sum_{j=1}^n x_j)x_1^{a_1-1}\ldots x_n^{a_n-1}}{[x_1 D_1 + \ldots + x_n D_n]^{a_1+\ldots+a_n}} \,. \tag{7.9}$$

Hint: carry out the proof by induction, starting from $n = 2$; to prove the $n = 2$ case, you will need the following definition for the Gaußian hypergeometric function ${}_2\mathrm{F}_1$ (for $\mathrm{Re}(c) > \mathrm{Re}(b)$, $\mathrm{Re}(a) > 0$, $|z| < 1$)

$$\begin{aligned} {}_2\mathrm{F}_1(a, b; c; z) &:= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 \mathrm{d}x\, x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} \\ &= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!} \,. \end{aligned} \tag{7.10}$$

It makes sense to show that

$${}_2\mathrm{F}_1(a+b, a; a+b; z) = (1-z)^{-a}. \tag{7.11}$$

7.3. Renormalisation of cubic scalar theory in 6D

Consider a scalar theory in $D = 6$ with cubic interaction

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{6}\lambda_0\phi_0^3. \tag{7.12}$$

In this problem we want to derive the one-loop renormalisation of the bare couplings in dimensional regularisation at $D = 6 - 2\epsilon$.

a) Show, by power counting, that the theory is renormalisable in $D = 6$.

b) Rewrite the Lagrangian in terms of the renormalised field $\phi = \kappa^{-1}\phi_0$. Then adjust the interaction term by a suitable power of the scale parameter μ for dimensional regularisation such that the coupling constant λ_0 becomes dimensionless.

c) Derive the Feynman rules for the theory.

d) Compute the one-loop self energy contribution to the effective action. Express the final result as a function of ϵ, and extract the divergent contribution at $\epsilon \to 0$.

Hint: Perform the integral over the Feynman parameter at the very end.

Next we introduce the renormalised mass m and coupling λ, and express the bare constants κ, m_0, λ_0 as functions of them

$$\kappa = 1 + c_1\lambda^2 + \mathcal{O}(\lambda^4), \quad m_0^2 = m^2\big(1 + c_2\lambda^2 + \mathcal{O}(\lambda^4)\big), \quad \lambda_0 = \lambda + c_3\lambda^3 + \mathcal{O}(\lambda^5). \tag{7.13}$$

e) Write the two-point contributions to the effective action (tree level plus one loop) as a function of the renormalised mass and coupling. Determine the one-loop contributions to (7.13) such that the divergences due to the self energy cancel.

f) Compute the one-loop correction to the vertex. Identify the divergent part, and determine the remaining coefficient in (7.13) to make the effective action finite.

8.1. Ward–Takahashi identity

The Ward–Takahashi identities in QED are the extension of the conservation of the Noether current for the U(1) global symmetry

$$N_\mu(x) = \mathring{\imath} q \bar{\psi}(x)\gamma_\mu \psi(x) \tag{8.1}$$

to correlation functions. The Ward–Takahashi identity reads

$$\begin{aligned}&\partial_z^\mu \big\langle N_\mu(z)\, \psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)\, A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\big\rangle \\ &= -q\big\langle \psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)\, A_{\nu_1}(z_1)\ldots A_{\nu_p}(z_p)\big\rangle \sum_{i=1}^{n}\big(\delta^4(z-y_i) - \delta^4(z-x_i)\big).\end{aligned} \tag{8.2}$$

Define the full propagator for the electron in momentum space as

$$-\mathring{\imath} G(k)\delta^4(k-p) := \int \mathrm{d}y^4\, \mathrm{d}z^4\, \big\langle \psi(y)\bar{\psi}(z)\big\rangle\, \mathrm{e}^{\mathring{\imath} p\cdot z - \mathring{\imath} k\cdot y}\,. \tag{8.3}$$

and the electromagnetic vertex function $V^\mu(k,l)$ as

$$\int \mathrm{d}x^4\, \mathrm{d}y^4\, \mathrm{d}z^4\, \mathrm{e}^{-\mathring{\imath}(p\cdot x + k\cdot y - l\cdot z)}\, \big\langle N^\mu(x)\psi(y)\bar{\psi}(z)\big\rangle =: \mathring{\imath} q G(k) V^\mu(k,l) G(l)\delta^4(p+k-l). \tag{8.4}$$

a) Show that the Ward–Takahashi identity implies that

$$p_\mu V^\mu(k, k+p) = \mathring{\imath} G^{-1}(p+k) - \mathring{\imath} G^{-1}(k). \tag{8.5}$$

b) Define the counterterms Z_1 and Z_2 as

$$V^\mu(k,k) = (Z_1)^{-1}\gamma^\mu, \tag{8.6}$$

and

$$-\mathring{\imath} Z_2 = \lim_{k^2\to -m^2} (\mathring{\imath}\gamma^\mu k_\mu + m)G(k). \tag{8.7}$$

Note that $V^\mu(k,k)$ is the complete vertex function when the momentum of the external photon goes to zero, and Z_2 is defined for on-shell electrons with momentum k_μ. Show that, with these definitions,

$$Z_1 = Z_2. \tag{8.8}$$

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8.2. Axial anomaly in two dimensions

Consider a massless Dirac spinor field ψ in two spacetime dimensions coupled to a fixed background gauge field A

$$\mathcal{L} = \bar{\psi}\gamma^\mu(\partial_\mu - \mathring{\imath} A_\mu)\psi. \tag{8.9}$$

The Dirac matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The matrix γ^3 is the two-dimensional analog of γ^5 in four dimensions

$$\gamma^3 = -\tfrac{1}{2}\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu = -\gamma^0\gamma^1, \tag{8.10}$$

where $\varepsilon_{\mu\nu}$ is the totally anti-symmetric tensor in two dimensions with $\varepsilon_{01} := +1$. It satisfies

$$\gamma^3\gamma^\mu = -\gamma^\mu\gamma^3 = \eta^{\mu\nu}\varepsilon_{\nu\rho}\gamma^\rho. \tag{8.11}$$

a) Choose the spinor field and gamma-matrices as follows:

$$\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \quad \gamma^0 = \begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}. \tag{8.12}$$

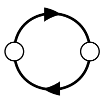
Write the Lagrangian (8.9) in terms of these components, and derive the equations of motion. What do you observe?

b) Use the equations of motion to show that the vector and axial currents

$$N_{\mathrm{V}}^\mu = -\mathring{\imath}\bar{\psi}\gamma^\mu\psi, \qquad N_{\mathrm{A}}^\mu = -\mathring{\imath}\bar{\psi}\gamma^3\gamma^\mu\psi. \tag{8.13}$$

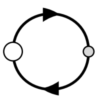
are separately conserved.

c) *optional:* Compute the correlator of two currents at $A = 0$

$$\big\langle N_{\mathrm{V}}^\mu(p)\, N_{\mathrm{V}}^\nu(q)\big\rangle_{A=0} = N_{\mathrm{V}}^\mu(p)$$  $$N_{\mathrm{V}}^\nu(q) = (2\pi)^2\delta^2(p+q)\,\frac{\mathring{\imath}}{\pi}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \tag{8.14}$$

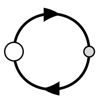
Hint: Use dimensional regularisation and assume massless tadpole integrals to vanish.

d) Express the vector current expectation value $\langle N_{\mathrm{V}}^\mu(p)\rangle$ at one loop and to linear order in the vector field A in terms of the current correlator (8.14). In other words, compute the graph

$$N_{\mathrm{V}}^\mu(p)$$  $$A\ . \tag{8.15}$$

Check that the current is conserved at this order.

e) Show that the axial current is no longer conserved at the quantum level. To this end compute the expectation value $\langle N_{\mathrm{A}}^\mu(p)\rangle$ at one loop and at linear order in A

$$N_{\mathrm{A}}^\mu(p)$$  $$A \tag{8.16}$$

using your results of part d) together with (8.11) to relate the currents.

f) Can you obtain the anomaly from a computation in position space? What do you find for $\langle\partial{\cdot}N_{\mathrm{V}}(x)\rangle$ and $\langle\partial{\cdot}N_{\mathrm{A}}(x)\rangle$? Can you make it agree with parts d) and e)?
Hint: No regularisation is required here.

9.1. The Higgs mechanism in the standard model

In this exercise we review the Higgs mechanism in the standard model. Let us consider first of all the mass terms of matter fields. Take the first lepton generation, i.e. the electron and its neutrino. The electroweak gauge group is $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ (weak isospin and hypercharge). Left- and right-handed fermions appear in different structures in the Lagrangian. The left-handed leptons form a doublet with respect to $\mathrm{SU}(2)_I$

$$L_\mathrm{L} := \begin{pmatrix} \nu_\mathrm{L} \\ e_\mathrm{L} \end{pmatrix}, \qquad L'_\mathrm{L} = \mathrm{e}^{i\theta^a I_a}\, L_\mathrm{L}, \tag{9.1}$$

where the generators of $\mathrm{SU}(2)_I$ in the fundamental representation may be chosen to be $I_a = \frac{1}{2}\sigma_a$ (σ_a are the Pauli matrices). The right-handed electron is a singlet under $\mathrm{SU}(2)_I$.

$$e'_\mathrm{R} = e_\mathrm{R}. \tag{9.2}$$

Neutrinos are taken to be massless: the corresponding right-handed field is thus not needed.

Under $\mathrm{U}(1)_Y$ the fields transform as

$$L'_\mathrm{L} = \mathrm{e}^{-i\theta/2}\, L_\mathrm{L}, \qquad e'_\mathrm{R} = \mathrm{e}^{-i\theta}\, e_\mathrm{R}. \tag{9.3}$$

a) Why is a mass term of the form

$$\mathcal{L}_\mathrm{mass} := -m_e(\bar{e}_\mathrm{L} e_\mathrm{R} + \mathrm{h.c.}), \tag{9.4}$$

not allowed?

b) In order to assign a mass to our fields we introduce a new scalar field H. This field couples to the electron and neutrino as

$$\mathcal{L}_\mathrm{Yukawa} := -y(\bar{L}_\mathrm{L} H e_\mathrm{R} + \mathrm{h.c.}), \tag{9.5}$$

where y is the coupling constant. How should H change with respect to a gauge transformation?

c) Show that it is always possible to find a gauge transformation U that sets the first component of H to zero and makes the second one real,

$$UH = U \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} 0 \\ H_\mathrm{r} \end{pmatrix}. \tag{9.6}$$

The gauge that brings H in this form is called *unitary gauge*.

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d) Consider a gauge-invariant, renormalisable potential term for the scalar field

$$\mathcal{L}_{\text{pot}} = \mu^2(H^\dagger H) - \tfrac{1}{2}\lambda(H^\dagger H)^2. \tag{9.7}$$

Find the minima of this potential and show that for $\mu^2 > 0$ they correspond to field configuration(s) $H \neq 0$.

e) Assume $\mu^2 > 0$ and redefine

$$H_{\text{r}} = \frac{1}{\sqrt{2}}(v + \eta), \tag{9.8}$$

to make excitations of H with respect to the ground state more explicit. η is thus the new physical Higgs boson field. Write down (9.5) in terms of the Higgs field and the vacuum expectation value v. Note that the fermions acquire a mass term, and express the fermion mass in terms of y and v. What is the coupling of the Higgs boson to the fermions in terms of m_e and v?

The standard model Lagrangian density also contains a kinetic term for the Higgs doublet

$$\mathcal{L}_{\text{kin}} := -(D_\mu H)^\dagger(D^\mu H). \tag{9.9}$$

The covariant derivative acting on the Higgs doublet involves the gauge fields for the $\mathrm{SU}(2)_I$ and $\mathrm{U}(1)_Y$ symmetry W^a_μ and B_μ, respectively

$$D_\mu = \partial_\mu - \mathring{\imath} g I_a W^a_\mu - \mathring{\imath} g' Y B_\mu. \tag{9.10}$$

In the following you should try to find the masses and the couplings to the Higgs boson of the physical standard model fields.

f) Diagonalise the quadratic term by introducing the physical fields

$$W^+_\mu = (W^-_\mu)^\dagger = \frac{1}{\sqrt{2}}(W^1_\mu - \mathring{\imath} W^2_\mu), \quad Z^0_\mu = \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2 + g'^2}}, \quad A_\mu = \frac{gB_\mu + g'W^3_\mu}{\sqrt{g^2+g'^2}}. \tag{9.11}$$

Read off the masses and couplings of the theory in terms of g, g', v, λ and μ by comparing the resulting expression for $\mathcal{L}_{\text{kin}}$ to

$$\begin{aligned}(D_\mu H)^\dagger(D^\mu H) = {}& \tfrac{1}{2}(\partial_\mu \eta)^2 \\ & + \tfrac{1}{2}m_Z^2 Z^0_\mu Z^{0,\mu} + \tfrac{1}{2}\rho_Z \eta Z^0_\mu Z^{0,\mu} + \tfrac{1}{4}\lambda_Z \eta^2 Z^0_\mu Z^{0,\mu} \\ & + m_W^2 W^+_\mu W^{-,\mu} + \rho_W \eta W^+_\mu W^{-,\mu} + \tfrac{1}{2}\lambda_W \eta^2 W^+_\mu W^{-,\mu}.\end{aligned} \tag{9.12}$$

g) The original doublet H included four degrees of freedom. Which symmetry was broken? Where are the degrees of freedom hidden after spontaneous symmetry breaking? Why did the photon field A_μ not acquire a mass term? Why does it couple to the fermion vector currents, but not to the axial currents?

10.1. The QED running coupling at one loop

Consider the Lagrangian of quantum electrodynamics,

$$\mathcal{L} = \bar{\psi}_0(\gamma_\mu D_0^\mu - m_0)\psi_0 - \tfrac{1}{4}(F_0^{\mu\nu})^2, \tag{10.1}$$

where the covariant derivative is given by

$$D_0^\mu = \partial^\mu - \mathring{\imath} q_0 A_0^\mu. \tag{10.2}$$

The gauge may be fixed in a covariant way by adding the term

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi_0}\,(\partial_\mu A_0^\mu)^2. \tag{10.3}$$

This theory may be renormalised via a multiplicative redefinition of each field and parameter appearing in the Lagrangian,

$$\psi_0 = Z_\psi^{1/2}\psi, \quad A_0 = Z_A^{1/2}A, \quad m_0 = \frac{Z_m}{Z_\psi}\,m, \quad q_0 = \frac{Z_q}{Z_A^{1/2}Z_\psi}\,q, \quad \xi_0 = \frac{Z_A}{Z_\xi}\,\xi. \tag{10.4}$$

a) Determine which one-particle irreducible correlation functions are superficially divergent in the UV and thus need to be renormalised.

The Ward–Takahashi identity for the connected generating functional can be obtained from the one for the partition function Z, by expressing expectation values as functional derivatives and substituting $Z = \exp(\mathring{\imath} W)$. With sources

$$\mathcal{L}_{\text{src}} = \bar{\psi}\rho + \bar{\rho}\psi - J{\cdot}A, \tag{10.5}$$

it reads

$$-\frac{1}{\xi_0}\,\partial_x^2\partial_x^\mu\,\frac{\mathring{\imath}\delta W}{\delta J^\mu(x)} + \mathring{\imath} q_0\left(\bar{\rho}(x)\,\frac{1}{\mathring{\imath}}\,\frac{\delta W}{\delta\bar{\rho}(x)} - \frac{1}{\mathring{\imath}}\,\frac{\delta W}{\delta\rho(x)}\,\rho(x)\right) - \mathring{\imath}\partial_x^\mu J_\mu(x) = 0. \tag{10.6}$$

b) Generate an identity for the photon two-point function by taking a functional derivative of equation (10.6) with respect to $J_\nu(y)$ and setting sources to zero. Use it to show that the gauge-fixing term does not need renormalisation, i.e. that it is consistent to set $Z_\xi = 1$ to all orders.

c) Compute the one-loop correction to the photon propagator. Use dimensional regularisation in $D = 4 - 2\epsilon$ to handle UV-divergences, and let μ_0 be the regularisation scale that is introduced by this procedure.

Hint: While it is necessary to carry out the integral over the loop momentum, evaluating the last integral over Feynman parameters in a closed form is not required.

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d) Determine the renormalisation constant Z_A by imposing the condition that the propagator for a photon of momentum p equals its tree-level value for a euclidean point $p^2 = \mu^2$. This renormalisation prescription is known as *momentum subtraction*. Note that, within such scheme, the *renormalisation scale* μ that is introduced is distinct from the regularisation scale μ_0.

e) Find the anomalous dimension of the photon field at one loop

$$\gamma_A := \frac{\partial \log Z_A^{1/2}}{\partial \log \mu}\,, \tag{10.7}$$

in the high energy limit $\mu^2 \gg m^2$.

From this point onwards, it is convenient to use Feynman gauge ($\xi = 1$) to make calculations less cumbersome.

f) Compute the one-loop correction to the electron propagator. Determine the renormalisation constants Z_ψ and Z_m by imposing the condition that the propagator for an electron of momentum p equals its tree-level value for a euclidean point $p^2 = \mu^2$.

g) Write down a Ward identity for the photon-electron-positron three-point function by taking two consecutive functional derivatives of equation (10.6) with respect to $\rho(y)$ and $\bar{\rho}(z)$ and setting sources to zero. Use it to show that Z_ψ and Z_q have the same divergent part to all orders.

h) Compute the one-loop correction to the QED vertex with on-shell fermions. Take the limit of massless electrons to make this part of the calculation easier. Determine the renormalisation constant Z_q by imposing the condition that the vertex for a photon momentum p equals its tree-level value for a euclidean point $p^2 = \mu^2$.

i) Show that, consistently with the Ward identity,

$$\frac{\partial \log Z_\psi}{\partial \log \mu} = \frac{\partial \log Z_q}{\partial \log \mu}\,. \tag{10.8}$$

j) Using the fact that the bare coupling cannot depend on the renormalisation scale, compute the beta-function of QED

$$\beta(\mu) := \frac{\partial \log q}{\partial \log \mu}\,. \tag{10.9}$$

Solve this differential equation in the limit of high energies $\mu^2 \gg m^2$ to find the QED running coupling $\alpha(\mu)$ and plot the corresponding function. Assuming $\alpha(m) \simeq 1/137$ with $m \simeq 0.5$ MeV the electron mass, compute the value of $\alpha(\mu_{\text{EW}})$ where $\mu_{\text{EW}} \simeq M_W \simeq M_Z \simeq 100$ GeV is the electroweak scale.

k) Does the fermion mass also depend on the renormalisation scale in this scheme?

11.1. The Coleman–Weinberg Potential

The Coleman–Weinberg model represents quantum electrodynamics of a scalar field in four dimensions. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D^\mu\phi)^* D_\mu\phi + \mu^2\phi^*\phi - \tfrac{1}{4}\lambda(\phi^*\phi)^2, \tag{11.1}$$

where $\phi(x)$ is a complex scalar field and $D^\mu = \partial^\mu - i q A^\mu$ is the covariant derivative.

a) Assume that $\mu^2 > 0$, so that the U(1) symmetry of the Lagrangian is spontaneously broken by means of the Higgs mechanism. Define the complex field ϕ in such a way, that one of its two degrees of freedom is gauged away by a proper gauge transformation and expand the other degree of freedom around its saddle point. Finally by expanding the Lagrangian itself, show that the field A_μ acquires a mass.

b) Consider a generic action $S[\phi]$ of a scalar field ϕ. Show that the one-loop effective action takes the form

$$G[\phi] = S[\phi] + \frac{i}{2}\log \operatorname{Det}\left[-i\,\frac{\delta^2 S}{\delta\phi^2}\right] + \ldots . \tag{11.2}$$

In order to compute the underlying partition function Z, we employ the so-called background field method and write the quantum scalar field $\hat{\phi}(x)$ as a classical background field $\phi(x)$ plus quantum fluctuations $\eta(x)$

$$\hat{\phi}(x) = \phi(x) + \eta(x). \tag{11.3}$$

Hint: Use the argument ϕ of the effective action $G[\phi]$ as the background field, expand the exponent of $Z[j]$ to quadratic order in η and choose j and ϕ in accordance with the saddle point condition

$$\frac{\delta S}{\delta\phi} + j = 0. \tag{11.4}$$

c) Let us return to the Lagrangian (11.1). Compute the one-loop correction to the effective potential

$$V_{\text{eff}}(\phi) = -G[\phi], \tag{11.5}$$

where ϕ is a constant background field. Eliminate the arising divergences by adjusted minimal subtraction (MS-bar) at the renormalisation scale M.

Hint: After performing the gauge transformation of part a), expand the remaining scalar degree of freedom according to (11.3). Then use (11.2) to compute the one-loop effective action due to the scalar and vector degrees of freedom for which the formula holds equivalently. Note, that only terms quadratic in the fluctuating fields A_μ and η contribute to the functional determinant of (11.2).

d) In the result of part b), take the limit $\mu^2 \to 0$. The result should be an effective potential that is scale-invariant up to logarithms of M. For λ very small, of order q^4, show that $V(\phi)$ has a symmetry-breaking minimum at a value of ϕ for which no logarithm is large and so the perturbation theory analysis is still valid. Thus the $\mu^2 = 0$ theory, for this choice of coupling constants, still has spontaneously broken symmetry, due to the influence of first-order quantum corrections.

$\longrightarrow$

e) The beta-functions for the running couplings q and λ and the gamma-function of ϕ are defined by the equations

$$M\,\frac{\partial\bar{\lambda}}{\partial M} = \beta_\lambda(\bar{\lambda},\bar{q}), \qquad M\,\frac{\partial\bar{q}}{\partial M} = \beta_q(\bar{\lambda},\bar{q}), \qquad M\,\frac{\partial\bar{\phi}}{\partial M} = -\gamma_\phi(\bar{\lambda},\bar{q})\bar{\phi}. \tag{11.6}$$

Explicit calculation yields the following one-loop results

$$\beta_q = \frac{q^3}{48\pi^2}\,, \qquad \beta_\lambda = \frac{15\lambda^2 - 36\lambda q^2 + 72q^4}{32\pi^2}\,, \qquad \gamma_\phi = -\frac{2q^2}{16\pi^2}\,. \tag{11.7}$$

Construct the RG-improved effective potential for the $\mu^2 = 0$ model and discuss qualitatively whether the spontaneous symmetry breaking still occurs.

Finally, compute the mass of the scalar particle m as a function of q^2, λ and M and determine the ratio m/m_A to leading order in q^2, for $\lambda \ll q^2$.

Hint: The RG-improved effective potential should be such that when expanded in terms of the coupling constants around the energy M, it will recover the result in part d).