

Quantum Field Theory I

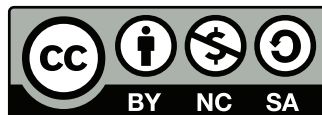
Lecture Notes

ETH Zurich, 2019 HS

PROF. N. BEISERT

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0 Overview

Quantum field theory is the quantum theory of fields just like quantum mechanics describes individual quantum particles. Here, the term “field” refers to one of the following:

- A field of a classical field theory, such as electromagnetism.
- A wave function of a particle in quantum mechanics. This is why QFT is sometimes called “second quantisation”.
- A smooth approximation to some property in a solid, e.g. the displacement of atoms in a lattice.
- Some function of space and time describing some physics.

Usually, excitations of the quantum field will be described by “particles”. In QFT the number of these particles is not conserved, they are created and annihilated when they interact. It is precisely what we observe in elementary particle physics, hence QFT has become the mathematical framework for this discipline.

This lecture series gives an introduction to the basics of quantum field theory. It describes how to quantise the basic types of fields, how to handle their quantum operators and how to treat (sufficiently weak) interactions. We will focus on relativistic models although most methods can in principle be applied to non-relativistic condensed matter systems as well. Furthermore, we discuss symmetries, infinities and running couplings. The goal of the course is a derivation of particle scattering processes in basic QFT models.

This course focuses on canonical quantisation along the lines of ordinary quantum mechanics. The continuation of this lecture course, QFT II, introduces an alternative quantisation framework: the path integral.¹ It is applied towards formulating the standard model of particle physics by means of non-abelian gauge theory and spontaneous symmetry breaking.

What Else is QFT? There are many points of view.

After attending this course, you may claim QFT is all about another 1000 ways to treat free particles and harmonic oscillators. True, these are some of the few systems we can solve exactly in theoretical physics; almost everything else requires approximation. After all, this is a physics course, not mathematics!

If you look more carefully you will find that QFT is a very rich subject, you can learn about many aspects of physics, some of which have attained a mythological status:

¹The path integral is much more convenient to use than canonical quantisation discussed here. However, some important basic concepts are not as obvious as in canonical quantisation, e.g. the notion of particles, scattering and, importantly, unitarity.

- anti-particles, anti-matter,
- vacuum energy,
- tachyons,
- ghosts,
- infinities,
- mathematical (in)consistency.

Infinities. How to deal with infinities?

There is a famous quote due to Dirac about QED (1975): “I must say that I am very dissatisfied with the situation, because this so-called ‘good theory’ does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!”

This is almost true, but QFT is neither neglecting infinities nor in an arbitrary way.

Infinities are one reason why QFT is claimed to be mathematically ill-defined or even inconsistent. Yet QFT is a well-defined and consistent calculational framework to predict certain particle observables to extremely high precision.

Many points of view; one is that it is our own fault: QFT is somewhat idealised; it assumes infinitely extended fields (IR) with infinite spatial resolution (UV);² there is no wonder that the theory produces infinities. Still, it is better to stick to idealised assumptions and live with infinities than use some enormous discrete systems (actual solid state systems).

There is also a physics reason why these infinities can be absorbed somehow: Our observable everyday physics should neither depend on how large the universe actually is (IR) nor on how smooth or coarse-grained space is (UV).

We can in fact use infinities to learn about allowable particle interactions. This leads to curious effects: running coupling and quantum anomalies.

More later, towards the end of the semester.

Uniqueness. A related issue is uniqueness of the formulation. Alike QM, QFT does not have a unique or universal formulation.

For instance, many meaningful things in QM/QFT are actually equivalence classes of objects. It is often more convenient or tempting to work with specific representatives of these classes. However, one has to bear in mind that only the equivalence class is meaningful, hence there are many ways to describe the same physical object.

The usage of equivalence classes goes further, it is not just classes of objects. Often we have to consider classes of models rather than specific models. This is something we have to accept, something that QFT forces upon us.

²The UV and the IR are the two main sources for infinities.

We will notice that QFT does what it wants, not necessarily what we want. For example, we cannot expect to get what we want using bare input parameters. Different formulations of the same model naively may give different results. We must learn to adjust the input to the desired output, then we shall find agreement. We just have to make sure that there is more output than input; otherwise QFT would be a nice but meaningless exercise because of the absence of predictions. Another nice feature is that we can hide infinities in these ambiguities in a self-consistent way.

Enough of Talk. Just some words of warning: We must give up some views on physics you have become used to, only then you can understand something new. For example, a classical view of the world makes understanding quantum mechanics harder. Nevertheless, one can derive classical physics as an approximation of quantum physics, once one understands the latter sufficiently well.

Let us start with something concrete, we will discuss the tricky issues when they arise.

Important Concepts. Some important concepts of QFT that will guide our way:

- unitarity – probabilistic framework.
- locality – interactions are strictly local.
- causality – special relativity.
- symmetries – exciting algebra and geometry.
- analyticity – complex analysis.

0.1 Prerequisites

Prerequisites for this course are the core courses in theoretical physics of the bachelor syllabus:

- Classical mechanics (brief review in first lecture)
- Quantum mechanics (brief review in first lecture)
- Electrodynamics (as a simple classical field theory)
- Mathematical methods in physics (HO, Fourier transforms, ...)

0.2 Contents

- | | |
|------------------------------------|-----------|
| 1. Classical and Quantum Mechanics | (105 min) |
| 2. Classical Free Scalar Field | (110 min) |
| 3. Scalar Field Quantisation | (210 min) |
| 4. Symmetries | (240 min) |
| 5. Free Spinor Field | (270 min) |

6. Free Vector Field	(250 min)
7. Interactions	(170 min)
8. Correlation Functions	(240 min)
9. Particle Scattering	(190 min)
10. Scattering Matrix	(200 min)
11. Loop Corrections	(225 min)

Indicated are the approximate number of lecture minutes for each chapter. Altogether, the course consists of 51 lectures of 45 minutes including one overview lecture.

0.3 Disclaimer

These are notes for an introductory lecture on quantum field theory. Their main purpose is to enable the lecturer to deliver the lectures. Consequently, the contents will overlap to a large extent with the oral and written parts of the lecture. This is not a bug, but a feature! Nevertheless, these notes may serve a variety of additional purposes. Please note the following hints, suggestions and disclaimers:

- Please accept that some explanations and derivations will remain obscure on a first, quick reading which could take place prior to the lecture. It may well make sense to skip over them in this situation.
- Following the ongoing lecture can be assisted by keeping a set of (printed/amendable) notes where one can keep track of additional details to be picked up during the lecture.
- The presentation of many calculations and derivations is incomplete and merely spells out the starting point and final result along with some key steps.³ It will be a useful exercise to fill in some gaps and work out the missing intermediate steps.
- The lecture notes contain many footnotes. They remind the lecturer of some relevant details, advanced aspects, connections and subtleties as well as issues that come up frequently in related discussions. It may be advantageous to disregard them in a basic reading towards gaining an average understanding of the topic. They may nevertheless serve as useful hints on exciting facts and as starting points to be followed up upon by more ambitious readers.
- Importantly, this document is not a textbook on the subject, and it should not be used as a substitute for the latter. The reader is urged to consult a proper textbook on QFT (or two) in parallel.

³A lecture should keep a balance between being detailed, explicit, exciting, inspiring as well as concise and pedagogical, all of which cannot be satisfied at the same time. The chosen balance may not be one that is subjectively preferred.

0.4 References

There are many text books and lecture notes on quantum field theory. Here is a selection of well-known ones:

- M. E. Peskin, D. V. Schroeder, “An Introduction to Quantum Field Theory”, Westview Press (1995)
- C. Itzykson, J.-B. Zuber, “Quantum Field Theory”, McGraw-Hill (1980)
- P. Ramond, “Field Theory: A Modern Primer”, Westview Press (1990)
- M. Srednicki, “Quantum Field Theory”, Cambridge University Press (2007)
- M. Kaku, “Quantum Field Theory”, Oxford University Press (1993)
- online: D. Tong, “Quantum Field Theory”, lecture notes, <http://www.damtp.cam.ac.uk/user/tong/qft.html>
- online: M. Gaberdiel, “Quantenfeldtheorie”, lecture notes (in German), <http://people.phys.ethz.ch/~mrg/Lectures/QFT.pdf>
- ...

Peskin & Schroeder may be closest to this lecture course, but we will not follow it literally, and the notation differs significantly.

0.5 Acknowledgements

I thank the participants of the course in 2012, 2014 and 2019 (including Michal Baczkyk, Vasileios Belis, Egor Im, Melina Lüthi, Alessio Ruggeri, Zihang Qiu, Dong Zhu) for directly and indirectly pointing out various typos, inconsistencies and shortcomings of the presentation. These have helped substantially in improving the lecture notes and making them more accessible.

1 Classical and Quantum Mechanics

To familiarise ourselves with the basics of quantum field theory, let us review some elements of classical and quantum mechanics. Then we shall discuss some issues of combining quantum mechanics with special relativity.

1.1 Classical Mechanics

Consider a classical non-relativistic particle in a potential. In Lagrangian mechanics it is described by the position variables $q^i(t)$ and the action functional $S[q]$ ^{1 2}

$$S[q] = \int_{t_1}^{t_2} dt L(q^i(t), \dot{q}^i(t); t). \quad (1.1)$$

A typical Lagrangian function is

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2} m \dot{\vec{q}}^2 - V(\vec{q}). \quad (1.2)$$

with mass m and $V(q)$ as the external potential.

A classical path extremises (minimises) the action S . One therefore determines the saddle point $\delta S = 0$ by variation of the action³

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \left(\delta q^i(t) \frac{\partial L}{\partial q^i} + \delta \dot{q}^i(t) \frac{\partial L}{\partial \dot{q}^i} \right) \\ &= \int_{t_1}^{t_2} dt \delta q^i(t) \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) + \int_{t=t_1}^{t=t_2} d \left(\delta q^i(t) \frac{\partial L}{\partial \dot{q}^i} \right). \end{aligned} \quad (1.3)$$

The first term is the equation of motion (Euler–Lagrange)

$$\frac{\delta S}{\delta q^i(t)} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0. \quad (1.4)$$

The second term due to partial integration is the boundary equation of motion; usually we ignore it.⁴

¹In many cases, L is time-independent: $L(q^i, \dot{q}^i; t) = L(q^i, \dot{q}^i)$.

²A single time derivative \dot{q}^i usually suffices.

³Einstein summation convention: there is an implicit sum over all index values for pairs of matching upper/lower indices.

⁴More precisely, we usually fix the position $q_i(t_k) = \text{const}$ (Dirichlet) or the momentum $\partial L / \partial \dot{q}^i(t_k) = 0$ (Neumann) at the boundary.

Example. Throughout this chapter we will use the harmonic oscillator and the free particle as an example to illustrate the abstract formalism. The harmonic oscillator is described by the following Lagrangian function and corresponding equation of motion

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2}m\dot{\vec{q}}^2 - \frac{1}{2}m\omega^2\vec{q}^2, \quad -m(\ddot{\vec{q}} + \omega^2\vec{q}) = 0. \quad (1.5)$$

For $\omega = 0$ this system becomes a free particle.

1.2 Hamiltonian Formulation

The Hamiltonian framework is the next step towards canonical quantum mechanics.

First, define the momentum p_i conjugate to q^i as⁵

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad (1.6)$$

and solve for $\dot{q}^i = \dot{q}^i(q, p; t)$.⁶ Phase space is defined as the space of the position and momentum variables (q^i, p_i) .

The Lagrangian function $L(q, \dot{q}; t)$ is replaced by the Hamiltonian function $H(q, p; t)$ on phase space. We define $H(q^i, p_i; t)$ as the Legendre transformation of L

$$H(q, p; t) = p_i \dot{q}^i(q, p; t) - L(q, \dot{q}(q, p; t); t). \quad (1.7)$$

Let us express the equations of motion through H : A variation of the Hamiltonian function w.r.t. all q^i and p_i reads

$$\delta H = \delta p_i \dot{q}^i - \delta q^i \frac{\partial L}{\partial q^i}, \quad (1.8)$$

where we substituted the definition of the momenta p_i twice to cancel four further terms that arise. We use the Euler–Lagrange equation and momenta to simplify the variation further

$$\delta H = \delta p_i \dot{q}^i - \delta q^i \dot{p}_i. \quad (1.9)$$

Comparing this expression to the general variation

$\delta H = \delta q^i (\partial H / \partial q^i) + \delta p_i (\partial H / \partial p_i)$ we obtain the Hamiltonian equations of motion

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (1.10)$$

Next, we introduce the Poisson bracket for two functions f, g on phase space

$$\{f, g\} := \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \quad (1.11)$$

⁵This is a choice, one might also use different factors or notations.

⁶We suppose the equation can be solved for \dot{q} .

The Poisson bracket allows to express the time evolution for phase space functions $f(q, p; t)$ ⁷

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \{H, f\}. \quad (1.12)$$

In particular, this works well for the functions $f(q, p; t) = q^i$ and $f(q, p; t) = p_i$, and yields the canonical equations of motion.

Example. For the harmonic oscillator we find $\vec{p} = m\dot{\vec{q}}$ and

$$H = \vec{p} \cdot \dot{\vec{q}} - \frac{1}{2}m\dot{\vec{q}}^2 + \frac{1}{2}m\omega^2\vec{q}^2 = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{q}^2. \quad (1.13)$$

The Hamiltonian equations of motion read

$$\begin{aligned} \dot{\vec{q}} &= -\{H, \vec{q}\} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\vec{p}, \\ \dot{\vec{p}} &= -\{H, \vec{p}\} = -\frac{\partial H}{\partial \vec{q}} = -m\omega^2\vec{q}. \end{aligned} \quad (1.14)$$

A convenient change of variables reads

$$\vec{a} = \frac{1}{\sqrt{2m\omega}}(m\omega\vec{q} + i\vec{p}), \quad \vec{a}^* = \frac{1}{\sqrt{2m\omega}}(m\omega\vec{q} - i\vec{p}). \quad (1.15)$$

Using these coordinates of phase space, the Poisson brackets read

$$\{f, g\} = -i \frac{\partial f}{\partial a^i} \frac{\partial g}{\partial a_i^*} + i \frac{\partial f}{\partial a_i^*} \frac{\partial g}{\partial a^i}. \quad (1.16)$$

We obtain a separated first-order time evolution for \vec{a}, \vec{a}^*

$$H = \omega \vec{a}^* \cdot \vec{a}, \quad \dot{\vec{a}} = -i\omega \vec{a}, \quad \dot{\vec{a}}^* = +i\omega \vec{a}^*. \quad (1.17)$$

Note that the Poisson bracket with this Hamiltonian simply measures the degree of a function in a vs. a^*

$$\{H, f\} = i\omega a^i \frac{\partial f}{\partial a^i} - i\omega a_i^* \frac{\partial f}{\partial a_i^*}. \quad (1.18)$$

E.g. for $f = (a)^m (a^*)^n$ one finds $\{H, f\} = i\omega(m - n)f$.

1.3 Quantum Mechanics

Let us revisit the canonical quantisation procedure highlighting the harmonic oscillator. Quantum field theory is built on the same methods, and we shall encounter the same kinds of problems, yet in some more elaborate fashion.

In canonical quantisation, classical objects are replaced by elements of linear algebra:

⁷The Hamiltonian H is itself a phase space function.

- the state (q^i, p_i) becomes a vector $|\psi\rangle$ in a Hilbert space V ;
- a phase space function f becomes a linear operator F on V ;
- Poisson brackets $\{f, g\}$ become commutators $-i\hbar^{-1}[F, G]$.⁸

The equation of motion for a state (Schrödinger) is a wave equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H|\psi(t)\rangle. \quad (1.19)$$

The (normalised) wave function of a state has a probabilistic interpretation: $|\langle\phi|\psi\rangle|^2$ is the probability of finding the system described by $|\psi\rangle$ in the state $|\phi\rangle$. This requires the following essential features of wave functions:

- $\langle\psi|\psi\rangle$ is positive;
- $\langle\psi|\psi\rangle$ can be normalised to 1 by scaling $|\psi\rangle$;
- $\langle\psi|\psi\rangle$ is conserved.

Conservation requires

$$\frac{d}{dt} [\langle\psi|\psi\rangle] = \frac{1}{i\hbar} \langle\psi|(H - H^\dagger)|\psi\rangle = 0. \quad (1.20)$$

Therefore, the Hamiltonian must be hermitian (self-adjoint). We then have time evolution with a unitary operator $U(t_2, t_1)$

$$|\psi(t_2)\rangle = U(t_2, t_1)|\psi(t_1)\rangle. \quad (1.21)$$

The matrix element $\langle\psi|F|\psi\rangle$ describes the expectation value of the operator F in state $|\psi\rangle$. Curiously, it obeys the quantum analog of the classical time evolution

$$\frac{d}{dt} \langle\psi|F|\psi\rangle = \langle\psi|\left(\frac{\partial F}{\partial t} - \frac{1}{i\hbar} [H, F]\right)|\psi\rangle. \quad (1.22)$$

Example. For the harmonic oscillator and free particle we need to represent the canonical commutation relations

$$[\hat{q}^i, \hat{p}_j] = i\hbar\{q^i, p_j\} = i\hbar\delta_j^i \quad (1.23)$$

on phase space. We introduce a basis of position eigenstates $|\vec{q}\rangle$. The position and momentum operators \hat{q}^i and \hat{p}_i then act as⁹

$$\hat{q}^i|\vec{q}\rangle = q^i|\vec{q}\rangle, \quad \hat{p}_i|\vec{q}\rangle = i\hbar \frac{\partial}{\partial q^i} |\vec{q}\rangle. \quad (1.24)$$

We introduce a wave function $\psi(t, \vec{q}) = \langle\vec{q}|\psi(t)\rangle$ to represent a general state $|\psi(t)\rangle$ on phase space

$$|\psi(t)\rangle = \int d\vec{q}^d \psi(t, \vec{q}) |\vec{q}\rangle. \quad (1.25)$$

⁸Poisson brackets cannot always be translated literally to commutators; the idea of quantisation is to represent them up to “simpler” terms, i.e. up to polynomials of lower degree in the operators and of higher orders in \hbar .

⁹As usual in quantum mechanics, the action of the operators on the states effectively inverts the order of terms in operator products. Plain insertion of $\hat{q}^i = q^i$ and $\hat{p}_i = i\hbar \partial/\partial q^i$ into the commutation relations leads to the wrong sign.

By construction, the position operator \hat{q}^i acts by multiplying the wave function by q^i . The momentum operator \hat{p}_i effectively acts by the derivative $-i\partial/\partial q^i$ on the wave function.¹⁰ For acting directly on the wave function $\psi(t, \vec{q})$ we can thus write

$$\hat{q}^i \psi(t, \vec{q}) \simeq q^i \psi(t, \vec{q}), \quad \hat{p}_i \psi(t, \vec{q}) \simeq -i\hbar \frac{\partial}{\partial q^i} \psi(t, \vec{q}). \quad (1.26)$$

The Hamiltonian acting on the wave function reads

$$H \simeq -\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial \vec{q}} \right)^2 + \frac{m\omega^2}{2} \vec{q}^2. \quad (1.27)$$

The free particle is solved exactly by momentum eigenstates (Fourier transformation)

$$|\vec{p}\rangle = \int d\vec{q}^d e^{i\hbar^{-1}\vec{p}\cdot\vec{q}} |\vec{q}\rangle, \quad |\vec{q}\rangle = \int \frac{d\vec{p}^d}{(2\pi\hbar)^d} e^{-i\hbar^{-1}\vec{p}\cdot\vec{q}} |\vec{p}\rangle. \quad (1.28)$$

The momentum eigenstate $|\vec{p}\rangle$ is an energy eigenstate with $E = \vec{p}^2/2m$.

For the harmonic oscillator we use the operators a^i and a_i^\dagger which act on a wave function as

$$\begin{aligned} \vec{a} &= \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} + i\vec{p}) \simeq \frac{1}{\sqrt{2m\omega}} \left(m\omega\vec{q} + \hbar \frac{\partial}{\partial \vec{q}} \right), \\ \vec{a}^\dagger &= \frac{1}{\sqrt{2m\omega}} (m\omega\vec{q} - i\vec{p}) \simeq \frac{1}{\sqrt{2m\omega}} \left(m\omega\vec{q} - \hbar \frac{\partial}{\partial \vec{q}} \right). \end{aligned} \quad (1.29)$$

They obey the commutation relations

$$[a^i, a_j^\dagger] = \hbar \delta_j^i. \quad (1.30)$$

The quantum Hamiltonian has an apparent extra vacuum energy $E_0 = \frac{1}{2}d\hbar\omega$ compared to its classical counterpart $H = \omega\vec{a}^\dagger \cdot \vec{a}$

$$H = \frac{1}{2}\omega a^i a_i^\dagger + \frac{1}{2}\omega a_i^\dagger a^i = \omega\vec{a}^\dagger \cdot \vec{a} + \frac{1}{2}d\hbar\omega = \omega\vec{a}^\dagger \cdot \vec{a} + E_0. \quad (1.31)$$

- One can add any numerical energy E_0 to the Hamiltonian. This has no effect on any commutation relations, it merely shifts the frequencies by a common amount. E_0 is largely irrelevant for physics.¹¹
- This has the same effect as adding the term $i\alpha(\vec{q}\cdot\vec{p} - \vec{p}\cdot\vec{q})$ to the Hamiltonian H .¹² Such a term is classically invisible, but it leads to a quantum energy shift $\Delta E_0 = -d\alpha\hbar$. There is a quantum ordering ambiguity. Here it is very harmless, it merely affects the trivial energy E_0 .

¹⁰The opposite sign arises from a partial integration. In this presentation direct substitution into commutators yields the desired result $[\hat{q}^i, \hat{p}_j] \simeq i\hbar\delta_j^i$.

¹¹This statement holds unless the energy couples to something else, e.g. in gravity theories. A similar case is the relativistic rest energy $E_0 = mc^2$ which is irrelevant unless particles are created or annihilated.

¹²There appears to be no ordering ambiguity for the terms \vec{p}^2 and \vec{q}^2 , but it is instructive to consider all ordered degree-two polynomials in \vec{p} and \vec{q} which include the terms $\vec{q}\cdot\vec{p}$ and $\vec{p}\cdot\vec{q}$.

- Quantum theory does as it pleases, e.g. it can introduce a non-trivial E_0 or shift its value. It is generally preferable to consider all allowable terms in the first place, i.e. also introduce an energy shift E_0 into the classical Hamiltonian (which has almost no consequences).
- The conventional vacuum energy $\frac{1}{2}\hbar\omega$ of the harmonic oscillator is *by no means uniquely determined* by quantum mechanics. Further input such as symmetry is needed to arrive at this result.
- The reason for being very picky about this minor issue is that some of the infinities of quantum field theory will be absorbed into a redefinition of this constant.

We can now construct the spectrum of the harmonic oscillator: Start from the vacuum state $|0\rangle$ which is defined to be annihilated by the lowering operator \vec{a}

$$a^i|0\rangle = 0. \quad (1.32)$$

This state has energy $E = E_0$ (although this information is irrelevant as argued above). Then add $n_i \geq 0$ excitations of the raising operator a_i^\dagger and normalise the state to $\langle \vec{n} | \vec{n} \rangle = 1$

$$|\vec{n}\rangle = \left(\prod_{i=1}^d \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!} \hbar^{n_i/2}} \right) |0\rangle. \quad (1.33)$$

This is an energy eigenstate with $E = E_0 + \hbar\omega N$ where $N = \sum_{i=1}^d n_i$ is the total excitation number. The crucial property to arrive at this result is

$$[H, \vec{a}_i^\dagger] = \hbar\omega \vec{a}_i^\dagger, \quad (1.34)$$

which follows directly from the commutator algebra.

1.4 Quantum Mechanics and Relativity

From now on, let us set $\hbar = 1$, $c = 1$ for convenience.¹³

Attempts to set up a relativistic version of quantum mechanics have failed. Let us see why. In investigating the reasons we shall encounter the key characteristics of a quantum (field) theory such as unitarity, locality and causality.

First, compare the typical non-relativistic and relativistic energy relations for a free particle

$$e = \frac{\vec{p}^2}{2m} \quad \text{vs.} \quad e^2 = \vec{p}^2 + m^2 \quad \text{or} \quad e = \sqrt{\vec{p}^2 + m^2}. \quad (1.35)$$

A natural guess for a relativistic wave equation is (Klein–Gordon)

$$\left(-\left(\frac{\partial}{\partial t} \right)^2 + \left(\frac{\partial}{\partial \vec{q}} \right)^2 - m^2 \right) \psi(t, \vec{q}) = 0. \quad (1.36)$$

This equation has several conceptual problems:

¹³The constants can always be recovered from considerations of physical units.

Probabilistic Properties. There is no notion of the norm of the state $|\psi\rangle$ which satisfies all the required properties of non-relativistic quantum mechanics:

- The norm $\langle\psi|\psi\rangle$ of non-relativistic quantum mechanics is conserved only for a first-order wave equation.
- There is a real conserved quantity

$$Q = \frac{i}{2m} \left(\langle\psi|\dot{\psi}\rangle - \langle\dot{\psi}|\psi\rangle \right). \quad (1.37)$$

This charge has a major problem: it is not positive definite. Consequently, it is not suitable for a probabilistic interpretation!¹⁴

- Alternatively, one can define a positive definite measure. Unfortunately, this measure is not local.

Altogether, one might ask why to consider probabilities defined in a time slice: A time slice is not a Lorentz invariant concept in a relativistic model.

Causality. Suppose, the issues of the probabilistic interpretation are not severe by themselves, and let us discuss causality instead. Consider the overlap

$$\langle\vec{q}_2|U(t_2, t_1)|\vec{q}_1\rangle \quad (1.38)$$

for a pair of spacetime points (t_1, q_1) and (t_2, q_2) . It describes the probability amplitude for a particle moving from 1 to 2.

For space-like separated points this leads to a conceptual problem in a relativistic model:

- the overlap is non-zero even if points are space-like separated;
- this is a forbidden region, non-zero probabilities appear to violate causality;
- at least there is exponential suppression in this region; this may be interpreted as a quantum tunnelling effect.

Negative-Energy Solutions. The relativistic wave equation is a second-order partial differential equation. For every positive-energy solution

$$|\vec{p}, +, t\rangle = \int d\vec{q}^d e^{i\vec{p}\cdot\vec{q} - ie(\vec{p})t} |\vec{q}\rangle \quad (1.39)$$

there is a negative-energy solution

$$|\vec{p}, -, t\rangle = \int d\vec{q}^d e^{i\vec{p}\cdot\vec{q} + ie(\vec{p})t} |\vec{q}\rangle. \quad (1.40)$$

This leads to the following problems:

- negative-energy particles have not been observed (they are not tachyons, though);

¹⁴As we shall see, Q is rather similar to an electric charge.

- one might extract energy from making this particle faster;
- a positive-energy particle could fall to a negative-energy state; such a process would release a lot of energy to produce further particles.

One could insist on positive energies by postulating the relativistic Schrödinger wave equation

$$i \frac{\partial}{\partial t} \psi(t, \vec{q}) = \sqrt{-\left(\frac{\partial}{\partial \vec{q}}\right)^2 + m^2} \psi(t, \vec{q}). \quad (1.41)$$

Likewise this leads to several problems:

- The square root of a (differential) operator is hard to define;
- any suitable definition would certainly imply a non-local wave-equation.

Particle Creation. Special relativity allows energy to be converted to rest mass of particles:

- relativistic quantum mechanics should allow such processes;
- ordinary quantum mechanics usually assumes a fixed particle number.

Dirac Equation. The Dirac equation was an attempt to overcome some of the above problems

$$\frac{\partial}{\partial t} \psi = \alpha^i \frac{\partial}{\partial q^i} \psi + m \beta \psi. \quad (1.42)$$

For a suitable choice of matrices α^i , β it is a relativistic wave equation which implies the Klein–Gordon equation. Let us consider the implications on the above problems.

Probabilistic interpretation:

- it is a first-order wave equation;
- the combination $\langle \psi | \psi \rangle$ is conserved and positive definite;
- positivity requires Bose statistics for the wave function.

Spin:

- ordinarily, operators α^k imply spin- $1/2$ particles;
- half-integer spin requires Fermi statistics.

Negative-energy solutions:

- they exist (with different spin d.o.f.);
- they are separated from positive-energy solution in a non-local fashion.

The Dirac equation essentially has the same problems as the Klein–Gordon equation.

Conclusion. Klein–Gordon and Dirac equations:

- they are perfectly acceptable relativistic wave equations;
- they do not offer a probabilistic interpretation;
- they represent a model without particle production.

1.5 Conventions

Units. We shall work with natural units $\hbar = c = 1$.

- $c = 299\,792\,458\,\text{m s}^{-1}$ therefore $\text{s} := 299\,792\,458\,\text{m}$.
- $\hbar = 1.055\ldots \cdot 10^{-34}\,\text{kg m}^2\text{s}^{-1}$ therefore $\text{kg} := 2.843 \cdot 10^{42}\,\text{m}^{-1}$.
- one can always reinstall appropriate units by inserting factors of $1 = c$ or $1 = \hbar$ until the units come out as desired;
- in particle physics one often uses electron Volt (eV) instead of length:
 $\text{m} = 5.068 \cdot 10^6\,\text{eV}^{-1}$, $\text{s} = 1.519 \cdot 10^{15}\,\text{eV}^{-1}$, $\text{kg} = 5.610 \cdot 10^{35}\,\text{eV}$;
- one can convert back to SI units:
 $\text{eV} = 5.068 \cdot 10^6\,\text{m}^{-1} = 1.519 \cdot 10^{15}\,\text{s}^{-1} = 1.783 \cdot 10^{-36}\,\text{kg}$.

Euclidean Space. We write a three-vector x alternatively as

- $x^j = x_j$ with Latin indices $k, l, \dots = 1, 2, 3$;
- $\vec{x} = (x^1, x^2, x^3) = (x, y, z)$.

Scalar product between two vectors:

$$\vec{a} \cdot \vec{b} := \sum_{k=1}^3 a^k b^k = a^1 b^1 + a^2 b^2 + a^3 b^3. \quad (1.43)$$

Vector square:

$$\vec{a}^2 := \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2. \quad (1.44)$$

Totally anti-symmetric tensor ε^{ijk} with normalisation:

$$\varepsilon^{123} = +1. \quad (1.45)$$

Cross product (in terms of ε):

$$(a \times b)^k = \varepsilon^{ijk} a^i b^j. \quad (1.46)$$

Minkowski Space. Four-vectors, Greek indices $\mu, \nu, \dots = 0, 1, 2, 3$:

- position vector $x^\mu := (x^0, x^1, x^2, x^3) = (+t, \vec{x})$.
- momentum covector $p_\mu := (p_0, p_1, p_2, p_3) = (-e, \vec{p})$.

Summation convention: repeated index μ means implicit sum over $\mu = 0, 1, 2, 3$

$$x^\mu p_\mu := \sum_{\mu=0}^3 x^\mu p_\mu = -et + \vec{x} \cdot \vec{p}. \quad (1.47)$$

Minkowski metric: signature $(-+++)$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1). \quad (1.48)$$

Raise and lower indices (wherever needed):

$$x_\mu := \eta_{\mu\nu} x^\mu = (-t, \vec{x}), \quad p^\mu := \eta^{\mu\nu} p_\mu = (+e, \vec{p}). \quad (1.49)$$

Scalar products of two vectors or two covectors, e.g.

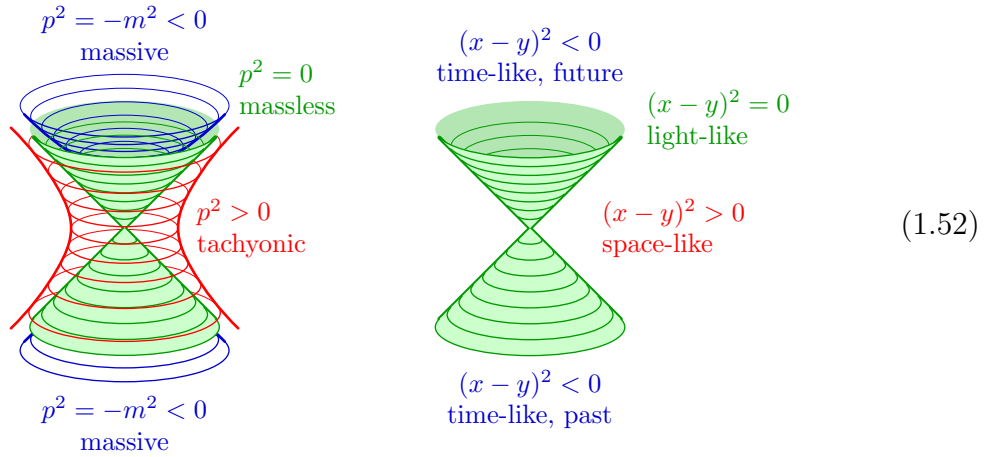
$$p \cdot p := -e^2 + \vec{p}^2. \quad (1.50)$$

Totally anti-symmetric tensor $\varepsilon_{\mu\nu\rho\sigma}$ with normalisation:

$$\varepsilon_{0123} = +1, \quad \varepsilon^{0123} = -1. \quad (1.51)$$

Our conventions:

- Mass shell $p^2 = -m^2$: $p^2 < 0$ massive, $p^2 = 0$ massless, $p^2 > 0$ tachyonic.
- Light cone: $(x - y)^2 < 0$ time-like, $(x - y)^2 = 0$ light-like, $(x - y)^2 > 0$ space-like.



Why $(-+++)$ signature?

- notation follows space (not time): $\underline{x}^\mu = (t, \underline{x})$, $\underline{p}^\mu = (e, \underline{p})$;
- $x^i = x_i$ but $x^0 = -x_0 = t$;
- $p^i = p_i$ but $p^0 = -p_0 = e$;
- Wick rotations more natural: just rotate time $t \rightarrow it$ and obtain Euclidean metric.

How to convert from $(+----)$ to $(-+++)$ signature?

- flip sign of every $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$;
- find out which (co)vectors match: x^μ , p^μ and ∂_μ agree literally, x_μ , p_μ and ∂^μ flip the sign;
- flip sign for every scalar product of vectors of same type: e.g. $p^2 + m^2 \leftrightarrow -p^2 + m^2$;
- preserve the scalar product between vectors of opposite type, e.g. $x^\mu \partial_\mu$.

Clifford Algebra. We use the standard form of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (1.53)$$

In comparison to gamma-matrices $\tilde{\gamma}$ in $(+----)$ signature the gamma-matrices in $(-+++)$ signature carry a factor of $\pm i$ to compensate for the overall factor of -1

in the metric η : $\tilde{\gamma} = -i\gamma$. For the Weyl representation it makes sense to apply a further change of basis

$$\tilde{\gamma}^\mu = -iU^{-1}\gamma^\mu U, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (1.54)$$

Then the various conjugation matrices are related by

$$\begin{aligned} \tilde{\gamma}_P &= U^{-1}\gamma_P U, \\ \tilde{\gamma}_C &= U^{-1}\gamma_C U^*, \\ \tilde{\gamma}^5 &= U^{-1}\gamma^5 U, \\ \tilde{\gamma}_\dagger &= U^{-1}\gamma_\dagger U^{-\dagger}, \\ \tilde{\gamma}_{\bar{T}} &= U^{-1}\gamma_{\bar{T}} U^*. \end{aligned} \quad (1.55)$$

Finally, our definition of the extended Pauli matrices $\sigma, \bar{\sigma}$ in signature $(-+++)$ relates to the conventional one $\tilde{\sigma}, \tilde{\bar{\sigma}}$ in signature $(+---)$

$$\tilde{\sigma} = \sigma, \quad \tilde{\bar{\sigma}} = -\bar{\sigma}. \quad (1.56)$$

This not only yields the conventional expression for the Weyl representation of gamma-matrices, but also has the benefit that $\sigma^k = \sigma_k = \bar{\sigma}^k = \bar{\sigma}_k$ irrespectively of the index position and the conjugation (whereas $\sigma^0 = -\sigma_0 = -\bar{\sigma}^0 = \bar{\sigma}_0 = 1$).

Name Spaces. We have only 26 Latin letters at our disposal and some are more attractive than others. We have to recycle:

- e may be $2.71\dots$, but also energy e (no distinction made on blackboard);
- π may be $3.14\dots$, but also momentum conjugate to field ϕ ;
- i may be $\sqrt{-1}$, but also useful for counting i (no distinction made on blackboard);
- κ may look like k or K on the blackboard;
- H may be Hamilton function or operator;
- \dots

We will typically not say explicitly which letter means what:

- we may even use same letter for different meanings in one formula;
- one can guess meaning from the context, e.g. i in $\exp(\pi i \dots)$ vs. $\sum_{i=1}^n$;
- indices typically do not mix with other symbols;
- one may try to avoid clashes, but this may end up cluttering notation;
- it's a fact of life (and the literature) and one needs to get used to it.

2 Classical Free Scalar Field

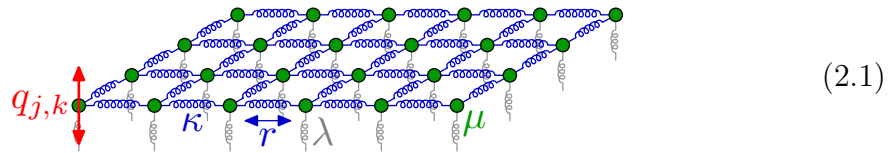
In the following chapter we will discuss the transition from classical mechanics to the mechanics of fields, and introduce one of the simplest field theory models, the classical non-interacting relativistic scalar field.¹

2.1 Spring Lattice

Before considering a field, we start with an approximation we can certainly handle: a lattice.

Consider an atomic lattice:

- 1D or 2D cubic lattice,
- atoms are coupled to neighbours by springs,²
- atoms are coupled to rest position by springs,
- atoms can move only orthogonally to lattice (transverse),
- boundaries: periodic identification.



The model has the following parameters and variables:

- lattice separation r ,
- number of atoms N (in each direction),
- mass of each atom μ ,
- lattice spring constant κ ,
- return spring constant λ ,
- shift orthogonal to lattice $q_{j,k}$

Lagrangian Formulation. The Lagrange function reads

$$L = L_{\text{kin}} - V_{\text{lat}} - V_{\text{rest}}. \quad (2.2)$$

¹This chapter will be somewhat repetitive and present similar relationships from different points of view. These are the basic relationships of relativistic quantum field theory, and it helps to be able to recognise and interpret them.

²Springs are useful approximations because they model first deviation from rest position; always applies to small excitations.

We make the standard non-relativistic ansatz for the kinetic terms, where μ is the mass of an atom

$$L_{\text{kin}} = \frac{1}{2}\mu \sum_{i,j=1}^N \dot{q}_{i,j}^2. \quad (2.3)$$

The potential for the springs with spring constants κ between the atoms reads³

$$V_{\text{lat}} = \frac{1}{2}\kappa \sum_{i,j=1}^N (q_{i-1,j} - q_{i,j})^2 + \frac{1}{2}\kappa \sum_{i,j=1}^N (q_{i,j-1} - q_{i,j})^2. \quad (2.4)$$

Finally, there is some spring potential with constant λ to drive the atoms back to their rest position

$$V_{\text{rest}} = \frac{1}{2}\lambda \sum_{i,j=1}^N q_{i,j}^2. \quad (2.5)$$

Altogether the interactions are quadratic in q 's. In other words, we have a bunch of coupled harmonic oscillators. The equations of motion read

$$\mu \ddot{q}_{i,j} - \kappa(q_{i-1,j} - 2q_{i,j} + q_{i+1,j}) - \kappa(q_{i,j-1} - 2q_{i,j} + q_{i,j+1}) + \lambda q_{i,j} = 0. \quad (2.6)$$

Note that these are spatially homogeneous equations. We can use a discrete Fourier transform (respecting periodicity) to solve them

$$q_{i,j}(t) = \frac{1}{N^2} \sum_{k,l=1}^N \frac{\gamma_{k,l}}{\sqrt{2\mu\omega_{k,l}}} \exp\left(\frac{2\pi i}{N}(ki + lj) - i\omega_{k,l}t\right) + \frac{1}{N^2} \sum_{k,l=1}^N \frac{\gamma_{k,l}^*}{\sqrt{2\mu\omega_{k,l}}} \exp\left(-\frac{2\pi i}{N}(ki + lj) + i\omega_{k,l}t\right). \quad (2.7)$$

We have used the freedom to (re)define the Fourier coefficients $\gamma_{k,l}$ in order to introduce prefactors $1/\sqrt{2\mu\omega_{k,l}}N^2$ which will later lead to a canonical normalisation of all harmonic oscillators.^{4 5} The complex conjugate coefficients $\gamma_{k,l}^*$

³In fact, we can ignore the contribution of the spring extension along the x and y directions of the lattice. This is because a three-dimensional distance is given by $d^2 = d_x^2 + d_y^2 + d_z^2$ and therefore the extensions in the three spatial directions decouple in the spring potential. Moreover, d_x and d_y describe the distance of two lattice sites. In our model we assume transverse excitations only, hence d_x and d_y are constant leading to an irrelevant constant shift of the potential. Even if d_x and d_y are considered dynamical, the results for the transverse excitations would not change because longitudinal and transverse excitations decouple exactly.

⁴We would lose nothing by using different prefactors, only the resulting expressions will have a (slightly) more complicated form.

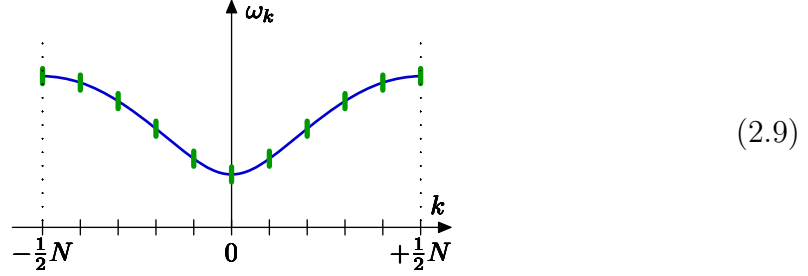
⁵Admittedly, this is violating causality. A priori, there is little motivation to put precisely these factors here right now. In practice, one would start with trivial factors, go through the calculation and then redefine the coefficients. Out of lazyness, nostalgia or a different interpretation of being pedagogical, one might stick to the old coefficients for part of the calculation, state the translation to new coefficients, and use the latter for the rest. Tampering with causality may, however, improve the readability even though it leaves the reader somewhat puzzled about the starting point. Thank you for your attention and back to the subject.

in the second term ensure reality of $q_{i,j}$. Note that $\gamma_{k,l}^*$ represents a Fourier mode of opposite momentum and energy as compared to $\gamma_{k,l}$!

The equation of motion translates to a dispersion relation:

$$\mu\omega_{k,l}^2 = \lambda + 4\kappa \sin^2 \frac{\pi k}{N} + 4\kappa \sin^2 \frac{\pi l}{N}. \quad (2.8)$$

Along one axis it takes the following shape:



Hamiltonian Formulation. First, define canonical momentum variables

$$p_{i,j} := \frac{\partial L}{\partial \dot{q}_{i,j}} = \mu \dot{q}_{i,j}. \quad (2.10)$$

Then derive the Hamiltonian function as the Legendre transform of L

$$H = H_{\text{kin}} + V_{\text{lat}} + V_{\text{rest}} \quad \text{with} \quad H_{\text{kin}} = \frac{1}{2\mu} \sum_{i,j=1}^N p_{i,j}^2. \quad (2.11)$$

Finally, define canonical Poisson brackets

$$\{f, g\} := \sum_{i,j=1}^N \left(\frac{\partial f}{\partial q_{i,j}} \frac{\partial g}{\partial p_{i,j}} - \frac{\partial f}{\partial p_{i,j}} \frac{\partial g}{\partial q_{i,j}} \right). \quad (2.12)$$

In other words $\{q_{i,j}, p_{k,l}\} = \delta_{i,k} \delta_{j,l}$ and $\{q, q\} = \{p, p\} = 0$.

Fourier Modes. Next we introduce new complex variables $c_{k,l}$ as the discrete Fourier transformation of the position and momentum variables $q_{i,j}$ and $p_{i,j}$

$$c_{k,l} = \frac{1}{\sqrt{2\mu\omega_{k,l}}} \sum_{i,j=1}^N \exp\left(-\frac{2\pi i}{N} (ki + lj)\right) (\mu\omega_{k,l} q_{i,j} + i p_{i,j}). \quad (2.13)$$

The transformed Hamiltonian becomes very simple

$$H = \frac{1}{N^2} \sum_{k,l=1}^N \omega_{k,l} c_{k,l}^* c_{k,l}. \quad (2.14)$$

The Poisson brackets for new variables are simple, too

$$\begin{aligned}\{c_{i,j}, c_{k,l}^*\} &= -i N^2 \delta_{i,k} \delta_{j,l}, \\ \{c_{i,j}, c_{k,l}\} &= \{c_{i,j}^*, c_{k,l}^*\} = 0.\end{aligned}\tag{2.15}$$

One can convince oneself that the canonical equations of motion apply

$$\begin{aligned}\dot{c}_{k,l} &= -\{H, c_{k,l}\} = -i\omega_{k,l} c_{k,l}, \\ \dot{c}_{k,l}^* &= -\{H, c_{k,l}^*\} = +i\omega_{k,l} c_{k,l}^*.\end{aligned}\tag{2.16}$$

They are solved by the above solution in the Lagrangian framework

$$c_{k,l}(t) = \gamma_{k,l} \exp(-i\omega_{k,l}t), \quad c_{k,l}^*(t) = \gamma_{k,l}^* \exp(+i\omega_{k,l}t).\tag{2.17}$$

2.2 Continuum Limit

Now turn this spring lattice into a smooth field φ :

- send the number of sites $N \rightarrow \infty$;
- consider a box of size L in all directions; lattice separation $r = L/N \rightarrow 0$;
- positions $x = ir = iL/N$;
- field $q_{i,\dots} = \varphi(\vec{x})$;
- generalise to d spatial dimensions, e.g. $d = 1, 2, 3$.

Two useful rules in applying this limit are as follows:

$$\sum_{i=1}^N \rightarrow \frac{1}{r} \int dx, \quad q_i - q_{i-1} \rightarrow r(\partial\varphi)\tag{2.18}$$

Lagrangian Formulation. Substitute the above rules in the Lagrangian function

$$L \rightarrow \int d\vec{x}^d \left(\frac{\mu}{2r^d} \dot{\varphi}^2 - \frac{\kappa}{2r^{d-2}} (\vec{\partial}\varphi)^2 - \frac{\lambda}{2r^d} \varphi^2 \right).\tag{2.19}$$

This expression diverges as $r \rightarrow 0$, but we can rescale the parameters to compensate the divergences. Suitable rescalings are given by^{6 7}

$$\mu = r^d \bar{\mu}, \quad \kappa = r^{d-2} \bar{\kappa}, \quad \lambda = r^d \bar{\lambda}.\tag{2.20}$$

⁶A crucial step in formulating a continuum limit is to choose the limiting behaviour of constants: Typically there are two dominant domains. If the scaling is too weak (e.g. $\lambda \sim r^a$ with $a < d$), the resulting limit is inconsistent (e.g. infinities remain). If the scaling is too strong (e.g. $\lambda \sim r^a$ with $a > d$), some terms are discarded from the resulting model (e.g. $m = 0$). However, the most interesting physical models (e.g. with $0 < m < \infty$) can be found at the interface of the two domains where the scaling is fine-tuned (e.g. $\lambda \sim r^a$ with $a = d$). Note that the location of this interface in parameter space is defined by the discrete model itself, but often some freely tunable parameters remain on the interface.

⁷Such rescalings are closely linked to what we will later understand as renormalisation. In fact, the lattice serves as a UV regulator for field theory, and this is a very basic example of renormalisation.

The parameters μ, κ, λ become densities $\bar{\mu}, \bar{\kappa}, \bar{\lambda}$, and the Lagrangian functional takes the form

$$L[\varphi, \dot{\varphi}](t) = \int d\vec{x}^d \left(\frac{1}{2} \bar{\mu} \dot{\varphi}^2 - \frac{1}{2} \bar{\kappa} (\vec{\partial} \varphi)^2 - \frac{1}{2} \bar{\lambda} \varphi^2 \right). \quad (2.21)$$

We can furthermore rescale the field $\varphi = \bar{\kappa}^{-1/2} \phi$ to remove the coefficient in front of the momentum term and make the field canonically normalised

$$L[\phi, \dot{\phi}](t) = \int d\vec{x}^d \left(\frac{1}{2} \bar{\mu} \bar{\kappa}^{-1} \dot{\phi}^2 - \frac{1}{2} (\vec{\partial} \phi)^2 - \frac{1}{2} \bar{\lambda} \bar{\kappa}^{-1} \phi^2 \right). \quad (2.22)$$

From here, we can derive the equations of motion for the field: We start with the action functional $S[\phi]$

$$S[\phi] = \int dt d\vec{x}^d L[\phi](t) = \int dt d\vec{x}^d \mathcal{L}(\phi(\vec{x}, t), \partial_i \phi(\vec{x}, t), \dot{\phi}(\vec{x}, t)); \quad (2.23)$$

it is useful to express the (homogeneous) Lagrangian functional $L[\phi](t)$ through the Lagrangian density $\mathcal{L}(\phi, \vec{\partial} \phi, \dot{\phi})$ (the Lagrangian).

We vary the action functional (and discard boundary terms)

$$\delta S[\phi] = \int dt d\vec{x}^d \delta \phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial (\partial_i \phi)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) + \dots \stackrel{!}{=} 0. \quad (2.24)$$

We thus write the general Euler–Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial \phi}(\vec{x}, t) - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial (\partial_i \phi)}(\vec{x}, t) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x}, t) = 0. \quad (2.25)$$

In our case the result agrees with the continuum limit of the discrete equations of motion

$$-\bar{\mu} \bar{\kappa}^{-1} \ddot{\phi} + \vec{\partial}^2 \phi - \bar{\lambda} \bar{\kappa}^{-1} \phi = 0. \quad (2.26)$$

Now denote the speed of sound by c and the mass by m

$$\bar{\mu} \bar{\kappa}^{-1} = c^{-2} = 1, \quad \bar{\lambda} \bar{\kappa}^{-1} = m^2; \quad (2.27)$$

we discover the Klein–Gordon equation (set $c = 1$)

$$-c^{-2} \ddot{\phi} + \vec{\partial}^2 \phi - m^2 \phi = 0. \quad (2.28)$$

Plane Wave Solutions. Consider solutions on infinitely extended space and time. The homogeneous equation is solved by a Fourier transformation

$$\begin{aligned} \phi(\vec{x}, t) &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \alpha(\vec{p}) \exp(i\vec{p} \cdot \vec{x} - ie(\vec{p})t) \\ &+ \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \alpha^*(\vec{p}) \exp(-i\vec{p} \cdot \vec{x} + ie(\vec{p})t) \end{aligned} \quad (2.29)$$

with (positive) energy $e(\vec{p})$ on the mass shell (the energy $e(\vec{p})$ is often denoted as the angular velocity $\omega(\vec{p})$)

$$e(\vec{p}) = \sqrt{\vec{p}^2 + m^2}. \quad (2.30)$$

This agrees with the discrete solution when identifying momenta as

$$p = 2\pi k/L, \quad \sum_{k=1}^N \rightarrow \frac{L}{2\pi} \int dp, \quad \gamma_{k,\dots} = \frac{\alpha(\vec{p})}{\sqrt{2e(\vec{p})r^d}}. \quad (2.31)$$

Some remarks on factors and conventions:

- Fourier transforms on \mathbb{R} produce factors of 2π ; they must be put somewhere. The dominant convention in physics is to associate $(2\pi)^{-1}$ to every dp : $\mathring{d}p := dp/2\pi$. There are no factors of 2π associated to dx . There are no factors of 2π in the exponent.
- The combination $\mathring{d}\vec{p}^d/2e(\vec{p})$ is useful for relativistic covariance. This is also the reason for the conversion factor for $\gamma_{k,\dots}$ vs. $\alpha(\vec{p})$.

2.3 Relativistic Covariance

The Klein–Gordon equation is invariant under Lorentz and Poincaré symmetry. If we interpret c as the speed of light rather than the speed of sound, we obtain a relativistic theory.⁸

The Klein–Gordon equation can be written manifestly relativistically^{9 10}

$$-\partial^\mu \partial_\mu \phi + m^2 \phi = -\partial^2 \phi + m^2 \phi = 0. \quad (2.32)$$

Also the Lagrangian density and the action have a manifestly relativistic form with $D = d + 1$

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2, \quad S = \int dx^D \mathcal{L}. \quad (2.33)$$

To understand the relativistic behaviour of the general solution derived above, we take a look at the momentum space representation of the field

$$\phi(x) = \int \frac{dp^D}{(2\pi)^D} \exp(i p \cdot x) \phi(p) \quad (2.34)$$

and its corresponding equation of motion

$$(p^2 + m^2)\phi(p) = 0. \quad (2.35)$$

⁸There appears to be no deeper reason why the limiting model should be relativistic. However, as long as the lattice spacing is finite, there is a periodic band structure with Brillouin zones which clearly violates Poincaré symmetry.

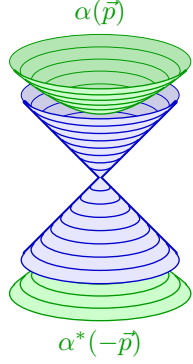
⁹ ∂^2 is often written as the D'Alembertian operator \square .

¹⁰Our signature of spacetime is $-+++!$

The solution of this equation of motion is thus a delta-function supported on the solution of the algebraic equation $p^2 + m^2 = 0$

$$\phi(p) = 2\pi\delta(p^2 + m^2)(\theta(p^0)\alpha(\vec{p}) + \theta(-p^0)\alpha^*(-\vec{p})). \quad (2.36)$$

The fields $\alpha(\vec{p})$ and $\alpha^*(\vec{p})$ define amplitudes on the forward/backward mass shells

$$\begin{aligned} e &= \pm e(\vec{p}), \\ e(\vec{p}) &:= \sqrt{\vec{p}^2 + m^2}. \end{aligned} \quad (2.37)$$


Note that the reality condition in position space translates to a non-local reality condition in momentum space

$$\phi(x)^* = \phi(x) \quad \Longleftrightarrow \quad \phi(p)^* = \phi(-p). \quad (2.38)$$

To investigate the relativistic behaviour of the solution, we perform the integration of a function $f(p)$ over the mass shell $p^2 + m^2 = 0$ with positive energy

$$\begin{aligned} & \int \frac{dp^D}{(2\pi)^D} 2\pi\delta(p^2 + m^2)\theta(p^0) f(p) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{de}{2\pi} 2\pi\delta(-e^2 + \vec{p}^2 + m^2)\theta(e) f(e, \vec{p}) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{de}{2\pi} 2\pi\delta(-e^2 + e(\vec{p})^2)\theta(e) f(e, \vec{p}) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{de}{2\pi} 2\pi\delta((e(\vec{p}) + e)(e(\vec{p}) - e))\theta(e) f(e, \vec{p}) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{de}{e(\vec{p}) + e} \delta(e - e(\vec{p})) f(e, \vec{p}) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{1}{2e(\vec{p})} f(e(\vec{p}), \vec{p}). \end{aligned} \quad (2.39)$$

Since we started from a manifestly covariant integral, the above measure on the positive mass shell is Poincaré covariant. This is precisely the measure used in the formulation of the general solution.

2.4 Hamiltonian Field Theory

Now that we have a nice relativistic formulation for the Klein–Gordon field $\phi(x)$, let us separate space from time.¹¹

Position Space. Define the momentum variables $\pi(x)$ (field) conjugate to the field $\phi(x)$

$$\pi(\vec{x}, t) = \frac{\delta L}{\delta \dot{\phi}(\vec{x})}(t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x}, t) = \dot{\phi}(\vec{x}, t). \quad (2.40)$$

Next, determine the Hamiltonian function

$$\begin{aligned} H[\phi, \pi] &= \int d\vec{x}^d \pi \dot{\phi} - L[\phi, \dot{\phi}] \\ &= \int d\vec{x}^d \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\partial} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right). \end{aligned} \quad (2.41)$$

This expression is not relativistically covariant, but it was not designed to be.¹²

Finally, define Poisson brackets for two phase space functionals f, g with $f = f[\phi, \pi]$

$$\{f, g\} = \int d\vec{z}^d \left(\frac{\delta f}{\delta \phi(\vec{z})} \frac{\delta g}{\delta \pi(\vec{z})} - \frac{\delta f}{\delta \pi(\vec{z})} \frac{\delta g}{\delta \phi(\vec{z})} \right). \quad (2.42)$$

The Poisson brackets of the fundamental fields yield delta-functions¹³

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \int d\vec{z}^d \delta^d(\vec{x} - \vec{z}) \delta^d(\vec{y} - \vec{z}) = \delta^d(\vec{x} - \vec{z}). \quad (2.43)$$

Momentum Space. Now introduce momentum modes by a suitable Fourier transformation¹⁴

$$\begin{aligned} a(\vec{p}) &= \int d\vec{x}^d \exp(-i\vec{p} \cdot \vec{x}) (e(\vec{p}) \phi(\vec{x}) + i\pi(\vec{x})), \\ a^*(\vec{p}) &= \int d\vec{x}^d \exp(+i\vec{p} \cdot \vec{x}) (e(\vec{p}) \phi(\vec{x}) - i\pi(\vec{x})), \end{aligned} \quad (2.44)$$

¹¹The Hamiltonian formalism breaks manifest relativistic invariance, but the physics remains (secretly) relativistic.

¹²The Hamiltonian as a phase space function governs translation in time. The notion of a specific time is not relativistically covariant.

¹³Use the formula for variation $\delta \phi(\vec{z}) / \delta \phi(\vec{x}) = \delta^d(\vec{z} - \vec{x})$.

¹⁴We have additional factors compared to some literature.

and the inverse Fourier transformation

$$\begin{aligned}
\phi(\vec{x}) &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}) \\
&\quad + \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a^*(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}), \\
\pi(\vec{x}) &= -\frac{i}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}) \\
&\quad + \frac{i}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a^*(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}).
\end{aligned} \tag{2.45}$$

We compute the Poisson brackets for the Fourier modes^{15 16}

$$\{a(\vec{p}), a^*(\vec{q})\} = -i 2e(\vec{p}) (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{2.46}$$

In other words,

$$\{f, g\} = -i(2\pi)^d \int d\vec{p}^d 2e(\vec{p}) \left(\frac{\delta f}{\delta a(\vec{p})} \frac{\delta g}{\delta a^*(\vec{p})} - \frac{\delta f}{\delta a^*(\vec{p})} \frac{\delta g}{\delta a(\vec{p})} \right). \tag{2.47}$$

The Hamiltonian translates to¹⁷

$$H = \frac{1}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a^*(\vec{p}) a(\vec{p}) = \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} e(\vec{p}) a^*(\vec{p}) a(\vec{p}). \tag{2.48}$$

Hence the equations of motion for momentum space fields take the simple form of a harmonic oscillator

$$\begin{aligned}
\dot{a}(\vec{p}) &= -\{H, a(\vec{p})\} = -ie(\vec{p}) a(\vec{p}), \\
\dot{a}^*(\vec{p}) &= -\{H, a^*(\vec{p})\} = +ie(\vec{p}) a^*(\vec{p}).
\end{aligned} \tag{2.49}$$

There is one harmonic oscillator for each momentum \vec{p} . The solution reads

$$\begin{aligned}
a(\vec{p}, t) &= \alpha(\vec{p}) \exp(-ie(\vec{p})t), \\
a^*(\vec{p}, t) &= \alpha^*(\vec{p}) \exp(+ie(\vec{p})t).
\end{aligned} \tag{2.50}$$

¹⁵There is a conventional factor of 2π for the delta-function in momentum space complementary to the factor of $1/2\pi$ for the momentum space measure.

¹⁶The factor $2e(\vec{p})$ is the appropriate relativistic measure for the mass shell complementary to the factor of $1/2e$ for the relativistic measure on the mass shell.

¹⁷The second form highlights the relativistic covariance of the mass shell measure as well as the fact that the Hamiltonian measures the integral of energy $e(\vec{p})$.

3 Scalar Field Quantisation

We can now go ahead and try to quantise the classical scalar field using the canonical procedure described before. We will encounter some infinities, and discuss how to deal with them. Then we shall investigate a few basic objects in quantum field theory.

3.1 Quantisation

We start with the Hamiltonian formulation of the scalar field discussed earlier.

Equal-Time Commutators. Phase space consists of the field $\phi(\vec{x})$ and the conjugate momentum $\pi(\vec{x})$ with Poisson bracket¹

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \delta^d(\vec{x} - \vec{y}). \quad (3.1)$$

Hence the canonical quantisation implies operators $\hat{\phi}(\vec{x})$ and $\hat{\pi}(\vec{x})$

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^d(\vec{x} - \vec{y}). \quad (3.2)$$

Note that ϕ and π are now operator-valued fields (rather: distributions).²

Field States. Next we have to define some states. Straight application of quantum mechanics would lead to a state $|\phi\rangle$ for every field configuration $\phi(\vec{x})$ such that

$$\hat{\phi}(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle, \quad \hat{\pi}(\vec{x})|\phi\rangle = i\frac{\delta}{\delta\phi(\vec{x})}|\phi\rangle. \quad (3.3)$$

This can be done formally, but it is not very convenient. For example, the ground state wave functional takes the following form with a suitable function Ω

$$|0\rangle = \int D\phi \exp\left(-\frac{1}{2} \int d\vec{x}^d d\vec{y}^d \Omega(\vec{x}, \vec{y}) \phi(\vec{x}) \phi(\vec{y})\right) |\phi\rangle. \quad (3.4)$$

This is to be compared to the wave function of a harmonic oscillator

$$|0\rangle = \int d\vec{x}^d \exp\left(-\frac{1}{2} \sum_{i,j} \omega_{ij} x^i x^j\right) |\vec{x}\rangle. \quad (3.5)$$

¹At the moment, the fields are defined on a common time slice t , e.g. $t = 0$. Later we discuss unequal times.

²The delta-function is a distribution, also the fields should be considered distributions. Distributions are linear maps from test functions to numbers (or operators in this case). In physics, we write them as integrals with a distributional kernel (e.g. the delta-function). Sometimes we also perform illegal operations (e.g. evaluate delta-function $\delta(x)$ at $x = 0$).

Momentum Space. The classical field is a bunch of coupled harmonic oscillators, let us diagonalise them and use creation and annihilation operators.

We go to momentum space, and pick a and a^\dagger appropriately, using the same transformation as above³

$$\begin{aligned}
a(\vec{p}) &= \int d\vec{x}^d \exp(-i\vec{p}\cdot\vec{x}) (e(\vec{p})\phi(\vec{x}) + i\pi(\vec{x})), \\
\phi(\vec{x}) &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}) \\
&\quad + \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a^\dagger(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}), \\
\pi(\vec{x}) &= -\frac{i}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}) \\
&\quad + \frac{i}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a^\dagger(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}).
\end{aligned} \tag{3.6}$$

The canonical commutation relations in momentum space read

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p}) (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{3.7}$$

We substitute these fields into the Hamiltonian of the scalar field paying attention to ordering

$$\begin{aligned}
H &= \int d\vec{x}^d \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\partial}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \\
&= \frac{1}{4} \int \frac{d\vec{p}^d}{(2\pi)^d} (a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})) \\
&= \frac{1}{4} \int \frac{d\vec{p}^d}{(2\pi)^d} (2a^\dagger(\vec{p})a(\vec{p}) + [a(\vec{p}), a^\dagger(\vec{p})]) \\
&= \frac{1}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2} \int d\vec{p}^d e(\vec{p}) \delta^d(\vec{p} - \vec{p}).
\end{aligned} \tag{3.8}$$

Vacuum Energy. We introduce a vacuum state $|0\rangle$ annihilated by all $a(\vec{p})$. There are two problems with the vacuum energy $H|0\rangle = E_0|0\rangle$ where

$$E_0 = \varepsilon_0 \delta^d(\vec{p} - \vec{p}), \quad \varepsilon_0 = \frac{1}{2} \int d\vec{p}^d e(\vec{p}). \tag{3.9}$$

Namely:

- evaluation of the delta-function at the origin, $\delta^d(\vec{p} - \vec{p}) = \delta^d(0)$, is ill-defined;
- the integral $\varepsilon_0 = \frac{1}{2} \int d\vec{p}^d e(\vec{p})$ diverges.

These are self-made problems:

³We will henceforth drop the indication of quantum operators for the fields $\phi(\vec{x})$, $\pi(\vec{x})$.

- We consider an infinite volume. It is not reasonable to expect a finite overall energy. This is an IR problem! The delta-function has units of volume. If one traces its appearance carefully, it turns out to measure the volume of the system $\delta^d(\vec{p} - \vec{p}) \sim V \rightarrow \infty$. We should consider the energy density ε_0 instead!
- The integral $\varepsilon_0 = \frac{1}{2} \int_0^P d\vec{p}^d e(\vec{p})$ represents the vacuum energy density. There are infinitely many oscillators per volume element, and it is not reasonable to expect a finite energy density. This is a UV problem! We can introduce a momentum cutoff $|\vec{p}| < P$ to obtain a finite result $\frac{1}{2} \int_0^P d\vec{p}^d e(\vec{p}) \sim P^{d+1} \rightarrow \infty$ which diverges as the cutoff is sent to infinity.

The finite lattice has a similar effect to regularise the IR and the UV. There are many other ways to avoid the infinities that arise in a QFT.

To avoid IR infinities in QFT: One may put the system in a finite box. This is possible but inconvenient, e.g. one would have to deal with non-trivial boundary conditions. One would rather work with an infinite system and cope with the arising IR divergences in a different manner. E.g. one might consider the energy density instead of the overall energy.

To avoid UV infinities in QFT: By definition we want a *field* theory, not a discrete model. The intuition from physics is that one should be able to approximate a field by a discrete model (or vice versa). The procedure involves several steps:

1. regularisation; there are several options: impose a momentum cut-off, consider a lattice, use other tricks without physical motivation, ...
2. renormalisation: absorb the terms that would otherwise diverge.
3. remove regularisation: obtain finite results.

In our case we simply drop E_0 . We can do it because:

- There is no meaning to an absolute vacuum energy. It is only a philosophical or religious issue.
- We may add any constant to the Hamiltonian to compensate E_0 (renormalisation).
- This makes no observable difference in any physical process.
- Here we keep in mind that, later on, infinities may also lead to interesting effects.

Renormalised Hamiltonian H_{ren}

$$H_{\text{ren}} := H - E_0 = \frac{1}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} a^\dagger(\vec{p}) a(\vec{p}). \quad (3.10)$$

This result is nice, the vacuum has zero energy. Note that this result or definition is essential for Poincaré invariance.

Normal Ordering. The ordering of variables in the classical Hamiltonian H plays no role. In quantum theory it does! It is responsible for the vacuum energy E_0 .

How to map some classical observable O to a quantum operator?

A possible map is normal ordering $N(O)$ which is defined as follows:

- express O in terms of a and $a^* \rightarrow a^\dagger$;
- write all a^\dagger 's to the left of all a 's.

Here, the renormalised Hamiltonian is the normal ordering of H

$$H_{\text{ren}} = N(H). \quad (3.11)$$

There are other ordering prescriptions for operators:

- Normal ordering depends on the choice of the vacuum state; there may be other normal-ordering prescriptions associated to different “vacuum” states.
- There are other useful ordering prescriptions which we will encounter, e.g. time ordering, symmetric ordering.
- Note that one cannot consistently lift classical to quantum observables while respecting their Poisson brackets lifted to commutators in all cases.

3.2 Fock Space

We already have:

- a collection of HO's operators $a(\vec{p}), a^\dagger(\vec{p})$ labelled by the momentum \vec{p} ,
- a vacuum state $|0\rangle$ with energy $E = 0$.

We can now discuss the other related states.

Single-Particle States. We can excite the vacuum

$$|\vec{p}\rangle := a^\dagger(\vec{p})|0\rangle. \quad (3.12)$$

This is an energy eigenstate with energy

$$E = +e(\vec{p}). \quad (3.13)$$

It is precisely the relativistic energy of a particle with momentum \vec{p} and mass m .

Negative-Energy Solutions. Please note:

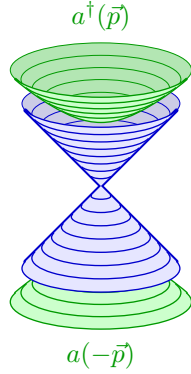
- The energy of the above state is positive definite $E > 0$.
- The state $a(\vec{p})|0\rangle$ would have negative energy. Gladly, it does not exist by construction.

Our earlier problem of negative-energy particles is solved!

- $a^\dagger(\vec{p})$ creates a particle of momentum \vec{p} and positive energy $+e(\vec{p})$ from the vacuum. $a^\dagger(\vec{p})$ is the particle creation operator.
- $a(\vec{p})$ removes a particle of momentum \vec{p} and thus removes the positive energy $+e(\vec{p})$ from the state.

$$\begin{aligned} a(\vec{p})|\vec{q}\rangle &= a(\vec{p})a^\dagger(\vec{q})|0\rangle = [a(\vec{p}), a^\dagger(\vec{q})]|0\rangle \\ &= 2e(\vec{p})(2\pi)^d \delta^d(\vec{p} - \vec{q})|0\rangle. \end{aligned} \quad (3.14)$$

$a(\vec{p})$ is the particle annihilation operator.⁴



$$(3.15)$$

The interpretation of the position space operators is not as straight-forward: $\phi(\vec{x}) = \phi^\dagger(\vec{x})$ as well as $\pi(\vec{x}) = \pi^\dagger(\vec{x})$ *either create or annihilate* a particle at position \vec{x} . The result of application of this operator to a state is typically a superposition of states with different particle numbers.

Normalisation. Let us declare a proper normalisation for the vacuum state

$$\langle 0|0\rangle = 1. \quad (3.16)$$

The normalisation of a single-particle state follows:

$$\langle \vec{p}|\vec{p}\rangle = 2e(\vec{p}) (2\pi)^d \delta^d(\vec{p} - \vec{p}) = \infty. \quad (3.17)$$

This is a known problem from quantum mechanics: plane-wave states have an infinite extent; they are smeared over all space. This is an unphysical assumption. Recall that $\delta^d(\vec{p} - \vec{p})$ typically represents some volume of space.

Consider instead a peaked wave packet state $|\psi\rangle$ defined by the test function $\psi(\vec{p})$

$$|\psi\rangle := \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{\psi(\vec{p})}{2e(\vec{p})} |\vec{p}\rangle. \quad (3.18)$$

For a suitable test function $\psi(\vec{p})$ this state has a finite normalisation

$$\langle \psi|\psi\rangle = \int \frac{d\vec{p}^d}{(2\pi)^d} \frac{|\psi(\vec{p})|^2}{2e(\vec{p})}. \quad (3.19)$$

Importantly, the normalisation is positive definite, which overcomes one of the difficulties of relativistic quantum mechanics.

⁴ $a(\vec{p})$ is not the generator of an anti-particle (as sometimes claimed). It has negative energy while anti-particles (like particles) have positive energy. For the real scalar, the particle is its own anti-particle.

Multi-Particle States and Fock Space. Now we excite more than one harmonic oscillator

$$|\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n\rangle := a^\dagger(\vec{p}_1) a^\dagger(\vec{p}_2) \cdots a^\dagger(\vec{p}_n) |0\rangle. \quad (3.20)$$

The energy is given by the sum of particle energies

$$E = \sum_{k=1}^n e(\vec{p}_k). \quad (3.21)$$

To arrive at this result, use the elementary commutator

$$[H, a^\dagger(\vec{p})] = e(\vec{p}) a^\dagger(\vec{p}) \quad (3.22)$$

to show

$$\begin{aligned} H|n\rangle &= H a_1^\dagger \dots a_n^\dagger |0\rangle \\ &= [H, a_1^\dagger] a_2^\dagger \dots a_n^\dagger |0\rangle + \dots + a_1^\dagger \dots a_{n-1}^\dagger [H, a_n^\dagger] |0\rangle \\ &= (e_1 + \dots + e_n) a_1^\dagger a_2^\dagger \dots a_n^\dagger |0\rangle \\ &= E|n\rangle. \end{aligned} \quad (3.23)$$

Moreover all particles are freely interchangeable

$$|\dots, \vec{p}, \vec{q}, \dots\rangle = |\dots, \vec{q}, \vec{p}, \dots\rangle \quad (3.24)$$

because creation operators commute

$$[a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0. \quad (3.25)$$

This corresponds to Bose statistics for indistinguishable particles: In QFT the wave function is automatically totally symmetric under particle exchange.

A generic QFT state (based on the vacuum $|0\rangle$) is a linear combination of k -particle states with k not fixed. This vector space is called Fock space. It is the direct sum

$$\mathbb{V}_{\text{Fock}} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3 \oplus \dots, \quad \mathbb{V}_n = (\mathbb{V}_1)^{\otimes n} \quad (3.26)$$

of n -particle spaces \mathbb{V}_n where

- $\mathbb{V}_0 = \mathbb{C}$ merely contains the vacuum state $|0\rangle$;
- $\mathbb{V}_1 = \mathbb{V}_{\text{particle}}$ is the space of single particle states $|\vec{p}\rangle$ with positive energy;
- \mathbb{V}_n is the symmetric tensor product of n copies of \mathbb{V}_1 .

To understand Fock space better, consider non-relativistic physics:

- In practice, the amount of available energy is bounded from above. It is much smaller than the particle rest mass $m = e(0)$.
- The relevant part of Fock space has n bounded from above. For example: \mathbb{V}_1 , \mathbb{V}_2 or $\mathbb{V}_1 \oplus \mathbb{V}_2$.
- Multiple-particle quantum mechanics is a low-energy limit of quantum field theory. It becomes quantum field theory when the number of particles is unbounded.

Charges and Conservation Laws. The Hamiltonian H measures the total energy E .

There is also a set of operators to measure total momentum \vec{P}

$$\vec{P} := \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \vec{p} a^\dagger(\vec{p}) a(\vec{p}). \quad (3.27)$$

with eigenvalue $\vec{P} = \sum_{k=1}^n \vec{p}_k$ on state $|\vec{p}_1, \dots, \vec{p}_n\rangle$. The vacuum state carries no momentum. We can combine this into a relativistic vector $P^\mu = (H, \vec{P})$ of operators

$$P^\mu := \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} p^\mu(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}), \quad p^\mu(\vec{p}) := (e(\vec{p}), \vec{p}). \quad (3.28)$$

Another useful operator is the particle number operator

$$N := \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a^\dagger(\vec{p}) a(\vec{p}). \quad (3.29)$$

It measures the number of particles n in a state $|\vec{p}_1, \dots, \vec{p}_n\rangle$

$$N \mathbb{V}_n = n \mathbb{V}_n. \quad (3.30)$$

The relativistic momentum vector and the number operator are conserved under time evolution due to the relationships

$$[H, P_\mu] = [H, N] = 0. \quad (3.31)$$

Equivalently, one can say that these operators do not carry energy in the sense that acting on any state does not change its energy. Equivalently, the operators carry no momentum

$$[P^\mu, P^\nu] = [P^\mu, N] = 0. \quad (3.32)$$

In fact, there are plenty of similar conservation laws. Any operators composed from number density operators commute

$$n(\vec{p}) := a^\dagger(\vec{p}) a(\vec{p}), \quad [n(\vec{p}), n(\vec{q})] = 0. \quad (3.33)$$

Hence such operators are conserved, they carry no momentum and no particle number:

$$[H, n(\vec{p})] = [P_\mu, n(\vec{p})] = [N, n(\vec{p})] = 0. \quad (3.34)$$

In a free theory there are infinitely many conservation laws.⁵

⁵I should not say this: the free theory is trivial.

3.3 Complex Scalar Field

Let us discuss a slightly more elaborate case of the scalar field, the complex scalar, where we first encounter anti-particles.

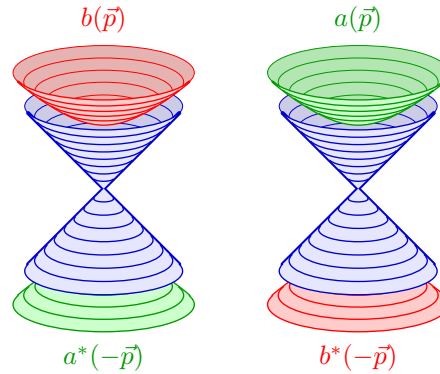
The complex scalar field $\phi(x)$ has the Lagrangian^{6 7}

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi = -|\partial\phi|^2 - m^2 |\phi|^2. \quad (3.35)$$

The equation of motion associated to the above Lagrangian is the very same Klein–Gordon equation. However, now complex solutions $\phi(x)$ are allowed. The general solution is given by

$$\begin{aligned} \phi(x) &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} b(\vec{p}) \exp(+ip \cdot x) \\ &\quad + \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a^*(\vec{p}) \exp(-ip \cdot x), \\ \phi^*(x) &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} a(\vec{p}) \exp(+ip \cdot x) \\ &\quad + \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} b^*(\vec{p}) \exp(-ip \cdot x). \end{aligned} \quad (3.36)$$

Note the skewed appearance of the coefficients a and b . For $a \neq b$ we have $\phi \neq \phi^*$ while $a = b$ implies a real field $\phi = \phi^*$.



$$(3.37)$$

For the conjugate momentum we obtain⁸

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}. \quad (3.38)$$

⁶The prefactor $1/2$ of the real scalar field (ϕ^2) is now absent. It is a convenient symmetry factor. Here the appropriate symmetry factor is 1 because $\phi^* \neq \phi$. More on symmetry factors later.

⁷One can also set $\phi(x) = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^*(x) = (\phi_1 - i\phi_2)/\sqrt{2}$ and obtain two independent scalar fields with *equal mass*.

⁸This implies that π is canonically conjugate to ϕ^* . One may just as well define $\pi = \partial \mathcal{L} / \partial \dot{\phi} = \dot{\phi}^*$ in which case π is conjugate to ϕ . This is a matter of convention, and it makes no difference if applied consistently.

In the quantum theory we then impose the canonical commutators

$$[\phi(\vec{x}), \pi^\dagger(\vec{y})] = [\phi^\dagger(\vec{x}), \pi(\vec{y})] = i\delta^d(\vec{x} - \vec{y}). \quad (3.39)$$

Equivalently, we quantise the coefficients $a(\vec{p}), b(\vec{p})$ to operators. The non-trivial commutation relations of these operators read:

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = 2e(\vec{p}) (2\pi)^d \delta^d(\vec{p} - \vec{q}). \quad (3.40)$$

The quantum Hamiltonian takes the form

$$H_{\text{ren}} := \frac{1}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} (a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})). \quad (3.41)$$

The complex scalar carries a charge, $+1$ for ϕ and -1 for ϕ^\dagger .

- The operator a^\dagger creates a particle with charge $+1$.
- The operator b has negative energy, it should remove a particle. The operator carries the same charge as a^\dagger , hence the corresponding particle should have charge -1 .

There are two types of particles: the particle and the anti-particle. They have opposite charges, but equal masses and all positive energies. In conclusion:

- a^\dagger creates a particle,
- b annihilates an anti-particle,
- b^\dagger creates an anti-particle,
- a annihilates a particle,
- the vacuum is annihilated by a 's and b 's.

3.4 Correlators

We have now quantised the scalar field. The states have an adjustable number of indistinguishable particles with definite momenta. Now what? Let us consider particle propagation in space and time.

Schrödinger Picture. For particle propagation we consider the following steps:

- create a particle at $x^\mu = (t, \vec{x})$;
- let the state evolve for some time $s - t$;
- measure the particle at $y^\mu = (s, \vec{y})$.

We could use $\phi(\vec{x})$ or $\pi(\vec{x})$ to create the particle from the vacuum.⁹ Let us use ϕ because $\pi = \dot{\phi}$ can be obtained from its time derivative.

In the Schrödinger picture, the states evolve in time

$$i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle. \quad (3.42)$$

⁹These operators also annihilate particles, but evidently, there are none present in the vacuum.

We can solve this equation as¹⁰

$$|\Psi(s)\rangle = \exp(-iH(s-t))|\Psi(t)\rangle. \quad (3.43)$$

We substitute the field ϕ in terms of creation and annihilation operators, and altogether we find for the correlator

$$\begin{aligned} \Delta_+(y, x) &:= i\langle 0|\phi(\vec{y})\exp(-i(s-t)H)\phi(\vec{x})|0\rangle \\ &= i\int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \exp(i\vec{p}\cdot(\vec{y}-\vec{x}) - i(s-t)e(\vec{p})) \\ &= i\int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \exp(ip\cdot(y-x)), \end{aligned} \quad (3.44)$$

where $p^0 = e(\vec{p})$ is the relativistic energy of the particle. In this derivation, space and time take different roles. Nevertheless, the final answer is manifestly Poincaré covariant.

Heisenberg Picture. Before we continue to investigate this function, let us go to a different formulation of quantum mechanics, which makes the relativistic properties more manifest: the Heisenberg picture. We translate the time dependence of the state to time dependence of operators¹¹

$$F_H(t) := \exp(+iH(t-t_0))F_S(t)\exp(-iH(t-t_0)), \quad (3.45)$$

where t_0 is the reference time slice on which quantum states are defined, commonly $t_0 = 0$. States are therefore time-independent in the Heisenberg picture. Let us compare the application of an operator to a state

$$\begin{aligned} F_H(t)|\Psi(t_0)\rangle &= \exp(iH(t-t_0))F_S(t)\exp(-iH(t-t_0))|\Psi(t_0)\rangle \\ &= \exp(iH(t-t_0))F_S(t)|\Psi(t)\rangle. \end{aligned} \quad (3.46)$$

The difference between the applications is the factor $\exp(iH(t-t_0))$ which is required to translate between the two time slices but which eventually drops out in correlators.

The field operator $\phi(x) := \phi_H(t, \vec{x})$ in the Heisenberg picture recovers a dependence on time compared to the field operator $\phi(\vec{x}) := \phi_S(\vec{x}) = \phi_H(0, \vec{x})$

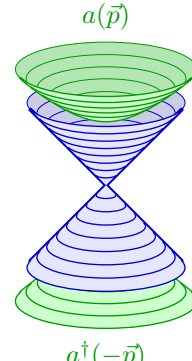
$$\begin{aligned} \phi(x) &= \exp(+iHt)\phi(\vec{x})\exp(-iHt) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} (e^{+ip\cdot x} a(\vec{p}) + e^{-ip\cdot x} a^\dagger(\vec{p})) \end{aligned} \quad (3.47)$$

again with $p^0 = e(\vec{p})$ implied. Some features:

¹⁰The Hamiltonian H is time-independent, hence time translation is governed simply by $\exp(-iH\Delta t)$. Later we shall encounter a more difficult situation.

¹¹In most relevant cases, the operator $F_S(t) = F_S$ has no explicit time dependence. The implicit time dependence from applying the operator F_S to a time-dependent state is made explicit in $F_H(t)$.

- it has complete spacetime dependence;
- it is manifestly relativistic;
- there is no need to consider $\pi = \dot{\phi}$ separately;
- it obeys the Klein–Gordon equation $(\partial^2 - m^2)\phi = 0$;
- it has the same form as the solution of the Euler–Lagrange equations.
- in momentum space it takes the form¹²

$$\begin{aligned}\phi(x) &= \int \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} \phi(p), \\ \phi(p) &= \delta(p^2 + m^2) \\ &\quad \cdot (\theta(p^0) a(\vec{p}) + \theta(-p^0) a^\dagger(-\vec{p})).\end{aligned}\tag{3.48}$$


In the Heisenberg picture the correlator takes the following form¹³

$$\begin{aligned}\Delta_+(y, x) &= i \langle 0 | \phi(\vec{y}) \exp(iH(t - s)) \phi(\vec{x}) | 0 \rangle \\ &= i \langle 0 | \phi(y) \phi(x) | 0 \rangle.\end{aligned}\tag{3.49}$$

This yields the same result, but by means of a more immediate and relativistic derivation.

Correlator. Let us discuss the correlation function

$$\Delta_+(y, x) = i \int \frac{d^d \vec{p}}{(2\pi)^d 2e(\vec{p})} \exp(ip \cdot (y - x)).\tag{3.50}$$

We know the function in momentum space from the derivation of the integration measure over the mass shell

$$\begin{aligned}\Delta_+(y, x) &= \int \frac{d^{d+1} p}{(2\pi)^{d+1}} \Delta_+(p) \exp(ip \cdot (y - x)), \\ \Delta_+(p) &= 2\pi i \delta(p^2 + m^2) \theta(p^0).\end{aligned}\tag{3.51}$$

¹²It is curious to notice that the particle *creation* operator a^\dagger is formally the contribution with *negative* energy in the Fourier transform $\phi(x) = \int d^4 p (2\pi)^{-4} e^{ip \cdot x} \phi(p)$. This is because a^\dagger in $\phi(x)$ creates a particle at time x^0 , whereas a quantum mechanical state $|\Psi(t)\rangle$ describes the evolution of a particle (which has been created earlier). The latter is determined by the Schrödinger equation which defines the notion and sign of energy according to $|\Psi(t)\rangle \sim e^{-iet}$. This time evolution is appropriately *measured* by the particle *annihilation* operator a which therefore is attributed a *positive* energy e in the Fourier transform; correspondingly, the particle *creation* operator resides at *negative* energy.

¹³The vacuum state is the same in both pictures; it is not time dependent because the vacuum energy E_0 is (was defined to be) zero. Therefore $\exp(iHt)|0\rangle = |0\rangle$.

How about position space? Translational symmetry implies that only differences of positions can matter in agreement with the above momentum space representation

$$\Delta_+(y, x) = \Delta_+(y - x). \quad (3.52)$$

Furthermore, Lorentz invariance implies that the result can only depend on Lorentz invariant quantities. The only Lorentz invariant quantity that can be constructed from the vector x^μ is x^2 , hence the function can only depend on the single variable x^2

$$\Delta_+(x) = \Delta_+(x^2). \quad (3.53)$$

The mass is the only dimensionful constant at our disposal, and therefore dimensional analysis implies

$$\Delta_+(x^2) = m^{d-1} F(m^2 x^2), \quad (3.54)$$

where F is an numerical function. The above considerations hold locally, we should discuss three distinct regions of spacetime: future, past, elsewhere.¹⁴

The Klein–Gordon equation for F becomes

$$4rF''(r) + 2(d+1)F'(r) - F(r) = 0. \quad (3.55)$$

This is a differential equation for Bessel functions $J_\alpha(z)$.¹⁵ There are two solutions:

$$F_\pm(r) = r^{-(d-1)/4} J_{\pm(d-1)/2}(i\sqrt{r}). \quad (3.56)$$

First, consider the future with $x = (t, 0)$ where $t = \pm\|x\|$ with $\|x\| = \sqrt{-x^2}$.¹⁶ Substitute this into the above momentum space expression

$$\begin{aligned} \Delta_+(x) &= i \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} \exp(-ite(\vec{p})). \\ &= \frac{i \text{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^\infty \frac{dp p^{d-1}}{\sqrt{p^2 + m^2}} \exp(-it\sqrt{p^2 + m^2}). \\ &= \frac{i \text{Vol}(S^{d-1})}{2(2\pi)^d} \int_m^\infty de (e^2 - m^2)^{(d-2)/2} \exp(-ite) \\ &\sim e^{-imt} \quad \text{for } t \rightarrow \pm\infty. \end{aligned} \quad (3.57)$$

One can see that the function oscillates with a positive frequency m . The same conclusion holds for the past. This fixes the relevant linear combination of F_\pm , which is in fact the Hankel function H_α .¹⁷

$$\Delta_+(y - x) \sim \frac{m^{(d-1)/2}}{(y - x)^{(d-1)/2}} H_{(d-1)/2}(m\sqrt{-(y - x)^2}). \quad (3.58)$$

¹⁴These regions are disconnected in the sense that they are not related by orthochronous Lorentz transformations.

¹⁵Bessel functions are well-known solutions for spherical waves.

¹⁶For time-like separation one can go to such a frame with $x = (t, 0)$.

¹⁷The asymptotic behaviour e^{-imt} determines which of the two Hankel functions $H^{(1)}$ and $H^{(2)}$ applies to the past and the future.

For space-like separation, however, one finds the asymptotic behaviour

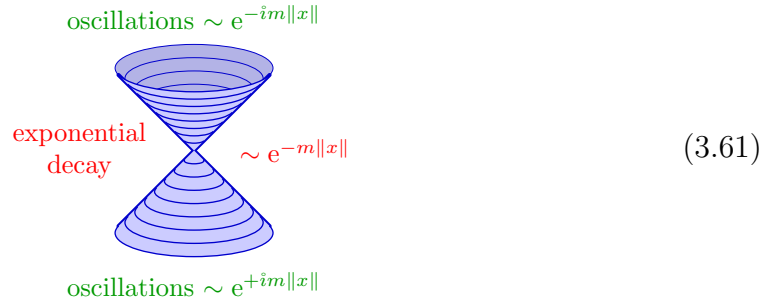
$$\Delta_+(x) \sim e^{-mr} \quad \text{for } r = \|x\| := \sqrt{x^2} \rightarrow \infty. \quad (3.59)$$

The result is non-zero, but it decays exponentially with range $1/m$. This behaviour fixes the linear combination of F_\pm which is a modified Bessel function K_α

$$\Delta_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} K_{(d-1)/2}(m\sqrt{(y-x)^2}). \quad (3.60)$$

This non-vanishing result is the same as in relativistic quantum mechanics.

Before we discuss causality, let us summarise in this figure:



Note that there are additional delta-function contributions for light-like separation in $\Delta^+(y, x)$ which we will not discuss here.

Unequal-Time Commutator. It is in fact acceptable to violate causality as long as this effect is never measured.

The correct question to ask is: Can one measurement at spacetime point x influence the other at spacetime point y ? To that end, consider the commutator

$$\Delta(y-x) := i[\phi(y), \phi(x)]. \quad (3.62)$$

We can relate it to the above correlators by inserting the relationship in between two vacua

$$\Delta(y-x) = i\langle 0 | [\phi(y), \phi(x)] | 0 \rangle = \Delta_+(y-x) - \Delta_+(x-y). \quad (3.63)$$

We obtain the following observations for space-like separations:

- Δ_+ is a symmetric function;
- the commutator Δ vanishes;
- $\phi(x)$ and $\phi(y)$ commute;
- this insight follows also from invariance and the equal-time commutator which is a delta-function at coincident points;
- causality is preserved!

Note: In the calculation, we observe a cancellation. The particle created at x and annihilated at y cancels against the particle created at y and annihilated at x ; one of these evidently travels backwards in time.

However, the commutator is non-trivial for time-like separations; one finds the Bessel functions of the first kind J_α

$$\Delta(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} J_{(d-1)/2}(m\sqrt{-(y-x)^2}). \quad (3.64)$$

This means that time-like separated measurements can indeed influence each other.

Finally, we can recover the equal-time commutators. For two fields ϕ the commutator follows from asymmetry of the integrand

$$\begin{aligned} [\phi(\vec{y}), \phi(\vec{x})] &= -i\Delta(0, \vec{y} - \vec{x}) \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} (e^{+i\vec{p}\cdot(\vec{y}-\vec{x})} - e^{-i\vec{p}\cdot(\vec{y}-\vec{x})}) \\ &= 0. \end{aligned} \quad (3.65)$$

The commutator between a field ϕ and its conjugate momentum π yields

$$\begin{aligned} [\phi(\vec{y}), \pi(\vec{x})] &= -i \frac{\partial}{\partial t} \Delta(s-t, \vec{y} - \vec{x}) \Big|_{s=t} \\ &= \frac{i}{2} \int \frac{d\vec{p}^d}{(2\pi)^d} (e^{+i\vec{p}\cdot(\vec{y}-\vec{x})} + e^{-i\vec{p}\cdot(\vec{y}-\vec{x})}) \\ &= i\delta^d(\vec{y} - \vec{x}) \end{aligned} \quad (3.66)$$

in agreement with the fundamental commutation relations.

3.5 Sources

We have quantised a free field, we have discussed correlators of two fields, but there is not much else we can do besides adding interactions (later).

As a first step towards interactions, let us discuss driving the field by an external source $\rho(x)$

$$-\partial^2 \phi(x) + m^2 \phi(x) = \rho(x). \quad (3.67)$$

We make the assumption that the source field is non-zero only for some finite interval of time.

Question: Given an initial field $\phi_i(x)$ obeying the homogeneous Klein–Gordon equation, how to determine the solution $\phi(x)$ of the inhomogeneous Klein–Gordon equation such that $\phi(x) = \phi_i(x)$ for all $t < t_0$ before the activation of the source at t_0 ? In particular, what is the final field $\phi_f(x)$ after deactivation of the source at t_1 ? Note that the latter must also obey the homogeneous Klein–Gordon equation.

Due to linearity and translation invariance we make the general ansatz^{18 19}

$$\phi(x) = \phi_i(x) + \Delta\phi(x), \quad \Delta\phi(x) = \int dy^{d+1} G_R(x-y) \rho(y), \quad (3.69)$$

where G_R is the retarded propagator (Green function)

$$-\partial^2 G_R(x) + m^2 G_R(x) = \delta^{d+1}(x), \quad G_R(x) = 0 \text{ for } x^0 < 0. \quad (3.70)$$

The first equation is conveniently solved in momentum space

$$G(x) = \int \frac{dp^{d+1}}{(2\pi)^{d+1}} e^{ip \cdot x} G(p) \quad (3.71)$$

with the inhomogeneous Klein–Gordon equation and its solution²⁰

$$p^2 G(p) + m^2 G(p) = 1, \quad G(p) = \frac{1}{p^2 + m^2}. \quad (3.72)$$

Let us see how to incorporate the second relation. Write the function as

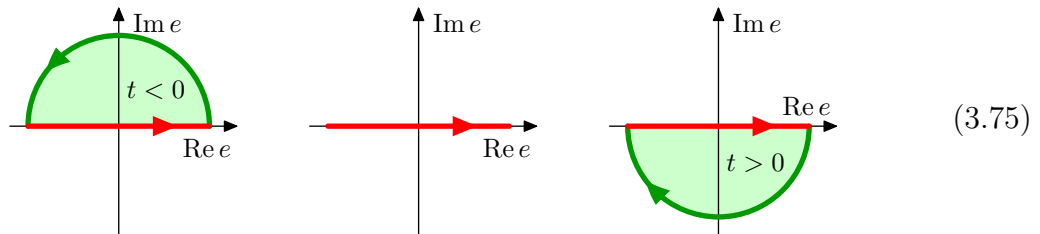
$$\begin{aligned} G(x) &= \int \frac{d\vec{p}^d}{(2\pi)^d} e^{i\vec{p} \cdot \vec{x}} \int \frac{de}{2\pi} e^{-iet} \frac{-1}{e^2 - e(\vec{p})^2} \\ &= \int \frac{d\vec{p}^d}{(2\pi)^d} e^{i\vec{p} \cdot \vec{x}} \int \frac{de}{2\pi} \frac{-1}{2e(\vec{p})} \left(\frac{e^{-iet}}{e - e(\vec{p})} - \frac{e^{-iet}}{e + e(\vec{p})} \right). \end{aligned} \quad (3.73)$$

Then solve the Fourier integrals by the residue theorem in the complex plane. The integral over e runs from $-\infty$ to $+\infty$; close the contour! This is done by a semi-circle in the complex plane with very large radius. Its contribution must vanish in order not to alter the overall integral; therefore consider the exponent:

$$\exp(-iet) = \exp(-it \operatorname{Re} e) \exp(t \operatorname{Im} e). \quad (3.74)$$

Only the second term is able to suppress the contribution:

- For $t > 0$ we need $\operatorname{Im} e < 0$: close contour in lower half.
- For $t < 0$ we need $\operatorname{Im} e > 0$: close contour in upper half.



¹⁸Admittedly, this is a perfectly classical problem already encountered in electrodynamics.

¹⁹Due to linearity each element of the source field ρ will influence the field ϕ independently. Furthermore the influence will be invariant under translations.

²⁰Due to the poles on the mass shell this solution is in fact ill-defined; in particular, it is left unspecified how to treat them in the Fourier transformation. This corresponds to the freedom to add a solution of the homogeneous equation, $\delta(p^2 + m^2)f(p)$, which is fixed by imposing the second condition as we shall see.

We have two poles at $e = \pm e(\vec{p})$ on the real axis i.e. on the integration contour. We need to decide how they contribute to residues.

For the retarded propagator, we want $G_R(x) = 0$ for $t < 0$. This is achieved by shifting the poles slightly into the lower half plane

$$G_R(p) = \frac{1}{p^2 + m^2 - i p^0 \epsilon} \cdot \begin{array}{c} \text{Im } e \\ \text{Re } e \\ -e(\vec{p}) \quad +e(\vec{p}) \end{array} \quad (3.76)$$

Alternatively, we can deform the contour slightly to close above the poles.²¹

$$\begin{array}{c} \text{Im } e \\ \text{Re } e \\ -e(\vec{p}) \quad +e(\vec{p}) \end{array} \quad \begin{array}{c} \text{Im } e \\ \text{Re } e \\ -e(\vec{p}) \quad +e(\vec{p}) \end{array} \quad (3.77)$$

There are no poles in the upper half plane, hence $G_R(x) = 0$ for $t < 0$. For $t > 0$, however, both poles contribute a residue,²² and we obtain

$$\begin{aligned} G_R(x) &= i\theta(t) \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} (e^{+ip \cdot x} - e^{-ip \cdot x}) \\ &= \theta(t) (\Delta_+(x) - \Delta_+(-x)) \\ &= \theta(t) \Delta(x). \end{aligned} \quad (3.78)$$

This is nice: we have found a relationship between correlation functions and propagators. It yields the position space form for the propagator.

We can confirm that it satisfies the defining relation:²³

$$\begin{aligned} &(-\partial^2 + m^2)G_R(x) \\ &= \theta(t) (-\partial^2 + m^2)\Delta(x) + \frac{\partial}{\partial t} (\delta(t) \Delta(x)) + \delta(t) \dot{\Delta}(x) \\ &= \delta^{d+1}(x). \end{aligned} \quad (3.79)$$

- the first term vanishes because Δ satisfies the equations of motion;
- the second term vanishes because $[\phi(\vec{x}), \phi(\vec{y})] = 0$;
- the third term uses $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^d(\vec{x} - \vec{y})$.

We can now determine the contribution to ϕ from the source. Let us focus on the future after the source is switched off²⁴

$$\Delta\phi(x) = \phi_f(x) - \phi_i(x) = \int dy^{d+1} \Delta(x - y) \rho(y). \quad (3.80)$$

²¹Another option is to write the function as the principal value of the pole $1/(p^2 + m^2)$ and a distributional contribution $i\pi\delta(p^2 + m^2) \text{sign}(p^0)$ representing a homogeneous solution.

²²Note that the contour is clockwise.

²³One has to distribute the derivatives in a suitable way between θ and Δ to arrive at the intermediate expression.

²⁴We can therefore set $\theta(y^0 - x^0) = 1$.

Transform this expression to momentum space with the Fourier transformation

$$\rho(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} e^{ip \cdot x} \rho(p), \quad \rho(p)^* = \rho(-p). \quad (3.81)$$

It yields after substitution and evaluation of two integrals

$$\Delta\phi(x) = \int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})} (\dot{\rho}(p) e^{ip \cdot x} - \dot{\rho}^*(p) e^{-ip \cdot x}). \quad (3.82)$$

As we know, the homogeneous Klein–Gordon equation is solved by

$$\phi(x) = \int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})} (\alpha(\vec{p}) e^{ip \cdot x} + \alpha^*(\vec{p}) e^{-ip \cdot x}). \quad (3.83)$$

Let this represent the solution in the distant past. Then the solution ϕ_f in the distant future is obtained by replacing

$$\alpha_f(\vec{p}) = \alpha_i(\vec{p}) + \dot{\rho}(e(\vec{p}), \vec{p}). \quad (3.84)$$

We notice that only the Fourier modes of the source ρ on the mass shell can actually drive the field ϕ .

We can now ask how much energy, momentum or particle number the source ρ transfers to the field. We compare the corresponding charges for ϕ_f and ϕ_i to find

$$\begin{aligned} \Delta P^\mu &= \int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})} p^\mu [|\rho(p)|^2 + 2 \operatorname{Im}(\alpha_i(\vec{p}) \rho^*(p))], \\ \Delta N &= \int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})} [|\rho(p)|^2 + 2 \operatorname{Im}(\alpha_i(\vec{p}) \rho^*(p))]. \end{aligned} \quad (3.85)$$

The contribution proportional to $|\rho|^2$ originates from the source alone, and it is manifestly positive for the energy $\Delta E = \Delta P^0$ and particle number ΔN .²⁵ The other contribution is an interference term with the initial field configuration which can raise or lower these charges.

²⁵ ΔN is not manifestly integer for an classical external field. However, if ρ is itself a quantum operator related to another field, the combination $\rho^\dagger(\vec{p})\rho(\vec{p})$ should again lead to an integer N .

4 Symmetries

So far we have not discussed symmetries. QFT does not actually need symmetries, but they help very much in restricting classes of models, providing stability and simplifying calculations as well as results.¹

For example, in most cases QFT's have some symmetry of space and time. Particularly in fundamental particle physics all models have relativistic invariance or Poincaré symmetry.

Symmetries are some transformations of the fields $\phi \rightarrow \phi'$ that map solutions of the equations of motion to other solutions. Hence they can be used to generate a whole class of solutions from a single one.

We shall discuss the action of various types of symmetries, their groups and representations, and the resulting conserved charges via Noether's theorem. Most of the discussion applies to classical and quantum field theories.

4.1 Internal Symmetries

Let us first discuss internal symmetries. In a QFT with several fields, these typically transform the fields into each other in some way without making reference to their dependence on space or time.

Internal Transformations. The simplest example is a complex scalar field $\phi(x)$ with Lagrangian and corresponding equation of motion

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi, \quad \begin{aligned} \partial^2 \phi - m^2 \phi &= 0, \\ \partial^2 \phi^* - m^2 \phi^* &= 0. \end{aligned} \quad (4.1)$$

Consider a global transformation of the fields

$$\phi'(x) = e^{+i\alpha} \phi(x), \quad \phi'^*(x) = e^{-i\alpha} \phi^*(x). \quad (4.2)$$

It maps a solution of the equations of motion to another solution²

$$\partial^2 \phi' - m^2 \phi' = e^{i\alpha} (\partial^2 \phi - m^2 \phi) = 0. \quad (4.3)$$

Moreover the symmetry leaves the Lagrangian and the action invariant

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi), \quad S[\phi'] = S[\phi]. \quad (4.4)$$

¹For free particles symmetries are not that helpful, the true power of symmetries arises in interacting situations.

²The transformed field ϕ' satisfies the equation of motion because ϕ does.

Any such transformation must be a symmetry because it maps extrema of the action to extrema and hence solutions to solutions. Symmetries of the action are more powerful than mere symmetries of the equations of motion. In the following we will only consider symmetries of the action.³

Noether's Theorem. Every continuous global symmetry of the action leads to a conserved current and thus a conserved charge for solutions of the equations of motion.

Let us derive the theorem: Consider a solution ϕ of the equations of motion. By construction, any variation of the Lagrangian evaluated on this solution is a total derivative⁴

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi} \delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \partial_\mu\delta\phi \\ &= \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \partial_\mu\delta\phi = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \delta\phi \right).\end{aligned}\quad (4.5)$$

Suppose now $\delta\phi$ is the infinitesimal field variation generated by a continuous symmetry. We know that $\delta S = 0$, hence the Lagrangian can only change by some total derivative

$$\delta\mathcal{L} = \delta\alpha \partial_\mu K^\mu. \quad (4.6)$$

Equating the two expressions for $\delta\mathcal{L}$ we find a current⁵

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \frac{\delta\phi}{\delta\alpha} - K^\mu, \quad (4.7)$$

which is conserved for every solution ϕ

$$\partial_\mu J^\mu = 0. \quad (4.8)$$

Furthermore, a conserved current implies a conserved charge

$$Q(t) = \int d\vec{x}^d J^0(t, \vec{x}) \quad (4.9)$$

if we assume that the field vanishes sufficiently fast at spatial infinity

$$\dot{Q} = \int d\vec{x}^d \partial_0 J^0 = - \int d\vec{x}^d \partial_k J^k = 0. \quad (4.10)$$

³For example the scaling transformation $\phi(x) \rightarrow e^\beta \phi(x)$ also maps solutions to solutions, but it rescales the Lagrangian $\mathcal{L}' = e^{2\beta} \mathcal{L}$ and likewise the action. If one considers QFT's to be specified by their Lagrangians, then this symmetry of the equations of motion relates two different models \mathcal{L} and \mathcal{L}' . We typically use the freedom to redefine the fields to bring the Lagrangian to some canonical form.

⁴Usually we can ignore the total derivative term, here it is relevant.

⁵Any term of the form $\partial_\nu B^{\mu\nu}$ with anti-symmetric indices on $B^{\mu\nu}$ can be added to J^μ without modifying any of the following relations.

The conserved charge actually generates an infinitesimal symmetry transformation via the Poisson brackets

$$\{Q, F\} = -\frac{\delta F}{\delta \alpha} \quad (4.11)$$

as can be shown using its defining relations.⁶

Example. Let us consider the complex scalar field. The field variation is defined by

$$\delta \phi = i\phi \delta \alpha, \quad \delta \phi^* = -i\phi^* \delta \alpha. \quad (4.12)$$

The Lagrangian is invariant under the transformation $\delta \mathcal{L} = 0$, hence $K^\mu = 0$. The other term reads

$$\begin{aligned} J^\mu &= \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \frac{\delta \phi}{\delta \alpha} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta \alpha} \\ &= (-\partial^\mu \phi^*) (i\phi) + (-\partial^\mu \phi) (-i\phi^*) \\ &= -i(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi). \end{aligned} \quad (4.13)$$

The naive divergence of the current reads

$$\partial_\mu J^\mu = -i(\partial^2 \phi^* \phi - \phi^* \partial^2 \phi). \quad (4.14)$$

This indeed vanishes for a solution of the equations of motion.

The conserved charge reads

$$Q = i \int d\vec{x}^d (\dot{\phi}^* \phi - \phi^* \dot{\phi}) = i \int d\vec{x}^d (\pi^* \phi - \phi^* \pi). \quad (4.15)$$

Transformed to momentum space we get

$$Q = \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} (a^*(\vec{p})a(\vec{p}) - b^*(\vec{p})b(\vec{p})). \quad (4.16)$$

This charge is indeed time-independent and (Poisson) commutes with the Hamiltonian. As expected, it obeys

$$\{Q, \phi\} = -i\phi = -\frac{\delta \phi}{\delta \alpha}, \quad \{Q, \phi^*\} = +i\phi^* = -\frac{\delta \phi^*}{\delta \alpha}. \quad (4.17)$$

We furthermore observe a relation to the number operators

$$Q = N_a - N_b. \quad (4.18)$$

Despite the similarities, there is a crucial difference to the number operator: the charge Q is associated to a symmetry, whereas the individual number operators N_a, N_b are not.⁷ In a symmetric theory with interactions, Q is conserved while N is in general not.

⁶The difficulty is to deal with the term K which is only implicitly defined. However, it suffices to show the relationship for the canonical fields $F = \phi, \pi$.

⁷One might construct a non-local symmetry transformation corresponding the number operator in a free field theory. However, this symmetry would typically not survive when interaction terms are introduced.

Quantum Action. Let us briefly state how to represent this symmetry in the quantum theory where $Q = N_a - N_b$ becomes a quantum operator.⁸ The operator Q therefore measures the number of particles created by a^\dagger minus the number of particles created by b^\dagger . It is obviously hermitian

$$Q^\dagger = Q. \quad (4.19)$$

It obeys the following commutation relations with creation and annihilation operators

$$\begin{aligned} [Q, a(\vec{p})] &= -a(\vec{p}), & [Q, b(\vec{p})] &= +b(\vec{p}), \\ [Q, a^\dagger(\vec{p})] &= +a^\dagger(\vec{p}), & [Q, b^\dagger(\vec{p})] &= -b^\dagger(\vec{p}). \end{aligned} \quad (4.20)$$

This tells us that particles of type a carry positive unit charge while the anti-particles of type b carry negative unit charge.

The commutators of spacetime fields $\phi \sim a^\dagger + b$ read

$$[Q, \phi(x)] = +\phi(x), \quad [Q, \phi^\dagger(x)] = -\phi^\dagger(x), \quad (4.21)$$

which tell us that ϕ and ϕ^\dagger carry charges $+1$ and -1 , respectively. The commutators are also in agreement with the classical result that charges generate infinitesimal transformations, i.e.

$$[Q, \phi] = +\phi = -i \frac{\delta \phi}{\delta \alpha}, \quad [Q, \phi^\dagger] = -\phi^\dagger = -i \frac{\delta \phi^\dagger}{\delta \alpha}. \quad (4.22)$$

For finite transformations we introduce the operator

$$U(\alpha) = \exp(i\alpha Q). \quad (4.23)$$

We can convince ourselves that it obeys the following algebra with the fields⁹

$$\begin{aligned} U(\alpha) \phi(x) U(\alpha)^{-1} &= e^{+i\alpha} \phi(x) = \phi'(x), \\ U(\alpha) \phi^\dagger(x) U(\alpha)^{-1} &= e^{-i\alpha} \phi^\dagger(x) = \phi^{\dagger'}(x). \end{aligned} \quad (4.24)$$

So $U(\alpha)$ generates a finite symmetry transformation by means of conjugation while Q generates the corresponding infinitesimal transformation by means of commutators.

Note that the operator $U(\alpha)$ is unitary because Q is hermitian

$$U(\alpha)^\dagger = \exp(-i\alpha Q^\dagger) = \exp(-i\alpha Q) = U(-\alpha) = U(\alpha)^{-1}. \quad (4.25)$$

⁸The lifting of Q from the classical to the quantum theory is subject to an ordering ambiguity for the products $\phi^\dagger \pi$ and $\pi^\dagger \phi$ or, equivalently, $a^\dagger a$ and $b^\dagger b$. These amount to a shift of Q by some indefinite constant. The latter affects the absolute charges of states, and it could be fixed by defining the charge of the ground state. Importantly, the charge of (particle) excitations is a relative quantity and therefore unaffected by the ambiguity.

⁹Note that $Q\phi = \phi(Q+1)$ implies $\exp(i\alpha Q)\phi = \phi \exp(i\alpha(Q+1)) = e^{i\alpha} \phi \exp(i\alpha Q)$.

A crucial property of symmetries in QFT is that they are represented by *unitary* operators. This is required to make expectation values invariant under symmetry.

The symmetry group for the complex scalar is simply $U(1)$.

The above discussions only apply to operators; let us finally discuss transformations for states. States transform under finite transformations as

$$|\Psi'\rangle = U(\alpha)|\Psi\rangle. \quad (4.26)$$

Typically, the vacuum is uncharged under symmetries¹⁰

$$Q|0\rangle = 0. \quad (4.27)$$

The transformation for all other states in the Fock space then follows from the transformation of creation operators.

4.2 Spacetime Symmetries

Next we shall consider symmetries related to space and time. In relativistic theories these are the spatial rotations and Lorentz boosts (altogether called Lorentz symmetries) as well as spatial and temporal translations. In total they form the Poincaré group. We will see different ways in which the symmetry is implemented in QFT.

Translations. Let us start with simple translations in space and time

$$(x')^\mu = x^\mu + a^\mu. \quad (4.28)$$

We demand that the fields merely change by shifting the position argument

$$\phi'(x') = \phi(x). \quad (4.29)$$

In other words the new field evaluated at the new position equals the old field at the old position.¹¹ Explicitly,¹²

$$\phi'(x) = \phi(x - a) \quad \text{or} \quad \delta\phi(x) = -\delta a^\mu \partial_\mu \phi(x). \quad (4.30)$$

In order for translations to be a symmetry, we have to require that the Lagrangian does not explicitly depend on the position

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \quad \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) = \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \quad (4.31)$$

¹⁰An uncharged vacuum is not a requirement in QFT. In fact, a charged vacuum is related to spontaneous symmetry breaking and Goldstone particles, see QFT II. Note that ordering ambiguities can arise in the determination of the charges, and are resolved by specifying the intended charge of the vacuum.

¹¹Alternatively, one might define $\phi'(x) = \phi(x')$ resulting in the opposite transformation rules.

¹²The variation is defined via $\phi'(x) = \phi(x) + \delta\phi(x)$.

Energy and Momentum. The Noether theorem equally applies to this situation, let us derive the associated currents and charges. The symmetry variation of the Lagrangian is (by definition) through the field variables only

$$\delta\mathcal{L} = -\delta a^\mu \left(\frac{\delta\mathcal{L}}{\delta\phi} \partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi} \partial_\mu\partial_\nu\phi \right). \quad (4.32)$$

This expression can be written as a total derivative because there is no other (explicit) dependence on spacetime

$$\delta\mathcal{L} = -\delta a^\mu \partial_\mu (\mathcal{L}(\phi, \partial\phi)). \quad (4.33)$$

We therefore obtain a contribution $K^\mu{}_\nu = -\delta^\mu_\nu \mathcal{L}$. Together with the ordinary contribution $(\delta\mathcal{L}/\delta(\partial_\mu\phi))\delta\phi$ to the Noether current, we obtain a vector of conserved currents $J^\mu{}_\nu =: T^\mu{}_\nu$ (with $\partial_\mu T^\mu{}_\nu = 0$) where the index ν labels the $d+1$ dimensions for shifting

$$T^{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \partial^\nu\phi + \eta^{\mu\nu} \mathcal{L}. \quad (4.34)$$

This object is called stress-energy (or energy-momentum) tensor. For a real scalar field it reads

$$T^{\mu\nu} = \partial^\mu\phi \partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu} ((\partial\phi)^2 + m^2\phi^2). \quad (4.35)$$

The corresponding conserved charge is the momentum vector

$$P^\mu = \int d\vec{x}^d T^{0\mu} = \int d\vec{x}^d \left(-\pi \partial_\mu\phi - \frac{1}{2}\eta^{0\mu} ((\partial\phi)^2 + m^2\phi^2) \right). \quad (4.36)$$

We recover the Hamiltonian as its time component

$$H = P^0 = \int d\vec{x}^d \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\partial}\phi)^2 + \frac{1}{2}m^2\phi^2 \right), \quad (4.37)$$

while the total spatial momentum simply reads

$$\vec{P} = - \int d\vec{x}^d \pi \vec{\partial}\phi. \quad (4.38)$$

Quantum Action. We have already encountered the quantum operators for energy and momentum. Recall that in momentum space they read

$$P^\mu = \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} p^\mu(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}). \quad (4.39)$$

Performing a quantum commutator with the field yields the expected shift

$$[P^\mu, \phi(x)] = i\partial^\mu\phi(x). \quad (4.40)$$

As before, we can introduce an operator $U(a)$ for finite shift transformations as the exponential

$$U(a) = \exp(ia^\mu P_\mu). \quad (4.41)$$

Conjugating a field with it yields the shifted field¹³

$$U(a)\phi(x)U(a)^{-1} = \exp(-a^\mu \partial_\mu)\phi(x) = \phi(x - a) = \phi'(x). \quad (4.42)$$

Note that the operator $U(a)$ is unitary because P^μ is hermitian.

Lorentz Transformations. Next, consider Lorentz transformations

$$(x')^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu. \quad (4.43)$$

All upper (contravariant) indices transform according to the same rule as x^μ under Lorentz transformations, whereas lower (covariant) indices transform with the matrix Λ , just as ∂_μ does, e.g.

$$(\partial')_\mu = \Lambda^\nu{}_\mu \partial_\nu. \quad (4.44)$$

A product between a covariant and contravariant index is Lorentz invariant

$$x' \cdot \partial' = (x')^\mu (\partial')_\mu = \Lambda^\rho{}_\mu (\Lambda^{-1})^\mu{}_\nu x^\nu \partial_\rho = x^\nu \partial_\nu = x \cdot \partial. \quad (4.45)$$

The matrix Λ has the defining property that it leaves the metric $\eta_{\mu\nu}$ invariant

$$\eta'_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}. \quad (4.46)$$

We can write this relation also as

$$(\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\sigma} \Lambda^\sigma{}_\rho \eta^{\rho\mu} =: \Lambda_\nu{}^\mu. \quad (4.47)$$

It implies that it makes no difference whether indices are raised or lowered before or after a Lorentz transformation. Correspondingly, scalar products between equal types of vectors are invariant.

Lorentz transformations combine spatial rotations (the matrix acts on two of the spatial dimensions)

$$\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \quad (4.48)$$

and Lorentz boosts (the matrix acts on time and one of the spatial dimensions)

$$\begin{pmatrix} \cosh \vartheta & \sinh \vartheta \\ \sinh \vartheta & \cosh \vartheta \end{pmatrix} = \exp \begin{pmatrix} 0 & \vartheta \\ \vartheta & 0 \end{pmatrix}. \quad (4.49)$$

There are also some discrete transformations which we shall discuss below. Here we restrict to proper orthochronous Lorentz transformations which form the Lie group $\text{SO}^+(d, 1)$.

We note that spatial rotations are generated by anti-symmetric matrices while Lorentz boosts are generated by symmetric matrices. Composing various such

¹³The exponentiated derivative $\exp(-a^\mu \partial_\mu)\phi(x)$ generates all the terms in the Taylor expansion of $\phi(x - a)$ for small a .

transformations in two-dimensional subspaces of spacetime we conclude that Lorentz rotations are generated as

$$\Lambda^\mu{}_\nu = \exp(\omega)^\mu{}_\nu \quad (4.50)$$

where $\omega^\mu{}_\nu$ is a matrix satisfying

$$\omega^k{}_l = -\omega^l{}_k, \quad \omega^0{}_k = \omega^k{}_0, \quad \omega^0{}_0 = \omega^k{}_k = 0. \quad (4.51)$$

Lowering the first index $\omega_{\mu\nu} = \eta_{\mu\rho}\omega^\rho{}_\nu$, this is equivalent to an anti-symmetric matrix

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (4.52)$$

Angular Momentum. For a scalar field the transformation reads

$$\phi'(x) = \phi(\Lambda x), \quad \delta\phi = \delta\omega^\mu{}_\nu x^\nu \partial_\mu \phi. \quad (4.53)$$

Lorentz invariance of the action requires the Lagrangian to transform in the same way¹⁴

$$\delta\mathcal{L} = \delta\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = \delta\omega_{\mu\nu} \partial^\mu (x^\nu \mathcal{L}). \quad (4.54)$$

Note that the measure dx^{d+1} is Lorentz invariant. Comparing this to an explicit variation of $\mathcal{L}(\phi, \partial\phi)$ in terms of the field variables implies the relation

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \partial^\nu \phi = \frac{\delta\mathcal{L}}{\delta(\partial_\nu\phi)} \partial^\mu \phi. \quad (4.55)$$

This relation holds whenever $\partial_\mu\phi$ appears only in the Lorentz-invariant combination $(\partial\phi)^2 = \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$. For the stress-energy tensor it implies symmetry in both indices

$$T^{\mu\nu} = T^{\nu\mu}. \quad (4.56)$$

The currents $J^{\mu,\rho\sigma} = -J^{\mu,\sigma\rho}$ corresponding to the anti-symmetric matrix $\delta\omega_{\rho\sigma}$ can be expressed in terms of the stress-energy tensor T

$$J^{\mu,\rho\sigma} = -T^{\mu\rho}x^\sigma + T^{\mu\sigma}x^\rho. \quad (4.57)$$

Conservation of $J^{\mu,\rho\sigma}$ is then guaranteed by conservation and symmetry of T

$$\partial_\mu J^{\mu,\rho\sigma} = -T^{\sigma\rho} + T^{\rho\sigma} = 0. \quad (4.58)$$

The integral of J is the relativistic angular momentum tensor

$$M^{\mu\nu} = \int d\vec{x}^d J^{0,\mu\nu} = \int d\vec{x}^d (-T^{0\mu}x^\nu + T^{0\nu}x^\mu). \quad (4.59)$$

For a scalar field in $d = 3$ dimensional space we obtain the well-known spatial angular momentum

$$J^m = -\frac{1}{2}\varepsilon^{mkl}M^{kl} = \int d\vec{x}^3 \pi(\vec{x} \times (\vec{\partial}\phi))^m. \quad (4.60)$$

¹⁴The anti-symmetry of $\omega_{\mu\nu}$ allows to pull x^ν past the derivative.

Furthermore, the momentum for Lorentz boosts reads¹⁵

$$K^m = M^{m0} = \int d\vec{x}^3 (T^{00} x^m) - P^m t. \quad (4.61)$$

We can also write the Lorentz generators in momentum space¹⁶

$$M^{\mu\nu} = i \int \frac{d\vec{p}^d}{(2\pi)^d 2e(\vec{p})} (p^\mu \partial^\nu a^*(\vec{p}) a(\vec{p}) - p^\nu \partial^\mu a^*(\vec{p}) a(\vec{p})). \quad (4.62)$$

In the quantum theory, all components of the tensor $M^{\mu\nu}$ are hermitian operators. Consequently, the operators for finite transformations are unitary

$$U(\omega) = \exp(\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}), \quad U(\omega)^\dagger = U(\omega)^{-1}. \quad (4.63)$$

The interesting conclusion is that we have found a unitary representation of the Poincaré group. As the latter is a non-compact Lie group this representation is necessarily infinite-dimensional. Indeed, the field $\phi(x)$ and Fock space carry infinitely many degrees of freedom.

Flavours of Symmetries. The concept of spacetime symmetries and how they act within field theory is very rich and occasionally confusing. Let us step aside and review the notions of spacetime symmetry transformations that we encounter.

A spacetime symmetry is originally defined as a transformation of spacetime $x \mapsto x'$ which leaves the physical laws unchanged in some way. There are basically two ways in which to implement such a transformation in field theory:

The geometrical notion of the transformation for a scalar field $\phi(x)$ is to define the transformed field ϕ' by the relationship $\phi'(x') = \phi(x)$.¹⁷ Assuming the field $\psi^\alpha(x)$ also carries some spin degrees of freedom α , a spacetime transformation would have to take these into account by applying some transformation matrix¹⁸

$$\psi'^{\alpha'}(x') = \sum_{\alpha} M^{\alpha'}_{\alpha}(x) \psi^{\alpha}(x). \quad (4.64)$$

The geometrical notion works analogously for the classical and quantum theory because fields have the same dependencies in both formulations.

Conversely, the dynamical notion of a symmetry acts exclusively on the phase space in a classical theory or on the Hilbert space in the quantum theory. As these

¹⁵Conservation implies that the motion of the centre of gravity (first term) is governed by the momentum (second term).

¹⁶Its form is reminiscent of the position space form because Lorentz rotations in of x^μ and p^μ are practically the same.

¹⁷Confusingly, in a different convention one might as well define the transformation by $\phi'(x) = \phi(x')$ amounting to the inverse transformation.

¹⁸Again, depending on convention, the transformation matrix M could be replaced by its inverse M^{-1} . Furthermore, some fields can come with multiple indices and each index should be transformed with a different matrix.

spaces are not (immediately) labelled by a coordinate x , there is no prescribed way to define the symmetry action. For the classical theory one needs to construct an appropriate transformation on phase space. For a quantum theory the transformation should be given by a (unitary) quantum operator U acting on a Hilbert space state $|\Psi\rangle$ as $|\Psi'\rangle = U|\Psi\rangle$. The corresponding transformation on another quantum operator X is given by $X' = UXU^{-1}$. In particular, a field $\psi^\alpha(x)$ is a quantum operator, and its dynamical transformation law reads

$$\psi'^\alpha(x) = U \psi^\alpha(x) U^{-1}. \quad (4.65)$$

Superficially, this transformation takes a form very similar to the above geometrical variant, but it is worth pointing out the differences: the object M is a classical matrix acting on the spin degrees of freedom α ; in addition, it may depend on the position x , but it is blind to Hilbert space. Conversely, the object U is a quantum operator acting on the Hilbert space, i.e. on the creation and annihilation operators a, a^\dagger , but it leaves the spacetime position x and the spin degrees of freedom untouched.

Of course, both implementations of symmetries can and should have exactly the same effect on fields; in fact, their equivalence can be used to define the dynamical notion corresponding to a geometrical one. The appropriate formal tool to obtain one from the other is the Noether theorem. Given a geometrical symmetry transformation, the theorem provides a conserved charge which as a quantum operator serves as the generator of the infinitesimal dynamical transformation.¹⁹ Conversely, the latter can be converted back to a geometrical transformation by a reversal of the Noether theorem.

Beyond the above distinction, symmetry transformations can come in many different flavours, e.g.:

- finite vs. infinitesimal transformations;
- continuous vs. discrete transformations or a mixture of both concepts;
- compact vs. non-compact Lie groups;
- geometrical transformations may have a linear action on fields (as above), or it can be non-linear;
- with respect to the equations of motion: symmetry algebras may close universally or they may do so only on solutions, or the symmetry transformation might only be defined on the solutions in the first place;
- transformations can remain unchanged or get deformed by quantum effects, they can even be invalidated;
- transformations may map models to themselves (symmetries) or to other similar models or even very distinct ones (dualities);
- ...

These various optional features and their combinations make symmetries a very rich subject within QFT.

¹⁹Exponentiation of the charge describes a finite symmetry transformation. Similar constructions apply to the classical theory by using vector fields on phase space.

4.3 Poincaré Symmetry

Above we have derived Lorentz ($M^{\mu\nu}$) and momentum (P^μ) generators for relativistic transformations of a scalar field. Let us now discuss the algebraic foundations and generalisations.

Some Basic Definitions. Here are some sketches of basic definitions in group and representation theory.²⁰

Group. A set G with an associative composition law $G \times G \rightarrow G$ (usually called multiplication), a unit element and the inverse map $G \rightarrow G$.

Algebra. A vector space A with a bi-linear composition law $A \otimes A \rightarrow A$ (usually called multiplication).

Lie group. A group G that is also a manifold. Continuous symmetries in physics are realised as Lie groups.

Lie algebra. A non-associative algebra \mathfrak{g} with an anti-symmetric product $[\cdot, \cdot]$ (called Lie bracket) that satisfies the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0. \quad (4.66)$$

The tangent space of a Lie group G at the unit element is the corresponding Lie algebra \mathfrak{g} .

Quantum group, quantum algebra. The algebra of operators in quantum mechanics is called a quantum group or a quantum algebra. In QFT it is spanned by the field operators and their products modulo their commutation relations. In addition to being an algebra, it has a unit element and an inverse for most elements.

Interestingly, it can act as any of the above structures: It is an algebra by definition. It may contain Lie groups, e.g. the symmetries of a quantum theory. A Lie algebra can be realised by the map $[a, b] = ab - ba$ which automatically satisfies the Jacobi identity.

Representation. A map $R : X \rightarrow \text{End}(V)$ from a group or an algebra X to linear operators (matrices, endomorphism) on some vector space V . The representation must reflect X 's composition law by operator composition (matrix multiplication). If $ab = c$ then $R(a)R(b) = R(c)$.

Representation of a Lie algebra. The Lie bracket must be represented by a commutator: If $[a, b] = c$ then $R(a)R(b) - R(b)R(a) = R(c)$.

Representation space. The vector space V on which a representation $R : X \rightarrow \text{End}(V)$ can act. Also known as a module.

Physics. The notation in physics often does not distinguish between abstract Lie algebra generators a and their representations $R(a)$, both may be denoted simply by a . Likewise the distinction between Lie brackets and commutators may be dropped (this is perfectly reasonable in a quantum algebra). Moreover the term

²⁰See a textbook for proper definitions.

representation is used not only for an operatorial version of algebra generators, but also for the space on which these operators act.

Unitary groups and algebras. The simplest class of Lie groups are the unitary groups $U(N)$ consisting of unitary $N \times N$ matrices. In quantum field theory, they frequently appear as flavour and gauge symmetries. The unitary matrices themselves also form the fundamental or defining representation. The special unitary subgroup $SU(N)$ is given by matrices with unit determinant. The corresponding Lie algebras $\mathfrak{u}(N)$ are spanned by hermitian matrices.²¹ For the subalgebra $\mathfrak{su}(N)$ the matrices must furthermore be traceless.

Orthogonal groups and algebras. Another large class of Lie groups are the special orthogonal groups $SO(N)$ which describe rotations in N -dimensional real space. The group $SO(N, M)$ is a generalisation which describes rotations in an $(N + M)$ -dimensional space of signature (N, M) . In particular, $SO(d, 1)$ is the Lorentz group for a spacetime of dimension $d + 1$. The corresponding Lie algebras $\mathfrak{so}(N)$ and $\mathfrak{so}(N, M)$ are spanned by anti-symmetric matrices (with respect to a metric of the given signature).

Double cover of the orthogonal groups. A peculiar feature of the orthogonal groups $SO(N, M)$ is that they have a double cover called $\text{Spin}(N, M)$. In the latter group, a rotation by an angle of 2π is a non-trivial central element. It can act as either $+1$ for vectorial representations (integer spin) or -1 for spinorial representations (half-integer spin). In $SO(N, M)$ this element is projected out reducing the set of allowable representations to the vectorial ones. The double cover is a topological issue which does not affect the local Lie algebra; it is $\mathfrak{so}(N, M)$ for both groups.

Algebra and Group. It is straight-forward to derive the algebra of infinitesimal transformations from the spacetime symmetry operators derived earlier

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= i\eta^{\nu\rho} M^{\mu\sigma} - i\eta^{\mu\rho} M^{\nu\sigma} - i\eta^{\nu\sigma} M^{\mu\rho} + i\eta^{\mu\sigma} M^{\nu\rho}, \\ [M^{\mu\nu}, P^\rho] &= i\eta^{\nu\rho} P^\mu - i\eta^{\mu\rho} P^\nu, \\ [P^\mu, P^\nu] &= 0. \end{aligned} \tag{4.67}$$

These define the so-called Poincaré algebra. The operators $M^{\mu\nu}$ generate the algebra $\mathfrak{so}(d, 1)$ of Lorentz (orthogonal) transformations in $d + 1$ spacetime dimensions. The spatial components M^{jk} generate the algebra $\mathfrak{so}(d)$ of rotations in d spatial dimensions.

The Poincaré group is obtained by exponentiating the algebra²²

$$U(\omega, a) = \exp\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu\right). \tag{4.68}$$

More precisely it is the component of the Poincaré group connected to the identity element. It includes $\text{Spin}^+(d, 1)$, the double cover of the proper orthochronous Lorentz group, along with translations.

²¹Depending on conventions for the basis of Lie algebras one uses either hermitian matrices (physics) or anti-hermitian matrices (mathematics).

²²The exponentiation can be specified in different forms such as $\exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})\exp(ia_\mu P^\mu)$ or $\exp(ia_\mu P^\mu)\exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$. These forms are all equivalent but a would take different values in the various cases.

Fock Space Representations. The above symbols (M, P) and their relationships can be interpreted in two ways:

They can serve as abstract elements of a Lie algebra without immediate connection to a physics problem. In that case the above relations should be viewed as the defining Lie brackets and group multiplication laws.

This is not the approach we had taken. Instead, we have derived them as quantum operators from Noether's theorem of a physical system. Their algebraic relations are expressed as commutators and products of operators. They form an explicit representation (M, P) of the Poincaré algebra and the corresponding representation $U(\omega, a)$ of the Poincaré group. The representation space is the Fock space of a scalar particle.

Since (M, P) commute with the number operator N , the representation is reducible.²³

- The most relevant representation is the one acting on single particle states.
- The multi-particle representations are symmetric tensor powers of it; they are generally reducible.
- The vacuum transforms in the trivial representation.
- The single-particle representation is complex, unitary, infinite-dimensional and irreducible.

4.4 Poincaré Representations

We have seen how the Poincaré algebra acts on the free Klein–Gordon field and its Fock space. In the following we will reverse the logic, and use the abstract Poincaré algebra to derive the concept of relativistic particles. We will see that the particles are characterised by their mass and spin, which leads to generalisations of the Klein–Gordon field in the following chapters.

Let us therefore investigate the unitary irreducible representations (UIR's) of the Poincaré group (Wigner's classification). These will be the elementary building blocks for physical theories with relativistic invariance. The derivation will parallel the derivation of unitary irreducible representations of the rotation group $SO(3) \simeq SU(2)$ (known from elementary quantum mechanics) which leads an understanding of spin. Here the result will characterise the types of admissible particles in a relativistic QFT.

The following construction will start with a set of states. The action of the symmetry operators requires the introduction of further states. As soon as the symmetry generators close on the space of states, we have found a complete representation and its corresponding representation space.

²³In other words, the action of (M, P) neither creates nor annihilates particles and therefore maps $\mathbb{V}_n \rightarrow \mathbb{V}_n$. The representation on Fock space thus splits into representations on the individual n -particle subspaces \mathbb{V}_n .

Mass. First, we should look for commuting combinations C of elements of the algebra (respectively commuting elements of the quantum algebra). Their eigenvalues classify irreducible representations because if measured on one state $|X\rangle$, any other state $|Y\rangle = J|X\rangle$ related to it by some symmetry operator J must have the same eigenvalue c :

$$C|Y\rangle = CJ|X\rangle = JC|X\rangle = cJ|X\rangle = c|Y\rangle. \quad (4.69)$$

The principal example in $\mathfrak{so}(3)$ is the total spin operator $C = J^2$. There, a representation of spin j is uniquely characterised by the eigenvalue $j(j+1)$ of J^2 .

We notice that the Poincaré algebra possesses a quadratic invariant

$$P^2 = P^\mu P_\mu. \quad (4.70)$$

This combination obviously commutes with all the momenta P^μ . It also commutes with the Lorentz generators $M^{\mu\nu}$ because it is constructed as a scalar product.

The combination P^2 must be represented by a unique number on an irreducible representation, see above. Clearly, P^2 measures the mass of a particle

$$P^2 = -m^2. \quad (4.71)$$

For unitary representations P^2 must be real. There are three cases to be distinguished:

- $P^2 < 0$, i.e. massive particles;
- $P^2 = 0$, i.e. massless particles;
- $P^2 > 0$, i.e. tachyons.

We shall discuss the massive case in detail and comment only briefly on the others.

Momentum. The next observation is that the momentum generators P^μ span an abelian ideal²⁴ This property is useful because we can treat the representation of the ideal independently of the other generators of the Poincaré algebra.²⁵

Moreover the ideal is abelian, and therefore we can choose simultaneous eigenstates $|p\rangle$ of all the generators P^μ as basis vectors for the representation space,

$$P^\mu |p\rangle = p^\mu |p\rangle. \quad (4.72)$$

As the representation of a finite translation by a_μ is given by $\exp(i a \cdot P)$, the representation is necessarily complex.

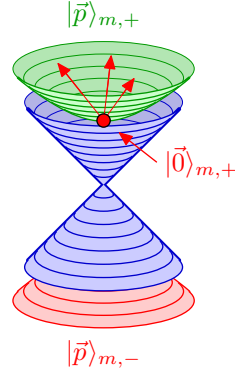
We have already fixed $P^2 = -m^2$ within our irreducible representation, and hence we must restrict the representation space to a mass shell $p^2 = -m^2$. For a given spatial momentum the constraint has two solutions for the energy

$$p^\mu = (\pm e(\vec{p}), \vec{p}). \quad (4.73)$$

²⁴An ideal is a subalgebra such that brackets between its elements and elements of the algebra always end up in the subalgebra, here $[M, P] \sim P$.

²⁵More precisely, individual states of the representation of the ideal will become subspaces of the representation of the complete algebra.

We can thus label the basis states for our representation space by means of their spatial momentum \vec{p} and the sign of energy.



$$(4.74)$$

This completes the discussion of the representation of momentum generators P^μ . What about the Lorentz generators $M^{\mu\nu}$?

Orthochronous Lorentz boosts can map between any two momentum vectors on the same mass shell. However, they cannot map between the forward and backward mass shells; this is achieved only by discrete time reversal transformations which we will consider later. Consequently, for an irreducible representation of the orthochronous Poincaré group, all energies must have the same sign. The irreducible representations are thus labelled by the sign of the energy in addition to the mass m . The positive-energy representation space is now spanned by the vectors

$$|\vec{p}\rangle_{m,+} = |e_m(\vec{p}), \vec{p}\rangle. \quad (4.75)$$

For negative energies the representation space is spanned by

$$|\vec{p}\rangle_{m,-} = |-e_m(\vec{p}), \vec{p}\rangle. \quad (4.76)$$

Note that, in physics, the negative-energy representation is typically obtained as the hermitian conjugate of the positive-energy representation

$$|\vec{p}\rangle_{m,-} \sim \langle +e_m(\vec{p}), \vec{p} |^\dagger. \quad (4.77)$$

Let us consider only positive energies from now on; negative energy representations are analogous.

Spin. Among the Lorentz generators, there are some which change a given vector $p^\mu = (e(\vec{p}), \vec{p})$ on the mass shell. For the particle at rest, $(m, \vec{0})$, these are the Lorentz boosts. The Lorentz boosts can be used to map the momentum vector $(m, \vec{0})$ to any other admissible momentum vector $(e(\vec{p}), \vec{p})$. Therefore, they ensure that the following discussion for $(m, \vec{0})$ equivalently applies to any other p^μ on the same mass shell.

The transformations which do not change $p^\mu = (m, \vec{0})$ are the spatial rotations which form the orthogonal group $\text{SO}(d)$ or its double cover $\text{Spin}(d)$.^{26 27} This

²⁶Reflections extend $\text{SO}(d)$ to $\text{O}(d)$ or $\text{Spin}(d)$ to $\text{Pin}(d)$, but they are not included in the identity component of the Poincaré algebra.

²⁷One obtains the same group $\text{SO}(d)$ for any time-like p , but they form different subgroups within $\text{SO}^+(d, 1)$. In fact, the mass shell is a coset space $\text{SO}^+(d, 1)/\text{SO}(d)$.

group is called the little group (physics) or stabiliser (mathematics) of p^μ . The representation subspace with fixed \vec{p} must therefore transform under a representation of $\text{Spin}(d)$.

For the most relevant case of $d = 3$ spatial dimensions, the unitary irreducible representations of $\text{Spin}(3) = \text{SU}(2)$ are labelled by a non-negative half-integer j . Their representation space is spanned by $2j + 1$ vectors $|-j\rangle_j, |-j - 1\rangle_j, \dots, |+j - 1\rangle_j, |+j\rangle_j$ with definite z -component of spin. An equivalent representation of $\text{Spin}(d)$ must apply to all momenta \vec{p} because it can be shifted to the point $\vec{p} = \vec{0}$.²⁸

We have now considered all algebra generators and hence the representation is complete. The representation space is thus spanned by the states

$$|\vec{p}, j_3\rangle_{(m, \pm, j)} = |\vec{p}\rangle_{m, \pm} \otimes |j_3\rangle_j. \quad (4.78)$$

Unitary Irreducible Representations. Altogether we find that the massive UIR's of the Poincaré algebra are labelled by their mass $m > 0$, the sign of energy and a unitary irreducible representation of $\text{Spin}(d)$. In the case of $d = 3$, the latter UIR are labelled by a non-negative half-integer j . The representation space for (m, \pm, j) is spanned by the vectors

$$|\vec{p}, j_3\rangle_{(m, \pm, j)} \quad (4.79)$$

with continuous \vec{p} and discrete $j_3 = -j, -j + 1, \dots, j - 1, j$.

For spin $j = 0$ the representation space is simply spanned by momentum eigenstates $|\vec{p}\rangle$ with arbitrary three-momentum \vec{p} . These are just the single-particle states of a scalar field. The conjugate states $\langle \vec{p}|$ also transform in a UIR, but one with negative momentum.

The next interesting case is $j = 1/2$ which we shall discuss in the following chapter of the lecture.

In addition, there are massless representations of positive or negative energy. They are classified by a representation of $\text{Spin}(d - 1)$.²⁹ For $d = 3$ the massless representations of $\text{Spin}(2) = \text{U}(1)$ are labelled by a positive or negative half-integer h known as helicity. There is only one state in the representation $(0, \pm, h)$ with given helicity

$$|\vec{p}\rangle_{(0, \pm, h)}. \quad (4.80)$$

This includes the case $h = \pm 1$ which is used for the photon excitations of the electromagnetic field.

Last but not least, there is the trivial representation with $P = 0$. The Fock space vacuum transforms under it. Finally, there are tachyonic representations with

²⁸Alternatively, one can introduce the Pauli–Lubanski vector $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$. It generates the appropriate $\mathfrak{so}(3)$ stabiliser subalgebra of $\mathfrak{so}(3, 1)$ for arbitrary momentum \vec{p} . Its invariant square $W^2 = m^2j(j + 1)$ yields the total spin j .

²⁹In fact, the stabiliser is the Euclidean group in $d - 1$ dimensions which also allows for so-called continuous spin representations.

$P^2 > 0$, but they are clearly unphysical. In particular, the stabiliser of a tachyonic momentum vector is the non-compact group $\text{SO}(d-1, 1)$ whose unitary representations are necessarily trivial or infinite-dimensional.

4.5 Discrete Symmetries

In addition to the continuous symmetries discussed above, there are also relevant discrete symmetries. The most prominent ones are parity, time reversal and charge conjugation. Let us discuss them for the example of a quantised complex scalar field $\phi(t, \vec{x})$.

Parity. Spatial rotations in d dimensions form the special orthogonal group $\text{SO}(d)$. However, also spatial reflections preserve all distances, and it is natural to consider them among the symmetries, too. Reflections were long believed to be a symmetry of nature, until the electroweak interactions were shown to violate parity symmetry. On the mathematical side, reflections flip the orientation and together with the rotations they form the general orthogonal group $\text{O}(d)$.³⁰

For an odd number of spatial dimensions d , it is convenient to introduce parity P as the unique transformation which inverts all spatial components of the position vector³¹

$$P : (t, \vec{x}) \mapsto (t, -\vec{x}). \quad (4.81)$$

It is an element of the group $\text{O}(d)$, but not of $\text{SO}(d)$, and it is convenient to choose this element because it does not introduce any preferred directions. There are many more orientation-inverting elements in $\text{O}(d)$; these can be obtained as products PR of parity P with elements R of $\text{SO}(d)$. Hence it is sufficient to consider only P . This parity reflection acts as an involution on spacetime,

$$(t, \vec{x}) \mapsto (t, -\vec{x}) \mapsto (t, \vec{x}), \quad (4.82)$$

i.e. it makes sense to demand $P^2 = 1$ at the level of the operator. In spacetime, introducing parity enlarges the proper orthochronous Lorentz group $\text{SO}^+(d, 1)$ to the orthochronous Lorentz group $\text{O}^+(d, 1)$.

In the quantum theory, P is represented by a linear operator. A scalar field $\phi(t, \vec{x})$ is also an operator, and as such it transforms under parity by conjugation with P . By definition, the result of this operation should flip the position vector \vec{x} and we expect the following relationships

$$\begin{aligned} P\phi(t, \vec{x})P^{-1} &= \eta_P \phi(t, -\vec{x}), \\ P\phi^\dagger(t, \vec{x})P^{-1} &= \eta_P^* \phi^\dagger(t, -\vec{x}), \end{aligned} \quad (4.83)$$

³⁰The double cover of $\text{O}(d)$ is called $\text{Pin}(d)$ in analogy to $\text{Spin}(d)$ which is the double cover of $\text{SO}(d)$.

³¹For even d this transformation would be a rotation rather than a reflection, and there is no reflection that commutes with all rotations.

where the constant η_P is the intrinsic parity of the field ϕ . We want that two consecutive parity transformations equal the identity $P^2 = 1$, therefore the parity of scalar fields can be either positive or negative, $\eta_P = \pm 1$. Typically, the vacuum state has positive parity, $P|0\rangle = |0\rangle$.

For the creation and annihilation operators the above relationship implies a transformation which reverses the momentum

$$\begin{aligned} Pa(\vec{p})P^{-1} &= \eta_P a(-\vec{p}), & Pa^\dagger(\vec{p})P^{-1} &= \eta_P a^\dagger(-\vec{p}), \\ Pb(\vec{p})P^{-1} &= \eta_P b(-\vec{p}), & Pb^\dagger(\vec{p})P^{-1} &= \eta_P b^\dagger(-\vec{p}). \end{aligned} \quad (4.84)$$

This is a unitary operation because $P^\dagger = P = P^{-1}$.

Time Reversal. The other discrete transformation of the Lorentz group is time reversal

$$T : (t, \vec{x}) \mapsto (-t, \vec{x}). \quad (4.85)$$

It enlarges the orthochronous Lorentz group $O^+(d, 1)$ to the complete Lorentz group $O(d, 1)$.

Time reversal is a rather special transformation due to the distinguished role of time in quantum mechanics and special relativity.

For a complex scalar field ϕ we expect

$$\begin{aligned} T\phi(t, \vec{x})T^{-1} &= \eta_T \phi(-t, \vec{x}), \\ T\phi^\dagger(t, \vec{x})T^{-1} &= \eta_T^* \phi^\dagger(-t, \vec{x}). \end{aligned} \quad (4.86)$$

Comparing to the mode expansion of fields, this could be implemented by a linear transformation of the type $a^\dagger(\vec{p}) \mapsto b(-\vec{p})$. In fact, it makes perfect sense that time reversal interchanges particle creation and annihilation as much as it flips the sign of energy E . However, such a transformation could not act consistently within a Fock space because all states of the latter have positive definite energy, and naively, T would seem to annihilate all states but the vacuum.³²

Instead, time reversal (sometimes called motion reversal) is defined by an anti-linear operator \bar{T} ,³³ which also conjugates plain complex numbers

$$\bar{T} a|X\rangle = a^* \bar{T}|X\rangle. \quad (4.87)$$

This inverts the plane wave factors $e^{\pm i p \cdot x}$ of the mode expansion of $\phi(t, \vec{x})$, and allows to map creation to creation operators $a^\dagger \mapsto a^\dagger$ such that the structure of

³²More accurately, time reversal would map the ordinary vacuum $|0\rangle$ to the vacuum $|0'\rangle$ of a conjugate Fock space with negative definite energy. Such an alternative vacuum is annihilated by all creation operators a^\dagger, b^\dagger and therefore it is analogous to the hermitian conjugate vacuum $\langle 0|$. In fact, this identification naturally makes the linear operation equivalent to the conventional anti-linear operation upon hermitian conjugation.

³³We denote this operation by \bar{T} in order to emphasise its anti-linearity; more commonly it is denoted by T in the literature.

Fock space is preserved.³⁴ More explicitly

$$\begin{aligned}\bar{T}a(\vec{p})\bar{T}^{-1} &= \eta_{\bar{T}}^* a(-\vec{p}), & \bar{T}a^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{T}} a^\dagger(-\vec{p}), \\ \bar{T}b(\vec{p})\bar{T}^{-1} &= \eta_{\bar{T}} b(-\vec{p}), & \bar{T}b^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{T}}^* b^\dagger(-\vec{p}).\end{aligned}\tag{4.88}$$

The difference with respect to parity is merely the anti-linear feature of \bar{T} . Time reversal actually allows for a complex $\eta_{\bar{T}}$ only restricted by $|\eta_{\bar{T}}|^2 = 1$. For the vacuum we may define $\bar{T}|0\rangle = |0\rangle$.

Charge Conjugation. Also the internal symmetry groups can come along with several connected components. For example the complex scalar field has a global $U(1) = SO(2)$ symmetry. This can be extended to $O(2)$ by adding a charge conjugation symmetry.

We already know that the complex conjugate scalar field ϕ^* or ϕ^\dagger satisfies the same equations of motion as the original field ϕ . Charge conjugation symmetry thus maps between the fields ϕ and ϕ^\dagger

$$\begin{aligned}C\phi(x)C^{-1} &= \eta_C \phi^\dagger(x), \\ C\phi^\dagger(x)C^{-1} &= \eta_C^* \phi(x).\end{aligned}\tag{4.89}$$

Requiring that two charge conjugations square to unity, the parity η_C must be on the complex unit circle $|\eta_C|^2 = 1$.

$$\begin{aligned}Ca(\vec{p})C^{-1} &= \eta_C^* b(\vec{p}), & Ca^\dagger(\vec{p})C^{-1} &= \eta_C b^\dagger(\vec{p}), \\ Cb(\vec{p})C^{-1} &= \eta_C a(\vec{p}), & Cb^\dagger(\vec{p})C^{-1} &= \eta_C^* a^\dagger(\vec{p}).\end{aligned}\tag{4.90}$$

Although C maps $\phi \mapsto \phi^\dagger$, it is a perfectly linear map. Charge conjugation is not complex conjugation. One might as well make an anti-linear ansatz for C , but it would lead to a transformation of the kind $a^\dagger \mapsto a$ which could not act within Fock space.

There are several conceptual difficulties with charge conjugation parity:

- In the presence of a corresponding internal symmetry the parity η_C actually does not have deeper meaning. In this case one can define a new charge conjugation operation C' by conjugating C with the internal symmetry. This would lead to a different η_C , and it makes sense to choose C' such that $\eta_C = 1$.
- Even if there is no continuous internal symmetry, there can be discrete internal symmetries. For example, a possible transformation for a real scalar field is $\phi \mapsto -\phi$.
- In a model with multiple fields, several independent internal parities can coexist, and there may not be a distinguished charge conjugation symmetry. In general, one would expect C to invert all internal charges.

³⁴The idea underlying the anti-linear operation \bar{T} is to view the reverse time evolution of the original system as forward time evolution of some mirror system. The spaces of states for these two systems are formally not the same, but they should be the equivalent, hence we can identify their Fock spaces and let \bar{T} act within it.

- In the presence of some internal parity C , the spacetime parities P and T become somewhat ambiguous, as one could define $P' = PC$.³⁵

Hence the choice of discrete symmetries C , P , T can be ambiguous, and it is sometimes tricky to identify the most suitable (or the established) one.³⁶

Implications. A discrete transformation is a symmetry of a Hamiltonian system if it commutes with the Hamiltonian. It is natural to assume parity and time reversal as symmetries of relativistic QFT models and of nature. For a single scalar field, it appears impossible to violate parity or time reversal. However, as we shall see, this need not be so for other types of fields. In nature, indeed, some of these symmetries are violated.

Discrete symmetries also lead to conservation laws in the quantum theory. States can be classified by their eigenvalue (parity) under the discrete symmetry. Typically these parities take only finitely many values (e.g. $+1$ or -1).

³⁵This may appear strange at first sight as P' would conjugate the field ϕ . But by writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ we merely get two fields with opposite parities η_P .

³⁶For example, the statement that a system has a parity symmetry does not imply that the parity acts in the conventional fashion; it merely meant that there is some definition of parity that is a symmetry, e.g. by transforming the fields in a non-diagonal fashion.

5 Free Spinor Field

We have seen that next to the scalar field there exist massive representations of the Poincaré algebra with spin. The next higher case is spin $j = 1/2$. It is described by the Dirac equation, and as a field with half-integer spin it should obey Fermi statistics.

5.1 Dirac Equation and Clifford Algebra

Dirac Equation. Dirac attempted to overcome some of the problems of relativistic quantum mechanics by introducing a first-order wave equation.^{1 2}

$$\gamma^\mu \partial_\mu \psi - m\psi = 0. \quad (5.1)$$

Here, the γ^μ are some suitably chosen operators acting locally on the wave function ψ . This wave equation can be viewed as a factorisation of the second-order Klein–Gordon equation as follows:

$$(\gamma^\nu \partial_\nu + m)(\gamma^\mu \partial_\mu - m)\psi = (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m^2)\psi = 0. \quad (5.2)$$

The latter form becomes the Klein–Gordon equation provided that the γ 's satisfy the Clifford algebra^{3 4}

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \quad (5.3)$$

This means that every solution of the Dirac equation also satisfies the Klein–Gordon equation and thus describes a particle of mass m .

Clifford Algebra. The Clifford algebra obviously cannot be realised in terms of plain numbers, but finite-dimensional matrices suffice. The realisation of the Clifford algebra strongly depends on the dimension and signature of spacetime.

¹The combination of a gamma-matrix and an ordinary vector $\gamma^\mu B_\mu$ is often denoted by a slashed vector \not{B} ; however, it is questionable whether this notation improves readability. As a less common alternative one could simply write B (without vector index) as a short form of $\gamma^\mu B_\mu$.

²The sign in front of the mass term has no physical significance as long as chosen consistently. Analogously, the mass m could be a negative number, but for simplicity will always assume $m \geq 0$.

³The indices of the two derivatives are automatically symmetric, hence only the symmetrisation of $\gamma^\mu \gamma^\nu$ must equal $\eta^{\mu\nu}$.

⁴We use gamma-matrices for signature $-+++$ which implies that some factors of -1 and i are needed to match with other conventions used in the literature.

The simplest non-trivial case is three-dimensional space (without time). A representation of the corresponding Clifford algebra is given by the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.4)$$

One can convince oneself that these matrices obey the algebra⁵

$$\sigma^j \sigma^k = \delta^{jk} + i \varepsilon^{jkl} \sigma^l. \quad (5.5)$$

This also implies the three-dimensional Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \quad (5.6)$$

In this course we will be predominantly interested in the case of $d = 3$ spatial dimensions plus time, i.e. spacetime with $D = d + 1 = 4$ dimensions. There, the smallest non-trivial representation of the Clifford algebra is four-dimensional (coincidence!). The elements of this four-dimensional vector space are called spinors, more precisely, Dirac spinors or 4-spinors.

There are many equivalent ways to write this representation as 4×4 matrices. The best-known ones are the Dirac, Weyl and Majorana representations. These are often presented in a block form of 2×2 matrices whose elements are again 2×2 matrices. The latter are written using the Pauli matrices σ^i or the 2×2 unit matrix 1. We shall mainly use the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}. \quad (5.7)$$

One can easily confirm that these matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ by means of the three-dimensional Clifford algebra. A useful property of the Weyl representation is that all four gamma-matrices are block off-diagonal. The Dirac and Majorana representations have different useful properties. In most situations, it is however convenient not to use any of the explicit representations, but work directly with the abstract Clifford algebra.

Solutions. The Dirac equation is homogeneous, therefore it is conveniently solved by Fourier transformation

$$\psi(x) = \int d^4p \, e^{ip \cdot x} \psi(p), \quad (ip_\mu \gamma^\mu - m)\psi(p) = 0. \quad (5.8)$$

To construct the solutions, let us introduce the matrices

$$\Pi^\pm = \frac{1}{2m} (m \mp ip \cdot \gamma) \quad (5.9)$$

⁵We will not introduce a distinguished symbol for unit matrices. 1 is the unit element. Here, the term δ^{ij} has an implicit 2×2 unit matrix.

such that the Dirac equation becomes $\Pi^+\psi = 0$. We are interested in the kernel of Π^+ .

As noted above, we have the identity

$$\Pi^-\Pi^+\psi = \frac{1}{4m^2}(m^2 + p^2)\psi. \quad (5.10)$$

This operator acts identically on all components of ψ . Any solution therefore requires the mass shell condition $p^2 = -m^2$.

On the mass shell $p^2 = -m^2$, the operators Π^\pm act as a complete set of orthogonal projectors:

$$\Pi^\pm\Pi^\pm = \Pi^\pm, \quad \Pi^\pm\Pi^\mp = 0, \quad \Pi^+ + \Pi^- = 1. \quad (5.11)$$

Both of them have half-maximal rank, $\text{tr}\Pi^\pm = 2$. The Dirac equation thus has two solutions for each on-shell momentum p .

A basis of two positive-energy solutions is denoted by

$$u_\alpha(\vec{p}), \quad \alpha = \pm, \quad (i\vec{p}\cdot\gamma - m)u_\alpha(\vec{p}) = 0. \quad (5.12)$$

Instead of introducing negative-energy solutions, we prefer to consider equivalent positive-energy solutions of the opposite Dirac equation $\Pi^-\psi = 0$ ⁶

$$v_\alpha(\vec{p}), \quad \alpha = \pm, \quad (i\vec{p}\cdot\gamma + m)v_\alpha(\vec{p}) = 0. \quad (5.13)$$

To write such solutions explicitly, we can recycle the projectors Π^\pm and set

$$u = \Pi^-\lambda, \quad v = \Pi^+\lambda, \quad (5.14)$$

where λ is some spinor. The properties of the projectors immediately show that u and v are solutions to their respective equations. Note, however, that some components of λ are projected out in u and in v .

Let us consider explicitly solutions in the Weyl representation. E.g. setting $\lambda = (\kappa, 0)$ with κ some 2-spinor, we find

$$\begin{aligned} u(\vec{p}) &= \frac{1}{2m} \begin{pmatrix} m & -i\vec{e}(\vec{p}) + i\vec{p}\cdot\vec{\sigma} \\ i\vec{e}(\vec{p}) + i\vec{p}\cdot\vec{\sigma} & m \end{pmatrix} \begin{pmatrix} \kappa \\ 0 \end{pmatrix} \\ &= \frac{1}{2m} \begin{pmatrix} m\kappa \\ i\vec{e}(\vec{p})\kappa + i\vec{p}\cdot\vec{\sigma}\kappa \end{pmatrix}, \\ v(\vec{p}) &= \frac{1}{2m} \begin{pmatrix} m\kappa \\ -i\vec{e}(\vec{p})\kappa - i\vec{p}\cdot\vec{\sigma}\kappa \end{pmatrix}. \end{aligned} \quad (5.15)$$

There are two independent choices for the 2-spinor κ , hence there are two solutions for u and v , respectively. One typically considers $u_\gamma(\vec{p})$, $v_\gamma(\vec{p})$, $\gamma = \pm$, as two pairs of fixed basis vectors for each momentum \vec{p} .

⁶At the level of the Dirac equation the replacement $m \rightarrow -m$ is equivalent to $p \rightarrow -p$.

Altogether, the general solution can now be expanded as⁷

$$\psi(x) = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} \sum_{\gamma=\pm} (e^{ip \cdot x} u_\gamma(\vec{p}) b_\gamma(\vec{p}) + e^{-ip \cdot x} v_\gamma(\vec{p}) a_\gamma^\dagger(\vec{p})). \quad (5.16)$$

Here, the coefficient b_γ is chosen differently from a_γ because gamma-matrices are generally complex and therefore also the Dirac spinor ψ is complex. This means that the anti-particles (e.g. positrons) are different from the particles (e.g. electrons).

5.2 Poincaré Symmetry

The Dirac equation is a relativistic wave equation. Translational invariance is evident, but we have not yet shown its Lorentz covariance (although the resulting Klein–Gordon equation certainly is covariant).

Lorentz Symmetry. Let us therefore consider a Lorentz transformation $x' = \Lambda^{-1}x$ with $\Lambda(\omega) = \exp(\omega)$. Suppose ψ is a solution of the Dirac equation. It is not sufficient to use the transformation rule for scalar fields $\psi'(x') = \psi(x)$. In analogy to vectors we should also transform spinors. We make the ansatz

$$\psi'(x') = S(\omega)\psi(x), \quad (5.17)$$

where $S(\omega)$ is a matrix that acts on Dirac spinors. We then substitute $\psi'(x) = S\psi(\Lambda x)$ into the Dirac equation

$$\begin{aligned} 0 &= (\gamma^\mu \partial_\mu - m)\psi'(x) = (\gamma^\mu \partial_\mu - m)S\psi(\Lambda x) \\ &= (\gamma^\nu S \Lambda^\mu{}_\nu \partial_\mu \psi - S m \psi)(\Lambda x) \\ &= S(S^{-1} \gamma^\nu S \Lambda^\mu{}_\nu \partial_\mu \psi - \gamma^\mu \partial_\mu \psi)(\Lambda x) \\ &= S(\Lambda^\mu{}_\nu S^{-1} \gamma^\nu S - \gamma^\mu)(\partial_\mu \psi)(\Lambda x). \end{aligned} \quad (5.18)$$

So the term in the bracket must vanish for invariance of the Dirac equation.

Indeed, the canonical Lorentz transformation of gamma-matrices

$$\gamma'^\mu = (\Lambda^{-1})^\mu{}_\nu S \gamma^\nu S^{-1}, \quad (5.19)$$

where not only the vector index is transformed by Λ^{-1} , but also the spinor matrix is conjugated by the corresponding spinor transformation S .⁸ In analogy to the invariance of the Minkowski metric, $\eta' = \eta$, the Dirac equation is invariant if the gamma-matrices are invariant

$$\gamma'^\mu = \gamma^\mu. \quad (5.20)$$

⁷By convention p is integrated over the positive mass shell with $p^0 = +e(\vec{p})$. We use the positive-energy solution $v_\alpha(\vec{p})$ of the opposite Dirac equation as a substitute for the equivalent negative-energy solution of the original Dirac equation using the map $p \rightarrow -p$, $m \rightarrow -m$.

⁸In general, spinors are transformed by S from the left, co-spinors by the S^{-1} from the right.

This condition relates S to the Lorentz transformation Λ .

The infinitesimal form of the invariance condition reads

$$[\delta S, \gamma^\mu] - \delta\omega^\mu{}_\nu \gamma^\nu = 0. \quad (5.21)$$

This implies that δS must be proportional to $\delta\omega_{\mu\nu}$. The latter carries two vector indices, while δS carries none. The only possibility is to contract the vector indices to gamma-matrices, and we make the ansatz $\delta S = \frac{1}{2}\alpha \delta\omega_{\mu\nu} \gamma^\mu \gamma^\nu$. Substituting this into the invariance condition and using

$$[\gamma^\rho \gamma^\sigma, \gamma^\mu] = \gamma^\rho \{\gamma^\sigma, \gamma^\mu\} - \{\gamma^\rho, \gamma^\mu\} \gamma^\sigma, \quad (5.22)$$

we arrive at $-(2\alpha + 1)\delta\omega^\mu{}_\nu \gamma^\nu = 0$. We conclude that a Lorentz transformation for spinors is given by the matrix

$$\delta S = -\frac{1}{4}\delta\omega_{\mu\nu} \gamma^\mu \gamma^\nu \quad \text{or} \quad S(\omega) = \exp\left(-\frac{1}{4}\omega_{\mu\nu} \gamma^\mu \gamma^\nu\right). \quad (5.23)$$

Comparing this result to the abstract form of finite Lorentz transformations as $U(\omega) = \exp(\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu})$ we have derived a new representation on spinors⁹

$$M^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (5.24)$$

This representation obeys the Lorentz algebra derived above, i.e.

$$[M^{\mu\nu}, M^{\rho\sigma}] = iM + \dots$$

Double Cover. Spinor representations exist only for the double cover $\text{Spin}(N)$ of an orthogonal group $\text{SO}(N)$. Let us observe this fact in a simple example.

Consider a rotation in the x, y -plane with angle $\omega_{12} = -\omega_{21} = \varphi$. The associated finite Lorentz transformation matrix in the x, y -plane reads

$$\Lambda(\varphi) = \exp(\omega) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}. \quad (5.25)$$

The associated spinor transformation reads

$$S(\varphi) = \text{diag}(e^{-i\varphi/2}, e^{+i\varphi/2}, e^{-i\varphi/2}, e^{+i\varphi/2}). \quad (5.26)$$

The vector rotation $\Lambda(\varphi)$ is 2π -periodic in φ whereas the spinor rotation is merely 4π -periodic. The rotation by $\varphi = 2\pi$ is represented by the unit matrix for vectors, but for spinors it is the negative unit matrix

$$\Lambda(2\pi) = 1 = (-1)^F, \quad S(2\pi) = -1 = (-1)^F. \quad (5.27)$$

The spin group thus has an element which represents a rotation by 2π (irrespective of the direction). On vector representations (integer spin) it acts as the identity, on spinor representations (half-integer spin) it acts as -1 . Due to the relation between spin and statistics, the extra element is equivalent to $(-1)^F$ where F measures the number of fermions (odd for spinors, even for vectors).

⁹We have to make sure that $M^{\mu\nu}$ is anti-symmetric in its vector indices, hence the explicit anti-symmetrisation.

Chiral Representation. There is an important feature of the spin representation $M^{\mu\nu}$ which is best observed in the Weyl representation of gamma-matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (5.28)$$

Here we have introduced the sigma-matrices $\sigma^\mu, \bar{\sigma}^\mu$ as an extension of the Pauli matrices σ^k to four spacetime dimensions as follows

$$\sigma^0 = -\bar{\sigma}^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\sigma}^k := \sigma^k. \quad (5.29)$$

The Lorentz representation now reads

$$M^{\mu\nu} = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}. \quad (5.30)$$

This representation has block-diagonal form and therefore reduces to two independent representations $M^{\mu\nu} = \text{diag}(M_L^{\mu\nu}, M_R^{\mu\nu})$ with

$$M_L^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \quad M_R^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu). \quad (5.31)$$

In other words, the Dirac spinor $\psi = (\psi_L, \psi_R)$ transforms in the direct sum of two (irreducible) representations of the Lorentz group. The 2-spinors ψ_L and ψ_R are called left-chiral and right-chiral spinors. The massive Dirac equation, however, mixes these two representations

$$\begin{aligned} \sigma^\mu \partial_\mu \psi_R - m \psi_L &= 0, \\ \bar{\sigma}^\mu \partial_\mu \psi_L - m \psi_R &= 0. \end{aligned} \quad (5.32)$$

It is therefore convenient to use Dirac spinors for massive spinor particles whereas massless spinor particles can also be formulated using 2-spinors; we shall discuss the massless case later on.

The decomposition into chiral parts is not just valid in the Weyl representation of the Clifford algebra. More abstractly, it is due to the existence of the matrix

$$\gamma^5 = \frac{i}{24} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = i \gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (5.33)$$

In the Weyl representation it reads $\gamma^5 = \text{diag}(-1, +1)$, it therefore measures the chirality of spinors. In general, it anti-commutes with all the other gamma-matrices,

$$\{\gamma^5, \gamma^\mu\} = 0. \quad (5.34)$$

This property implies that a single gamma-matrix maps between opposite chiralities, i.e. it inverts chirality. The property is also sufficient to prove commutation with $M^{\mu\nu}$. Alternatively, it follows by construction of γ^5 as a (pseudo)-scalar combination of gamma-matrices.

A further useful property is

$$\gamma^5 \gamma^5 = 1. \quad (5.35)$$

It can be used to show that the combinations $\frac{1}{2}(1 \pm \gamma^5)$ are two orthogonal projectors to the chiral subspaces.

Sigma-Matrices. Let us briefly discuss the sigma-matrices which are chiral analogs of the gamma-matrices. The sigma-matrices obey an algebra reminiscent of the Clifford algebra^{10 11}

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu. \quad (5.36)$$

The two variants of the sigma-matrices are related by transposition (modulo conjugation by the anti-symmetric matrix ε)

$$\varepsilon(\sigma^\mu)^\top \varepsilon^{-1} = -\bar{\sigma}^\mu, \quad \varepsilon = -\varepsilon^{-1} := i\sigma^2 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \quad (5.37)$$

Inspection shows that all sigma-matrices are hermitian

$$(\sigma^\mu)^\dagger = \sigma^\mu, \quad (\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu. \quad (5.38)$$

Since there are 4 independent 2×2 hermitian matrices, the four sigma-matrices σ^μ (or equivalently $\bar{\sigma}^\mu$) form a real basis for such matrices. Likewise, the 6 matrices $M_L^{\mu\nu}$ (or equivalently $M_R^{\mu\nu}$) form a real basis of 2×2 complex traceless matrices.¹² Furthermore,

$$(M_L^{\mu\nu})^\dagger = M_R^{\mu\nu}. \quad (5.39)$$

These are just the defining relations for the fundamental representation of $\mathfrak{sl}(2, \mathbb{C})$ along with its conjugate representation.¹³

The Lorentz algebra $\mathfrak{so}(3, 1)$ is indeed equivalent to the algebra $\mathfrak{sl}(2, \mathbb{C})$. At the level of groups, $\text{Spin}^+(3, 1) = \text{SL}(2, \mathbb{C})$ is the double cover of $\text{SO}^+(3, 1)$.

- A chiral 2-spinor of $\text{Spin}^+(3, 1)$ transforms in the fundamental representation of $\text{SL}(2, \mathbb{C})$.
- Similarly, a 2-spinor of opposite chirality transforms in the conjugate fundamental representation of $\text{SL}(2, \mathbb{C})$.
- Spinor representations exist only for the double-cover group $\text{Spin}^+(3, 1)$, but not for the original Lorentz group $\text{SO}^+(3, 1)$.

5.3 Discrete Symmetries

In addition to the continuous Poincaré symmetry and an obvious $\text{U}(1)$ internal symmetry, there are several discrete symmetries and transformations which we shall now discuss. These are also needed to formulate a Lagrangian.

¹⁰The assignment of bars enables a 2-dimensional representation for this algebra unlike the Clifford algebra which requires a larger 4-dimensional representation.

¹¹Note that for any reasonable product of sigma-matrices the sequence of factors will alternate between σ and $\bar{\sigma}$. This agrees with the fact that a single γ maps between the two chiralities.

¹²A complex traceless 2×2 matrix has $2 \cdot 2 - 1 = 3$ complex degrees of freedom which are equivalent to 6 real ones.

¹³The group $\text{SL}(N)$ of matrices with unit determinant is generated by the algebra $\mathfrak{sl}(N)$ of traceless matrices.

Parity. Spatial parity $\vec{x}' = -\vec{x}$ is the simplest discrete symmetry. We make the usual ansatz

$$\psi'(t, -\vec{x}) = \gamma_P \psi(t, \vec{x}), \quad (5.40)$$

where γ_P is a matrix that induces the reflection on spinors.

The new field obeys the same old Dirac equation provided that the gamma-matrices are invariant

$$\gamma'^{\mu} := \Lambda^{\mu}_{\nu} \gamma_P \gamma^{\nu} \gamma_P^{-1} \stackrel{!}{=} \gamma^{\mu}. \quad (5.41)$$

We need to find a matrix γ_P that

- commutes with γ^0 (because $\Lambda^0_0 = 1$),
- anti-commutes with γ^k (to compensate $\Lambda^k_k = -1$),
- squares to unity (because $P^2 = 1$).¹⁴

This matrix is easily identified (up to a sign) as

$$\gamma_P = -i\gamma^0. \quad (5.42)$$

Note that γ_P interchanges the two chiralities. Hence the Dirac spinor is

- reducible to chiral spinors under proper orthochronous Lorentz rotations,
- but irreducible under orthochronous Lorentz rotations which include the spatial reflections.

Time Reversal. Anti-linear time (motion) reversal also has a representation on spinors

$$\psi'(-t, \vec{x}) = \gamma_{\bar{T}} \psi(t, \vec{x}). \quad (5.43)$$

The anti-linear nature of \bar{T} implies that a solution of the Dirac equation should be mapped to a solution of the complex conjugated Dirac equation. In the Weyl representation this is achieved by the matrix

$$\gamma_{\bar{T}} = i\gamma^0\gamma^2. \quad (5.44)$$

The gamma-matrices satisfy the following identity with the time reversal matrix

$$\Lambda^{\mu}_{\nu} \gamma_{\bar{T}} (\gamma^{\nu})^* \gamma_{\bar{T}}^{-1} = \gamma^{\mu}. \quad (5.45)$$

Charge Conjugation. The Dirac field is charged, it therefore makes sense to define charge conjugation. We will use it later to investigate the statistics associated to spinor fields.

Linear charge conjugation maps a field to its conjugate field¹⁵

$$\psi'(x) = \gamma_C \psi^{\dagger\top} \quad (5.46)$$

¹⁴In fact, it may also square to a rotation of 2π , i.e. the element $(-1)^F$, which is achieved by the replacement by $\gamma_P \rightarrow i\gamma_P$.

¹⁵The composition of adjoint and transpose operations is almost the same as complex conjugation. There is however a slight difference which becomes relevant only later.

such that ψ' solves the same wave equation as ψ . Let us substitute

$$(\gamma^\mu \partial_\mu - m)\psi' = (\gamma^\mu \partial_\mu - m)\gamma_C \psi^{\dagger\tau} = [((\gamma^\mu)^* \partial_\mu - m)\gamma_C^* \psi]^{\dagger\tau}. \quad (5.47)$$

This vanishes if

$$\gamma_C (\gamma^\mu)^* \gamma_C^{-1} = \gamma^\mu. \quad (5.48)$$

In the Weyl representation only γ^2 is imaginary, and the condition is solved by the matrix

$$\gamma_C = i\gamma^0 \gamma^1 \gamma^3. \quad (5.49)$$

CPT-Transformation In QFT a discrete transformation of fundamental importance is the combination of charge conjugation, parity and time reversal, called CPT. Effectively, it flips the sign of all coordinates¹⁶ and performs a complex conjugation.

A spinor transforms according to

$$\psi'(x) = \gamma_{\bar{T}} \gamma_P \gamma_C \psi^{\dagger\tau}(-x). \quad (5.50)$$

We find that the combination of matrices is just the additional gamma-matrix γ^5

$$\gamma_{\bar{T}} \gamma_P \gamma_C = i\gamma^0 \gamma^2 \gamma^0 \gamma^0 \gamma^1 \gamma^3 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5. \quad (5.51)$$

This anti-commutes with all gamma-matrices

$$\gamma^5 \gamma^\mu \gamma^5 = -\gamma^\mu. \quad (5.52)$$

The sign is compensated by flipping the sign of all vectors.

The CPT-theorem states that all reasonable relativistic QFT's must be invariant under the CPT-transformation. They need not be invariant under any of the individual transformations.

Hermitian Conjugation. The Dirac spinor ψ is complex. To construct real quantities for use in the Lagrangian or the Hamiltonian one typically uses hermitian conjugation. However, the various gamma-matrices transform differently under this operation.

The transformation can be uniformised by conjugation with some matrix γ_{\dagger} such that¹⁷

$$\gamma_{\dagger} (\gamma^\mu)^\dagger \gamma_{\dagger}^{-1} = -\gamma^\mu. \quad (5.53)$$

In most relevant representations, in particular in the chiral one, one finds

$$\gamma_{\dagger} = -i\gamma^0. \quad (5.54)$$

¹⁶This is an orientation-preserving transformation which belongs to $\text{Spin}(3,1)$, but not to $\text{Spin}^+(3,1)$.

¹⁷Hermitian conjugation can be defined with a plus or minus on the r.h.s. leading to two distinct operations. Consistent physics requires the minus sign.

Therefore, one should modify hermitian conjugation for a spinor ψ and likewise for a spinor matrix X as

$$\bar{\psi} = \psi^\dagger \gamma_4^{-1}, \quad \bar{X} = \gamma_4 X^\dagger \gamma_4^{-1}. \quad (5.55)$$

The gamma-matrices are anti-hermitian under hermitian conjugation, $\bar{\gamma}^\mu = -\gamma^\mu$, with respect to the scalar product defined by γ_4 .

5.4 Spin Statistics

So far we have only considered the Dirac equation. For quantisation, conserved charges and later for adding interactions we should construct a Lagrangian.

Lagrangian. It is straight-forward to guess

$$\mathcal{L} = \bar{\psi}(\gamma^\mu \partial_\mu - m)\psi. \quad (5.56)$$

The variation with respect to ψ^\dagger obviously yields the Dirac equation. Variation with respect to ψ gives the hermitian conjugate equation

$$-\partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi} = (\gamma^\mu \partial_\mu \psi - m\psi)^\dagger \gamma_4^{-1} = 0. \quad (5.57)$$

In fact, the Lagrangian is almost real

$$\begin{aligned} \mathcal{L}^\dagger &= \psi^\dagger ((\gamma^\mu)^\dagger \partial_\mu^\dagger - m) \gamma_4^{-1} \psi = \bar{\psi} (-\gamma^\mu \partial_\mu^\dagger - m) \psi \\ &= \mathcal{L} - \partial_\mu (\bar{\psi} \gamma^\mu \psi). \end{aligned} \quad (5.58)$$

There is merely an imaginary topological term left,¹⁸ the action is manifestly real.

Hamiltonian Formulation. To go to the Hamiltonian framework we compute the conjugate momenta $\pi = \partial \mathcal{L} / \partial \dot{\psi}^\dagger = 0$ and $\pi^\dagger = \partial \mathcal{L} / \partial \dot{\psi} = i\psi^\dagger$. It turns out that the conjugate momenta are proportional to the fields without (time) derivatives:¹⁹

- Dirac equation is a first-order differential equation.
- The dynamical data on a time slice consists of the fields ψ and $\bar{\psi}$ alone.
- There is no need for independent momenta keeping track of time derivatives; the latter are determined by the equations of motion.
- Phase space equals position space.
- ψ and ψ^\dagger are canonically conjugate fields.

¹⁸The topological term can be removed from the Lagrangian to obtain a manifestly real $\mathcal{L}' = \mathcal{L} - \frac{1}{2} \partial_\mu (\bar{\psi} \gamma^\mu \psi)$.

¹⁹Here, π^\dagger is not the complex conjugate of π because \mathcal{L} is not real. For the real \mathcal{L}' we get instead $\pi = \frac{i}{2} \psi^\dagger$ and $\pi^\dagger = -\frac{i}{2} \psi$ in which case the definition of the canonical structure is even less evident. The point is that the Lagrangian is already in the first-order form of the Hamiltonian with ψ^\dagger and ψ being canonically conjugate.

We shall not follow the canonical framework towards quantisation because it has some complications that we cannot yet understand. Instead, we can compute the stress-energy tensor of which the Hamiltonian is a component

$$T^{\mu\nu} = -\bar{\psi}\gamma^\mu\partial^\nu\psi + \eta^{\mu\nu}\mathcal{L}. \quad (5.59)$$

Unfortunately this tensor is not symmetric as it should be due to Lorentz invariance. Gladly, the anti-symmetric part can be written as (making use of the equations of motion)²⁰

$$T^{[\mu\nu]} = \partial_\rho(\bar{\psi}\gamma^{[\rho}\gamma^\mu\gamma^{\nu]}\psi) = \partial_\rho K^{\rho\mu\nu}. \quad (5.60)$$

The contribution from K is a boundary term for the integral defining the total momentum integral P^μ . We can thus subtract $\partial_\rho K^{\rho\mu\nu}$ from $T^{\mu\nu}$ to make the latter symmetric.

The Hamiltonian for the Dirac equation now reads

$$H = \int d\vec{x}^3 T^{00} = \int d\vec{x}^3 \bar{\psi}(-\vec{\gamma}\cdot\vec{\partial} + m)\psi. \quad (5.61)$$

Charge Conjugation. A naive treatment and quantisation of the above framework of the Dirac equation leads to several undesirable features.

For example, there is an issue for charge conjugation. For every solution ψ of the Dirac equation, there is a charge conjugate solution $\psi_C = \gamma_C\psi^{\dagger\top}$. Let us compute its energy²¹

$$\begin{aligned} H[\psi_C] &= \int d\vec{x}^3 \psi^\top \gamma_C^\dagger \gamma_\dagger^{-1} (-\vec{\gamma}\cdot\vec{\partial} + m) \gamma_C \psi^{\dagger\top} \\ &= \int d\vec{x}^3 \psi^\top \gamma_C^\dagger \gamma_\dagger^{-1} \gamma_C (-\vec{\gamma}^* \cdot \vec{\partial} + m) \psi^{\dagger\top} \\ &= \int d\vec{x}^3 \psi^\top \gamma_\dagger^{-1} (-\vec{\gamma}^* \cdot \vec{\partial} + m) \psi^{\dagger\top} \\ &= \int d\vec{x}^3 \psi^\top (+\vec{\gamma}^\top \cdot \vec{\partial} + m) \gamma_\dagger^{-1} \psi^{\dagger\top} \\ &\stackrel{*}{=} - \int d\vec{x}^3 \psi^\dagger \gamma_\dagger^{-1} (+\vec{\gamma} \cdot \vec{\partial}^\top + m) \psi \\ &= - \int d\vec{x}^3 \bar{\psi} (-\vec{\gamma}\cdot\vec{\partial} + m) \psi = -H[\psi]. \end{aligned} \quad (5.62)$$

We notice that the charge conjugate solution has the opposite energy

$$H[\psi_C] = -H[\psi]. \quad (5.63)$$

²⁰Here, the square brackets within indices imply total anti-symmetrisation of all the enclosed indices.

²¹We denote by ∂^\top a derivative operator which acts towards the left. The ordinary derivative operator is obtained by partial integration and in the absence of boundary contributions we have $\partial^\top \simeq -\partial$.

This implies that there must exist solutions with positive and negative energy.

We compare this result to the complex scalar field where charge conjugation is defined by $\phi_C = \phi^*$. There we obtain a similar relationship for the energy,

$$H[\phi_C] = +H[\phi], \quad (5.64)$$

but the sign is the same for both solutions. This is consistent with the Hamiltonian being a positive definite functional and with the physical concept of positivity of the energy. Conversely, the naive field ψ does not have positive definite energy.

Related issues arise for the naive evaluation of propagators and for causality.

Almost all of the above steps are elementary and have to be accepted. Only the step marked by $*$ can be altered: transposition. We have used

$$\psi^\top X \psi^{\dagger\top} = \psi^a X_a{}^b \psi_b^\dagger = \psi_b^\dagger X_a{}^b \psi^a = \psi^\dagger X^\top \psi. \quad (5.65)$$

Instead of $\psi^a \psi_b^\dagger = \psi_b^\dagger \psi^a$ we could use a different rule²²

$$\psi^a \psi_b^\dagger = -\psi_b^\dagger \psi^a. \quad (5.66)$$

This change inserts a minus sign at $*$ and the energy of a solution and its charge conjugate become equal

$$H[\psi_C] = +H[\psi]. \quad (5.67)$$

This modification of the algebra of the fields ψ actually solves all the other issues of the spinor field as well.

Spin-Statistics Theorem. The spin-statistics theorem states that consistent quantisation of fields with half-integer spin requires the use of anti-commutation relations²³

$$\{\psi, \psi^\dagger\} \sim \hbar. \quad (5.68)$$

Such fields are called fermionic, they obey the Fermi–Dirac statistics. Multi-particle wave functions will be totally anti-symmetric.

Conversely, fields with integer spin require commutation relations

$$[\phi, \phi^\dagger] \sim \hbar. \quad (5.69)$$

These fields are called bosonic, they obey the Bose–Einstein statistics. Multi-particle wave functions will be totally symmetric.

²²In view of quantum mechanics, changing the commutation relationship is not as severe as it may seem: eventually, quantisation will make ψ 's become operators which do not commute either.

²³ $\{A, B\} := AB + BA$ denotes the anti-commutator of two operators A and B .

5.5 Graßmann Numbers

Quantisation can be viewed as a deformation of classical physics. Therefore, the anti-commutation relations $\{\psi, \psi^\dagger\} \sim \hbar$ for fermionic quantum operators (ψ, ψ^\dagger) should be reflected by anti-commuting fields $\{\psi, \psi^\dagger\} = 0$ in the classical theory. This is in direct analogy to the fact that the commutation relations $[\phi, \pi] \sim \hbar$ for bosonic quantum operators (ϕ, π) are reflected by commuting fields $[\phi, \pi] = 0$ in the classical theory. More explicitly, the various spin components of the classical fields should obey the relations

$$\psi^a \psi^b = -\psi^b \psi^a, \quad \psi^a \psi_b^\dagger = -\psi_b^\dagger \psi^a, \quad \psi_a^\dagger \psi_b^\dagger = -\psi_b^\dagger \psi_a^\dagger. \quad (5.70)$$

Besides these additional signs, the fields ψ will commute with numbers and scalar fields.

We therefore cannot use ordinary commuting numbers to represent the field ψ and its components in the classical Lagrangian; we need something else.²⁴

Description. The required extension of the concept of numbers is called Graßmann numbers:

- Graßmann numbers form a non-commutative ring.
- Graßmann numbers are \mathbb{Z}_2 -graded, they can be even or odd: $|a| = 0, 1$.²⁵
- Sums and products respect the even/odd grading

$$|a + b| = |a| = |b|, \quad |ab| = |a| + |b|. \quad (5.71)$$

- The product is commutative unless both factors are odd in which case it is anti-commutative:

$$ab = (-1)^{|a||b|}ba. \quad (5.72)$$

- Ordinary numbers are among the even Graßmann numbers.
- The field ψ takes values in odd Graßmann numbers.
- Real and complex Graßmann numbers can be defined. Graßmann numbers then form an algebra over the respective field.

A basis a^n of odd Graßmann numbers can be constructed out of a Clifford algebra $\{\gamma^j, \gamma^k\} = 2\delta^{jk}$

$$a^n = \frac{1}{\sqrt{2}} (\gamma^{2n} + i\gamma^{2n+1}). \quad (5.73)$$

In other words, Graßmann numbers can be represented in terms of (exponentially large) matrices. One should view the basis a^n to be sufficiently large or infinite.²⁶

²⁴Put differently, classical physics arises as a limit of quantum physics. The precise definition of the limit is important. For the Dirac equation there is a choice. Formulation of the fields in terms of ordinary commuting numbers leads to an ill-defined classical limit. Formulation of the fields in terms of anti-commuting numbers, as described below, leads to a well-defined classical limit, albeit with an extended notion of numbers. The new class of numbers allow to work with the Lagrangian and canonical framework with minor alterations.

²⁵Linear combinations of even and odd numbers could be defined, but usually they do not appear.

²⁶There is no distinguished element such as i which extends the real numbers to complex

Calculus. One can do calculus with Graßmann numbers much like ordinary numbers, but note:

- odd numbers square to zero: $(a^n)^2 = \frac{1}{2}\{a^n, a^n\} = 0$.
- the square root of zero is ill-defined.
- odd numbers have no inverse.
- some even numbers (e.g. products of two odd numbers) have no inverse.

A derivative for odd numbers can be defined as usual

$$\frac{\partial}{\partial a^m} a^n = \delta_m^n. \quad (5.74)$$

Note that derivatives are also odd objects

$$\{\partial/\partial a^m, \partial/\partial a^n\} = 0. \quad (5.75)$$

The above derivative relation can be written as an anti-commutator

$$\{\partial/\partial a^m, a^n\} = \delta_m^n. \quad (5.76)$$

We can also define the derivatives as elements of the same Clifford algebra

$$\frac{\partial}{\partial a^n} = \frac{1}{\sqrt{2}} (\gamma^{2n} - i\gamma^{2n-1}). \quad (5.77)$$

This leads to the same anti-commutation relations as above.

Complex Conjugation. A complex Graßmann number a can be written as a combination of the real Graßmann numbers a_r, a_i as

$$a = a_r + i a_i. \quad (5.78)$$

Spinor fields are typically complex and we often need to complex conjugate them. Confusingly, there are two equivalent definitions of complex conjugation for Graßmann numbers.

One is reminiscent of complex conjugation

$$a^* = a_r - i a_i. \quad (5.79)$$

It obviously satisfies

$$(ab)^* = a^* b^*. \quad (5.80)$$

The other conjugation is reminiscent of hermitian conjugation. It satisfies the rule that the order of factors within a product is flipped

$$(ab)^\dagger = b^\dagger a^\dagger. \quad (5.81)$$

numbers. We therefore do not have universal means to assign a value to a Graßmann variable. We will mainly use Graßmann variables to describe classical (fermionic) fields without assigning values.

For ordinary numbers it would be the same as complex conjugation, but this cannot hold for odd Grassmann numbers which do not commute. The two definitions are in fact related as follows

$$a^\dagger = \begin{cases} a^* & \text{if } a \text{ is even,} \\ -i a^* & \text{if } a \text{ is odd.} \end{cases} \quad (5.82)$$

Quantum mechanics and QFT frequently use the adjoint operation, hence it is convenient to stick to hermitian conjugation for Grassmann numbers as well.

One should pay close attention in defining real and odd Grassmann numbers: An odd number which satisfies $a^\dagger = a$ is not real. In particular, the even product of two such numbers is imaginary

$$(ab)^\dagger = b^\dagger a^\dagger = ba = -ab. \quad (5.83)$$

Instead, real odd numbers are defined by the relationship

$$a^\dagger = -i a \quad \text{or} \quad a^* = a. \quad (5.84)$$

5.6 Quantisation

Poisson Brackets. We have also seen that ψ and ψ^\dagger are canonically conjugate fields, there is no need to introduce additional conjugate momenta. The Poisson bracket for the spinor field should read²⁷

$$\{F, G\} = i \int d\vec{x}^3 \left(\frac{\delta F}{\delta \psi^a(\vec{x})} \frac{\delta G}{\delta \psi_a^\dagger(\vec{x})} + \frac{\delta F}{\delta \psi_a^\dagger(\vec{x})} \frac{\delta G}{\delta \psi^a(\vec{x})} \right). \quad (5.85)$$

Alternatively, the Poisson brackets can be expressed in the canonical form as^{28 29}

$$\{\psi^a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \{\psi_b^\dagger(\vec{y}), \psi^a(\vec{x})\} = i \delta_b^a \delta^3(\vec{x} - \vec{y}). \quad (5.86)$$

Anti-Commutators. Upon quantisation, these Poisson brackets are replaced by an anti-commutator³⁰

$$\{\psi^a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_b^a \delta^3(\vec{x} - \vec{y}). \quad (5.87)$$

By Fourier transformation to momentum space^{31 32 33}

$$\psi(x) = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} \left(e^{ip \cdot x} u_\alpha(\vec{p}) b_\alpha(\vec{p}) + e^{-ip \cdot x} v_\alpha(\vec{p}) a_\alpha^\dagger(\vec{p}) \right) \quad (5.88)$$

²⁷The correct normalisation can be derived from $\dot{F} = -\{H, F\}$.

²⁸Poisson brackets involving Grassmann numbers are not strictly anti-symmetric but rather graded anti-symmetric. In particular, they are symmetric for two Grassmann odd elements.

²⁹Unfortunately, the notation for Poisson brackets clashes with the one for anti-commutators.

³⁰The sign can be determined from the relation $[H, \psi^a(\vec{x})] = -i \dot{\psi}^a(\vec{x})$.

³¹The field $\psi(x)$ is now extended to all times by means of the equations of motion. In other words, $\psi(x)$ satisfies the equations of motion by construction.

³²A sum over repeated indices $\gamma, \delta, \dots = \pm$ is implicit.

³³Here, the factors u_α and v_α are ordinary even functions while the a_α and b_α are quantum operators obeying anti-commutation relations (or Grassmann odd fields in a classical setting).

we obtain anti-commutation relations for the Fourier modes³⁴

$$\begin{aligned}\{u_\gamma^a(\vec{p})b_\gamma(\vec{p}), \bar{u}_{\delta,b}(\vec{q})b_\delta^\dagger(\vec{q})\} &= (i\vec{p}\cdot\vec{\gamma} + m)^a_b 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}), \\ \{\bar{v}_{\gamma,b}(\vec{p})a_\gamma(\vec{p}), v_\delta^a(\vec{q})a_\delta^\dagger(\vec{q})\} &= (i\vec{p}\cdot\vec{\gamma} - m)^a_b 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}).\end{aligned}\quad (5.89)$$

It is convenient to split these relations into contributions from quantum operators a, b and contributions from spinor solutions u, v . The spinors u and v form a complete set of basis vectors for the space of spinors. Therefore, they obey certain completeness relations. In a typical normalisation, these read

$$\begin{aligned}u_\alpha(\vec{p})\bar{u}_\alpha(\vec{p}) &= i\vec{p}\cdot\vec{\gamma} + m, \\ v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) &= i\vec{p}\cdot\vec{\gamma} - m.\end{aligned}\quad (5.90)$$

In the Weyl representation these relations are reproduced by the following choice for the spinors u, v ³⁵

$$u_\alpha(\vec{p}) = \begin{pmatrix} (-\vec{p}\cdot\vec{\sigma})^{1/2}\xi_\alpha \\ i(\vec{p}\cdot\vec{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \quad v_\alpha(\vec{p}) = \begin{pmatrix} (-\vec{p}\cdot\vec{\sigma})^{1/2}\xi_\alpha \\ -i(\vec{p}\cdot\vec{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \quad (5.91)$$

where the ξ_α form an orthonormal basis of 2-spinors.

With the above normalisation and completeness relations, we obtain simple anti-commutation relations for the creation and annihilation operators that mimic their scalar field counterparts

$$\{a_\alpha(\vec{p}), a_\beta^\dagger(\vec{q})\} = \{b_\alpha(\vec{p}), b_\beta^\dagger(\vec{q})\} = \delta_{\alpha\beta} 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}). \quad (5.92)$$

Dirac Sea. The Pauli exclusion principle for fermions states that each state can be occupied only once. It follows from the (implicit) anti-commutators

$$\{a_\alpha^\dagger(\vec{p}), a_\beta^\dagger(\vec{q})\} = \{b_\alpha^\dagger(\vec{p}), b_\beta^\dagger(\vec{q})\} = 0 \quad (5.93)$$

that

$$(a_\alpha^\dagger(\vec{p}))^2 = (b_\alpha^\dagger(\vec{p}))^2 = 0. \quad (5.94)$$

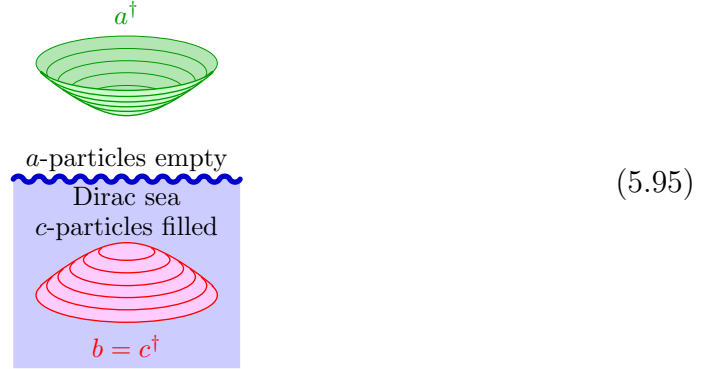
Dirac used the exclusion principle to make useful proposals concerning negative-energy states in relativistic quantum mechanics and the prediction of anti-particles.

The Dirac equation has positive and negative solutions. Furthermore, the solutions carry an (electrical) charge. The positive-energy states are regular, and we do not need to discuss them. Dirac proposed that all negative-energy states are already

³⁴This form is determined by the fact that the field ψ obeys the Dirac equation.

³⁵The square roots of matrices are defined such that $(-\vec{p}\cdot\vec{\sigma})^{1/2}(-\vec{p}\cdot\vec{\sigma})^{1/2} = -\vec{p}\cdot\vec{\sigma}$ and $(-\vec{p}\cdot\vec{\sigma})^{1/2}(\vec{p}\cdot\vec{\sigma})^{1/2} = m$, etc.. Note that this definition has no branch cuts on the mass shell because the eigenvalues $e \pm \|\vec{p}\|$ of $-\vec{p}\cdot\vec{\sigma}$ and $\vec{p}\cdot\vec{\sigma}$ are positive definite.

occupied in the vacuum and cannot be excited further. This picture is called the Dirac sea, and it explained how to avoid negative-energy solutions.



Continuing this line of thought, there is now the option to remove an excitation from one of the occupied states. This hole state would not only have positive energy, but also carry charges exactly opposite to the ones of the regular positive-energy solutions. In this way he predicted the existence of positrons as the anti-particles of electrons. The prediction was soon thereafter confirmed in experiment.

Our view of QFT today is analogous, but also different, so let us compare:

- Positive-energy solutions of ψ are associated to a^\dagger .
- Negative-energy solutions of ψ are associated to b .
- We may define an operator c such that $c^\dagger := b$.
- The vacuum is annihilated by c^\dagger . All c -states are occupied.
- A hole in the Dirac sea $c = b^\dagger$ creates an anti-particle.

Applying the QFT framework to the Dirac equations works as predicted, but:

- There is no need for a Dirac sea.
- Negative-energy solutions are commonly defined as annihilation operators, not as creation operators with a sea of occupied states.
- Dirac's argument relies on the exclusion principle, it works for fermions only. QFT deals with bosons in the very same fashion.
- The Dirac equation has real solutions (see later) just as well as the Klein–Gordon equation has complex solutions. The existence of anti-particles is unrelated to spin and the Dirac equation. It is a consequence of the CPT theorem.

Correlators and Propagators. We now have all we need to compute correlators of the free quantum fields. There are two non-vanishing correlators of two fields

$$\begin{aligned}\Delta_+^{\text{D}a}{}_b(x-y) &= +i\langle 0|\psi^a(x)\bar{\psi}_b(y)|0\rangle, \\ \Delta_-^{\text{D}a}{}_b(x-y) &= -i\langle 0|\bar{\psi}_b(y)\psi^a(x)|0\rangle.\end{aligned}\tag{5.96}$$

Evaluation of the correlators yields

$$\begin{aligned}\Delta_+^{\text{D}a}{}_b(x) &= i \int \frac{d\vec{p}^3 e^{ip \cdot x}}{(2\pi)^3 2e(\vec{p})} (ip \cdot \gamma + m)^a{}_b, \\ \Delta_-^{\text{D}a}{}_b(-x) &= i \int \frac{d\vec{p}^3 e^{ip \cdot x}}{(2\pi)^3 2e(\vec{p})} (-ip \cdot \gamma + m)^a{}_b.\end{aligned}\tag{5.97}$$

They can be expressed in terms of the correlator Δ_+ of two scalar fields with an additional operator acting on spinors

$$\begin{aligned}\Delta_+^{\text{D}a}{}_b(x) &= (\gamma \cdot \partial + m)^a{}_b \Delta_+(x), \\ \Delta_-^{\text{D}a}{}_b(-x) &= (-\gamma \cdot \partial + m)^a{}_b \Delta_+(x).\end{aligned}\tag{5.98}$$

When acting with the Dirac equation on the correlator, it combines with the operator to give the Klein–Gordon equation acting on Δ_+ , e.g.

$$\begin{aligned}(\gamma^\mu \partial_\mu - m)^a{}_b \Delta_+^{\text{D}b}{}_c(x) \\ = (\gamma \cdot \partial - m)^a{}_b (\gamma \cdot \partial + m)^b{}_c \Delta_+(x) \\ = \delta_c^a (\partial^2 - m^2) \Delta_+(x) = 0.\end{aligned}\tag{5.99}$$

Likewise, the unequal-time anti-commutator

$$i\{\psi^a(x), \bar{\psi}_b(y)\} = \Delta^{\text{D}a}{}_b(x - y)\tag{5.100}$$

can be written in terms of the one for the scalar field

$$\Delta^{\text{D}}(x) = (\gamma \cdot \partial + m) \Delta(x).\tag{5.101}$$

As such it satisfies the Dirac equation and vanishes for space-like separations³⁶

$$\{\psi^a(x), \bar{\psi}_b(y)\} = 0 \quad \text{for } (x - y)^2 > 0.\tag{5.102}$$

For the Dirac equation with a source, the same methods we introduced earlier for the scalar field apply. The propagator is a spinor matrix and it is defined via the equations

$$\begin{aligned}(-\gamma \cdot \partial + m)^a{}_b G^{\text{D}b}{}_c(x) &= \delta_c^a \delta^4(x), \\ (-\gamma \cdot \partial + m)^b{}_c G^{\text{D}a}{}_b(x) &= \delta_c^a \delta^4(x),\end{aligned}\tag{5.103}$$

supplemented by suitable boundary conditions. By the same reasons as above, we can express the Dirac propagator through the scalar propagator

$$G^{\text{D}a}{}_b(x) = (\gamma \cdot \partial + m)^a{}_b G(x).\tag{5.104}$$

³⁶The fact that the anti-commutator rather than the commutator vanishes is not in contradiction with causality. Typically we can observe only fermion bilinears which are bosonic and which do commute.

Obviously, one has the same relations as before, e.g. for the retarded propagator

$$G_R^D(x) = \theta(t)\Delta^D(x). \quad (5.105)$$

There are some other useful relationships between correlators and propagators in momentum space which are worth emphasising because they hold generally.

First of all, by construction the propagator is the inverse of the kinetic term in the action³⁷

$$G^D(p) = (-i\gamma \cdot p + m)^{-1} = \frac{i\gamma \cdot p + m}{p^2 + m^2}. \quad (5.106)$$

The corresponding correlators and unequal-time commutators take the form

$$\begin{aligned} \Delta_{\pm}^D(p) &= 2\pi i \delta(p^2 + m^2) \theta(\pm p^0) (i\gamma \cdot p + m), \\ \Delta^D(p) &= 2\pi i \delta(p^2 + m^2) \text{sign}(p^0) (i\gamma \cdot p + m). \end{aligned} \quad (5.107)$$

These reflect precisely the residues times a delta-function localised at the position of the pole along with some restriction to positive or negative energies.

The construction of the propagator and its relationship to correlators and commutators can be used as a shortcut in deriving the latter. Large parts of the canonical quantisation procedure can thus be avoided in practice.

5.7 Complex and Real Fields

Complex Field. The Dirac spinor is complex and the Lagrangian has an obvious U(1) global symmetry

$$\psi'(x) = e^{i\alpha} \psi, \quad \bar{\psi}'(x) = e^{-i\alpha} \bar{\psi}. \quad (5.108)$$

The symmetry has a corresponding conserved Noether current³⁸

$$J^\mu = \frac{\delta\psi^a}{\delta\alpha} \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^a} = i\bar{\psi}\gamma^\mu\psi. \quad (5.109)$$

The time component of the current was used earlier to define a positive definite probability density, $-J^0 = \psi^\dagger\psi$. However, if one follows the spin-statistics theorem and let ψ be Grassmann odd, the density is not positive. In particular, it changes sign for the charge conjugate solution

$$J_C^\mu = i\bar{\psi}_C\gamma^\mu\psi_C = -i\bar{\psi}\gamma^\mu\psi = -J^\mu. \quad (5.110)$$

Nevertheless, the current is conserved and it defines a conserved Noether charge

$$Q = \int d\vec{x}^3 J^0 = - \int d\vec{x}^3 \psi^\dagger\psi. \quad (5.111)$$

It leads to the usual charge assignments for a complex field

$$[Q, \psi(x)] = +\psi(x), \quad [Q, \bar{\psi}(x)] = -\bar{\psi}(x). \quad (5.112)$$

³⁷The poles should be shifted away from the real axis to accommodate for the desired boundary conditions.

³⁸Note the order of terms.

Reality Condition. The Dirac field has four independent particle modes, $a_\alpha^\dagger(\vec{p})$ and $b_\alpha^\dagger(\vec{p})$, for each three-momentum. From the classification of Poincaré UIR's we know that the irreducible representation for spin $j = 1/2$ has only two spin orientations for each three-momentum.

This discrepancy is associated to the existence of charge conjugate solutions. We can remove the additional solutions by imposing a reality condition on ψ , namely³⁹

$$\psi_C = \psi. \quad (5.113)$$

A spinor which satisfies this condition is called a Majorana spinor. There exist representations of the Clifford algebra where all γ^μ are purely real. In this basis the Dirac equation is real, and it makes sense to restrict ψ to real (Graßmann odd) numbers.

For the momentum modes we can use the identity (which also involves a change of basis between α and α')

$$u_\alpha(\vec{p}) = \gamma_C v_{\alpha'}^*(\vec{p}), \quad (5.114)$$

to show that the identification $\psi_C = \psi$ implies

$$a_\alpha(\vec{p}) = b_{\alpha'}(\vec{p}). \quad (5.115)$$

It reduces the modes of the Dirac field by a factor of two.

2-Spinors. Let us consider a real spinor $\psi = (\psi_L, \psi_R)$ in the Weyl representation. The reality condition implies

$$\psi_L = -i\varepsilon\psi_R^{\dagger\tau} = \frac{1}{\sqrt{2}}\chi. \quad (5.116)$$

This allows to write the Lagrangian in terms of the 2-spinor field χ as⁴⁰

$$\mathcal{L} = -\chi^\dagger i\vec{\sigma} \cdot \partial \chi + m\chi^\tau \varepsilon \chi - m\chi^\dagger \varepsilon \chi^{\dagger\tau}. \quad (5.117)$$

The Lagrangian for the Dirac field can be written as two identical copies of this.⁴¹

Parity. Note that parity interchanges ψ_L and ψ_R . The reality condition relates the two, hence

$$\chi'(t, -\vec{x}) = \varepsilon\chi^{\dagger\tau}(t, \vec{x}). \quad (5.118)$$

As this transformation also sends the field χ to its complex conjugate $\chi^{\dagger\tau}$, it is usually viewed as the combination CP of charge conjugation C and parity P

$$CP\chi(t, \vec{x})(CP)^{-1} = \varepsilon\chi^{\dagger\tau}(t, -\vec{x}). \quad (5.119)$$

³⁹One could also use any other complex phase $e^{i\alpha}$ between ψ_C and ψ .

⁴⁰The two mass terms are anti-symmetric in χ , which requires the classical field χ to be an odd Graßmann number.

⁴¹The $U(1)$ global symmetry of the Dirac equation is recovered as a $SO(2)$ rotation symmetry of the two fields χ .

In that sense there cannot be individual C and P transformations and only CP can be a symmetry.

An alternative point of view is that C was used to define the reality condition. Hence C is preserved by construction, and the parity operation P is well-defined on its own.

Technically, both points of view have the same content: They merely use the same words to refer to different operators. They are related by identifying $C' = 1$ and $P' = CP$, where the primed operations refer to the latter approach.

5.8 Massless Field and Chiral Symmetry

So far we have assumed a non-zero mass m . Let us now consider the massless case which has some special features. We will assume a real (Majorana) field.

First, let us compare to the irreps of the Poincaré group: The massless real spinor field gives rise to two particle states $a_\alpha^\dagger(\vec{p})$ for each momentum. Conversely, a massless irrep with fixed helicity has merely one state for each momentum. The two particles correspond to irreps with helicity $h = \pm 1/2$. In fact, helicity states must always come in pairs in QFT due to the CPT theorem. One cannot construct a real Lagrangian which describes just one helicity.

Interestingly, the splitting of representations leads to an enhancement of symmetry. The massive real spinor field has no symmetry beyond the Poincaré transformations. For $m = 0$, the Lagrangian in terms of 2-spinors simplifies to

$$\mathcal{L} = -\chi^\dagger i\bar{\sigma} \cdot \partial \chi. \quad (5.120)$$

Quite obviously, this Lagrangian has a global $U(1)$ symmetry⁴²

$$\chi' = e^{i\alpha} \chi. \quad (5.121)$$

It is called chiral symmetry. The associated Noether current reads

$$J^\mu = \chi^\dagger \bar{\sigma}^\mu \chi. \quad (5.122)$$

At the level of 4-spinors, chiral symmetry is represented by the transformation

$$\psi' = \exp(-i\alpha\gamma^5)\psi, \quad \bar{\psi}' = \bar{\psi} \exp(-i\alpha\gamma^5). \quad (5.123)$$

These equal transformation factors cancel when they are separated by a single gamma-matrix as in the kinetic term

$$\exp(-i\alpha\gamma^5)\gamma^\mu = \gamma^\mu \exp(i\alpha\gamma^5). \quad (5.124)$$

The massless Lagrangian is therefore invariant under chiral transformations.⁴³

Here, the conserved current is the so-called axial vector current

$$J^\mu = i\bar{\psi}\gamma^5\gamma^\mu\psi. \quad (5.125)$$

⁴²The mass terms $\chi^\top \varepsilon \chi$ and $\chi^\dagger \varepsilon \chi^{\dagger\top}$ evidently break this symmetry.

⁴³In the presence of masses the chiral transformation converts between the two allowable mass terms $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^5\psi$.

6 Free Vector Field

Next we want to find a formulation for vector fields. This includes the important case of the electromagnetic field with its photon excitations as massless relativistic particles of helicity ± 1 . This field will be the foundation for a QFT treatment of electrodynamics called quantum electrodynamics (QED). Here we will encounter a new type of symmetry which will turn out to be extremely powerful but at the price of new complications.

6.1 Classical Electrodynamics

We start by recalling electrodynamics which is the first classical field theory most of us have encountered in theoretical physics.

Maxwell Equations. The electromagnetic field consists of the electric field $\vec{E}(t, \vec{x})$ and the magnetic field $\vec{B}(t, \vec{x})$. These fields satisfy the four Maxwell equations (with $\varepsilon_0 = \mu_0 = c = 1$)

$$\begin{aligned} 0 &= \text{div } \vec{B} := \vec{\partial} \cdot \vec{B} = \partial_k B_k, \\ 0 &= \text{rot } \vec{E} + \dot{\vec{B}} := \vec{\partial} \times \vec{E} + \dot{\vec{B}} = \varepsilon_{ijk} \partial_j E_k + \dot{B}_i, \\ \rho &= \text{div } \vec{E} = \vec{\partial} \cdot \vec{E} = \partial_k E_k, \\ \vec{j} &= \text{rot } \vec{B} - \dot{\vec{E}} = \vec{\partial} \times \vec{B} - \dot{\vec{E}} = \varepsilon_{ijk} \partial_j B_k - \dot{E}_i. \end{aligned} \tag{6.1}$$

The fields ρ and \vec{j} are the electrical charge and current densities.

The solutions to the Maxwell equations without sources are waves propagating with the speed of light. The Maxwell equations were the first relativistic wave equations that were found. Eventually their consideration led to the discovery of special relativity.

Relativistic Formulation. Lorentz invariance of the Maxwell equations is not evident in their usual form. Let us transform them to a relativistic form.

The first step consists in converting B_i to an anti-symmetric tensor of rank 2

$$B_i = -\frac{1}{2} \varepsilon_{ijk} F_{jk}, \quad F = \begin{pmatrix} 0 & -B_z & +B_y \\ +B_z & 0 & -B_x \\ -B_y & +B_x & 0 \end{pmatrix}. \tag{6.2}$$

Then the Maxwell equations read

$$\begin{aligned} 0 &= -\varepsilon^{ijk} \partial_k F_{ij}, & \rho &= \partial_k E_k, \\ 0 &= \varepsilon^{ijk} (2\partial_j E_k - \dot{F}_{jk}), & j_i &= \partial_j F_{ji} - \dot{E}_i. \end{aligned} \quad (6.3)$$

These equations are the 1 + 3 components of two 4-vectors which can be seen by setting

$$E_k = F_{0k} = -F_{k0}, \quad J^\mu = (\rho, \vec{j}). \quad (6.4)$$

Now the Maxwell equations simply read

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\nu F^{\nu\mu} = J^\mu. \quad (6.5)$$

Electromagnetic Potential. For QFT purposes we need to write a Lagrangian from which the Maxwell equations follow. This is however not possible using $F_{\mu\nu}$ as the fundamental degrees of freedom. A Lagrangian can be constructed by the help of the electromagnetic vector potential A_μ . To some extent the potential is a convenient technical tool which is also needed to couple the field to charged matter.¹

The first (homogeneous) equation is an integrability condition for the field $F_{\mu\nu}$. It implies that it can be integrated consistently to an electromagnetic potential A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6.6)$$

With this parametrisation of F the homogeneous equation is automatically satisfied.

The electromagnetic potential is not uniquely defined by the electromagnetic fields F . For any solution A , we can add the derivative of an arbitrary scalar field $\alpha(x)$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x). \quad (6.7)$$

The extra term cancels out when anti-symmetrising the two indices of $\partial_\mu A_\nu$ and hence

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x). \quad (6.8)$$

This freedom in defining A_μ is called a gauge symmetry or gauge redundancy. It is called a local symmetry because the transformation can be chosen independently for every point of spacetime as opposed to globally for all of spacetime. Gauge symmetry will turn out to be very important in quantising the vector field.

¹This fact can be observed in the Aharonov–Bohm effect, where the phase of a charged quantum particle feels the presence of a non-trivial electromagnetic potential A , although it is confined to a region of spacetime where the field strength F vanishes. Nevertheless, the field strength F is the primary physical quantity whereas A is secondary concept because the physical content of A can be derived from F unambiguously. Note that this requires integration and therefore the vector potential contains non-local information on the physics. Even though non-local interactions are undesired in classical physics, they do influence phases in quantum mechanics.

Lagrangian. A Lagrangian for the electromagnetic fields can now be formulated in terms of the potential A_μ

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}[A]F_{\mu\nu}[A] = \frac{1}{2}\vec{E}[A]^2 - \frac{1}{2}\vec{B}[A]^2. \quad (6.9)$$

Here and in the following, $F_{\mu\nu}[A]$ is not considered a fundamental field, but merely represents the combination

$$F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6.10)$$

The equation of motion yields the second (inhomogeneous) Maxwell equation (here, with a trivial source term)

$$\partial_\nu F^{\nu\mu} = 0. \quad (6.11)$$

The first (homogeneous) Maxwell equation is already implied by the definition of F in terms of A .

Due to Poincaré symmetry we can also derive a conserved stress-energy tensor $T^{\mu\nu}$. It takes the form²

$$T^{\mu\nu} = F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \quad (6.12)$$

6.2 Gauge Fixing

Hamiltonian Framework. Towards quantisation we should proceed to the Hamiltonian framework. The canonical momentum Π conjugate to the field A reads

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{0\mu}. \quad (6.13)$$

Here, a complication arises because the component Π_0 is strictly zero and the field A_0 has no conjugate momentum. The non-zero spatial components form the electrical field $F_{0k} = E_k$. Moreover, the equations of motion imply $\partial_k \Pi_k = 0$ which is an equation without time derivative. This so-called constraint has to be implemented on the dynamical data provided in each time slice.³ It is related to Gauß' law which determines the electrical charge density (zero) from a field configuration.

Here, the missing of the momentum Π_0 and the constraint for Π_k are related to gauge redundancy of A . Although A_μ has four components, two of them can be chosen arbitrarily using gauge symmetry. Effectively A_μ has only two physically relevant components, which is matched by only two independent conjugate momenta. Under-determined and constrained variables require extra work in the Hamiltonian framework. Here we can avoid the complication by fixing a gauge.

²A naive derivation from the Lagrangian yields $T^{\mu\nu} = F^{\mu\rho}\partial^\nu A_\rho + \eta^{\mu\nu}\mathcal{L}$ which is neither symmetric nor gauge-invariant. Equivalently, we can use a gauge-invariant form of the translation $\delta A_\rho = -\delta a^\nu F_{\nu\rho}$ which differs from the naive translation $\delta A_\rho = -\delta a^\nu \partial_\nu A_\rho$ by a gauge transformation with the field $\delta\alpha = \delta a^\nu A_\nu$.

³Typically it suffices to satisfy constraints on the initial time slice.

Coulomb Gauge. A simple ansatz to resolve the problem of $\Pi_0 = 0$ is to demand that

$$A_0 = 0. \quad (6.14)$$

This can always be achieved by a suitable gauge transformation. It eliminates the need for the problematic canonical Poisson bracket between A_0 and Π_0 .

Fixing A_0 in the Lagrangian leads to a completely determined dynamical system. However, since A_0 is now missing from the set of dynamical variables, the corresponding equation of motion is absent. The latter amounts to the Gauß law constraint

$$\vec{\partial} \cdot \dot{\vec{A}} = \vec{\partial} \cdot \vec{\Pi} = 0. \quad (6.15)$$

This effectively implies that the gauge-fixed system is bigger than electrodynamics. Our dynamical system will coincide with electrodynamics only if the initial data expressed in terms of $(\vec{A}, \vec{\Pi})$ satisfies the above constraint $\vec{\partial} \cdot \vec{\Pi} = 0$, along with its time derivative which amounts to $\vec{\partial}^2 \vec{\partial} \cdot \vec{A} = 0$ upon using the equations of motion.

The choice $A_0 = 0$ does not completely eliminate all gauge freedom for A_k , a time-independent gauge redundancy $\alpha(\vec{x})$ remains. It can be eliminated by demanding

$$\vec{\partial} \cdot \vec{A} = 0. \quad (6.16)$$

This is called the Coulomb gauge-fixing condition.

Now $\Pi_k = F_{0k} = E_k = \dot{A}_k$ and for the Hamiltonian we obtain

$$H = \int d\vec{x}^3 \frac{1}{2} (\vec{E}^2 + \vec{B}^2), \quad (6.17)$$

which indeed represents the energy of the electromagnetic field.

With the Coulomb gauge, we can now quantise the electromagnetic field. The gauge is however not always convenient, since it specialises the time direction and therefore breaks relativistic invariance. For instance, it leads to instantaneous contributions to field correlators, which appears odd in a relativistic model. However, these contributions are gauge artifacts of the gauge potentials. In physical gauge-invariant observables, eventually such instantaneous or causality-violating contributions will always cancel.

Lorenz Gauges. A more general class of gauge-fixing conditions are the Lorenz gauges

$$\partial^\mu A_\mu = 0. \quad (6.18)$$

They are particularly convenient in a relativistic theory because they respect Poincaré symmetry. They do not completely fix the gauge freedom since any gauge transformation with $\partial^2 \alpha = 0$ will preserve the Lorenz gauge condition.⁴ For example, one may furthermore demand $A_0 = 0$ to recover the Coulomb gauge.

⁴This degree of freedom will later serve as an additional scalar field without physical relevance. Gladly, the field decouples from all physical processes.

Gauge-Fixing Term. Instead of choosing a particular gauge, we can alter the Lagrangian such that it imposes a gauge on its own. To that end, we can add a gauge-fixing term $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\xi^{-1}(\partial \cdot A)^2$ to the Lagrangian which is known as the R_ξ gauge⁵

$$\mathcal{L} = \mathcal{L}_{\text{ED}} + \mathcal{L}_{\text{gf}} \simeq -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}(1 - \xi^{-1})\partial^\mu A_\mu \partial^\nu A_\nu. \quad (6.19)$$

The equations of motion now read

$$\partial^2 A_\mu - (1 - \xi^{-1})\partial_\mu \partial^\nu A_\nu = 0. \quad (6.20)$$

The gauge-fixing term spoils gauge invariance and makes time evolution for all four gauge potentials well-defined. As before, the new system displays more degrees of freedom than plain electrodynamics. However, the additional gauge-fixing term vanishes quadratically in the vicinity of the Lorenz gauge,⁶ so that in this gauge we recover electrodynamics. In particular, we have to implement the Lorenz gauge on the initial data by hand

$$\begin{aligned} \partial \cdot A &= -\dot{A}_0 + \vec{\partial} \cdot \vec{A} = 0, \\ \partial \cdot \dot{A} &\sim -\vec{\partial}^2 A_0 + \vec{\partial} \cdot \dot{\vec{A}} = 0. \end{aligned} \quad (6.21)$$

The equations of motion propagate the gauge condition to all other time slices. Therefore, the extended model with constrained phase space yields electrodynamics.

Generators of Residual Symmetries. The gauge fixing also has a relevant effect on the canonical structure. Let us determine the latter to see this effect. The canonical momenta now read⁷⁸

$$\Pi_0 = \xi^{-1}\dot{A}_0 + (1 - \xi^{-1})\vec{\partial} \cdot \vec{A}, \quad \vec{\Pi} = \dot{\vec{A}}, \quad (6.22)$$

and we define canonical Poisson brackets

$$\{A_\mu(t, \vec{x}), \Pi_\nu(t, \vec{y})\} = \eta_{\mu\nu} \delta^3(\vec{x} - \vec{y}). \quad (6.23)$$

The constraint function $\partial \cdot A$ has non-trivial Poisson brackets with some phase space function F

$$\begin{aligned} \{\partial \cdot A, F\} &= \xi \{-\Pi_0 + \vec{\partial} \cdot \vec{A}, F\}, \\ \{\partial \cdot \dot{A}, F\} &= \xi \{-\vec{\partial}^2 A_0 + \vec{\partial} \cdot \vec{\Pi}, F\}. \end{aligned} \quad (6.24)$$

⁵Note that the equations of motion imply $\partial^2 \partial^\mu A_\mu = 0$. This condition holds in the Lorenz gauge, but it does actually not fix it; instead, it also holds for general massless gauge potentials not satisfying the gauge. To impose the Lorenz gauge one can take a Lagrange multiplier field B and add the term $B \partial \cdot A$ to the Lagrangian (or alternatively take the limit $\xi \rightarrow 0$ with the above term).

⁶This strong asymptotic behaviour could be viewed to remove the effect of the gauge-fixing term once the Lorenz gauge is imposed on the initial conditions.

⁷All time derivatives can be solved for canonical momenta unless $\xi = \infty$.

⁸The definition of the canonical momenta is ambiguous due to the equivalence of the terms $\partial^\mu A_\nu \partial^\nu A_\mu$ and $\partial^\mu A_\mu \partial^\nu A_\nu$ in the Lagrangian by partial integration. This ambiguity, however, does not make any difference for the Poisson brackets.

It is desirable to be able to set the left-hand sides to zero, but for a general function F the right-hand side does not vanish. However, one can convince oneself that for all components of the field strength tensor, i.e. $F = E_k$ and $F = B_k$, the right-hand side is zero. In fact, the functions $\partial \cdot A$ and $\partial \cdot \dot{A}$ generate the residual gauge transformations with $\partial^2 \alpha = 0$ given by their initial data α and $\dot{\alpha}$. For gauge-invariant observables F , the canonical structure thus becomes consistent with the gauge-fixing condition, but we will also have to deal with gauge-dependent quantities at intermediate stages of some calculations, and thus we cannot impose the Lorenz gauge in a strict sense.

Feynman Gauge. To simplify the subsequent analysis, we shall set $\xi = 1$. This so-called Feynman gauge has a simple Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu \quad (6.25)$$

with simple equations of motion

$$\partial^2 A_\mu = 0. \quad (6.26)$$

Effectively, it describes four massless scalar fields A_μ with the peculiarity that the sign of the kinetic term for A_0 is wrong. With the canonical momenta $\Pi_\mu = \dot{A}_\mu$ the Poisson brackets read

$$\{A_\mu(\vec{x}), \Pi_\nu(\vec{y})\} = \eta_{\mu\nu} \delta^3(\vec{x} - \vec{y}), \quad (6.27)$$

where again the relation for A_0 has the opposite sign. Likewise all correlation functions and propagators equal their scalar counterparts times $\eta_{\mu\nu}$.

As such, the model described by the above simple Lagrangian is not electrodynamics. Only when taking into account the constraint $\partial \cdot A = 0$ it becomes electrodynamics. Moreover, the constraint will be crucial in making the QFT model physically meaningful. Nevertheless, we have to be careful in implementing the constraint since it is inconsistent with the Poisson brackets.

Light Cone Gauge. The above R_ξ gauge does not eliminate all unphysical degrees of freedom, which introduce some complications later. There are other useful gauges which avoid these problems, but they trade them in for different problems. A prominent example is the light cone gauge which eliminates a light-like component $A_- = A_0 - A_3 = 0$ of the gauge potential A_μ . The equations of motion then allow to solve for a non-collinear like-like component of $A_+ = A_0 + A_3$. The remaining two degrees of freedom of A_μ then represent the two helicity modes of the electromagnetic field. Let us nevertheless continue in the Feynman gauge.

6.3 Particle States

Next we quantise the model and discuss its particle states. The construction of Fock space is the same as for a set of four massless scalar fields, but we need to implement the gauge-fixing constraint.

Quantisation. We quantise the vector field $A_\mu(x)$ in Feynman gauge analogously to four independent scalar fields where merely one of the kinetic term has the opposite sign. This leads to the equal-time commutation relations

$$[A_\mu(t, \vec{x}), \dot{A}_\nu(t, \vec{y})] = i\eta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \quad (6.28)$$

We then solve the equation of motion $\partial^2 A_\mu = 0$ in momentum space

$$A_\mu(x) = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} (e^{ip \cdot x} a_\mu(\vec{p}) + e^{-ip \cdot x} a_\mu^\dagger(\vec{p})) \quad (6.29)$$

and translate the above field commutators to commutators for creation and annihilation operators

$$[a_\mu(\vec{p}), a_\nu^\dagger(\vec{q})] = \eta_{\mu\nu} 2e(\vec{p})(2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (6.30)$$

Fock Space. We define the vacuum state $|0\rangle$ to be annihilated by all $a_\mu(\vec{p})$

$$a_\mu(\vec{p})|0\rangle = 0. \quad (6.31)$$

As before, multi-particle states are constructed by acting with the creation operators $a_\mu^\dagger(\vec{p})$ on the vacuum $|0\rangle$.

There are two problems with this naive Fock space. The first is that there ought to be only two states (with helicity $h = \pm 1$) for each momentum. Here we have introduced four states. The other problem is that one of these states has a negative norm: To see this we prepare a wave packet for a_0^\dagger

$$|f\rangle = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} f(\vec{p}) a_0^\dagger(\vec{p}) |0\rangle. \quad (6.32)$$

The norm of this state is negative definite

$$\langle f|f\rangle = - \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} |f(\vec{p})|^2 < 0. \quad (6.33)$$

A negative-norm state violates the probabilistic interpretation of QFT, hence it must be avoided at all means.⁹

Physical States. The above problems are eventually resolved by implementing the gauge-fixing constraint $\partial \cdot A = 0$ which we have not yet considered. This is not straight-forward:

- The non-trivial commutation relations of $\partial \cdot A$ prevent us from implementing the constraint $\partial \cdot A = 0$ at an operatorial level.

⁹Alternatively, one might define the vacuum state to be annihilated by a_0^\dagger instead of a_0 . In that case, all states have a positive norm. However, the particles created by a_0 now have a negative energy, leading to an equally undesirable situation.

- We cannot implement it directly on states: E.g. requiring the vacuum $|0\rangle$ to be physical means setting $p \cdot a^\dagger(\vec{p})|0\rangle = 0$ which is inconsistent with the commutation relations.
- The weakest implementation is to demand that the expectation value of $\partial \cdot A$ vanishes for all physical states. This is the Gupta–Bleuler formalism.

For two physical states $|\Psi\rangle, |\Phi\rangle$ we thus demand

$$\langle \Phi | \partial \cdot A | \Psi \rangle = 0. \quad (6.34)$$

This is achieved by demanding

$$p \cdot a(\vec{p})|\Psi\rangle = 0 \quad (6.35)$$

for any physical state. An adjoint physical state then obeys $\langle \Phi | p \cdot a^\dagger(\vec{p}) = 0$.

- Both conditions together ensure that $\langle \Phi | \partial \cdot A | \Psi \rangle = 0$.
- Moreover, the vacuum is physical by construction.

We conclude that Fock space is too large in agreement with the discussion at the classical level. The space of physical states $|\Psi\rangle$ is a subspace of Fock space such that for all \vec{p}

$$p \cdot a(\vec{p})|\Psi\rangle = 0. \quad (6.36)$$

Nevertheless, we cannot completely abandon the larger Fock space in favour of the smaller space of physical states. For instance, the action of $A_\mu(x)$ cannot be confined to the physical subspace since it does not commute with the operator $p \cdot a(\vec{p})$.

Evidently, the negative-norm state $|f\rangle$ discussed above is not physical since

$$p \cdot a(\vec{p})|f\rangle = -f(\vec{p})e(\vec{p})|0\rangle. \quad (6.37)$$

The right-hand side vanishes for all \vec{p} only if the function f is identically zero, in which case the state $|f\rangle = 0$ is trivial. Therefore, the state is an element of Fock space, but not of its physical subspace.

Basis of Polarisation Vectors. To investigate the space of physical states further, we introduce a convenient basis for polarisation vectors $\epsilon_\alpha^\mu(\vec{p})$ of the vector field $a_\mu(\vec{p})$ and $a_\mu^\dagger(\vec{p})$ on the light cone $p^2 = 0$.¹⁰

We denote the four polarisations α by G for gauge, L for longitudinal and 1, 2 for the two transverse directions.¹¹

- We first define $\epsilon_{(G)}$ as a light-like vector in the direction of p . For definiteness, we set $\epsilon_{(G)} = p$.
- We construct another light-like vector $\epsilon_{(L)}$ which has unit scalar product with $\epsilon_{(G)}$, i.e. $\epsilon_{(L)} \cdot \epsilon_{(G)} = 1$.

¹⁰The polarisation vectors are similar to the spinors $u(\vec{p})$ and $v(\vec{p})$ for the Dirac equation.

¹¹There is a lot of arbitrariness in defining the polarisation vectors $\epsilon^{(L)}, \epsilon^{(1)}, \epsilon^{(2)}$ for each momentum p . Moreover, there is no universal Lorentz-invariant choice of basis. Gladly, the arbitrariness does not affect physics.

- We then construct two orthonormal space-like vectors $\epsilon_{(1,2)}$ which are also orthogonal to $\epsilon_{(G)}$ and $\epsilon_{(L)}$.

For example, suppose the light-like momentum is given by

$$p^\mu = (e, 0, 0, e). \quad (6.38)$$

Then we can define the following four vectors¹²

$$\begin{aligned} \epsilon_{(G)}^\mu(\vec{p}) &= (e, 0, 0, e), \\ \epsilon_{(L)}^\mu(\vec{p}) &= (-1/2e, 0, 0, 1/2e), \\ \epsilon_{(1)}^\mu(\vec{p}) &= (0, 1, 0, 0), \\ \epsilon_{(2)}^\mu(\vec{p}) &= (0, 0, 1, 0). \end{aligned} \quad (6.39)$$

These four polarisations define a complete basis for the vector space. We can thus decompose the creation and annihilation operators as follows

$$a_{(\alpha)} = \epsilon_{(\alpha)}^\mu a_\mu, \quad a_{(\alpha)}^\dagger = \epsilon_{(\alpha)}^\mu a_\mu^\dagger. \quad (6.40)$$

Likewise, we can write the non-trivial commutation relations¹³

$$\begin{aligned} [a_{(1)}(\vec{p}), a_{(1)}^\dagger(\vec{q})] &= [a_{(2)}(\vec{p}), a_{(2)}^\dagger(\vec{q})] \\ &= [a_{(L)}(\vec{p}), a_{(G)}^\dagger(\vec{q})] = [a_{(G)}(\vec{p}), a_{(L)}^\dagger(\vec{q})] \\ &= 2e(\vec{p}) (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \end{aligned} \quad (6.41)$$

By construction we know that

$$p \cdot a(\vec{p}) = \epsilon_{(G)} \cdot a(\vec{p}) = a_{(G)}(\vec{p}). \quad (6.42)$$

hence the physical state condition in this basis reads

$$a_{(G)}(\vec{p})|\Psi\rangle = 0. \quad (6.43)$$

The physical state condition together with the commutation relations implies that a physical state cannot have any longitudinal excitations $a_{(L)}^\dagger(\vec{p})$. It must be of the form¹⁴

$$|\Psi\rangle = a_{(G)}^\dagger \cdots a_{(G)}^\dagger a_{(1,2)}^\dagger \cdots a_{(1,2)}^\dagger |0\rangle. \quad (6.44)$$

Since negative norm states can originate exclusively from the commutators $[a_{(L)}, a_{(G)}^\dagger]$ and $[a_{(G)}, a_{(L)}^\dagger]$, and since the $a_{(L)}^\dagger$'s are absent, the norm of any such state is positive semi-definite

$$\langle\Psi|\Psi\rangle \geq 0. \quad (6.45)$$

The modes $a_{(1,2)}^\dagger$ have a positive norm while $a_{(G)}^\dagger$ is null.¹⁵

¹²The vectors $\epsilon_{(1)}$ and $\epsilon_{(2)}$ define a basis with linear polarisation. The complex combinations $\epsilon_{(\pm)} \sim \epsilon_{(1)} \pm i\epsilon_{(2)}$ define circular polarisation with definite helicity.

¹³The non-trivial overlap between L and G is due to the construction of the basis using two light-like directions.

¹⁴Note that $a_{(G)}$ commutes with $a_{(G)}^\dagger$ and $a_{(1,2)}^\dagger$ but not with $a_{(L)}^\dagger$.

¹⁵Full Minkowski space has the indefinite signature $(-+++)$ while the subspace spanned by $\epsilon_{(G,1,2)}$ has positive semi-definite signature $(0++)$.

Null States. Let us discuss a physical state $|\Psi\rangle$ which contains an excitation of type $a_{(G)}^\dagger = p \cdot a^\dagger$, i.e. a state which can be written as¹⁶

$$|\Psi\rangle = p \cdot a^\dagger(\vec{p})|\Omega\rangle \quad (6.46)$$

with some other physical state $|\Omega\rangle$. This state has zero norm by the physical state condition

$$\langle\Psi|\Psi\rangle = \langle\Omega|p \cdot a(\vec{p}) p \cdot a^\dagger(\vec{p})|\Omega\rangle = \langle\Omega|p \cdot a^\dagger(\vec{p}) p \cdot a(\vec{p})|\Omega\rangle = 0. \quad (6.47)$$

Null states are not normalisable and therefore have to be interpreted appropriately. Typically null states are considered irrelevant because QM is a probabilistic framework. Something that takes place with probability zero does not happen in practice. Nevertheless, some consistency requirements have to be fulfilled:

By the same argument as above, we can show that a null state

$$|\Psi\rangle = p \cdot a^\dagger(\vec{p})|\Omega\rangle \quad (6.48)$$

actually has vanishing scalar products with any physical state $|\Phi\rangle$ due to the physicality condition of the latter

$$\langle\Phi|\Psi\rangle = \langle\Phi|p \cdot a^\dagger(\vec{p})|\Omega\rangle = 0. \quad (6.49)$$

In particular, this implies that the sum $|\Psi'\rangle$ of a physical state $|\Psi\rangle$ and some null state

$$|\Psi'\rangle = |\Psi\rangle + p \cdot a^\dagger(\vec{p})|\Omega\rangle \quad (6.50)$$

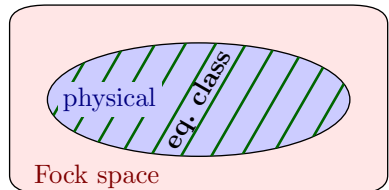
behaves just like the original physical state $|\Psi\rangle$ in scalar products

$$\langle\Phi|\Psi'\rangle = \langle\Phi|\Psi\rangle + \langle\Phi|p \cdot a^\dagger(\vec{p})|\Omega\rangle = \langle\Phi|\Psi\rangle. \quad (6.51)$$

We should thus impose an equivalence relation on the physical Fock space

$$|\Psi\rangle \simeq |\Psi'\rangle = |\Psi\rangle + p \cdot a^\dagger(\vec{p})|\Omega\rangle. \quad (6.52)$$

Any two states which differ by a state which is in the image of some $p \cdot a^\dagger$ are physically equivalent. In other words, physical states of the gauge field are not described by particular states but by equivalence classes of states.



$$(6.53)$$

¹⁶For simplicity, we assume the gauge excitation to be a plane wave. As such the norm will contain a factor of ∞ leading to some potential ambiguities. Nevertheless, all the following statements will apply if the plane wave is replaced by a normalisable wave packet.

We may use states which have no contribution of $p \cdot a^\dagger$ as reference states of the equivalence classes¹⁷

$$|\Psi\rangle = a_{(1,2)}^\dagger \cdots a_{(1,2)}^\dagger |0\rangle. \quad (6.54)$$

These representatives show that we have two states for each momentum \vec{p} . It matches nicely with the massless UIR's of the Poincaré group with positive and negative helicity $h = \pm 1$. The particle excitations of the electromagnetic field are the photons.

Gauge Transformations. However, inserting some gauge potentials $A_\mu(x)$ into the scalar product may actually lead to dependence on null states. Let us therefore compute

$$\begin{aligned} \langle \Phi | A_\mu(x) | \Psi' \rangle &= \langle \Phi | A_\mu(x) | \Psi \rangle + \langle \Phi | A_\mu(x) p \cdot a^\dagger(\vec{p}) | \Omega \rangle \\ &= \langle \Phi | A_\mu(x) | \Psi \rangle + \langle \Phi | [A_\mu(x), p \cdot a^\dagger(\vec{p})] | \Omega \rangle. \end{aligned} \quad (6.55)$$

The commutator evaluates to

$$[A_\mu(x), p \cdot a^\dagger(\vec{p})] = p_\mu e^{ip \cdot x} = -i \partial_\mu e^{ip \cdot x}. \quad (6.56)$$

The expectation value of $A_\mu(x)$ thus changes effectively by a derivative term

$$A_\mu(x) \mapsto A_\mu(x) - \frac{\langle \Phi | \Omega \rangle}{\langle \Phi | \Psi \rangle} i \partial_\mu e^{ip \cdot x}. \quad (6.57)$$

This is just a gauge transformation of the potential $A_\mu(x)$. We observe that the states $|\Psi\rangle$ and $|\Psi'\rangle$ lead to two expectation values which differ by a gauge transformation of the fields within the expectation value. Note that the gauge transformation does not leave the Lorenz gauges

$$[\partial \cdot A(x), p \cdot a^\dagger(\vec{p})] = -i \partial^2 e^{ip \cdot x} = 0. \quad (6.58)$$

Hence null states induce residual gauge transformation within the Lorenz gauges as discussed in the classical context.

Now it appears that the choice of representative in an equivalence class has undesirable impact on certain expectation values. Gladly, this does not apply to gauge-invariant observables. For instance, the electromagnetic field strength is unaffected

$$[F_{\mu\nu}(x), p \cdot a^\dagger(\vec{p})] = \partial_\mu(p_\nu e^{ip \cdot x}) - \partial_\nu(p_\mu e^{ip \cdot x}) = 0. \quad (6.59)$$

Moreover, the coupling of the gauge potential to a conserved current J^μ

$$J[A] = \int d^4x J^\mu(x) A_\mu(x) \quad (6.60)$$

¹⁷Although this appears to be a useful choice at first sight, it is not at all unique. By a change of basis for the polarisation vectors at any given \vec{p} we can add any amount of $a_{(G)}^\dagger$ to $a_{(1,2)}^\dagger$. The new states are certainly in the same equivalence class, but they are different representatives.

commutes with $p \cdot a^\dagger$

$$\begin{aligned} [J[A], p \cdot a^\dagger(\vec{p})] &= -i \int d^4x J^\mu(x) \partial_\mu e^{ip \cdot x} \\ &= i \int d^4x e^{ip \cdot x} \partial_\mu J^\mu(x) = 0. \end{aligned} \quad (6.61)$$

The expectation value of any gauge-invariant operator composed from $F_{\mu\nu}$, $J[A]$ or similar combinations thus does not depend on the choice of representatives, and it is consistent to define physical states as equivalence classes.

6.4 Casimir Energy

At this point we can already compute a quantum effect of the electromagnetic field, the Casimir effect. The Casimir effect is a tiny force between nearby conductors which exists even in the absence of charges, currents or medium. In classical electrodynamics no forces are expected in this setup. There are several alternative explanations for the quantum origin of the force. One is the exchange of virtual photons between the conductors. An equivalent explanation attributes the force to a change of vacuum energy of the electromagnetic field induced by the presence of the plates. The latter one has a quite efficient derivation, and we shall present it here.

Setup. We place two large planar metal plates at a small distance into the vacuum (much smaller than their size, but much larger than atomic distances). In our idealised setup, the plates extend infinitely along the x, y -directions. They are separated by the distance a in the z -direction. We will not be interested in the microscopic or quantum details of the metal objects. We simply assume that they are classical conductors and that they shield the electromagnetic field efficiently.



At the surface of the plates, the electric fields must be orthogonal $E_x = E_y = 0$ while the magnetic field must be parallel $B_z = 0$. In order to match these conditions simultaneously at both plates, the z -component of the wave vector (momentum) must be quantised

$$p_z \in \frac{\pi}{a} \mathbb{Z}. \quad \begin{array}{c} \text{---} B_z \text{---} \\ \text{---} \end{array} \quad (6.63)$$

Careful analysis shows that for $p_z = 0$ only one of the two polarisation vectors is permissible. Conversely, for $p_z \neq 0$ both polarisations are good. To achieve cancellations in this case, each wave must be synchronised to its reflected wave where $p_z \rightarrow -p_z$. Hence we should only count the contributions with $p_z > 0$.

Vacuum Energy. Just like the scalar field, the electromagnetic field carries some vacuum energy.¹⁸ The discretisation modifies the vacuum energy E_0 , which results in a force between the plates if the new vacuum energy depends on the distance a .

The sum and integral of all permissible modes between the plates yields the energy E per area A ¹⁹

$$E = \int \frac{A dp_x dp_y}{(2\pi)^2} \left(\frac{1}{2} e(p_x, p_y, 0) + 2 \sum_{n=1}^{\infty} \frac{1}{2} e(p_x, p_y, \pi n/a) \right). \quad (6.64)$$

For convenience we shall exploit the rotation symmetry in the x, y -plane to simplify the expression to

$$\frac{E}{A} = \int_0^{\infty} \frac{p dp}{2\pi} \left(\frac{1}{2} p + \sum_{n=1}^{\infty} \sqrt{p^2 + \pi^2 n^2 / a^2} \right). \quad (6.65)$$

As discussed earlier, this expression diverges due to UV contributions at large momenta.

Regularisation. We also emphasised earlier that infinities are largely our own fault. The idealised setup was somewhat too ideal.

For macroscopic electromagnetic waves, we certainly made the right assumption of total reflection. But it is also clear that the conducting plates will behave differently for hard gamma radiation. This is precisely where the problem arises, so we seem to be on the right track. Electromagnetic waves with wave length much smaller than atomic distances or energies much larger than atomic energy levels will pass the conducting plates relatively unperturbed. These modes therefore should be discarded from the above sum.²⁰

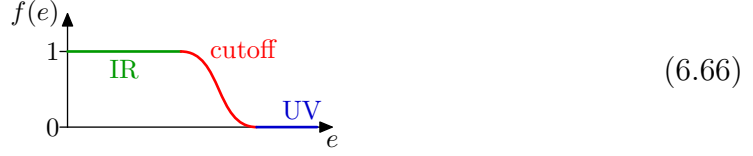
Therefore, we must introduce a UV cutoff for the modes. Define a function $f(e)$ which is constantly 1 for sufficiently small energies, constantly 0 for sufficiently

¹⁸Earlier, we had argued that vacuum energies are infinite, ambiguous and unobservable. Here, we will learn how to deal with the infinity. The ambiguity could in principle be resolved by considering how the electromagnetic field couples to the conductors. The standard expression for the vacuum energy $\frac{1}{2}\hbar\omega$ of harmonic oscillators from the symmetric ordering prescription turns out to do the job.

¹⁹A sum over the modes in some box of volume V in d dimensions turns into an integral over momenta when the volume is very large. In a box, the positive and negative modes are coupled, so the integral is over positive p only with integration measure $V d\vec{p}^d / \pi^d$. In the absence of boundary contributions, the integration domain extends to positive and negative p which is compensated by the measure $V d\vec{p}^d / (2\pi)^d$.

²⁰The modes do contribute to the vacuum energy between the plates. Importantly, the distance between the plates will hardly enter their contribution, and consequently they cannot contribute to forces.

large energy and which somehow interpolates between the 1 and 0 for intermediate energy.



The cutoff replaces each contribution $\frac{1}{2}e$ by $\frac{1}{2}f(e)e$

$$\frac{E_{\text{IR}}}{A} = \int_0^\infty \frac{p \, dp}{2\pi} \left(\frac{1}{2} p f(p) + \sum_{n=1}^\infty \sqrt{p^2 + \pi^2 n^2 / a^2} f(\sqrt{\dots}) \right). \quad (6.67)$$

Let us keep in mind the remaining contribution in the ultraviolet

$$\frac{E_{\text{UV}}}{A} = \int_0^\infty \frac{p \, dp}{2\pi} \int_0^\infty dn \sqrt{p^2 + \pi^2 n^2 / a^2} (1 - f(\sqrt{\dots})). \quad (6.68)$$

Here, we have converted the sum to an integral due to the absence of quantisation in the z -direction.²¹

Summation. The regularised expression E_{IR} is now finite, but it certainly depends on the cutoff in $f(e)$. We continue its evaluation and write it as a sum of integrals

$$\frac{E_{\text{IR}}}{A} = \frac{1}{2} F(0) + \sum_{n=1}^\infty F(n), \quad (6.69)$$

with

$$F(n) = \int_0^\infty \frac{p \, dp}{2\pi} \sqrt{p^2 + \pi^2 n^2 / a^2} f(\sqrt{p^2 + \pi^2 n^2 / a^2}). \quad (6.70)$$

It is convenient to use energy as the integration variable

$$e = \sqrt{p^2 + \pi^2 n^2 / a^2}, \quad p \, dp = e \, de, \quad (6.71)$$

and write the integral as

$$F(n) = \frac{1}{2\pi} \int_{\pi n/a}^\infty de \, e^2 f(e). \quad (6.72)$$

The Euler–MacLaurin summation formula writes the above sum for E_{IR}/A as an integral E_{int}/A plus discrete correction terms E_{C}/A

$$\begin{aligned} \frac{E_{\text{int}}}{A} &= \int_0^\infty dn \, F(n), \\ \frac{E_{\text{C}}}{A} &= - \sum_{k=1}^\infty (-1)^k \frac{B_{2k}}{(2k)!} F^{(2k-1)}(0), \end{aligned} \quad (6.73)$$

²¹The high-energy electromagnetic waves can effectively pass through the conducting plates.

where we have used that the function $F(n)$ is constantly zero at infinity due to the cutoff. Here B_n is the n -th Bernoulli number.

Let us analyse the two terms: The first term we can rewrite as

$$\frac{E_{\text{int}}}{A} = \frac{1}{2\pi} \int_0^\infty dn \int_{\pi n/a}^\infty de e^2 f(e) = \frac{2a}{4\pi^2} \int_0^\infty de' \int_{e'}^\infty de e^2 f(e). \quad (6.74)$$

It depends on the cutoff, but it is manifestly linear in a . In fact, it combines nicely with the contribution from UV modes above the cutoff that we dropped earlier

$$\frac{E_0}{V} = \frac{E_{\text{UV}} + E_{\text{int}}}{Aa} = \frac{2}{4\pi^2} \int_0^\infty de' \int_{e'}^\infty de e^2. \quad (6.75)$$

As such it represents the vacuum energy of the enclosed volume $V = Aa$ in the absence of plates. The same vacuum energy density is present outside the plates.²² This term therefore does not contribute to the force because any shift of the plate would merely transfer some vacuum energy from the inside to the outside leaving the overall energy invariant. The fact that E_0 is formally infinite does not play a role. We therefore consider only the change in energy $E_C = E - E_0$ arising from the second term of the Euler–MacLaurin summation.

The second term can be evaluated near $n = 0$ by extending the domain of integration down to $e = 0$ and subtracting the extra part

$$\begin{aligned} F(n) &= \frac{1}{2\pi} \int_0^\infty de e^2 f(e) - \frac{1}{2\pi} \int_0^{\pi n/a} de e^2 f(e) \\ &= F(0) - \frac{\pi^2 n^3}{6a^3}, \end{aligned} \quad (6.76)$$

where we used that $f(e) = 1$ for the second term near $e = 0$. Quite surprisingly, $F(n)$ is a polynomial with only two terms. All cutoff dependence is in the constant term $F(0)$ which does not appear in the summation formula.²³ The single correction term contributes the following vacuum energy ($B_4 = -1/30$)

$$\frac{E_C}{A} = -\frac{B_4}{4!} F^{(3)}(0) = -\frac{\pi^2}{720a^3}. \quad (6.77)$$

The presence of the conducting plates decreases the vacuum energy by some amount proportional to $1/a^3$.

Casimir Force. The Casimir force can be expressed as the pressure

$$P = \frac{F}{A} = \frac{E'_C(a)}{A} = \frac{\pi^2}{240a^4} = \frac{\hbar c \pi^2}{240a^4}. \quad (6.78)$$

Some properties:

²²The factor of $2/2\pi$ is interpreted as follows: $1/2\pi$ is the correct measure for integration over p_z and the factor of 2 compensates for the restricted integration region $p_z \geq 0$.

²³The reason is apparently that the cutoff is in a region of energies where the difference between a sum and an integral does not matter.

- Bringing the plates closer decreases the energy, hence the Casimir force is attractive.
- It increases with the fourth power of the inverse distance as the plates come closer.
- It is a quantum effect, and there are hidden factors of \hbar and c . Due to the fourth-power behaviour in $1/a$ it can nevertheless be detected at reasonable separations. It becomes relevant at micrometer distance.
- It does not depend on the coupling strength of the electromagnetic field or on the elementary charge.

6.5 Massive Vector Field

So far we have discussed the massless vector field. Among the UIR's of the Poincaré group there is also the massive representation with spin 1. Massive vector particles exist in nature as the W_{\pm} and Z_0 bosons transmitting the weak nuclear interactions.²⁴

Lagrangian. We can add a mass term to the vector Lagrangian to obtain the corresponding quantum field

$$\mathcal{L} = -\frac{1}{2}\partial_{\mu}V_{\nu}\partial^{\mu}V^{\nu} + \frac{1}{2}\partial_{\mu}V_{\nu}\partial^{\nu}V^{\mu} - \frac{1}{2}m^2V^{\mu}V_{\mu}. \quad (6.79)$$

The corresponding equation of motion reads

$$\partial^2V_{\mu} - \partial_{\mu}\partial^{\nu}V_{\nu} - m^2V_{\mu} = 0. \quad (6.80)$$

By taking the total derivative of this equation, we see that it implies the simpler equation $-m^2\partial^{\mu}V_{\mu} = 0$. Substituting this result in the original equation of motion then yields a system of two equations

$$\partial^2V_{\mu} - m^2V_{\mu} = 0, \quad \partial^{\mu}V_{\mu} = 0. \quad (6.81)$$

The first equation is the Klein–Gordon equation for each component of V_{μ} , the second equation removes one of the four potential orientations. The degrees of freedom agree with the classification of UIR's.

Correlators. We now want to quantise this system. In the canonical approach we first derive the conjugate momenta

$$\Pi_{\mu} = \dot{V}_{\mu} - \partial_{\mu}V_0. \quad (6.82)$$

As before, there is no conjugate momentum for the field V_0 hinting at the fact that the equation of motion for V_0 is a constraint²⁵

$$\partial_k\Pi_k + m^2V_0 = 0. \quad (6.83)$$

²⁴The implementation of interacting massive vector fields actually needs much more care. Interacting vectors fields can acquire mass only through the Higgs mechanism.

²⁵Note that unlike in the massless case it is possible to solve the constraint without integration by $V_0 = -m^{-2}\partial_k\Pi_k$.

Constrained systems are somewhat tedious to handle in the Hamiltonian framework and therefore in canonical quantisation.

Instead, let us take a shortcut. We consider the fields to be operators and cook up unequal-time commutation relations

$$\Delta_{\mu\nu}^V(x-y) = i[V_\mu(x), V_\nu(y)]. \quad (6.84)$$

Our previous experience has shown that correlators can be composed from derivatives acting on the correlator of the scalar field. This automatically implements the Klein–Gordon equation. Here we propose^{26 27}

$$\Delta_{\mu\nu}^V(x) = (\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu)\Delta(x). \quad (6.85)$$

The combination of derivatives was constructed such that Δ^V satisfies the polarisation equations

$$\partial^\mu\Delta_{\mu\nu}^V(x) = \partial^\nu\Delta_{\mu\nu}^V(x) = 0. \quad (6.86)$$

Equal-Time Commutators. Next let us see what this proposal implies for the equal-time commutators. The non-vanishing ones read as follows

$$\begin{aligned} [V_0(\vec{x}), V_k(\vec{y})] &= im^{-2}\partial_k\delta^3(\vec{x}-\vec{y}), \\ [V_0(\vec{x}), \dot{V}_0(\vec{y})] &= -im^{-2}\partial_k\partial_k\delta^3(\vec{x}-\vec{y}), \\ [V_k(\vec{x}), \dot{V}_l(\vec{y})] &= i\delta_{kl}\delta^3(\vec{x}-\vec{y}) - im^{-2}\partial_k\partial_l\delta^3(\vec{x}-\vec{y}), \\ [\dot{V}_0(\vec{x}), \dot{V}_k(\vec{y})] &= i\partial_k\delta^3(\vec{x}-\vec{y}) - im^{-2}\partial_k\partial_l\partial_l\delta^3(\vec{x}-\vec{y}). \end{aligned} \quad (6.87)$$

These relations appear somewhat unusual since they mix time and space components of V_μ .

Let us replace the time derivatives \dot{V}_k by the associated conjugate momenta Π_k . For the spatial components we recover the canonical commutator

$$[V_k(\vec{x}), \Pi_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x}-\vec{y}). \quad (6.88)$$

The commutators involving V_0 and \dot{V}_0 can be recovered using the equations of motion. The latter actually give an explicit solution for the field V_0 and its time derivative \dot{V}_0

$$V_0 = -m^{-2}\partial_k\Pi_k, \quad \dot{V}_0 = \partial_k V_k. \quad (6.89)$$

In other words, V_0 is not an elementary field and its commutation relations follow from the canonical one above

$$\begin{aligned} [\dot{V}_0(\vec{x}), \Pi_k(\vec{y})] &= i\partial_k\delta^3(\vec{x}-\vec{y}), \\ [V_k(\vec{x}), V_0(\vec{y})] &= im^{-2}\partial_k\delta^3(\vec{x}-\vec{y}), \\ [V_0(\vec{x}), \dot{V}_k(\vec{y})] &= -im^{-2}\partial_k\delta^3(\vec{x}-\vec{y}). \end{aligned} \quad (6.90)$$

²⁶The correlator Δ_+ and the propagator G take an equivalent form in terms of their scalar field counterparts.

²⁷The factor of $1/m^2$ is not as innocent as it may appear. When adding interactions, this term involving an inverse mass scale actually makes the theory behave badly for large momenta.

Hamiltonian Framework. We have obtained a reasonable QFT framework for our massive scalar field. Now we can revisit the Hamiltonian framework. First we perform a Legendre transformation of the Lagrangian for spatial components of the fields V_k ²⁸

$$\begin{aligned} H &= \int d\vec{x}^3 (\Pi_k \dot{V}_k - \mathcal{L}) \\ &= \int d\vec{x}^3 \left(\frac{1}{2} \Pi_k \Pi_k + \frac{1}{2} m^{-2} \partial_k \Pi_k \partial_l \Pi_l \right. \\ &\quad \left. + \frac{1}{2} \partial_k V_l \partial_k V_l - \frac{1}{2} \partial_l V_k \partial_k V_l + \frac{1}{2} m^2 V_k V_k \right). \end{aligned} \quad (6.91)$$

Here, we have also substituted the solution for the field V_0 and its time derivative.

We note that the Hamiltonian is slightly unusual in that it contains derivatives of the momenta along with inverse powers of the mass. The inverse powers of the mass in fact prevent us from taking the massless limit.²⁹

Gladly, this Hamiltonian implies the desired equations of motion

$$\begin{aligned} \dot{V}_k &= i[H, V_k] = \Pi_k - m^{-2} \partial_k \partial_l \Pi_l, \\ \dot{\Pi}_k &= i[H, \Pi_k] = \partial_l \partial_l V_k - \partial_k \partial_l V_l - m^2 V_k. \end{aligned} \quad (6.92)$$

It is not at all obvious that these equations imply the Klein–Gordon equation. However, their twisted form is required to be able to solve for the field V_0 easily and thereby obtain the correct energy.

²⁸The Hamiltonian is manifestly positive since $\frac{1}{2} \partial_k V_l \partial_k V_l - \frac{1}{2} \partial_l V_k \partial_k V_l = \frac{1}{4} (\partial_k V_l - \partial_l V_k)^2$.

²⁹We may impose a gauge by demanding $\partial_k \Pi_k = -m^{-2} V_0 = 0$. This eliminates the inverse mass from the Hamiltonian and validates the massless limit. Using $\partial_k \Pi_k = -m^{-2} V_0$ the gauge also implies $\dot{V}_0 = \partial_k V_k = 0$, i.e. the gauge is the Coulomb gauge.

7 Interactions

We have learned a lot about the three basic constituents of QFT in four dimensions:

- scalar fields (spin $j = 0$ or helicity $h = 0$),
- spinor fields (spin $j = 1/2$ or helicity $h = \pm 1/2$),
- vector fields (helicity $h = \pm 1$ or spin $j = 1$).

So far we considered only free fields. The particle number was conserved by all processes and most operators we encountered so far.

Now, we would like to introduce interactions between such fields. Unfortunately, interactions cannot be treated exactly, neither classically nor quantum mechanically.

We have to assume the strength of interactions to be sufficiently small. The well-understood free fields will dominate, and we insert interactions as small perturbations. This eventually leads us to Feynman diagrams to describe particle interactions order by order.

7.1 Interacting Lagrangians

One of the main reasons to consider QFT is its ability to deal with processes that do not conserve the number of particles.

Quantum fields are particle creation and annihilation operators: $\phi \rightarrow a, a^\dagger$. So far, we used them for two purposes:

- to build the multi-particle Fock space from a vacuum state, i.e. $a^\dagger \dots a^\dagger |0\rangle$;
- to write conserved charges as quadratic combinations of the fields which conspired to yield one creation and one annihilation operator $a^\dagger a$ and thus conserve the particle number.

Combining more than two fields typically yields a quantum operator which changes the particle number.

Time evolution of a quantum system is governed by its Hamiltonian, therefore it is natural to include such higher-order terms in it and consequently in the Lagrangian.

Scalar Interactions. For a real scalar field we could consider an interacting Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{6}\mu\phi^3 - \frac{1}{24}\lambda\phi^4. \quad (7.1)$$

This model called the ϕ^4 theory.¹ It is perhaps the conceptually simplest interacting QFT model, but it leads to very non-trivial physics. The ϕ^4 term is also an interaction of the scalar Higgs field which is essential for the Higgs mechanism.

We might also add higher-order terms or terms involving derivatives such as

$$\phi^5, \quad \phi(\partial\phi)^2, \quad \phi^2(\partial\phi)^2, \quad (\partial\phi)^4, \quad \dots \quad (7.2)$$

Such terms are in principle allowable in QFT, but they have some undesirable features regarding renormalisation which will be discussed later. We will, however, never add non-local terms of the type to the action

$$\int d^4x \phi(x)\phi(x+a), \quad \int d^4x d^4y f(x,y)\phi(x)\phi(y). \quad (7.3)$$

These terms represent some unphysical action at a distance; we consider only local interactions which can be written using a local Lagrangian²

$$S = \int d^4x \mathcal{L}(x), \quad \mathcal{L}(x) = \mathcal{L}[\phi(x), \partial\phi(x), \dots]. \quad (7.4)$$

The classical equation of motion for the above Lagrangian reads

$$\partial^2\phi - m^2\phi - \frac{1}{2}\mu\phi^2 - \frac{1}{6}\lambda\phi^3 = 0. \quad (7.5)$$

It is a non-linear differential equation. Our usual strategy to deal with the differential equation of motion was to go to momentum space

$$\begin{aligned} 0 = & -p^2\phi(p) - m^2\phi(p) - \frac{1}{2}\mu \int \frac{d^4q}{(2\pi)^4} \phi(q)\phi(p-q) \\ & - \frac{1}{6}\lambda \int \frac{d^4q_1 d^4q_2}{(2\pi)^8} \phi(q_1)\phi(q_2)\phi(p-q_1-q_2). \end{aligned} \quad (7.6)$$

Unfortunately, we obtain an integral equation instead of an algebraic equation. We cannot solve it in general, but for small $\mu \ll m$ and small $\lambda \ll 1$ we can try to find useful approximations.

Quantum Electrodynamics. Electrons and positrons carry an electrical charge $\pm q$, respectively,³ and their conserved current $J^\mu = iq\bar{\psi}\gamma^\mu\psi$ couples to the Maxwell equations. Putting together the Dirac and Maxwell Lagrangians we can simply add a source term for the electromagnetic potential $J^\mu A_\mu$

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - iq\bar{\psi}\gamma^\mu\psi A_\mu. \quad (7.7)$$

¹One often drops the term ϕ^3 and gains a discrete symmetry $\phi \mapsto -\phi$. A term ϕ^3 without a term ϕ^4 would lead to a potential unbounded from below.

²Non-local terms could be recovered as Taylor series involving derivatives of arbitrary order, e.g. $\phi(x)\phi(x+a) = \phi^2 + a\phi\partial\phi + \frac{1}{2}a^2\phi\partial^2\phi + \dots$. It is therefore desirable to restrict the number of derivatives that can appear in \mathcal{L} .

³We shall assume that $q = -e < 0$ is the charge of the electron where $e > 0$ is the unit electrical charge. For practical purposes, the overall sign of the charges does not matter.

This yields the desired inhomogeneous Maxwell equations, but also a modification of the Dirac equation

$$\begin{aligned}\partial_\mu F^{\mu\nu} &= iq\bar{\psi}\gamma^\nu\psi, \\ (\gamma_\mu\partial^\mu - m)\psi &= iq\gamma^\mu A_\mu\psi.\end{aligned}\tag{7.8}$$

The above model is called quantum electrodynamics (QED). It is a model that has been tested at a remarkable accuracy within its domain of validity, i.e. at low energies where the other elementary particles play no essential role. For instance, the electron anomalous magnetic dipole moment, also known as “ $g - 2$ ”, was predicted to more than 10 digits, and many of the leading digits are due to plain QED alone.

In the standard model, the above type of interaction between vectors and spinors is arguably the most important one because it couples matter in the form of spinor fields (leptons and quarks) to forces in the form of vector fields (photons, gluons and others).

Gauge Invariance. A crucial property of the electromagnetic potential is its gauge symmetry. In the quantisation procedure it eliminates an unphysical degree of freedom of the electromagnetic potential. We therefore want to preserve this symmetry in the presence of interactions.

The interaction term breaks the original gauge symmetry, but the latter can be restored by extending the symmetry to the Dirac field

$$\begin{aligned}A'_\mu(x) &= A_\mu(x) + \partial_\mu\alpha, \\ \psi'(x) &= \exp(iq\alpha(x))\psi(x).\end{aligned}\tag{7.9}$$

Note that the latter transformation rule is just the global U(1) symmetry of the Dirac field which is responsible for conservation of the current J^μ . This global symmetry is enhanced to a local transformation parameter $\alpha(x)$. The derivative terms of $\alpha(x)$ are now compensated by the inhomogeneous gauge transformation of the potential A_μ .

There is a construction which makes the gauge invariance more manifest. Introduce the gauge covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu.\tag{7.10}$$

Under gauge transformations this operator transforms homogeneously⁴

$$\begin{aligned}D'_\mu &= \partial_\mu - iqA'_\mu = \partial_\mu - iqA_\mu - iq\partial_\mu\alpha \\ &= D_\mu + [D_\mu, -iq\alpha] \\ &= \exp(+iq\alpha)D_\mu\exp(-iq\alpha).\end{aligned}\tag{7.11}$$

⁴This is an equation for operators. As such $O_1 = O_2$ is equivalent to the statement $O_1f = O_2f$ for all functions f .

In the QED Lagrangian written with a covariant derivative

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad (7.12)$$

the factors of $\exp(\pm iq\alpha)$ trivially cancel between $\bar{\psi}$, D_μ and ψ . Moreover the electromagnetic field strength can be written as

$$F_{\mu\nu} = \frac{i}{q} [D_\mu, D_\nu], \quad (7.13)$$

which makes manifest its invariance under gauge transformations.

Consequently, we can also couple the complex scalar field to the electromagnetic field via its kinetic term

$$\mathcal{L}_{\text{SQED}} = -(D^\mu \phi)^* D_\mu \phi - m^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \quad (7.14)$$

This model is called scalar QED.

Further Interactions. Let us list some other simple interactions. We want to consider only those interactions which respect Lorentz symmetry. Curiously, all of them appear in the standard model.

Two Dirac spinors can be multiplied to form a scalar combination. This can be multiplied by a scalar field

$$\bar{\psi}\psi\phi. \quad (7.15)$$

This term was originally proposed by Yukawa for the interaction between nucleons of spin $1/2$ and scalar pions. In the standard model such terms couple the Higgs field to the leptons and quarks.

A similar term coupling a scalar and two Dirac fermions but with opposite parity properties is

$$i\bar{\psi}\gamma^5\psi\phi. \quad (7.16)$$

There is also an analog of the spinor-vector coupling with opposite parity properties

$$-i\bar{\psi}\gamma^5\gamma^\mu\psi A_\mu. \quad (7.17)$$

This so-called axial vector coupling term is relevant to the weak nuclear interactions. Here, gauge invariance needs to extend to local chiral transformations of the spinors.

The above interactions for Dirac 4-spinors can be written in terms of more elementary chiral 2-spinor fields,

$$\chi^\dagger \varepsilon \chi \phi, \quad -\chi^\dagger \bar{\sigma}^\mu \chi A_\mu. \quad (7.18)$$

Note that the first interaction is complex, and therefore only some real projection can appear in the Lagrangian. This leads to two couplings, one for the real part and one for the imaginary part. The second term is perfectly real and requires a single real coupling constant.

Power Counting. We have encountered several types of interaction terms. These have a rather simple form with very few factors. Moreover, most of the simple terms have been observed directly or indirectly in nature. However, there are many more local terms one could imagine, but which have not been observed. What distinguishes the above interactions?

To answer this question, consider the mass dimension. The action S must be a dimensionless quantity.⁵ The action is the integral of the Lagrangian $S = \int d^4x \mathcal{L}$ and length counts as inverse mass, $dx \sim m^{-1}$, therefore the Lagrangian must have mass dimension 4,

$$\mathcal{L} \sim m^4. \quad (7.19)$$

The kinetic terms $(\partial\phi)^2$, $\bar{\psi}\partial\psi$ and F^2 where the derivative counts as a mass, $\partial \sim m$, determine the mass dimensions of the scalar, spinor and vector fields

$$\phi \sim A_\mu \sim m, \quad \psi \sim m^{3/2}. \quad (7.20)$$

The mass dimension of the remaining terms is now fixed, e.g. for the mass terms

$$\phi^2 \sim m^2, \quad \bar{\psi}\psi \sim m^3, \quad (7.21)$$

and for the simple interaction terms

$$\phi^3 \sim m^3, \quad \phi^4 \sim \bar{\psi}\gamma^\mu\psi A_\mu \sim \bar{\psi}\psi\phi \sim m^4. \quad (7.22)$$

All of these terms have mass dimension at most 4. When they appear in the Lagrangian $\mathcal{L} \sim m^4$, their coupling constant must compensate for the missing mass dimension. The scalar and fermion mass terms therefore read $m^2\phi^2$, $m\bar{\psi}\psi$. Among the interaction terms, only ϕ^3 requires a dimensionful coupling $\mu \sim m$. All the other terms have mass dimension 4 and their coupling constants are plain numbers.

We can take the bound of mass dimension 4⁶ as an experimentally observed principle. There are good reasons to consider only terms of this type:

- Such interactions are reasonably simple.
- There are only finitely many such terms, hence finitely many parameters for the model.
- All higher-dimensional terms require a coupling constant with negative mass dimension.
- Coupling constants with negative mass dimension lead to undesirable effects in the ultraviolet or short-distance regimes.
- Such theories are called non-renormalisable. Renormalisability will be considered later in QFT II.
- In the infrared or long-distance regime,⁷ only the interactions of mass dimension up to 4 are relevant. The higher-dimensional terms have small effects and are mostly irrelevant.

⁵Quantities that appear in an exponent must be dimensionless numbers. The action carries the same units as Planck constant \hbar which in natural units is a number $\hbar = 1$.

⁶More generally, the number of spacetime dimensions.

⁷The meaning of long-distance depends on the point of view. It can be astronomical units, everyday length scales, atomic scales or even less when interested in fundamental description of nature.

- The mass for a vector field reappears as an inverse power in the massive vector propagator. This also leads to a non-renormalisable model. To explain the mass of the W and Z vector bosons we have to rely on the Higgs mechanism.⁸

Symmetries. Unfortunately, it is generally hard to extract information from interacting QFT models. Usually we can only do certain approximations. Symmetries are powerful concepts in QFT because they can apply to interacting models as well:

- Free theories are somewhat trivial because they have infinitely many conservation laws. For example, the particle number is conserved, but it is related only to a non-local transformation.
- Only few of the conservation laws typically survive when interactions are added. Those are related to global symmetries preserved by the interactions.
- The conservation laws allow to make certain statements on the result of QFT observables even when actual computations are not feasible.
- Symmetries of the classical theory are not necessarily respected by the quantum theory. Such symmetries are called anomalous.
- Anomalies of local symmetries are typically bad because they spoil gauge redundancies which are required for consistency.

For example, consider ϕ^4 theory with a complex field given by the Lagrangian

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4. \quad (7.23)$$

It is invariant under global multiplication by a complex phase $\phi \rightarrow e^{i\alpha} \phi$. This leads to the same conserved current as for the complex scalar

$$J^\mu = -i(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi). \quad (7.24)$$

The associated current $Q = N_a - N_b$ is exactly conserved even in the presence of interactions. Conservation of the individual number operators N_a and N_b , however, is broken by interactions.

7.2 Interacting Field Operators

Consider an interacting field theory whose fields (and conjugate momenta) we will collectively denote by $\phi(x)$. More concretely, we can consider ϕ^4 theory.⁹

We want to compute some correlation function, for example a correlator of two fields at different times t_1, t_2

$$F(x_2, x_1) = \langle 0 | \phi(t_2, \vec{x}_2) \phi(t_1, \vec{x}_1) | 0 \rangle. \quad (7.25)$$

⁸Also known as the Englert–Brout–Higgs–Guralnik–Hagen–Kibble–Anderson–and–perhaps–also–’t–Hooft mechanism.

⁹The spatial dependence of the fields will hardly be relevant, and the following discussion applies just as well to any weakly interacting quantum mechanical system.

Interacting Field. As before, we can quantise the field ϕ on one time slice at some time t_0 . This step is identical to a free field ϕ_0 because the canonical Poisson brackets are the same for ϕ and ϕ_0 .¹⁰ The full time dependence of ϕ is recovered by conjugating with the Hamiltonian

$$\phi(t, \vec{x}) = \exp(i(t - t_0)H) \phi(\vec{x}) \exp(i(t_0 - t)H). \quad (7.26)$$

Supposing that the vacuum is time-invariant,¹¹ we can write the correlator as

$$F(x_2, x_1) = \langle 0 | \phi(\vec{x}_2) \exp(-i(t_2 - t_1)H) \phi(\vec{x}_1) | 0 \rangle. \quad (7.27)$$

Now everything is explicitly known except how to exponentiate H in practice. The latter is a hard problem.

Interaction Picture. We can do slightly better whenever the interactions are weak. In this case, the dominant contribution should come from the free Hamiltonian H_0 .¹² The quantisation of fields at a given time slice is the same. We can thus identify the fields at time t_0

$$\phi_0(\vec{x}) \stackrel{!}{=} \phi(\vec{x}). \quad (7.28)$$

Time evolution of the free field ϕ_0 is governed by the free Hamiltonian H_0

$$\phi_0(t, \vec{x}) = \exp(i(t - t_0)H_0) \phi_0(\vec{x}) \exp(i(t_0 - t)H_0). \quad (7.29)$$

We know almost everything about this field. For weak interactions and small times $t \approx t_0$, we expect the free field $\phi_0(t, \vec{x})$ to be a suitable approximation for the full field $\phi(t, \vec{x})$.

Comparing ϕ to ϕ_0 we can write

$$\phi(t, \vec{x}) = U(t)^{-1} \phi_0(t, \vec{x}) U(t) \quad (7.30)$$

with the transformation operator

$$U(t) = \exp(i(t - t_0)H_0) \exp(i(t_0 - t)H[\phi_0]). \quad (7.31)$$

Note that we expressed the full Hamiltonian H in terms of the free field $\phi_0 = \phi_0(t_0) = \phi(t_0)$. For small interactions and small times $t \approx t_0$, this operator is approximately the identity.

The correlator in question becomes

$$\langle 0 | U(t_2)^{-1} \phi_0(t_2, \vec{x}_2) U(t_2) U(t_1)^{-1} \phi_0(t_1, \vec{x}_1) U(t_1) | 0 \rangle. \quad (7.32)$$

¹⁰Interactions may alter the precise relationship between the conjugate momentum π and $\dot{\phi}$, but the Poisson brackets between ϕ and π retain their canonical form by construction.

¹¹A constant energy of the vacuum can always be eliminated by subtracting it from H .

¹²Any free Hamiltonian H_0 for the field ϕ_0 is permissible for the following construction of the interaction picture. In order for perturbation theory to be applicable, the interaction terms in $H - H_0$ must be small (but they may for instance include corrections to the mass of the free field, see the following chapters for applications).

Note that products of transformation operators can be combined in the pertinent time evolution operator

$$\begin{aligned} U(t_2, t_1) &:= U(t_2)U(t_1)^{-1} \\ &= \exp(i(t_2 - t_0)H_0) \exp(i(t_1 - t_2)H) \exp(i(t_0 - t_1)H_0) \\ &= \exp(i(t_2 - t_1)H_0) \exp(i(t_1 - t_2)H[\phi_0(t_1)]). \end{aligned} \quad (7.33)$$

In the last transformation we have used that $H = H[\phi] = H[\phi(t_0)] = H[\phi_0(t_0)]$ and the relationship

$$X[\phi_0(t_1)] \exp(i(t_1 - t_0)H_0) = \exp(i(t_1 - t_0)H_0) X[\phi_0(t_0)], \quad (7.34)$$

which holds for any quantum operator X composed from $\phi_0(t)$ without further dependence on time. This follows from (repeated) use of the time evolution of the free field $\phi_0(t)$.

Finally, we write the resulting correlator as

$$F(x_2, x_1) = \langle 0 | U(t_0, t_2) \phi_0(x_2) U(t_2, t_1) \phi_0(x_1) U(t_1, t_0) | 0 \rangle. \quad (7.35)$$

This is called the interaction picture, it is a mixture between the Schrödinger and the Heisenberg pictures

- In the Schrödinger picture, the field is defined on a constant time slice $\phi = \phi(t_0)$ and the operator $\exp(iH(t_2 - t_1))$ evolves states in time.
- In the Heisenberg picture, the field $\phi(t)$ carries the full time dependence, there is no need for a time evolution operator.
- In the interaction picture, the field $\phi_0(t)$ carries the time dependence of a free particle and the operator $U(t_2, t_1)$ evolves states in time.

Interacting Ground State. All the operators are expressed using the free field ϕ_0 , but the state $|0\rangle$ is a state of the interacting theory and we do not know how to act on it.

Luckily we can express the interacting ground state $|0\rangle$ in terms of the vacuum $|0_0\rangle$ of the free theory with a trick due to Gell-Mann and Low: The free vacuum $|0_0\rangle$ should be some linear combination of the interacting ground state $|0\rangle$ and excited eigenstates $|n\rangle$ with definite energy $E_n > E_0 = 0$ ¹³

$$|0_0\rangle = c_0|0\rangle + \sum_n c_n|n\rangle. \quad (7.36)$$

Letting this state evolve for some time T with the interacting Hamiltonian H we obtain

$$\exp(-iHT)|0_0\rangle = c_0|0\rangle + \sum_n c_n \exp(-iE_n T)|n\rangle. \quad (7.37)$$

All eigenstates oscillate with their respective frequencies. Suppose we give the time $T \in (1 - i\epsilon)\mathbb{R}$ some small negative imaginary part with $E_n^{-1} \ll |\text{Im } T| \ll |T|$.

¹³It is reasonable to assume $c_0 = \langle 0|0_0\rangle \neq 0$ when interactions are sufficiently small.

Then almost all eigenstates will get exponentially suppressed compared to the interacting ground state. The latter remains as the dominant contribution

$$\exp(-iHT)|0_0\rangle \approx c_0|0\rangle. \quad (7.38)$$

Primarily, this identification is a formal trick. In terms of physics, we let a system in some excited state $|0_0\rangle$ evolve for some long time and find it in its ground state $|0\rangle$. This is reasonable if we assume the system to be open or damped in some way. All real world systems, at least those we can expect to observe, are finite and open; the effect of $i\epsilon$ is to implement this assumption into our calculations.

We can thus express the interacting vacuum at time t_0 as the evolution of the free vacuum at time $-T$

$$|0\rangle \approx c_0^{-1} \exp(iH(-t_0 - T))|0_0\rangle = c_0^{-1} U(t_0, -T)|0_0\rangle \quad (7.39)$$

Analogously,

$$\langle 0| \approx (c_0^*)^{-1} \langle 0_0| U(+T, t_0). \quad (7.40)$$

Our final result for the correlation function $F(x_2, x_1)$ reads

$$\lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0_0| U(+T, t_2) \phi_0(x_2) U(t_2, t_1) \phi_0(x_1) U(t_1, -T) |0_0\rangle}{\langle 0_0| U(+T, -T) |0_0\rangle}. \quad (7.41)$$

We shall refer to it as the reduction formula for the interaction picture. One can also consider its denominator to implement the desired normalisation $\langle 0|0\rangle = 1$.

Interacting Correlators. In conclusion, the recipe for determining some correlation function in the interacting theory is the following reduction formula

$$\langle 0|X|0\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0_0| U(T, t_0) X U(t_0, -T) |0_0\rangle}{\langle 0_0| U(T, -T) |0_0\rangle}, \quad (7.42)$$

where all the interacting quantum operators in X are replaced by free fields evolved from time t_0 to the desired time slice

$$\phi(t, \vec{x}) \rightarrow U(t_0, t) \phi_0(t, \vec{x}) U(t, t_0). \quad (7.43)$$

Effectively, two consecutive time evolution operators can always be combined into one

$$U(t_2, t_0) U(t_0, t_1) = U(t_2, t_1). \quad (7.44)$$

7.3 Perturbation Theory

We still cannot evaluate the time evolution operator $U(t, t_0)$, but at least we know that it is close to the identity when interactions are sufficiently small

$$U(t, t_0) \approx 1. \quad (7.45)$$

This approximation is too crude, it is equivalent to computing the correlator in the free theory, and we gain nothing.

Schrödinger Equation. To improve the approximation, consider the time derivative of $U(t, t_0)$

$$\begin{aligned} i\partial_t U(t, t_0) &= \exp(i(t - t_0)H_0) (H[\phi_0(t_0)] - H_0) \\ &\quad \cdot \exp(-i(t - t_0)H_0) U(t, t_0) \\ &= (H[\phi_0(t)] - H_0) U(t, t_0). \end{aligned} \quad (7.46)$$

We see that the time evolution operator is determined by a differential equation and a trivial initial value condition

$$i\partial_t U(t, t_0) = H_{\text{int}}(t) U(t, t_0), \quad U(t_0, t_0) = 1. \quad (7.47)$$

This is a Schrödinger equation, and its Hamiltonian is the so-called interaction Hamiltonian

$$H_{\text{int}}(t) := H[\phi_0(t)] - H_0[\phi_0]. \quad (7.48)$$

This Hamiltonian is time-dependent, therefore the solution cannot be as simple as $\exp(-i(t - t_0)H_{\text{int}})$.¹⁴ For weak interactions, one can use the Dyson series to solve the equation perturbatively.

Dyson Series. The interaction Hamiltonian is the quantity which we should assume to be small. It appears in the Schrödinger equation, so at first order we can use the above approximation for U

$$i\partial_t U(t, t_0) = H_{\text{int}}(t) U(t, t_0) \approx H_{\text{int}}(t). \quad (7.49)$$

Integrating with proper initial value this yields

$$U(t, t_0) \approx 1 - i \int_{t_0}^t dt_1 H_{\text{int}}(t_1). \quad (7.50)$$

This is certainly better than before, it involves interactions at first order. Nevertheless we can do better.

To go further systematically, write the differential equation in integral form

$$U(t, t_0) = 1 - i \int_{t_0}^t dt_1 H_{\text{int}}(t_1) U(t_1, t_0). \quad (7.51)$$

Substitute the above solution yields a better solution

$$\begin{aligned} U(t, t_0) &\approx 1 - i \int_{t_0}^t dt_1 H_{\text{int}}(t_1) \\ &\quad - \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2). \end{aligned} \quad (7.52)$$

¹⁴ H_{int} is time-dependent because its time evolution is governed by H_0 with which it does not commute in general.

Now use the new solution instead

$$\begin{aligned}
U(t, t_0) \approx & 1 - i \int_{t_0}^t dt_1 H_{\text{int}}(t_1) \\
& - \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2) \\
& + i \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_{\text{int}}(t_1) H_{\text{int}}(t_2) H_{\text{int}}(t_3). \quad (7.53)
\end{aligned}$$

And so on.

The picture should be clear, we could go to arbitrarily high orders. More importantly, everything is expressed in terms of free fields ϕ_0 and the interaction Hamiltonian $H_{\text{int}}[\phi_0]$.

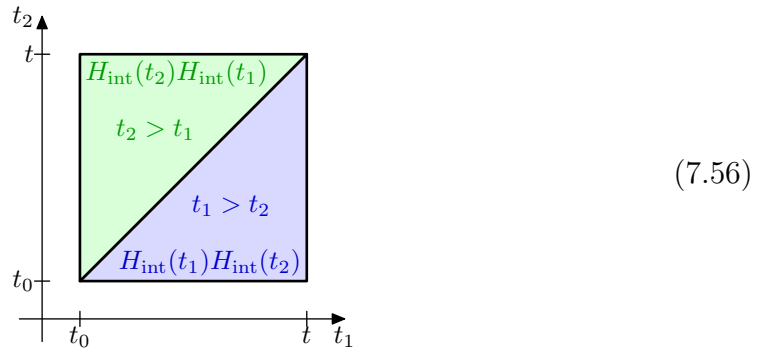
Time-Ordered Exponential. The multiple integral with a nested sequence of boundaries is hard to manage. We can formally improve the situation, and let us assume $t > t_0$ for definiteness. Consider the quadratic term:

$$- \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2). \quad (7.54)$$

We can also write it as

$$- \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 H_{\text{int}}(t_2) H_{\text{int}}(t_1). \quad (7.55)$$

The integration region assumes $t_1 \geq t_2$ in the first integral and $t_2 \geq t_1$ in the second integral. Importantly, in both integrands the operator $H_{\text{int}}(t_k)$ with larger t_k is to the left of the operator $H_{\text{int}}(t_j)$ with smaller t_j .



We introduce a time ordering symbol T which puts the affected operators in an order with time decreasing from left to right, e.g.¹⁵

$$T[X(t_1)Y(t_2)] := \begin{cases} X(t_1)Y(t_2) & \text{for } t_1 > t_2, \\ Y(t_2)X(t_1) & \text{for } t_1 < t_2, \end{cases} \quad (7.57)$$

¹⁵For fermionic operators X, Y one would insert suitable signs for flipping the order.

and similarly for multiple operators. This allows to write the integrand of both above integrals as the time-ordered product $T[H_{\text{int}}(t_1)H_{\text{int}}(t_2)]$. We can thus write the integral as the average of the two equivalent representations where the integration regions combine to a square

$$-\frac{1}{2} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T[H_{\text{int}}(t_1) H_{\text{int}}(t_2)]. \quad (7.58)$$

Even better, we can write this as the time-ordered square of a single integral

$$-\frac{1}{2} T \left[\int_{t_0}^t dt' H_{\text{int}}(t') \right]^2. \quad (7.59)$$

As all terms of the perturbative expansion of $U(t, t_0)$ are naturally in time ordering, the above construction generalises straight-forwardly to the n -th order term

$$\frac{1}{n!} T \left[-i \int_{t_0}^t dt' H_{\text{int}}(t') \right]^n. \quad (7.60)$$

Here the integration region is a hypercube in n dimensions. It contains $n!$ simplices,¹⁶ which form the integration regions for the terms in the Dyson series.

Summing up all terms yields the time-ordered exponential¹⁷

$$U(t_2, t_1) = T \exp(i S_{\text{int}}(t_2, t_1)), \quad (7.61)$$

where we introduced the interaction action S_{int} between times t_1 and t_2 ¹⁸

$$S_{\text{int}}(t_2, t_1) := - \int_{t_1}^{t_2} dt' H_{\text{int}}(t'). \quad (7.62)$$

The time-ordered exponential represents both the formal solution to the above Schrödinger equation for $U(t_2, t_1)$ as well as a concrete perturbative prescription to evaluate it.¹⁹

¹⁶A hypercube is the generalisation of a cube to n dimensions, a simplex is the generalisation of a triangle.

¹⁷Here, we have assumed $t_2 > t_1$. For the opposite order of external times $t_1 > t_2$, the reverse time ordering is required.

¹⁸It is clear that $L_{\text{int}} = -H_{\text{int}}$ unless the canonical momenta are non-linear functions of the fields. However, in the case of non-linear canonical momenta (such as in scalar QED), the definition of the canonical commutators involves additional terms which make up for the differences.

¹⁹In a mathematical sense, it represents a formal power series. It may or may not converge as an actual series; in a QFT setting it is typically merely an asymptotic series.

8 Correlation Functions

We have seen how to formally write the time evolution operator

$$U(t_1, t_0) = T \exp(iS_{\text{int}}(t_1, t_0)) \quad (8.1)$$

in an interacting QFT model based on the interaction picture and time-ordered products.

A particularly convenient correlator is one where the operators are already in proper time order

$$\langle X[\phi] \rangle := \langle 0 | T[X[\phi]] | 0 \rangle. \quad (8.2)$$

Such time-ordered correlation functions have multiple applications in QFT, for example, they can be used for particle scattering processes. In this chapter we will develop methods to compute them in more practical terms. The outcome will be a set of graphical rules, the Feynman rules.

For simplicity we will drop all free field indices $\phi_0 \rightarrow \phi$ from now on and instead mark interacting correlators and fields by an index “int”, $\phi \rightarrow \phi_{\text{int}}$.

8.1 Interacting Time-Ordered Correlators

Consider the correlator of two time-ordered fields with $t_1 > t_2$

$$F = \langle \phi(t_1, \vec{x}_1) \phi(t_2, \vec{x}_2) \rangle_{\text{int}} = \langle 0_{\text{int}} | \phi_{\text{int}}(t_1, \vec{x}_1) \phi_{\text{int}}(t_2, \vec{x}_2) | 0_{\text{int}} \rangle. \quad (8.3)$$

In the expression in terms of free fields

$$X = U(+T, t_1) \phi(x_1) U(t_1, t_2) \phi(x_2) U(t_2, -T), \quad (8.4)$$

we notice that all operators are in proper time order and we can extend the time ordering over all the operators

$$\begin{aligned} X &= [T \exp(iS_{\text{int}}(+T, t_1))] \phi(x_1) [T \exp(iS_{\text{int}}(t_1, t_2))] \\ &\quad \cdot \phi(x_2) [T \exp(iS_{\text{int}}(t_2, -T))] \\ &= T[\exp(iS_{\text{int}}(+T, t_1)) \phi(x_1) \exp(iS_{\text{int}}(t_1, t_2)) \\ &\quad \cdot \phi(x_2) \exp(iS_{\text{int}}(t_2, -T))]. \end{aligned} \quad (8.5)$$

Inside the time-ordering symbol the order of operators does not matter. The exponents can now be combined nicely

$$X = T[\phi(x_1) \phi(x_2) \exp(iS_{\text{int}}(+T, -T))]. \quad (8.6)$$

We thus find the correlation function $\langle \phi(x_1)\phi(x_2) \rangle_{\text{int}}$

$$F = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T [\phi(x_1)\phi(x_2) \exp(iS_{\text{int}}(+T, -T))] | 0 \rangle}{\langle 0 | T [\exp(iS_{\text{int}}(+T, -T))] | 0 \rangle}. \quad (8.7)$$

This reduction formula generalises to vacuum expectation values of arbitrary time-ordered combinations $X[\phi]$ of quantum operators

$$\begin{aligned} \langle T[X[\phi]] \rangle_{\text{int}} &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T[X[\phi] \exp(iS_{\text{int}}(+T, -T))] | 0 \rangle}{\langle 0 | T[\exp(iS_{\text{int}}(+T, -T))] | 0 \rangle} \\ &\simeq \frac{\langle 0 | T[X[\phi] \exp(iS_{\text{int}})] | 0 \rangle}{\langle 0 | T[\exp(iS_{\text{int}})] | 0 \rangle}. \end{aligned} \quad (8.8)$$

Here the complete interaction action S_{int} implies a small imaginary part for the time coordinate in the distant past and future. We can thus express time-ordered correlators in the interacting theory in terms of similar quantities in the free theory.

The reduction formula has several benefits and applications:

- Typically, there are no ordering issues within X because time ordering puts all constituent operators into some well-defined order. This is useful when interested in the quantum expectation value of some product of classical operators.
- It directly uses the interaction terms S_{int} in the action.
- Time-ordered products and expectation values can be evaluated conveniently.
- This expression appears in many useful observables, for example in particle scattering amplitudes.

8.2 Time-Ordered Products

We now look for a method to evaluate a time-ordered correlator of a combination of free-field operators $X[\phi]$

$$\langle X[\phi] \rangle := \langle 0 | T[X[\phi]] | 0 \rangle. \quad (8.9)$$

Feynman Propagator. We start with two fields

$$G_{\text{F}}(x_1, x_2) = i \langle 0 | T[\phi(x_1)\phi(x_2)] | 0 \rangle. \quad (8.10)$$

By construction and earlier results it reads

$$\begin{aligned} G_{\text{F}}(x_1, x_2) &= \begin{cases} i \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle & \text{for } t_1 > t_2, \\ i \langle 0 | \phi(x_2)\phi(x_1) | 0 \rangle & \text{for } t_2 > t_1 \end{cases} \\ &= \theta(t_1 - t_2) \Delta_+(x_1 - x_2) + \theta(t_2 - t_1) \Delta_+(x_2 - x_1). \end{aligned} \quad (8.11)$$

Comparing this to the retarded propagator $G_R(x)$

$$\begin{aligned} G_F(x) &= \theta(t)\Delta_+(x) + \theta(-t)\Delta_+(-x), \\ G_R(x) &= \theta(t)\Delta_+(x) - \theta(t)\Delta_+(-x), \end{aligned} \quad (8.12)$$

we can write

$$G_R(x) = G_F(x) - \Delta_+(-x). \quad (8.13)$$

As such it obeys the equation of a propagator,

$$-\partial^2 G_F(x) + m^2 G_F(x) = \delta^{d+1}(x), \quad (8.14)$$

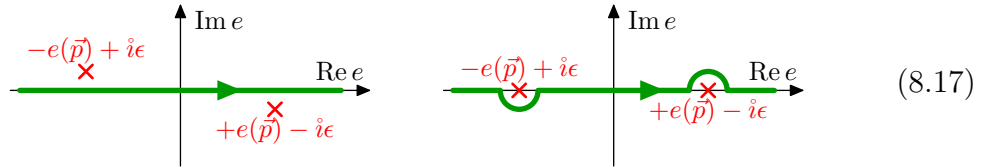
but with different boundary conditions than for the retarded propagator. It is called the Feynman propagator.

The momentum space representation of the Feynman propagator for the scalar field reads

$$G_F(p) = \frac{1}{p^2 + m^2 - i\epsilon}. \quad (8.15)$$

Here the two poles at $e = \pm e(\vec{p})$ are shifted up and down into the complex plane by a positive infinitesimal amount ϵ ¹

$$G_F(p) = \frac{1}{2e(\vec{p})} \left(\frac{1}{e - (-e(\vec{p}) + i\epsilon)} - \frac{1}{e - (+e(\vec{p}) - i\epsilon)} \right). \quad (8.16)$$



Concerning the relation to the position space representation:

- The positive energy pole $e = e(\vec{p}) - i\epsilon$ is below the real axis and thus relevant to positive times.
- The negative energy pole $e = -e(\vec{p}) + i\epsilon$ is above the real axis and thus relevant to negative times.

Alternatively, to obtain the correct contour around the two poles, we could integrate on a slightly tilted energy axis in the complex plane

$$e \in (1 + i\epsilon)\mathbb{R}. \quad (8.18)$$

Equivalently, we can assume times to be slightly imaginary, but with the time axis tilted in the opposite direction

$$t \in (1 - i\epsilon)\mathbb{R}. \quad (8.19)$$

¹The ϵ 's in the above and below expressions are not literally the same, but importantly they are both positive infinitesimals.

The $i\epsilon$ prescription of the Feynman propagators is thus directly related to and has the same effect as the $i\epsilon$ prescription for converting the free vacuum to the interacting one in the reduction formula.

$$(8.20)$$

Wick's Theorem. To evaluate more complex time-ordered vacuum expectation values one typically employs Wick's theorem. It relates a time-ordered product of operators $T[X[\phi]]$ to a normal-ordered product of operators $N[X[\phi]]$. The normal-ordered product is useful when evaluating vacuum expectation values because the vacuum expectation value picks out field-independent contributions only.

Let us recall the definition of normal ordering: Split up the free fields ϕ into pure creation operators ϕ^+ and pure annihilation operators ϕ^-

$$\phi = \phi^+ + \phi^-, \quad \phi^+ \sim a^\dagger, \quad \phi^- \sim a. \quad (8.21)$$

Normal ordering of a product is defined such that all factors of ϕ^+ are to the left of all factors ϕ^- . For example,

$$\begin{aligned} N[\phi(x_1)\phi(x_2)] &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\ &\quad + \phi^+(x_1)\phi^-(x_2) + \phi^+(x_2)\phi^-(x_1), \end{aligned} \quad (8.22)$$

where the latter two terms are in normal order and the ordering of the former two terms is irrelevant.

In comparison, time-ordering of the same product is defined as

$$\begin{aligned} T[\phi(x_1)\phi(x_2)] &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\ &\quad + \theta(t_1 - t_2)(\phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2)) \\ &\quad + \theta(t_2 - t_1)(\phi^+(x_2)\phi^-(x_1) + \phi^-(x_2)\phi^+(x_1)). \end{aligned} \quad (8.23)$$

The difference between the two expressions reads²

$$\begin{aligned} &(T - N)[\phi(x_1)\phi(x_2)] \\ &= \theta(t_1 - t_2)[\phi^-(x_1), \phi^+(x_2)] + \theta(t_2 - t_1)[\phi^-(x_2), \phi^+(x_1)] \\ &= -i\theta(t_1 - t_2)\Delta_+(x_1 - x_2) - i\theta(t_2 - t_1)\Delta_+(x_2 - x_1) \\ &= -iG_F(x_1 - x_2). \end{aligned} \quad (8.24)$$

Wick's theorem is a generalisation of this result to an arbitrary number of fields: It states that the time-ordered product of a set of fields equals the partially

²Note that the commutators are between creation and annihilation parts ϕ^\pm rather than the full fields ϕ . Therefore they yield the function Δ_+ rather than Δ as can be seen by taking the vacuum expectation value.

contracted normal-ordered products summed over multiple contractions between pairs of fields. A Wick contraction between two, not necessarily adjacent, fields $\phi(x_k)$ and $\phi(x_l)$ replaces the relevant two field operators by their Feynman propagator $-iG_F(x_k, x_l)$, in short:

$$\begin{aligned} & [\dots \phi_{k-1} \underbrace{\phi_k \phi_{k+1} \dots \phi_{l-1} \phi_l \phi_{l+1} \dots}] \\ & := -iG_F(x_k - x_l) [\dots \phi_{k-1} \phi_{k+1} \dots \phi_{l-1} \phi_{l+1} \dots]. \end{aligned} \quad (8.25)$$

For example:

$$\begin{aligned} T[\phi_1 \phi_2] &= N[\phi_1 \phi_2] + \underbrace{\phi_1 \phi_2}, \\ T[\phi_1 \phi_2 \phi_3] &= N[\phi_1 \phi_2 \phi_3] + \underbrace{\phi_1 \phi_2} \phi_3 + \underbrace{\phi_1 \phi_2} \phi_3 + \phi_1 \underbrace{\phi_2 \phi_3}, \\ T[\phi_1 \phi_2 \phi_3 \phi_4] &= N[\phi_1 \phi_2 \phi_3 \phi_4 + \underbrace{\phi_1 \phi_2} \phi_3 \phi_4 + \underbrace{\phi_1 \phi_2} \phi_3 \phi_4 \\ &\quad + \underbrace{\phi_1 \phi_2} \phi_3 \phi_4 + \phi_1 \underbrace{\phi_2 \phi_3} \phi_4 \\ &\quad + \phi_1 \underbrace{\phi_2 \phi_3} \phi_4 + \phi_1 \underbrace{\phi_2 \phi_3} \phi_4] \\ &\quad + \underbrace{\phi_1 \phi_2} \underbrace{\phi_3 \phi_4} + \underbrace{\phi_1 \phi_2} \underbrace{\phi_3 \phi_4} + \phi_1 \underbrace{\phi_2 \phi_3} \phi_4. \end{aligned} \quad (8.26)$$

To prove the statement by induction is straight-forward:

- Assume the statement holds for $n - 1$ fields.
- Arrange n fields in proper time order $\phi_n \dots \phi_1$ with $t_n > \dots > t_1$.
- Consider $T[\phi_n \dots \phi_1] = (\phi_n^+ + \phi_n^-) T[\phi_{n-1} \dots \phi_1]$ and replace $T[\phi_{n-1} \dots \phi_1]$ by contracted normal-ordered products.
- ϕ_n^+ is already in normal order, it can be pulled into $N[\dots]$.
- Commute ϕ_n^- past all the remaining fields in the normal ordering.
- For every uncontracted field ϕ_k in $N[\dots]$, pick up a term $[\phi_n^-, \phi_k] = -i\Delta_+(x_n - x_k) = -iG_F(x_n - x_k)$ because $t_n > t_k$.
- Convince yourself that all contractions of n fields are realised with unit weight.
- Convince yourself that for different original time-orderings of $\phi_n \dots \phi_1$, the step functions in G_F do their proper job.

Time-Ordered Correlators. To compute time-ordered correlators we can use the result of Wick's theorem. All the normal-ordered terms with remaining fields drop out of vacuum expectation values. The only terms to survive are those where all the fields are complete contracted in pairs

$$\langle \phi_1 \dots \phi_n \rangle := \sum_{\substack{\text{complete} \\ \text{contractions}}} \underbrace{\phi_1 \phi_2 \dots \phi_{n-1} \phi_n}_{\text{contractions}} \quad (8.27)$$

In particular, it implies that correlators of an odd number of fields must be zero.

This formula applies directly to a single species of real scalar fields, but for all the other fields and mixed products there are straight-forward equivalents:

- For fields with spin, use the appropriate propagator, e.g. $(G^D)^a_b$ for contracting the Dirac fields ψ^a and $\bar{\psi}_b$.
- For any crossing of lines attached to fermionic fields, multiply by a factor of (-1) .

8.3 Some Examples

We have learned how to reduce time-ordered correlators in a weakly interacting QFT to free time-ordered correlators using the reduction formula

$$\langle X[\phi] \rangle_{\text{int}} = \frac{\langle X[\phi] \exp(iS_{\text{int}}[\phi]) \rangle}{\langle \exp(iS_{\text{int}}[\phi]) \rangle}. \quad (8.28)$$

We have also learned how to evaluate the latter

$$\langle \phi_1 \dots \phi_n \rangle := \sum_{\substack{\text{complete} \\ \text{contractions}}} \phi_1 \phi_2 \dots \phi_{n-1} \phi_n. \quad (8.29)$$

We will now apply these formulas to some basic types of time-ordered correlators in order to develop an understanding for them.

Setup. We will consider ϕ^4 theory, i.e. a single real scalar field with a ϕ^4 interaction

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{24} \lambda \phi^4. \quad (8.30)$$

We define the interaction picture using the quadratic terms in the action

$$\mathcal{L}_0 = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (8.31)$$

What remains is the interaction term

$$\mathcal{L}_{\text{int}} = -\frac{1}{24} \lambda \phi^4, \quad (8.32)$$

whose coefficient, the coupling constant λ , is assumed to be small. The interaction part of the action is thus

$$S_{\text{int}} := \int dx^4 \mathcal{L}_{\text{int}}(x). \quad (8.33)$$

We would like to evaluate some correlators of the form

$$X = \langle \phi_1 \dots \phi_m \rangle_{\text{int}}, \quad (8.34)$$

where ϕ_k denotes the field $\phi(x_k)$ evaluated at position x_k . These are functions of the coupling constant λ , and we formally expand them for small λ as

$$X(\lambda) = \sum_{n=0}^{\infty} X^{(n)}, \quad X^{(n)} \sim \lambda^n, \quad (8.35)$$

and consider the leading two terms $X^{(0)}$ and $X^{(1)}$.

Two-Point Function. First, we would like to evaluate the correlator of two fields

$$T_{12}(\lambda) = \langle \phi_1 \phi_2 \rangle_{\text{int}}. \quad (8.36)$$

At leading order we simply set $\lambda = 0$ and obtain the correlator in the free theory

$$T^{(0)} = \langle \phi_1 \phi_2 \rangle. \quad (8.37)$$

Using Wick's theorem this evaluates to

$$T^{(0)} = \phi_1 \phi_2 = (-i) G_{12}, \quad (8.38)$$

where G_{kl} denotes the Feynman propagator $G_F(x_k - x_l)$.

The contributions to the interacting two-point function at the next perturbative order read³

$$\begin{aligned} T^{(1)} &= \langle \phi_1 \phi_2 iS_{\text{int}}[\phi] \rangle - \langle \phi_1 \phi_2 \rangle \langle iS_{\text{int}}[\phi] \rangle \\ &= -\frac{i}{24} \lambda \int dy^4 \langle \phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y \rangle \\ &\quad + \frac{i}{24} \lambda \int dy^4 \langle \phi_1 \phi_2 \rangle \langle \phi_y \phi_y \phi_y \phi_y \rangle. \end{aligned} \quad (8.39)$$

Using Wick's theorem the two terms expand to 15 and 3 contributions. Consider the first term only: The 15 contributions can be grouped into two types. The first type receives 12 identical contributions from contracting the 4 identical ϕ_y 's in superficially different ways. The remaining 3 terms in the second group are identical for the same reason. We summarise the groups as follows

$$\begin{aligned} T_{1a}^{(1)} &= -\frac{i}{2} \lambda \int dy^4 \phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y, \\ T_{1b}^{(1)} &= -\frac{i}{8} \lambda \int dy^4 \phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y. \end{aligned} \quad (8.40)$$

The term originating from the denominator of the reduction formula evaluates to

$$T_2^{(1)} = -\frac{1}{8} (-i\lambda) \int dy^4 \phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y = -T_{1b}^{(1)}. \quad (8.41)$$

It precisely cancels the second contribution to the first term. Altogether we find the following expression for the first-order correction to the two-point function

$$T^{(1)} = T_{1a}^{(1)} = \frac{1}{2} (-i)^4 \lambda \int dy^4 G_{1y} G_{2y} G_{yy}. \quad (8.42)$$

In fact, the term G_{yy} is problematic as it stands; we shall comment on this issue later.

³The first term originates from the expansion of the numerator of the reduction formula for the interaction picture, the second term from its denominator.

These expressions are not as innocent as they may look: Using Wick's theorem the two terms expand to $7 \cdot 5 \cdot 3 \cdot 1 = 105$ and $3 \cdot 3 = 9$ terms. Gladly, most of these terms are identical and can be summarised, we group them into 24, $6 \cdot 12$ and $3 \cdot 3$ terms from the first contribution and $3 \cdot 3$ terms from the second one

$$\begin{aligned}
F_{1a}^{(1)} &= (-i\lambda) \int dy^4 \underbrace{\phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y}_{\text{diagram}}, \\
F_{1b}^{(1)} &= \frac{1}{2}(-i\lambda) \int dy^4 \underbrace{\phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y}_{\text{diagram}} + 5 \text{ perm.}, \\
F_{1c}^{(1)} &= \frac{1}{8}(-i\lambda) \int dy^4 \underbrace{\phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y}_{\text{diagram}} + 2 \text{ perm.}, \\
F_2^{(1)} &= -\frac{1}{8}(-i\lambda) \int dy^4 \underbrace{\phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y}_{\text{diagram}} + 2 \text{ perm.}. \tag{8.50}
\end{aligned}$$

The graphical representation of these terms is

$$\begin{aligned}
F_{1a}^{(1)} &\simeq \text{diagram}, \\
F_{1b}^{(1)} &\simeq \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram} + \text{diagram}, \\
F_{1c}^{(1)} &\simeq \text{diagram} + \text{diagram} + \text{diagram}. \tag{8.51}
\end{aligned}$$

Let us now discuss the roles and features of the three terms.

Vacuum Bubbles. In the above result we notice that again the contribution $F_2^{(1)}$ from the denominator of the reduction formula cancels the term $F_{1c}^{(1)}$ from the numerator. This effect is general:

$$\text{diagram} \quad \text{diagram} \quad \text{diagram} \quad \text{diagram} \tag{8.52}$$

Some graphs have components which are coupled neither to the rest of the graph nor to the external points. Such parts of the graph are called vacuum bubbles.

- Vacuum bubbles represent some virtual particles which pop out of the quantum mechanical vacuum and annihilate among themselves. They do not interact with any of the physically observed particles, hence one should be able to ignore such contributions.

- Vacuum bubbles are usually infinite. Here we obtain as coefficient $G_F(0)^2 \int dy^4$. This contains two divergent factors of $G_F(0)$ and an infinite spacetime volume $\int dy^4$.
- Formally, we could remove such terms by adding a suitable field-independent term to the interaction Lagrangian. Alternatively, we could normal order it.⁴
- In any case, vacuum bubbles are generally removed by the denominator of the reduction formula for interacting correlation functions. This cancellation ensures that the interacting vacuum is properly normalised, $\langle 0|0 \rangle_{\text{int}} = 1$. Any diagram containing at least one vacuum bubble can be discarded right away.

Disconnected Graphs. The contribution from $F_{\text{lb}}^{(1)}$ is reminiscent of the correction $T^{(1)}$ to the two-point function. In fact it can be written as a sum of products of two-point functions

$$F_{\text{lb}}^{(1)} = T_{12}^{(0)} T_{34}^{(1)} + T_{13}^{(0)} T_{24}^{(1)} + T_{14}^{(0)} T_{23}^{(1)} + T_{12}^{(1)} T_{34}^{(0)} + T_{13}^{(1)} T_{24}^{(0)} + T_{14}^{(1)} T_{23}^{(0)}. \quad (8.53)$$

This combination is precisely the first-order contribution to a product of two $T(\lambda)$'s

$$x_1 \text{---} \text{---} x_2 = x_1 \text{---} \text{---} x_2 + \lambda \text{---} \text{---} + \dots \quad (8.54)$$

This is also a general feature of correlation functions:

- Correlation functions contain disconnected products of lower-point functions. The corresponding graphs contain disconnected components (each of which is connected to at least one external field).
- Such contributions are typically put aside because their form can be composed from correlators of fewer fields.⁵ Nevertheless, they are essential and non-negligible contributions to the correlation functions.
- Such disconnected contributions represent processes that take place simultaneously without interfering with each other.

Quite generally one can split the contributions into connected and disconnected terms. Here we know, to all orders in λ

$$F(\lambda) = T_{12}(\lambda) T_{34}(\lambda) + T_{13}(\lambda) T_{24}(\lambda) + T_{14}(\lambda) T_{23}(\lambda) + F_{\text{conn}}(\lambda),$$

⁴By construction, the normal ordering would prevent Wick contractions between the fields of the Lagrangian. However, the interplay between normal and time ordering is somewhat ambiguous for local products.

⁵When computing an n -point function one will typically already have computed all the k -point functions with $k < n$ anyway.

$$+ \text{ (diagram: a blue circle with four external lines labeled } x_1, x_2, x_3, x_4 \text{)} , \quad (8.55)$$

where $F_{\text{conn}}(\lambda)$ summarises all connected contributions. In our case

$$F_{\text{conn}}^{(1)} = F_{1a}^{(1)} = -i\lambda \int dy^4 G_{1y} G_{2y} G_{3y} G_{4y},$$

$$\text{ (diagram: blue circle with four external lines)} = \lambda \text{ (diagram: red circle with four external lines and a red dot labeled } y \text{)} + \dots \quad (8.56)$$

Tadpoles. We were careful enough not to write the leading correction to the two-point function too explicitly

$$T^{(1)} = \frac{1}{2}\lambda G_F(0) \int dy^4 G_F(x_1 - y) G_F(x_2 - y). \quad (8.57)$$

We notice that one of the propagators decouples from the function. Moreover its argument is precisely zero because the propagator connects a point to itself.

- The result is divergent, it is very similar to the vacuum energy we encountered much earlier in QFT.
- In our derivation of time ordering we were sloppy in that we did not discuss the case of equal times. In a local Lagrangian, however, all terms are defined at equal time, moreover at equal spatial position. It would make sense to employ normal ordering in this case, which eliminates the term right at the start.
- Whatever the numerical value of $G_F(0)$, even if infinite, it does not yield any interesting functional dependence to $T^{(1)}$. In fact it could be eliminated by formally adding a term $-\frac{i}{4}\lambda G_F(0)\phi^2$ to the interaction Lagrangian. This has the same effect as normal ordering the Lagrangian.

This term is called a tadpole term because the corresponding diagram looks like a tadpole sitting on the propagator line.

$$\text{ (diagram: a tadpole on a line)} \quad \text{ (diagram: a blue circle with four external lines and a tadpole on one of them)} \quad (8.58)$$

More generally, tadpoles are internal parts of a diagram which are attached to the rest of the diagram only via a single vertex. In most cases, they can be compensated by adding suitable local terms to the interaction Lagrangian. Even though this correction term is somewhat dangerous and somewhat trivial, let us pretend it is a regular contribution and carry it along.

Symmetry Factors. In our computation, we have encountered many equivalent contributions which summed up into a single term. We observe that these sums have conspired to cancel most of the prefactors of $1/24$. The purpose of having a

prefactor of $1/24$ for ϕ^4 in the action is precisely to be cancelled against multiplicities in correlators, where λ typically appears without or with small denominators.

We can avoid constructing a large number of copies of the same term by considering the symmetry of terms or the corresponding graphs. The symmetry factor is the inverse size of the discrete group that permutes the elements of a term or a graph while leaving its structure unchanged.

To make use of symmetry factors for the calculation of correlation functions, one should set up the Lagrangian such that every product of terms comes with the appropriate symmetry factor. For example, the term ϕ^4 allows arbitrary permutations of the 4 ϕ 's. There are $4! = 24$ such permutations, hence the appropriate symmetry factor is $1/24$.⁶

The crucial insight is the following: when the symmetry factors for the Lagrangian are set up properly, the summed contributions to correlation function also have their appropriate symmetry factors.

Occasionally, determining the symmetry factors correctly can be difficult, as one has to identify all permissible permutations. It is tricky, for example, when the graphical representation has a complicated topology or when it hides some relevant information.

Let us consider the symmetry factors of the terms we have computed so far. The contributions $T^{(0)}$, $F^{(0)}$ and $F_{\text{conn}}^{(1)}$ have trivial symmetry factors.


(8.59)

Permutations of any of the elements would change the labelling of the external fields. The symmetry factor for the tadpole diagram is $1/2$.


(8.60)

The relevant \mathbb{Z}_2 symmetry flips the direction of the tadpole line.⁷ Finally, the vacuum bubble diagram has a symmetry factor of $1/8$.


(8.61)

There are two factors of $1/2$ for flipping the direction of the tadpole lines. Then there is another factor of $1/2$ for permuting the two tadpole lines.

⁶After all, we are free to call the term that multiplies ϕ^4 either $\lambda/24$ or λ' . It is not even difficult to translate between them.

⁷In fact, the symmetry acts on the connections of lines to vertices. Here, exchanging the two endpoints of the tadpole line is the only symmetry.

8.4 Feynman Rules

We have seen how to evaluate some perturbative contributions to interacting correlators. Following the formal prescription leads to a lot of combinatorial overhead as the results tend to be reasonably simple compared to the necessary intermediate steps. Feynman turned the logic around and proposed a simple graphical construction of correlators:

The interacting correlator F of several fields can be expressed as a sum of so-called Feynman graphs

$$F = \sum_{\text{graphs}} F_{\text{graph}}. \quad (8.62)$$

Each Feynman graph represents a certain mathematical expression which can be evaluated from the graph by the Feynman rules. Moreover a Feynman graph nicely displays the physical process that leads to the corresponding term of the correlator.

For every weakly coupled QFT there is a set of Feynman rules to compute its correlators.⁸ Here we list the Feynman rules for the scalar ϕ^4 model.

Feynman Rules in Position Space. Consider a generic correlator of several fields

$$F(x_1, \dots, x_n) = \langle \phi(x_1) \dots \phi(x_n) \rangle_{\text{int}}. \quad (8.63)$$

A permissible graph for this correlator:

- has undirected and unlabelled edges;

$$z_k \text{---} z_l \quad (8.64)$$

- has n 1-valent (external) vertices labelled by x_j ;

$$x_j \text{---} \quad (8.65)$$

- has an arbitrary number m of 4-valent (internal) vertices labelled by y_j ;

$$\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \text{---} \end{array} y_j \quad (8.66)$$

- can have lines connecting a vertex to itself (tadpole);

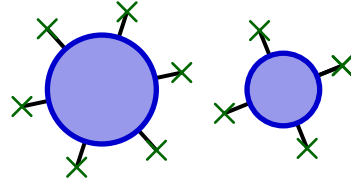
$$\begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array} \quad (8.67)$$

- can have multiple lines connecting two vertices;

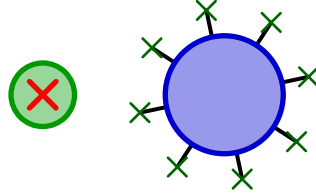
$$\text{---} \text{---} \quad (8.68)$$

⁸Similar graphs and rules can actually be set up and applied to a wide range of algebraic problems not at all limited to relativistic QFT's.

- can have several connection components;


(8.69)

- must not have components disconnected from all of the external vertices x_j (vacuum bubble).

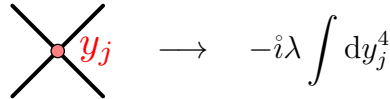

(8.70)

For each topologically distinct graph we can compute a contribution according to the following rules:

- For each edge connecting two vertices z_k and z_l write a factor

$$z_k \text{---} z_l \longrightarrow -iG_F(z_k - z_l). \quad (8.71)$$

- For each 4-valent vertex y_j , write a factor of $-i\lambda$ and integrate over dy_j^4


(8.72)

- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

Feynman Rules in Momentum Space. One of the problems we have not yet mentioned is that the Feynman propagator G_F is a complicated function in spacetime. Moreover, we need to compute multiple convolution integrals of these functions over spacetime, e.g. the integral defining $F_{\text{conn}}^{(1)}$. This soon enough exceeds our capabilities.

These computations can be simplified to some extent by going to momentum space. Such a momentum space representation will be particularly useful later when we compute the interaction between particles with definite momenta in particle scattering experiments.

The momentum space version is defined as follows⁹

$$F(p_1, \dots, p_n) = \int dx_1^4 \dots dx_n^4 e^{-ix_1 \cdot p_1 - \dots - ix_n \cdot p_n} \langle \phi_1 \dots \phi_n \rangle. \quad (8.73)$$

A Feynman graph in momentum space:

⁹Note that we are evaluating a time-ordered correlator. This is well-defined in position space and we have to perform the Fourier integrals *after* computing the correlator. It implies that the momenta p_j can and should be taken off-shell $p_j^2 + m^2 \neq 0$. This is different from computing a correlator such as $\langle 0|a(\vec{p}_1) \dots a^\dagger(\vec{p}_n)|0 \rangle$ where all the momenta are defined only on shell $p_j^2 + m^2 = 0$.

- has edges labelled by a directed flow of 4-momentum ℓ_j from one end to the other;

$$\begin{array}{c} \ell_j \longrightarrow \\ \bullet \text{-----} \bullet \end{array} \quad (8.74)$$

- has n 1-valent (external) vertices with an outflow of 4-momentum p_j ;

$$p_j \times \longleftarrow \quad (8.75)$$

- has an arbitrary number m of 4-valent (internal) vertices which conserve the flow of momentum;

$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \quad (8.76)$$

- shares the remaining attributes with the position space version.

The Feynman rules for evaluating a graph read:

- Work out the flow of momentum from the external vertices across the internal vertices. Label all edges with the appropriate momenta ℓ_j .

$$\begin{array}{c} \ell_i + \ell_j \\ \uparrow \\ \ell_i \quad \ell_j \\ \swarrow \quad \searrow \end{array} \quad (8.77)$$

- There is a momentum-conservation condition $p_{j_1} + \dots + p_{j_m} = 0$ for each connected component of the graph. It includes all contributing external momenta p_j . Write a factor enforcing momentum conservation

$$\begin{array}{c} \times \\ \times \\ \times \\ \times \\ \times \end{array} \text{ (around a blue circle)} \longrightarrow (2\pi)^4 \delta^4(p_{j_1} + \dots + p_{j_m}). \quad (8.78)$$

- For each internal loop of the graph, there is one undetermined 4-momentum ℓ_j . Integrate the final expression over all such momenta

$$\begin{array}{c} \text{loop with } \ell_j \end{array} \longrightarrow \int \frac{d\ell_j^4}{(2\pi)^4}. \quad (8.79)$$

- For each edge write a factor

$$\begin{array}{c} \ell_j \longrightarrow \\ \bullet \text{-----} \bullet \end{array} \longrightarrow \frac{-i}{\ell_j^2 + m^2 - i\epsilon}. \quad (8.80)$$

- For each 4-valent vertex, write a factor

$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \longrightarrow -i\lambda. \quad (8.81)$$

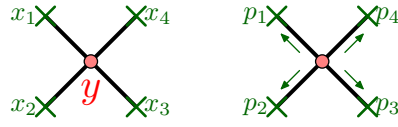
- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

General Models. We observe that the Feynman graphs and rules for a QFT model reflect quite directly the content of its action:

- In particular, the free part of the action S_0 determines the types and features of the fields and particles. These are reflected by the Feynman propagator G_F which is associated to the edges.
- The interaction part of the action S_{int} contains all the information about the set of interaction vertices.

Examples. Let us apply the Feynman rules to compute the mathematical expressions for a few Feynman graphs.

Consider first the graph for the leading connected contribution $F_{\text{conn}}^{(1)}$ to the four-point function.


(8.82)

Applying the rules for position space, we obtain right away

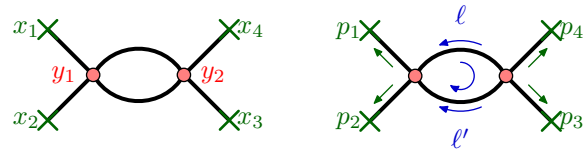
$$F_{\text{conn}}^{(1)} = (-i\lambda)(-i)^4 \int dy^4 \prod_{j=1}^4 G_F(x_j - y). \quad (8.83)$$

In momentum space, the corresponding result is

$$F_{\text{conn}}^{(1)} = (-i\lambda)(-i)^4 (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \prod_{j=1}^4 \frac{1}{p_j^2 + m^2 - i\epsilon}. \quad (8.84)$$

This expression is merely a rational function and does not contain any integrals. It is therefore conceivably simpler than its position space analog. Unfortunately, it is generally not easy to perform the Fourier transformation back to position space.¹⁰

Next, consider a slightly more complicated example involving an internal loop.


(8.85)

Evaluation of the Feynman graph in position space is straight-forward

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6 \int dy_1^4 dy_2^4 G_F(y_1 - y_2)^2 \cdot G_F(x_1 - y_1)G_F(x_2 - y_1)G_F(x_3 - y_2)G_F(x_4 - y_2). \quad (8.86)$$

The symmetry factor is $1/2$ because the two lines of the internal loop can be interchanged.

¹⁰A notable exception is the massless case where the correlation functions in position space has a manageable form.

For momentum space, we first have to label the remaining lines along the internal loop: The total flow of momentum *out of* the left vertex to the external lines is $p_1 + p_2$, whereas the momenta on both internal lines are yet undetermined. The sum of internal momenta flowing out of the vertex must therefore equal $-p_1 - p_2$ by momentum conservation.¹¹ One internal momentum remains undetermined, let us call it ℓ and eventually integrate over it. The other one must equal $\ell' = p_1 + p_2 - \ell$.

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4) \prod_{j=1}^4 \frac{1}{p_j^2 + m^2 - i\epsilon} \cdot \int \frac{d\ell^4}{(2\pi)^4} \frac{1}{\ell^2 + m^2 - i\epsilon} \frac{1}{(p_1 + p_2 - \ell)^2 + m^2 - i\epsilon}. \quad (8.87)$$

Now we are left with a multiple integral over a rational function. There exist techniques to deal with this sort of problem, we will briefly discuss some of the basic ones at the end of this course. Some integrals like this one can be performed, but most of them remain difficult and it is an art to evaluate them. Unfortunately, numerical methods are not applicable straight-forwardly either.¹² This is a generic difficulty of QFT with no hope for a universal solution. The Feynman rules are a somewhat formal method and it is hard to extract concrete numbers or functions from them.

8.5 Feynman Rules for QED

Finally, we would like to list the Feynman rules for the simplest physically relevant QFT model, namely quantum electrodynamics (QED). We shall use the Lagrangian in Feynman gauge

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi}(\gamma^\mu \partial_\mu - m)\psi - \frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu, \\ \mathcal{L}_{\text{int}} &= -iq \bar{\psi} \gamma^\mu \psi A_\mu. \end{aligned} \quad (8.88)$$

Feynman Graphs in Momentum Space. A non-trivial interacting correlation function in this model must contain as many fermionic fields ψ as conjugates $\bar{\psi}$ due to global U(1) symmetry. Consider such a correlation function

$$\langle A_{\mu_1}(k_1) \dots A_{\mu_m}(k_m) \bar{\psi}_{b_1}(q_1) \psi^{a_1}(p_1) \dots \bar{\psi}_{b_n}(q_n) \psi^{a_n}(p_n) \rangle. \quad (8.89)$$

Admissible Feynman graphs have the following properties in addition to or instead of the ones of the ϕ^4 model:

¹¹By considering the right vertex, it must also equal $p_3 + p_4$. This requirement is consistent by means of overall momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$.

¹²The integrals are over an unbounded space with a large dimension. Moreover the poles and asymptotic behaviour of the integrand lead to numerical instabilities and divergences. The latter are inherent to QFT and need to be dealt with.

- There are two types of edges: undirected wavy lines (photons) or directed straight lines (electrons and/or positrons).

$$c_k \bullet \xrightarrow{\ell_j} d_l \quad \nu_k \bullet \text{---} \nu_l \quad (8.90)$$

- The edges are labelled by a directed flow of 4-momentum ℓ_j .
- The ends of wavy lines are labelled by indices ν_k and ν_l ; the ends of straight lines are labelled by indices c_k and d_l in the direction of the arrow of the straight line.
- There is a 1-valent (external) vertex for each field in the correlator. The momentum outflow and the label at the end of the edge are determined by the corresponding field.

$$\psi: a_j \text{---} p_j \leftarrow \quad \bar{\psi}: b_j \text{---} q_j \leftarrow \quad A: \mu_j \text{---} k_j \leftarrow \quad (8.91)$$

- There is one type of (internal) vertex: It is 3-valent and connects an ingoing and an outgoing straight line (fermion) with a wavy (photon) line.

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} c_k \nu_j \\ d_l \end{array} \quad (8.92)$$

The QED-specific Feynman rules read as follows:

- The graph can have only fermion loops, which contribute an extra factor of (-1) due to their statistics

$$\begin{array}{c} \text{Feynman diagram of a fermion loop with external wavy lines} \end{array} \longrightarrow (-1) \int \frac{d\ell_j^4}{(2\pi)^4} \quad (8.93)$$

- For each wavy edge write a factor

$$\nu_k \bullet \text{---} \nu_l \xrightarrow{\ell_j} \frac{-i\eta_{\nu_k\nu_l}}{\ell_j^2 - i\epsilon}; \quad (8.94)$$

for each straight edge write a factor¹³

$$c_k \bullet \xrightarrow{\ell_j} d_l \longrightarrow \frac{-i(-i\ell_j \cdot \gamma + m)^{c_k d_l}}{\ell_j^2 + m^2 - i\epsilon}. \quad (8.95)$$

- For each 3-valent vertex, write a factor

$$\begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} c_k \nu_j \\ d_l \end{array} \longrightarrow -q(\gamma^{\nu_j})^{c_k d_l}. \quad (8.96)$$

Note that the arrows on the fermion lines conveniently describe the ordering of gamma-matrix factors from left to right and from ψ to $\bar{\psi}$.

¹³The signs in the numerator of the Feynman propagator match with the direction of momentum flow required for the Dirac equation to hold.

Example. Let us consider the simplest interacting correlator in QED

$$F = \langle A_\mu(k) \bar{\psi}_b(q) \psi^a(p) \rangle. \quad (8.97)$$

We shall restrict to the leading order with one interaction vertex.



$$(8.98)$$

Applying the Feynman rules leads to the following expression

$$F^{(1)} = (2\pi)^4 \delta^4(k + p + q) (-q)(\gamma^\nu)^c_d \cdot \frac{-i\eta_{\mu\nu}}{k^2 - i\epsilon} \frac{-i(\not{p} \cdot \gamma + m)^a_c}{p^2 + m^2 - i\epsilon} \frac{-i(-\not{q} \cdot \gamma + m)^d_b}{q^2 + m^2 - i\epsilon}. \quad (8.99)$$

Summing over the internal indices c, d, ν and combining the three spinor matrices we obtain a matrix product as the answer

$$F^{(1)} = (2\pi)^4 \delta^4(k + p + q) \frac{-iq[(\not{p} \cdot \gamma + m)\gamma_\mu(-\not{q} \cdot \gamma + m)]^a_b}{(k^2 - i\epsilon)(p^2 + m^2 - i\epsilon)(q^2 + m^2 - i\epsilon)}. \quad (8.100)$$

Note that the ordering of spinor matrix factors from left to right follows the flow of the spinor lines in the Feynman diagram.

QED and Gauge Invariance. Note that QED is a gauge theory which requires some gauge fixing. Feynman gauge is very convenient, but any other consistent gauge is acceptable, too. Different gauges imply different propagators which lead to non-unique results for correlation functions. Unique results are only to be expected when the field data within the correlation function is gauge invariant.

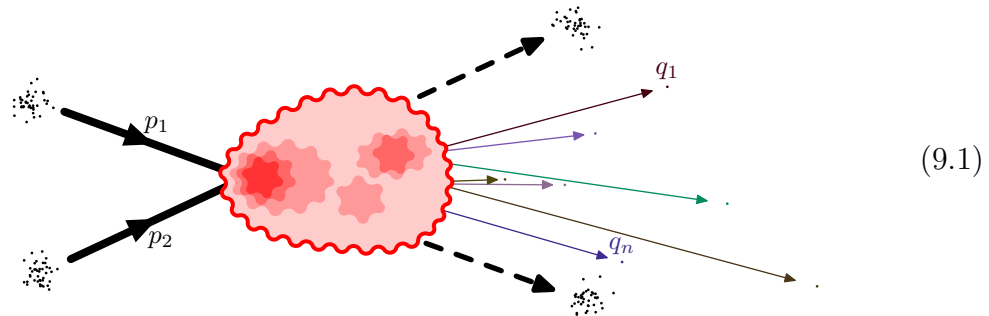
More precisely, the correlator should contain the gauge potential $A_\mu(x)$ only in the combination $F_{\mu\nu}(x)$ or as the coupling $\int dx^4 J^\mu A_\mu$ to some conserved current $J_\mu(x)$. Moreover, charged spinor fields should be combined into uncharged products, e.g. $\bar{\psi}(x) \dots \psi(x)$ potentially dressed with covariant derivatives.

9 Particle Scattering

A goal of this course is to understand how to compute scattering processes in particle physics.

9.1 Scattering Basics

Setup. The usual setup for scattering experiments at particle colliders is the following:



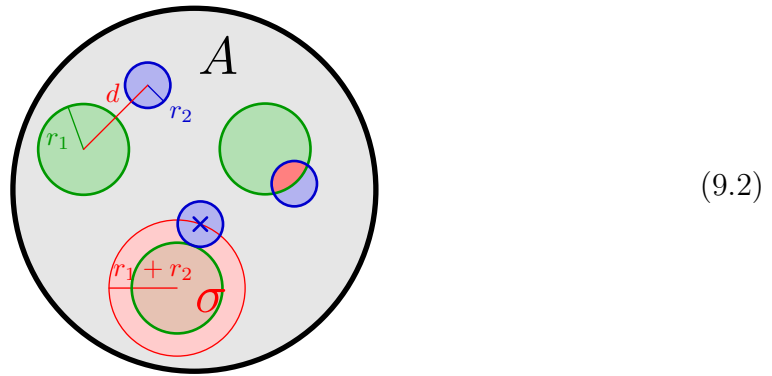
- Two bunches of particles are accelerated to high or relativistic velocities and made to collide.
- Whenever two particles from the bunches come very close, they produce some complicated interacting quantum state.
- After a while this state evolves into several particles moving away in various directions.
- The outgoing particles of each scattering event are measured and recorded.

Some additional comments:

- Quantum mechanics is probabilistic, so a large number of particle collisions must be measured.
- To measure collisions of three or more incoming particles is technically challenging because they would all have to be focused within a tiny region of space simultaneously. The likelihood for two particles to scatter is much higher.
- By Lorentz symmetry, the directions of the two ingoing momenta $\vec{p}_{1,2}$ can be adjusted arbitrarily. In some reference frame, the momenta will be parallel and along the z -axis. The relevant quantity is the centre-of-mass energy squared $s = -(p_1 + p_2)^2$. The highest energies \sqrt{s} are obtained where the collisions are head-on with equal but opposite momenta. For practical purposes, the particles can have momenta of different magnitude or one of the two bunches could be a fixed target at rest.

- The particle momenta in the beam are not perfectly aligned. By the uncertainty principle this is actually impossible if the beam is also focused on a finite area.
- The particle detectors are not perfect: They have a certain spatial and temporal resolution. They measure the energy and momenta at a certain resolution. They may not be able to detect and distinguish all kinds of particles; they may miss some particles; they may misidentify some. Scattered particles along the beam direction are hardest to detect.

Cross Sections. How to quantify scattering? Consider a simple classical scattering experiment:



- Take two hard balls of radii r_1, r_2 .
- Throw them towards each other along the z -axis in opposing directions.
- Depending on the transverse offset d , the balls will either hit ($d < r_1 + r_2$) or miss ($d > r_1 + r_2$).¹
- When the balls hit, they bounce off in different directions.

Quantum mechanics is probabilistic, there cannot be such deterministic output. One has to repeat the experiment many times or perform an experiment with many identical particles and count:

- Accelerate two bunches of n_1 and n_2 particles.
- Focus each bunch on a cross-sectional area A .
- Repeat the experiment n_{ex} times.
- Count the number of individual scattering events N .

The expectation value for N is

$$N = \frac{n_{\text{ex}} n_1 n_2 \sigma}{A}, \quad (9.3)$$

where the characteristic quantity is the scattering cross section σ . For two classical hard balls one obtains $\sigma = \pi(r_1 + r_2)^2$: Given the transverse position of the first ball, the centre of the second ball must be within an area of σ to make the two collide.

In collider experiments one measures scattering cross sections:

¹More accurately, at a near miss, the flow of air will also deform the balls' trajectories slightly.

- Total or inclusive cross sections σ simply count the number of overall collision events.
- Differential cross sections $d\sigma$ measure the number of events where the outgoing particles have predetermined momenta.² The definition of $d\sigma$ depends on the number of outgoing particles. The so-called phase space is furthermore constrained by Poincaré symmetry.
- One may even resolve the polarisation of the outgoing particles and measure polarised cross sections.

9.2 Cross Sections and Matrix Elements

The computation of the scattering cross section is not straight-forward. Naively, we prepare initial and final states with definite momenta p_1, p_2 and q_1, \dots, q_n at some times t_{in} and t_{out} in the distant past and distant future

$$\langle f | \sim \langle q_1, \dots, q_n |, \quad | i \rangle \sim | p_1, p_2 \rangle. \quad (9.4)$$

The probability is given by the square of the correlator $\langle f | \exp(-iH(t_{\text{out}} - t_{\text{in}})) | i \rangle$

$$\sigma \sim |\langle f | \exp(-iH(t_{\text{out}} - t_{\text{in}})) | i \rangle|^2. \quad (9.5)$$

For initial and final states with definite momenta, the correlator contains a delta-function $\delta^4(P_{\text{in}} - P_{\text{out}})$ to conserve momentum. It cannot be squared because this would result in a factor of $\delta^4(0) = \infty$. We know that such factors represent some volume of spacetime relevant to the problem. A proper treatment requires the use of wave packets. They actually account for the finite extent of the ingoing bunches, namely the cross-sectional area A , and for the finite resolution of the detector. The factor $\delta^4(0)$ represents this area A among others.

A somewhat tedious calculation in terms of wave packets yields a meaningful result for the differential cross section of $2 \rightarrow n$ scattering. At the end of the day, the wave packets can be focused to definite momenta³

$$d^{3n}\sigma = \frac{(2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}})}{4 \|e_2(\vec{p}_2)\vec{p}_1 - e_1(\vec{p}_1)\vec{p}_2\|} \prod_{k=1}^n \frac{d\vec{q}_k^3}{(2\pi)^3 2e'_k(\vec{q}_k)} |M|^2. \quad (9.6)$$

Here, M is the appropriate element of the scattering matrix with the momentum-conserving delta-function stripped off

$$\lim_{t_{\text{in}, \text{out}} \rightarrow \mp \infty} \langle f | \exp(-iH(t_{\text{out}} - t_{\text{in}})) | i \rangle = (2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}}) iM. \quad (9.7)$$

²The direction of the scattered classical balls is determined by the impact parameter d , and hence certain regions of the scattering cross section correspond to specific angles. In quantum mechanics this is mostly a matter of probability.

³This quantity is not invariant under Lorentz transformations due to the denominator $\|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1\|$. Nevertheless, it is covariant and transforms like an area as it should.

The normalisation is such that in the free theory the correlator for $n = 2$ final state particles equals⁴

$$2e_1(\vec{p}_1) 2e_2(\vec{p}_2) (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2). \quad (9.8)$$

For $2 \rightarrow 2$ scattering it makes sense to fix the outgoing overall momentum P_{out} , and express the result in terms of the spherical angle element $d^2\Omega$ of the direction of outgoing particle 1

$$\frac{d^2\sigma}{d^2\Omega} = \frac{1}{\|e_2(\vec{p}_2)\vec{p}_1 - e_1(\vec{p}_1)\vec{p}_2\|} \frac{\|\vec{q}_1\|^3}{(e'_2(\vec{q}_2)\vec{q}_1 - e'_1(\vec{q}_1)\vec{q}_2) \cdot \vec{q}_1} \frac{|M|^2}{64\pi^2}. \quad (9.9)$$

The formula simplifies in the centre-of-mass frame to

$$\frac{d^2\sigma}{d^2\Omega} = \frac{\|\vec{q}\|}{\|\vec{p}\|} \frac{|M|^2}{64\pi^2 s}, \quad (9.10)$$

where $s = -P_{\text{in}}^2 = -(p_1 + p_2)^2$ is the centre-of-mass energy squared. Its form becomes even simpler in case all four particles have identical masses

$$\frac{d^2\sigma}{d^2\Omega} = \frac{|M|^2}{64\pi^2 s}. \quad (9.11)$$

The total cross section is then obtained by integrating over all final states. For $n = 2$ outgoing particles this amounts to an integral over the two-sphere S^2 of spherical angles

$$\sigma = \int d^2\Omega \frac{d^2\sigma}{d^2\Omega}. \quad (9.12)$$

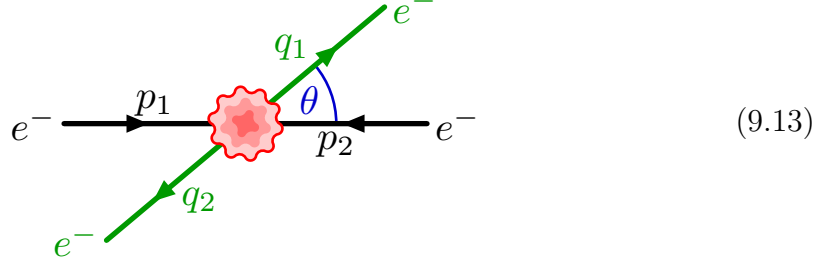
Note that in the case of identical outgoing particles, their exchange statistics must be taken into account: The state $|+\vec{p}, -\vec{p}\rangle$ is indistinguishable from $|-\vec{p}, +\vec{p}\rangle$, and integration for the total cross section should be over distinct final states only. This reduces the domain of integration to the real projective space S^2/\mathbb{Z}_2 . Equivalently, the integral can be extended to a full two-sphere S^2 with the appropriate symmetry factor of $1/2$ to compensate for the double counting.

9.3 Electron Scattering

We can now compute some realistic scattering event in quantum electrodynamics. We shall consider scattering of two electrons into two electrons (Møller

⁴This contribution representing no scattering is actually removed from M for $2 \rightarrow 2$ particle scattering.

scattering).⁵



Here, we will not distinguish the two polarisation modes of the electron spin. One might as well consider the polarised cross section, but the experimental setup as well as the theoretical calculation is more challenging.

Initial and Final States. To prepare the initial and final states we use the interaction picture. The free reference field provides the creation and annihilation operators for the in- and outgoing particles which are assumed not to interact when sufficiently far away. Moreover the initial and final states will be practically independent of t_{in} and t_{out} as long as the latter are sufficiently large.

The initial state is composed from two ingoing electrons

$$|i\rangle = a_{\alpha}^{\dagger}(\vec{p}_1) a_{\beta}^{\dagger}(\vec{p}_2) |0\rangle. \quad (9.14)$$

The electrons have definite momenta p_1, p_2 . Let us assume they are in their centre-of-mass frame with momenta aligned along the z -axis

$$p_{1,2} = (e, 0, 0, \pm p), \quad (9.15)$$

where $e^2 = p^2 + m^2$. The polarisations α, β are required to set up the state properly. We will not care about them, so we should eventually *average* over all ingoing polarisation configurations.

We want to probe the final state for two outgoing electrons

$$\langle f| = \langle 0| a_{\delta}(\vec{q}_2) a_{\gamma}(\vec{q}_1). \quad (9.16)$$

In the centre-of-mass frame they will escape in two opposite directions with the same magnitude p of momentum. Due to rotational symmetry around the z -axis,⁶ we only need to probe for particles in the x, z -plane

$$q_{1,2} = (e, \pm p \sin \theta, 0, \pm p \cos \theta). \quad (9.17)$$

Again we shall not care about the polarisations. We therefore have to *sum* over all outgoing polarisation configurations.

For a fixed particle momentum p or energy e , we will be interested in the angular distribution of outgoing particles. Due to rotational symmetry the differential

⁵Depending on conventions, our calculation may also represent positron-positron scattering. Obviously, the cross section is exactly the same by charge conjugation symmetry.

⁶We will not measure polarisations which would otherwise break the symmetry.

cross section $d^2\sigma/d^2\Omega$ must be an even function of the scattering angle θ alone. This function also has the symmetry $\theta \rightarrow \pi + \theta$ because the outgoing particles are indistinguishable

$$\frac{d^2\sigma}{d^2\Omega} = \frac{d^2\sigma}{d^2\Omega}(\theta) = \frac{d^2\sigma}{d^2\Omega}(-\theta) = \frac{d^2\sigma}{d^2\Omega}(\pi - \theta). \quad (9.18)$$

Time Evolution. We now insert the time evolution operator U_{int} of the interaction picture between the initial and final states to determine the probability amplitude⁷

$$F = \lim_{t_{\text{in}}, t_{\text{out}} \rightarrow \mp\infty} \langle f | U_{\text{int}}(t_{\text{out}}, t_{\text{in}}) | i \rangle = (2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}}) iM. \quad (9.19)$$

The matrix element is a function of the momenta and the polarisations $M_{\alpha\beta\gamma\delta}(p_1, p_2, q_1, q_2)$.⁸

The expansion of the amplitude at leading order reads simply

$$\begin{aligned} F^{(0)} &= \langle f | i \rangle \\ &= 2e(\vec{p}_1) 2e(\vec{p}_2) (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1) \delta^3(\vec{p}_2 - \vec{q}_2) \\ &\quad - 2e(\vec{p}_1) 2e(\vec{p}_2) (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_2) \delta^3(\vec{p}_2 - \vec{q}_1). \end{aligned} \quad (9.20)$$

The contribution from the free theory represents the situation when the two particles pass by each other without scattering at all. Note that there are two terms corresponding to the fact that the particles are indistinguishable.

At first perturbative order the matrix element vanishes

$$\begin{aligned} F^{(1)} &= i \langle f | S_{\text{int}} | i \rangle = i \int dx^4 \langle f | \mathcal{L}_{\text{int}}(x) | i \rangle \\ &= q \int dx^4 \langle f | A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) | i \rangle = 0 \end{aligned} \quad (9.21)$$

because there is a single electromagnetic field which cannot contract to anything else and thus directly annihilates either of the vacua.

Second Order. For the next order we insert two interaction Lagrangians

$$F^{(2)} = \frac{1}{2} i^2 \int dx^4 dy^4 \langle f | T[\mathcal{L}_{\text{int}}(x) \mathcal{L}_{\text{int}}(y)] | i \rangle. \quad (9.22)$$

Each of the interaction Lagrangians contains an electromagnetic field. As they would otherwise annihilate the vacua, they have to be contracted via Wick's theorem (for the field A) by a Feynman propagator

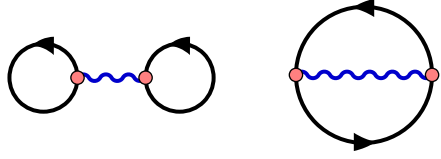
$$\frac{1}{2} q^2 \int dx^4 dy^4 \langle f | T[\underbrace{A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x)} A_\nu(y) \bar{\psi}(y) \gamma^\nu \psi(y)] | i \rangle. \quad (9.23)$$

Next, Wick's theorem should be applied to the time-ordered spinor fields yielding several contributions:

⁷The conventional factor of i typically makes the leading contributions to M (mostly) real.

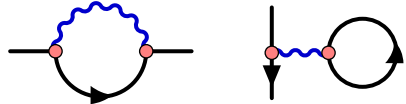
⁸We can write it as a function of all the external momenta noting that we shall only evaluate it for $p_1 + p_2 = q_1 + q_2$.

- There are two vacuum bubble contributions.


(9.24)

These vacuum processes take place everywhere and all the time, and they do not interact with the scattering process. As discussed earlier, they are discarded by the reduction formula of the interaction picture.

- There are two correction terms with two remaining external fields.


(9.25)

They contribute to two point functions of spinor fields, but cannot be non-trivial functions of all the four scattering particle momenta. Here they contribute only to forward scattering, and we can safely ignore their contribution. We will discuss their relevance later.

- Finally, there is one connected diagram.


(9.26)

This is the leading non-trivial contribution to the scattering process.

Connected Contribution. In our case, the latter diagram of interest is conveniently formulated by replacing time ordering by normal ordering in the above expression to avoid all internal connections

$$-\frac{1}{2}i q^2 \int d^4x d^4y G_{\mu\nu}^F(x-y) \langle f | N [\bar{\psi}(x) \gamma^\mu \psi(x) \bar{\psi}(y) \gamma^\nu \psi(y)] | i \rangle. \quad (9.27)$$

Now we need to contract the fields with the external particles. This is achieved by the following two anti-commutators which follow from the mode expansion of the free Dirac field

$$\begin{aligned} \{\bar{\psi}(x), a_\alpha^\dagger(\vec{p})\} &= e^{ip \cdot x} \bar{v}_\alpha(\vec{p}), \\ \{a_\alpha(\vec{q}), \psi(x)\} &= e^{-iq \cdot x} v_\alpha(\vec{q}). \end{aligned} \quad (9.28)$$

Putting everything together we obtain the matrix element

$$\begin{aligned}
F_{\text{conn}}^{(2)} &= F_t^{(2)} + F_u^{(2)} = \text{Diagram 1} - \text{Diagram 2}, \\
F_t^{(2)} &= -i q^2 \int d^4x d^4y G_{\mu\nu}^F(x-y) e^{ip_1 \cdot x + ip_2 \cdot y - iq_1 \cdot x - iq_2 \cdot y} \\
&\quad \cdot \bar{v}_\alpha(\vec{p}_1) \gamma^\mu v_\gamma(\vec{q}_1) \bar{v}_\beta(\vec{p}_2) \gamma^\nu v_\delta(\vec{q}_2), \\
F_u^{(2)} &= -\dots
\end{aligned} \tag{9.29}$$

The omitted term takes the same form, but with the two outgoing particles exchanged ($q_1 \leftrightarrow q_2$, $\gamma \leftrightarrow \delta$). The reason for the doubling of terms is that two identical types of particles are scattered. The two-particle wave function is anti-symmetric because the particles are fermionic.

The two remaining integrals are Fourier transforms. One of them transforms the Feynman propagator to momentum space. The other one generates the momentum conserving delta-function. Altogether the integral yields

$$\begin{aligned}
&\int d^4x d^4y G_{\mu\nu}^F(x-y) e^{ip_1 \cdot x + ip_2 \cdot y - iq_1 \cdot x - iq_2 \cdot y} \\
&= (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) G_{\mu\nu}^F(q_1 - p_1).
\end{aligned} \tag{9.30}$$

We now separate off the momentum conserving delta-function and write the matrix element M ⁹

$$\begin{aligned}
M_t &= -\frac{q^2 \eta_{\mu\nu}}{(p_1 - q_1)^2} \bar{v}_\alpha(\vec{p}_1) \gamma^\mu v_\gamma(\vec{q}_1) \bar{v}_\beta(\vec{p}_2) \gamma^\nu v_\delta(\vec{q}_2), \\
M_u &= -\dots
\end{aligned} \tag{9.31}$$

We could try to evaluate the various spinor products. It turns out to be much simpler to square the matrix element first

$$|M|^2 = \frac{1}{4} \sum_{\alpha, \beta, \gamma, \delta} M_{\alpha\beta\gamma\delta} M_{\alpha\beta\gamma\delta}^*. \tag{9.32}$$

The factor of $(1/2)^2$ originates from averaging over the polarisations of the ingoing particles.

Sum over Polarisations. A pleasant feature of the polarisation sums is that they can be performed by the completeness relations for spinor solutions

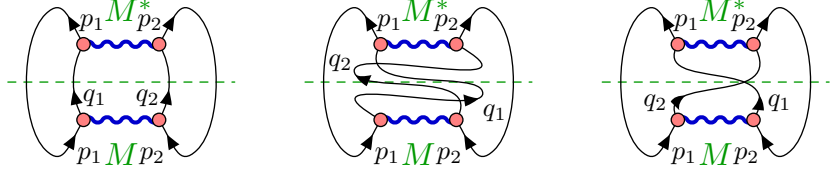
$$\sum_{\alpha} v_{\alpha}(\vec{p}) \bar{v}_{\alpha}(\vec{p}) = \not{p} \cdot \gamma - m. \tag{9.33}$$

⁹The $i\epsilon$ prescription for the Feynman propagator will not be relevant here.

We obtain the following three terms

$$|M|^2 = \frac{q^4 T_{tt}}{4(p_1 - q_1)^4} + \frac{q^4 T_{uu}}{4(p_1 - q_2)^4} - \frac{q^4 T_{tu}}{2(p_1 - q_1)^2(p_1 - q_2)^2}, \quad (9.34)$$

corresponding to the diagrams



$$(9.35)$$

The spinor products have turned into the traces

$$\begin{aligned} T_{tt} &= \text{tr}[(\not{p}_1 \cdot \gamma - m) \gamma_\mu (\not{q}_1 \cdot \gamma - m) \gamma_\nu] \\ &\quad \cdot \text{tr}[(\not{p}_2 \cdot \gamma - m) \gamma^\mu (\not{q}_2 \cdot \gamma - m) \gamma^\nu], \\ T_{uu} &= \text{tr}[(\not{p}_1 \cdot \gamma - m) \gamma_\mu (\not{q}_2 \cdot \gamma - m) \gamma_\nu] \\ &\quad \cdot \text{tr}[(\not{p}_2 \cdot \gamma - m) \gamma^\mu (\not{q}_1 \cdot \gamma - m) \gamma^\nu], \\ T_{tu} &= \text{tr}[(\not{p}_1 \cdot \gamma - m) \gamma_\mu (\not{q}_1 \cdot \gamma - m) \gamma^\nu] \\ &\quad \cdot (\not{p}_2 \cdot \gamma - m) \gamma^\mu (\not{q}_2 \cdot \gamma - m) \gamma_\nu. \end{aligned} \quad (9.36)$$

The double-trace terms are most conveniently evaluated using the spinor trace formulas¹⁰

$$\begin{aligned} \text{tr}(1) &= 4, \\ \text{tr}(\gamma^\mu) &= 0, \\ \text{tr}(\gamma^\mu \gamma^\nu) &= 4\eta^{\mu\nu}, \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) &= 0, \\ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) &= 4(\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}). \end{aligned} \quad (9.37)$$

This brings T_{tt} into the following form

$$\begin{aligned} T_{tt} &= 16(p_{1\mu} q_{1\nu} + q_{1\mu} p_{1\nu} - (p_1 \cdot q_1 + m^2) \eta_{\mu\nu}) \\ &\quad \cdot (p_2^\mu q_2^\nu + q_2^\mu p_2^\nu - (p_2 \cdot q_2 + m^2) \eta^{\mu\nu}) \\ &= 32(p_1 \cdot p_2)(q_1 \cdot q_2) + 32(p_1 \cdot q_2)(q_1 \cdot p_2) \\ &\quad + 32m^2(p_1 \cdot q_1 + p_2 \cdot q_2) + 64m^4. \end{aligned} \quad (9.38)$$

The other double-trace term takes a similar form with q_1 and q_2 interchanged. The crossed single-trace term can be simplified by means of the enveloping identities

$$\begin{aligned} \gamma_\mu \gamma^\mu &= 4, \\ \gamma_\mu \gamma^\nu \gamma^\mu &= -2\gamma^\nu, \\ \gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu &= 4\eta^{\nu\rho}, \\ \gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu &= -2\gamma^\sigma \gamma^\rho \gamma^\nu. \end{aligned} \quad (9.39)$$

¹⁰The latter of these formulas follow from anti-commuting one gamma-matrix past all the others.

After some algebra we obtain

$$T_{tu} = -32(q_1 \cdot q_2)(p_1 \cdot p_2) - 32m^4 \\ - 16m^2(p_1 \cdot p_2 + p_1 \cdot q_1 + p_1 \cdot q_2 + p_2 \cdot q_1 + p_2 \cdot q_2 + q_1 \cdot q_2). \quad (9.40)$$

Mandelstam Invariants. In order to simplify the expressions we introduce the Mandelstam invariants for $2 \rightarrow 2$ scattering processes¹¹

$$s = -(p_1 + p_2)^2 = -(q_1 + q_2)^2, \\ t = -(p_1 - q_1)^2 = -(p_2 - q_2)^2, \\ u = -(p_1 - q_2)^2 = -(p_2 - q_1)^2. \quad (9.41)$$

These variables describe the rest energy squared of an intermediate virtual particle composed from a pair of external particles, and as such they naturally appear in the virtual particle's propagator. Consequently, they lend the three pairwise partitions of external particles their established names: s -, t - and u -channel.¹²

Inverting the relations we can write all scalar products of momenta using the s, t, u

$$p_1 \cdot p_2 = q_1 \cdot q_2 = m^2 - \frac{1}{2}s, \\ p_1 \cdot q_1 = p_2 \cdot q_2 = \frac{1}{2}t - m^2, \\ p_1 \cdot q_2 = p_2 \cdot q_1 = \frac{1}{2}u - m^2. \quad (9.42)$$

Note furthermore that momentum conservation implies the relation¹³

$$s + t + u = 4m^2. \quad (9.43)$$

Using Mandelstam invariants, the traces can be expressed very compactly as¹⁴

$$T_{tt} = 8(t^2 - 2su + 8m^4), \\ T_{uu} = 8(u^2 - 2st + 8m^4), \\ T_{tu} = -8(s^2 - 8m^2s + 12m^4). \quad (9.44)$$

The squared matrix element now reads¹⁵

$$|M|^2 = q^4 \left(\frac{u-s}{t} + \frac{t-s}{u} \right)^2 + \frac{16q^4 m^2 (5m^2 - 2s)}{tu}. \quad (9.45)$$

This expression is symmetric under exchange of t and u as it should because the outgoing particles are of the same kind.

¹¹Note that the precise definition of the Mandelstam invariants depends on the labelling of external particles (and the chosen conventions). For ingoing external momenta p_i in cyclic order (here $p_3 := -q_2$, $p_4 := -q_1$), the Mandelstam invariants are given by $s = -(p_1 + p_2)^2$, $t = -(p_1 + p_4)^2$, $u = -(p_1 + p_3)^2$. In general, s is made from both ingoing (or outgoing) momentum; t is made from a pair of in- and outgoing momenta which are neighbouring (based on a graph representing the $2 \rightarrow 2$ process); u is made from two opposite momenta. As such $s \geq (m_1 + m_2)^2$ for physical processes, while t, u are mostly negative.

¹²The u -channel always refers to the combination of opposite particles within a four-particle diagram, and thus it necessarily has some crossing propagator lines, while the s - and t -channel diagrams typically go without crossings.

¹³This constraint implies that functions of s, t, u can be written in several alternative ways much alike functions of p_1, p_2, q_1, q_2 which are constrained by $p_1 + p_2 - q_1 - q_2 = 0$.

¹⁴It is not straight-forward to derive these particular expressions, but it is easy to confirm that they match with some other expression upon substituting, e.g. $s = 4m^2 - t - u$.

¹⁵We can identify the first term as the corresponding result in scalar QED.

Angular Distribution. In order to understand the angular distribution of scattered particles we express the invariants in terms of the scattering angle θ and the three-momentum magnitude $p = \|\vec{p}\|$ of the particles

$$s = 4p^2 + 4m^2, \quad t = -2p^2(1 - \cos \theta), \quad u = -2p^2(1 + \cos \theta). \quad (9.46)$$

and insert everything into the differential cross section

$$\frac{d^2\sigma}{d^2\Omega} = \frac{q^4}{64\pi^2 e^2} \left(\frac{(4p^2 + 2m^2)^2}{p^4 \sin^4 \theta} - \frac{8p^4 + 12m^2 p^2 + 3m^4}{p^4 \sin^2 \theta} + 1 \right). \quad (9.47)$$

This expression is the leading non-trivial contribution to the angular distribution of scattered electrons.

We can however notice a problem by inspecting the expression. It diverges when

- the electron momenta p are small or
- the scattering angle θ is close to 0 or π .

In those regions of parameter space, the formula cannot be trusted. The deeper reason for the divergences is that the photons which transmit the electromagnetic force are massless. Massless particles cause some conceptual problems in scattering processes.

The divergences are also relevant to the total cross section¹⁶

$$\sigma = \int_0^1 2\pi \, d\cos \theta \, \frac{d^2\sigma}{d^2\Omega}. \quad (9.48)$$

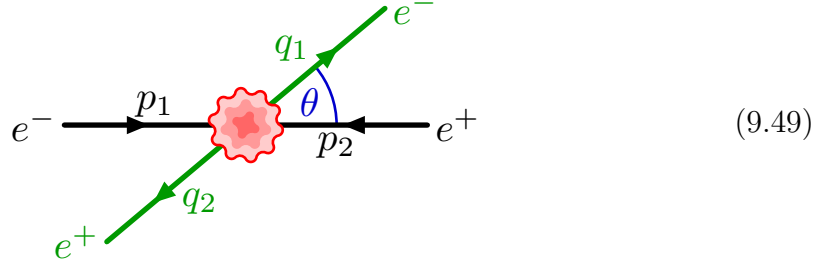
This integral diverges at $\cos \theta = 1$.

In order to properly address the above problematic regimes, one would have to take higher perturbative corrections and competing processes into account. However, only the full non-perturbative expression can provide exact results in those regimes.

Nevertheless one should not expect a meaningful result for the total cross section because the electromagnetic force is long-ranged: The photon propagator is not exponentially suppressed at long distances. Effectively all particles scatter at least by tiny amount and therefore the overall probability for scattering is 1. The scattering cross section σ is the complete area A of the bunches which is infinite due to our assumption of exactly defined momenta.

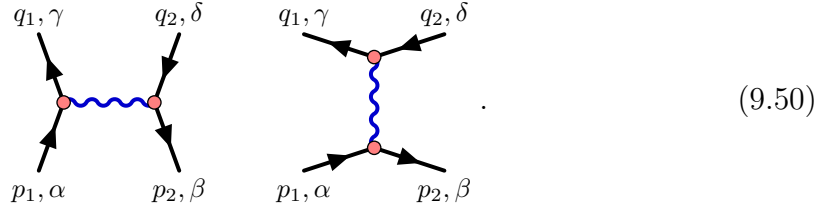
¹⁶The outgoing particles are indistinguishable, hence the integration extends only over one half of the spherical angles. This is sufficient since at leading order $\langle f|i \rangle$ has two terms one of which covers the opposite angles $\pi - \theta$. Alternatively the integral over all spherical angles must be multiplied by a factor of $1/2$.

Crossing Symmetry. A closely related process is the scattering of electrons and positrons (Bhabha scattering).



It can be computed in much the same way.

The relevant connected diagrams for electron-positron scattering are



The cross section turns out to be exactly the same as for electron scattering but with s and u interchanged

$$s \leftrightarrow u. \quad (9.51)$$

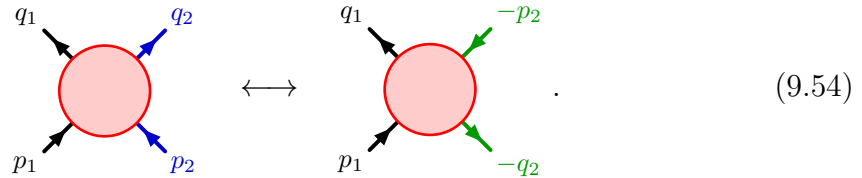
The resulting leading contribution to the squared matrix element is¹⁷

$$|M|^2 = q^4 \left(\frac{s-u}{t} + \frac{t-u}{s} \right)^2 + \frac{16q^4 m^2 (5m^2 - 2u)}{st}. \quad (9.52)$$

Indeed the computation is exactly the same when replacing

$$p_2 \leftrightarrow -q_2, \quad \sum_{\alpha} v_{\alpha}(\vec{p}_2) \bar{v}_{\alpha}(\vec{p}_2) \leftrightarrow - \sum_{\alpha} u_{\alpha}(\vec{q}_2) \bar{u}_{\alpha}(\vec{q}_2). \quad (9.53)$$

This relationship is called crossing symmetry. In terms of Feynman diagrams, the positron line is equivalent to the electron line in the reverse direction

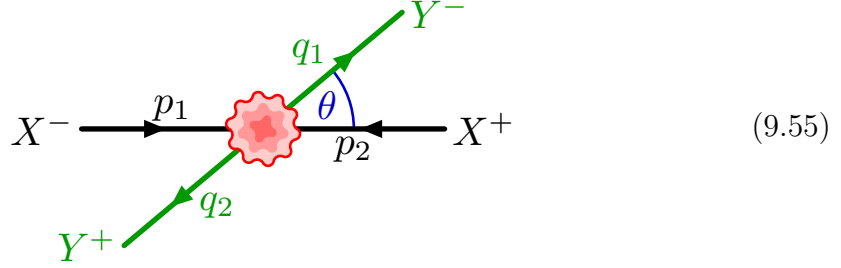


¹⁷Apart from effects due to identical particles, the electron-positron scattering cross section does not differ substantially from the case of electron-electron scattering. The difference between attraction and repulsion manifests in the phase of matrix elements rather than in their absolute value.

9.4 Pair Production

The above electron-positron scattering involves a process where the two particles combine into a photon and subsequently split up into a pair. This process is mixed with photon exchange in the t -channel.

A class of similar processes is pair production where the oppositely charged particles annihilate and create a pair of charged particles of a different kind.



Let us compute scattering cross sections for such processes.

We assume that the outgoing particles have a mass m_f and charge $\pm q_f$ which is different from the ingoing ones labelled by m_i and $\pm q_i$.

Spinor Processes. First we consider the case where all external particles are spinors.



There is now only one spinor trace to be evaluated

$$\begin{aligned}
 T &= \text{tr}[(i\not{p}_1 \cdot \gamma - m_i)\gamma_\mu(i\not{p}_2 \cdot \gamma + m_i)\gamma_\nu] \\
 &\quad \cdot \text{tr}[(i\not{q}_1 \cdot \gamma + m_f)\gamma^\mu(q_2 \cdot \gamma - m_f)\gamma^\nu], \\
 &= 16(-p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu} - (-p_1 \cdot p_2 + m_i^2)\eta_{\mu\nu}) \\
 &\quad (-q_1^\mu q_2^\nu - q_2^\mu q_1^\nu - (-q_1 \cdot q_2 + m_f^2)\eta^{\mu\nu}) \\
 &= 4(t - u)^2 + 4s^2 + 16(m_i^2 + m_f^2)s.
 \end{aligned} \tag{9.57}$$

The Mandelstam invariants are defined as above, but due to the different masses their relationships have to be adjusted

$$\begin{aligned}
 p_1 \cdot p_2 &= m_i^2 - \frac{1}{2}s, \\
 q_1 \cdot q_2 &= m_f^2 - \frac{1}{2}s, \\
 p_1 \cdot q_1 &= p_2 \cdot q_2 = \frac{1}{2}t - \frac{1}{2}m_i^2 - \frac{1}{2}m_f^2, \\
 p_1 \cdot q_2 &= p_2 \cdot q_1 = \frac{1}{2}u - \frac{1}{2}m_i^2 - \frac{1}{2}m_f^2, \\
 s + t + u &= 2m_i^2 + 2m_f^2.
 \end{aligned} \tag{9.58}$$

Next we express the invariants as functions of the scattering angle

$$s = 4e^2, \quad t = -2p_i p_f (1 - \cos \theta), \quad u = -2p_i p_f (1 + \cos \theta) \quad (9.59)$$

with the three-momentum magnitudes $p_{i,f} = \sqrt{e^2 - m_{i,f}^2}$ of the in- and outgoing particles. The unpolarised matrix element squared now reads

$$\begin{aligned} |M|^2 &= \frac{q_i^2 q_f^2 T}{4s^2} \\ &= q_i^2 q_f^2 \left(\frac{e^2 - m_i^2}{e^2} \frac{e^2 - m_f^2}{e^2} \cos^2 \theta + \frac{m_i^2 + m_f^2}{e^2} + 1 \right). \end{aligned} \quad (9.60)$$

The formula for the differential cross section for our configuration of masses and momenta in the centre-of-mass frame reads

$$\frac{d^2\sigma}{d^2\Omega} = \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \frac{|M|^2}{256\pi^2 e^2}. \quad (9.61)$$

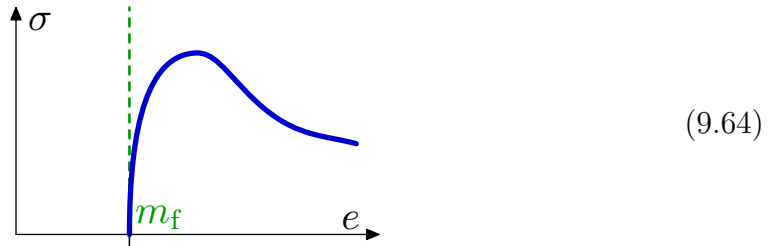
Total Cross Section. This expression is free from singularities and can be integrated to a total cross section

$$\sigma = 4\pi \int_{-1}^1 \frac{d \cos \theta}{2} \frac{d^2\sigma}{d^2\Omega} = \frac{1}{64\pi e^2} \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \int_{-1}^1 \frac{d \cos \theta}{2} |M|^2. \quad (9.62)$$

Upon integration we obtain the final result

$$\sigma = \frac{q_i^2 q_f^2}{48\pi e^2} \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \frac{e^2 + \frac{1}{2}m_i^2}{e^2} \frac{e^2 + \frac{1}{2}m_f^2}{e^2}. \quad (9.63)$$

We can plot the energy-dependence of this function.



Quite clearly the total energy $2e$ of the scattered particles must be at least as large as the sum of masses $2m_f$ of produced particles. There is a sharp increase above production threshold, a maximum slightly above threshold (for $m_i < m_f$), and a slow $1/e^2$ asymptotic descent.

Processes Involving Scalars. It is interesting to compare this process to the corresponding one of charged scalars. The matrix element reads

$$|M|^2 = q_i^2 q_f^2 \frac{(t-u)^2}{s^2} = q_i^2 q_f^2 \frac{e^2 - m_i^2}{e^2} \frac{e^2 - m_f^2}{e^2} \cos^2 \theta, \quad (9.65)$$

and after integration we obtain the total cross section

$$\sigma = \frac{q_i^2 q_f^2}{192\pi e^2} \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \frac{e^2 - m_i^2}{e^2} \frac{e^2 - m_f^2}{e^2}. \quad (9.66)$$

Let us finally consider a mixed process of spinors scattering into scalars for which the matrix element reads

$$\begin{aligned} |M|^2 &= q_i^2 q_f^2 \frac{-(t-u)^2 + s^2 - 4m_f^2 s}{s^2} \\ &= q_i^2 q_f^2 \frac{e^2 - (e^2 - m_i^2) \cos^2 \theta}{e^2} \frac{e^2 - m_f^2}{e^2}. \end{aligned} \quad (9.67)$$

For the total cross section we obtain

$$\sigma = \frac{q_i^2 q_f^2}{192\pi e^2} \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \frac{e^2 + \frac{1}{2}m_i^2}{e^2} \frac{e^2 - m_f^2}{e^2}. \quad (9.68)$$

The opposite process of scalars scattering into spinors merely has a different overall factor

$$\sigma = \frac{q_i^2 q_f^2}{48\pi e^2} \sqrt{\frac{e^2 - m_f^2}{e^2 - m_i^2}} \frac{e^2 - m_i^2}{e^2} \frac{e^2 + \frac{1}{2}m_f^2}{e^2}. \quad (9.69)$$

We observe that, although the matrix elements differ substantially, the final cross sections take a very predictable form: There are particular factors for scalars and spinors in the initial and final states, namely

- $e^2 - m_i^2$ for ingoing scalars,
- $e^2 + \frac{1}{2}m_i^2$ for ingoing spinors,
- $e^2 - m_f^2$ for outgoing scalars,
- $4(e^2 + \frac{1}{2}m_f^2)$ for outgoing spinors.

The square root on the other hand is a kinematical factor corresponding to the number/volume of initial and final states (phase space).

Let us discuss the physics of these factors in view of the total spin of the particle pairs: Two scalars form a spin-0 state, whereas the 4 polarisations of two spinors make up 1 spin-0 and 3 spin-1 states. We can deduce that a spin-0 state couples to the photon by a factor $e^2 - m^2$ whereas a spin-1 state must couple by the factor $e^2 + m^2$ such that the combination of two outgoing spinors yields the observed overall factor of

$$1(e^2 - m^2) + 3(e^2 + m^2) = 4(e^2 + \frac{1}{2}m^2). \quad (9.70)$$

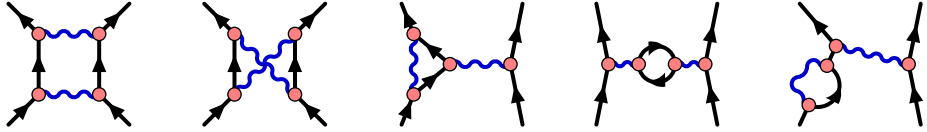
For ingoing spinors the factor of 4 is compensated by taking the average rather than a sum. It is also worth taking a look at the factors close to threshold

$e^2 = m^2$, where we find complete suppression for the spin-0 state with $e^2 - m^2 = 0$, whereas a spin-1 state couples with a non-zero strength $e^2 + m^2 = 2m^2$. This makes sense because the photon is a vector particle, and by conservation of angular momentum it can only couple to a state of overall angular momentum 1. At threshold, the outgoing particles have no orbital angular momentum, so their angular momentum equals the overall spin. A photon can therefore only transform into the spin-1 combination of two spinors, but not to pair of scalars. Above threshold this restriction is lifted by the availability of orbital angular momentum.

9.5 Loop Contributions

We have obtained the leading-order contributions to some particle scattering processes. Let us finally take a peek at contributions at higher orders in the perturbation series.

For the electron scattering process at the next order q^4 there are several classes of diagrams contributing.


(9.71)

Here we listed only the connected diagrams up to obvious symmetric copies. We clearly observe that these diagrams lead to two types of problems:

First, the diagram with a bubble on the external leg is ill-defined.

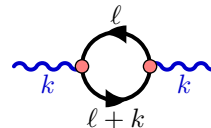

(9.72)

Due to momentum conservation the momentum on both sides of the bubble is the same. All external momenta originate from particle creation and annihilation operators $a^\dagger(\vec{p})$ and $a(\vec{p})$. These momenta are exactly on the mass shell $p^2 = -m^2$. Conversely, the internal line represents a Feynman propagator $1/(p^2 + m^2 - i\epsilon)$ which is to be evaluated right on the pole

$$\frac{1}{p^2 + m^2 - i\epsilon} \quad \text{at } p^2 = -m^2. \quad (9.73)$$

Therefore, this diagram as such makes no sense as a contribution to the scattering process.

Second, most of the integrals are actually divergent in the UV, i.e. where the loop momentum ℓ is very large. For example, the bubble on the photon line yields



$$\sim \int d^4\ell \frac{\ell^2 + \dots}{\ell^4 + \dots} \sim \int \frac{d^4\ell}{\ell^2} \rightarrow \infty. \quad (9.74)$$

In order to be able to compute quantum corrections to scattering processes consistently, we first have to understand how to deal with these two issues. This is going to be the subject of the final two chapters.

10 Scattering Matrix

When we computed some scattering processes in a straight-forward fashion, we did not really know what we were doing. At leading order our ignorance did not matter, but at higher orders complications arise, as we have just seen. Let us therefore discuss asymptotic particle states and their scattering matrix in detail. We shall start from scratch and use the results of the previous chapter only as a guideline.

10.1 Asymptotic States

First, we need to understand the notion of particle states which are to undergo scattering in the interacting theory

$$|p_1, p_2, \dots\rangle. \quad (10.1)$$

These states are called asymptotic particles because they describe independent particle excitations far away in time and space from the scattering region where the interactions are considered to happen. This is based on the assumption that in a local QFT the (effective) interactions between particles decay sufficiently fast with their distance.¹ Asymptotic particles are thus defined to behave like free particles at least in the absence of further nearby asymptotic particles.²

In particular, we need to understand how to include asymptotic particles in calculations. Up to now, we naively used free particle creation and annihilation operators to represent them; here we will learn how to express them properly in terms of the interacting field $\phi(x)$. For free fields we have seen how to encode the particle modes into two-point correlators, commutators and propagators. Let us therefore investigate these characteristic functions in the interacting model.

¹In QFT, interactions are understood to be mediated by fields or particles. Typically, the interaction strength relies on the propagator which decays exponentially with the distance where the effective range given by the inverse mass. Such exponential damping is sufficient to define asymptotic particles. This leaves a conceptual problem in defining asymptotic particles in the presence of massless fields (absence of a mass gap). Unfortunately, of course, our main case of QED describes the electromagnetic field and massless photons, and similar concerns apply to virtually all models of relevance in physics. In practice, this shortcoming leads to complications in describing scattering in the infra-red region which are understood but are also a nuisance.

²For any actual measurement this will be an approximation because particles will continue to interact at arbitrary distances, yet with extremely low strength or probability. In that sense, the precise mathematical formulation of scattering is an idealisation which cannot be achieved in an experiment (somewhat similar to Fourier transforms). In an experiment one has to make a choice at what point a particle is considered asymptotical. This introduces at least one extra length scale into the problem.

Two-Point Correlator. Consider first the correlator of two interacting fields $\phi(x)$

$$\Delta_+(x-y) := i\langle 0|\phi(x)\phi(y)|0\rangle. \quad (10.2)$$

Due to Poincaré symmetry, it must take the form

$$\Delta_+(x) = i \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot x} \theta(p^0) \rho(-p^2). \quad (10.3)$$

The factor $\theta(p^0)$ ensures that all excitations of the ground state $|0\rangle$ have positive energy. The function $\rho(s)$ parametrises our ignorance. We do not want tachyonic excitations, hence the function should be supported on positive values of $s = m^2$. We now insert a delta-function to express the correlator in terms of the free two-point correlator $\Delta_+(s; x, y)$ with mass \sqrt{s} (Källén, Lehmann)

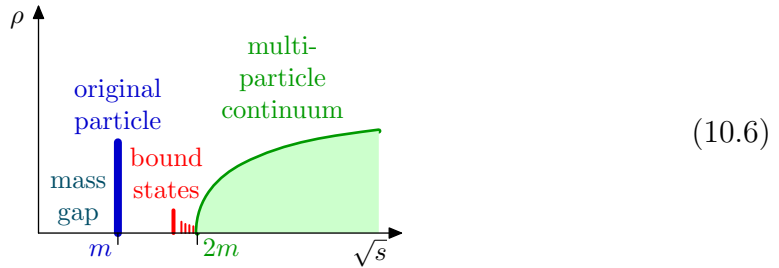
$$\begin{aligned} \Delta_+(x) &= i \int_0^\infty \frac{ds}{2\pi} \rho(s) \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot x} \theta(p^0) 2\pi \delta(p^2 + s) \\ &= \int_0^\infty \frac{ds}{2\pi} \rho(s) \Delta_+(s; x). \end{aligned} \quad (10.4)$$

This identifies $\rho(s)$ as the spectral function for the field $\phi(x)$: It tells us by what amount particle modes of mass \sqrt{s} will be excited by the field $\phi(x)$.³

Spectral Function. For a free field of mass m_0 we clearly have

$$\rho_0(s) = 2\pi \delta(s - m_0^2). \quad (10.5)$$

For weakly interacting fields, we should obtain a similar expression. In typical situations we expect the spectral function to have the following shape.



The sharp isolated peak represents a single particle excitation with mass m . Now the interacting field $\phi(x)$ may also excite multi-particle modes (with the same quantum numbers). Multi-particle modes in the free theory would have energy $e \geq 2m$: For example, a two-particle state $|\vec{p}_1, \vec{p}_2\rangle$ carries the overall momentum $P = p_1 + p_2$. Under Lorentz transformations it is related to similar states on the same mass shell given by the invariant mass $\sqrt{s} = \sqrt{-P^2} \geq 2m$ (equivalently,

³The spectral function describes the spectrum of all quantum states only to some extent. For example, not all states may be excited by the action of a single field $\phi(x)$. In particular, in a model with several kinds of fields, each field can excite only a subset of particles or states (e.g. the appropriate charges have to match).

energy $E = P^0$ at rest $\vec{P} = 0$).⁴ In the spectral function they form a continuum since the momenta of the individual particles can sum up to arbitrary energies in the frame at rest. In the presence of interactions, bound states may form whose rest energies are somewhat below $e = 2m$. Whenever these bound states are stable they will also be represented by sharp peaks.

We observe that our spectral function has at least two mass gaps: One separates the vacuum from the lowest excitation; the other separates the latter from bound states and the multi-particle continuum. The isolated modes are called asymptotic particles. This is the type of particle which we would like to collide. The assumption of a mass gap is crucial in this definition.

For weak interactions, we expect that the free particle mode approximates the asymptotic particle well.⁵ The interactions may shift the mass $m_0 \rightarrow m$ slightly; they may also change the strength with which this mode is excited by the field $\phi(x)$. Therefore, the weakly interacting spectral function takes the form

$$\rho(s) = 2\pi Z \delta(s - m^2) + \text{bound states} + \text{continuum}. \quad (10.7)$$

The factor Z is called field strength or wave function renormalisation.

Asymptotic Particles. Based on the above discussion we can expand the interacting field $\phi(x)$ as⁶

$$\phi(x) = \underbrace{\sqrt{Z} \phi_{\text{as}}(x)}_{\sim a^\dagger + a} + \underbrace{\text{bound states} + \text{continuum}}_{\sim a^{\dagger n} + a^n} + \underbrace{\text{operators}}_{\sim a^{\dagger m} a^n}. \quad (10.8)$$

Here $\phi_{\text{as}}(x)$ is a canonically normalised free field of the physical mass m expressed by means of creation and annihilation operators a^\dagger, a .⁷

$$\phi_{\text{as}}(x) = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} (e^{ip \cdot x} a(\vec{p}) + e^{-ip \cdot x} a^\dagger(\vec{p})). \quad (10.9)$$

These creation and annihilation operators satisfy the canonical commutation relations using the physical mass m for the determination of the relativistic energy, e.g. for a scalar excitation

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p}) (2\pi)^d \delta^d(\vec{p} - \vec{q}). \quad (10.10)$$

⁴In the Poincaré representation theory the invariant mass serves the same purpose as the mass of a particle. However, it makes sense to distinguish the two notions: In a multi-particle state, one can vary the momenta of the individual particles and thus change the invariant mass. Conversely, the mass of a particle is a fixed quantity and cannot be varied by changing the momentum.

⁵For reasonably strong interactions, bound states may approach the single particle states and even acquire lower energies. This case shows that the notion of fundamental particles is not evident in general QFT, but it belongs to weakly interacting models. In fact, some models may have alternative formulations where the fundamental degrees of freedom are some bound states of the original formulation.

⁶Note that the following discussion does not follow from a concrete Lagrangian. It rather makes use of the basic ingredients of QFT to formulate asymptotic particles based on their physical properties.

⁷We omit the label ‘as’ in order not to clutter the notation. These operators are meant to create and annihilate physical particle excitations.

The other terms in the field $\phi(x)$ are multiple creation and/or annihilation operators which we shall not consider in detail.

Single-particle asymptotic states are created simply by $a^\dagger(\vec{p})$ from the vacuum. The Hamiltonian H_{as} for the free asymptotic field reads

$$H_{\text{as}} = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} e(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}). \quad (10.11)$$

The characteristic property of H_{as} is that it reproduces exactly the time evolution of the vacuum and single-particle states

$$\begin{aligned} H_{\text{as}}|0\rangle &= 0 = H|0\rangle, \\ H_{\text{as}}a^\dagger(\vec{p})|0\rangle &= e(\vec{p})a^\dagger(\vec{p})|0\rangle = Ha^\dagger(\vec{p})|0\rangle. \end{aligned} \quad (10.12)$$

We shall use the free creation and annihilation operators as a convenient, distinguished basis to expand our interacting fields using the interaction picture.⁸ The extra operatorial terms in the field $\phi(x)$ are some higher-order polynomials in the operators a^\dagger, a which create and annihilate bound state particles and states from the multi-particle continuum. They cannot be observed in the spectral function.

Commutator and Normalisation. The other characteristic functions now follow from our expression for the correlator. As before these can be expressed as convolutions of the same spectral function $\rho(s)$ with their free counterparts.

The expectation value of the unequal-time commutator

$$\Delta(x - y) := i\langle 0 | [\phi(x), \phi(y)] | 0 \rangle \quad (10.13)$$

therefore reads

$$\Delta(x) = \int_0^\infty \frac{ds}{2\pi} \rho(s) \Delta(s; x). \quad (10.14)$$

We know that for a normalised free field the equal-time commutation relations imply $\dot{\Delta}(s; 0, \vec{x}) = \delta^3(\vec{x})$. Hence

$$\langle 0 | [\phi(\vec{x}), \dot{\phi}(\vec{y})] | 0 \rangle = i\dot{\Delta}(0, \vec{x} - \vec{y}) = i\delta^3(\vec{x} - \vec{y}) \int_0^\infty \frac{ds}{2\pi} \rho(s). \quad (10.15)$$

Assuming that the field $\phi(x)$ is canonically normalised,⁹ we have the constraint

$$\int_0^\infty \frac{ds}{2\pi} \rho(s) = 1. \quad (10.16)$$

When using the above expansion of the real field $\phi(x)$ in terms of creation and annihilation operators, it also follows that the function $\rho(s)$ must be positive. Hence the coefficient Z for the asymptotic modes should be between 0 and 1.

⁸The interaction picture is based on a reference free model which can be chosen freely. Typically one would choose it so that interactions represent small corrections, however that leaves the residual freedom of shifting small free contributions between the reference model and the interaction terms. The asymptotic model is fixed such that it is exact in the one-particle sector, i.e. all terms due to interactions cancel there.

⁹This is evident at least if the interaction terms do not contain derivatives.

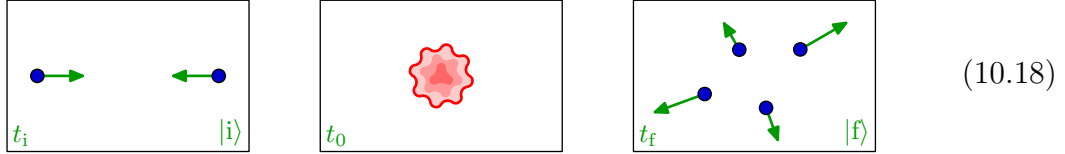
10.2 S-Matrix

Now that we have developed a robust notion of asymptotic particles and how it relates to two-point correlators, we can turn to scattering. For the scattering setup we define two asymptotic regions of spacetime, the initial one in the distant past $t_i \rightarrow -\infty$ and the final one in the distant future $t_f \rightarrow +\infty$.

Asymptotic Regions. On the initial time slice we create wave packets which are well separated in position space and narrowly peaked in momentum space. We let these quantum mechanical wave packets evolve in time. At some instance the wave packets collide. Then the state is evolved further until all outgoing wave packets are sufficiently well separated

$$|f\rangle = \exp(-iH(t_f - t_i))|i\rangle. \quad (10.17)$$

Now the initial and final states are in the Schrödinger picture and they evolve even at asymptotic times. It is hard to compare them to see what the effect of scattering is.¹⁰



At asymptotic times the wave packets are assumed to be sufficiently well separated such that they effectively do not interact. Therefore, we can use the asymptotic Hamiltonian of the asymptotic field ϕ_{as} ¹¹

$$H_{\text{as}} = \int \frac{d\vec{p}^3}{(2\pi)^3 2e(\vec{p})} e(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}). \quad (10.19)$$

to shift the two time slices onto a common one conventionally positioned at $t = 0$

$$|\text{out}\rangle = \exp(iH_{\text{as}}t_f)|f\rangle, \quad |i\rangle = \exp(-iH_{\text{as}}t_i)|\text{in}\rangle. \quad (10.20)$$

The relationship between the in and out states is the following

$$\begin{aligned} |\text{out}\rangle &= \exp(iH_{\text{as}}t_f) \exp(-iH(t_f - t_i)) \exp(-iH_{\text{as}}t_i)|\text{in}\rangle \\ &=: U_{\text{as}}(t_f, t_i)|\text{in}\rangle. \end{aligned} \quad (10.21)$$

The in and out states $|\text{in}\rangle$ and $|\text{out}\rangle$ are both defined at time $t = 0$. Consequently, they are elements of the same Hilbert space and can be compared directly. The

¹⁰The latter figure is somewhat misleading in a quantum mechanical setting: it merely shows one pure final state which in reality would rather be a superposition of all conceivable outcomes.

¹¹This asymptotic Hamiltonian is a specialisation of the free Hamiltonian H_0 used previously in the interaction picture. The free Hamiltonian was merely required to agree with the full Hamiltonian at leading order. The asymptotic Hamiltonian furthermore has to agree with the full Hamiltonian exactly when acting on the vacuum or one-particle states.

operator U_{as} is the time evolution operator for the interaction picture based on the asymptotic Hamiltonian H_{as} and the reference time slice at $t = 0$.

(10.22)

S-Matrix Definition. As interactions become negligible at asymptotic times, the in and out states are almost independent of t_i and t_f . It therefore makes sense to take the limit $t_{i,f} \rightarrow \mp\infty$. The limit of the time evolution operator for infinite times is called the S-matrix

$$\begin{aligned}
 S &= \lim_{t_{i,f} \rightarrow \mp\infty} \exp(iH_{\text{as}}t_f) \exp(iH(t_i - t_f)) \exp(-iH_{\text{as}}t_i) \\
 &= \lim_{t_{i,f} \rightarrow \mp\infty} U_{\text{as}}(t_f, t_i) = U_{\text{as}}(+\infty, -\infty).
 \end{aligned}
 \tag{10.23}$$

It transforms in states to out states

$$|\text{out}\rangle = S|\text{in}\rangle. \tag{10.24}$$

Note that the in and out Hilbert spaces are isomorphic.¹² This allows us to compare states between the two. To compute matrix elements of the S-matrix, we prepare definite in and out states¹³ using creation and annihilation operators a^\dagger , a for the reference time slice at $t = 0$

$$\begin{aligned}
 |\text{in}\rangle &= |p_1, \dots, p_m\rangle := a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_m)|0\rangle, \\
 \langle\text{out}| &= \langle q_1, \dots, q_n| := \langle 0|a(\vec{q}_1) \dots a(\vec{q}_n).
 \end{aligned}
 \tag{10.25}$$

Conventionally, scattering amplitudes M are defined as the matrix elements of $S - 1$ with the overall momentum-conserving delta-function stripped off

$$\langle\text{out}|(S - 1)|\text{in}\rangle = (2\pi)^4 \delta^4(P - Q) iM(p_1, \dots, p_m; q_1, \dots, q_n). \tag{10.26}$$

The combination $S - 1$ is particularly useful for $2 \rightarrow n$ scattering processes: It removes all immediate contractions between the in and out states as well as all other disconnected contributions.¹⁴

¹²It is natural to assume that outgoing particles of some scattering process can be used as ingoing particles of another scattering process. Therefore the in and out spaces must be isomorphic.

¹³These in and out states are not to be related by $|\text{out}\rangle = S|\text{in}\rangle$. The out and in states $\langle\text{out}|$, $|\text{in}\rangle$ rather define the basis for the matrix elements M .

¹⁴When one of the ingoing particles does not participate in the scattering, the S-matrix must act trivially on the other. For general $m \rightarrow n$ scattering, the matrix elements indeed contain direct contractions and disconnected contributions.

Properties of the S-Matrix. The S-matrix has a number of useful properties, let us list a few relevant ones.

First of all, the S-matrix is trivial for the ground state and for single-particle states

$$S|0\rangle = |0\rangle, \quad S|\vec{p}\rangle = |\vec{p}\rangle. \quad (10.27)$$

This follows from the definition of the asymptotic Hamiltonian to strictly emulate the action of the interacting Hamiltonian on these states.

The S-matrix is a unitary operator

$$S^\dagger = S^{-1}. \quad (10.28)$$

This property follows from the definition. It reflects the fact that probabilities are conserved across scattering processes.

The S-matrix is also Poincaré invariant

$$U(\omega, a) S U(\omega, a)^{-1} = S. \quad (10.29)$$

10.3 Time-Ordered Correlators

When we expressed the time-evolution operator in the interaction picture, we realised that time-ordered correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$ are very natural objects which can be expanded in terms of Feynman diagrams. The S-matrix is expressed in terms of the time-evolution operator for the interaction picture in terms of asymptotic states. Lehmann, Symanzik and Zimmermann derived a relationship between the S-matrix elements and time-ordered expectation values. In the following we will derive this relationship which we can subsequently exploit towards deriving Feynman rules for S-matrix elements.

Asymptotic States. First we need to understand how to represent particle creation and annihilation operators a^\dagger, a in terms of the field $\phi(x)$. Above, we have expanded the field $\phi(x)$ as

$$\phi(x) = \sqrt{Z} \int \frac{d\vec{q}^3}{(2\pi)^3 2e(\vec{q})} (e^{iq \cdot x} a(\vec{q}) + e^{-iq \cdot x} a^\dagger(\vec{q})) + \dots, \quad (10.30)$$

where the energy is on the mass shell, $q^0 = e(\vec{q})$. The omitted terms represent the contributions from multi-particle states and operators which annihilate the vacuum.

Previously, we were able to isolate $a_0^\dagger(\vec{q})$ from a time slice of the free field $\phi_0(x)$ as

$$a_0^\dagger(\vec{q}) = \int d\vec{x}^3 e^{iq \cdot x} (e(\vec{q})\phi_0(x) - i\dot{\phi}_0(x)). \quad (10.31)$$

This was easy because the free field ϕ_0 has only two modes with the energies $e = q^0 = \pm e(\vec{q})$, and the linear combination of ϕ_0 and $\dot{\phi}_0$ selects the desired one.

The interacting field, however, carries many other modes whose precise nature we do not understand a priori. To select the modes corresponding to a^\dagger and a we need to drive the field $\phi(x)$ for a sufficiently long time with a frequency that is in resonance with the relevant modes. Let us sketch the construction for a single oscillator $f(t) = c e^{i\omega t}$ with resonance frequency ω

$$F(e) = \int_{t_1}^{t_2} dt e^{-iet} f(t) = \frac{i c}{e - \omega} (e^{-i(e-\omega)t_2} - e^{-i(e-\omega)t_1}). \quad (10.32)$$

The longer the time, the stronger will be the amplitude at $e = \omega$. At infinite time the function $F(e)$ develops a pole at e , so we set $t_1 = -\infty$

$$F(e) = \int_{-\infty}^{t_2} dt e^{-iet} f(t) = \frac{i c e^{-i(e-\omega)t_2}}{e - \omega} = \frac{i c}{e - \omega} + \text{finite}. \quad (10.33)$$

Here we have discarded the term that keeps oscillating at asymptotic times.¹⁵ What remains is an isolated pole at $e = \omega$ whose residue is proportional to the amplitude c . The residue is in fact independent of the time t_2 where the driving stops.

Applied to the field $\phi(x)$ and the momentum p with energy $e = p^0$ independent of the spatial components \vec{p} as well as the resonance frequency $\omega = \pm e(\vec{q})$, we find the formula

$$\begin{aligned} & \int_{-\infty}^{t_2} dt \int d\vec{x}^3 e^{ip \cdot x} \phi(x) \\ &= \frac{i\sqrt{Z} a_{\text{in}}^\dagger(\vec{p})}{(p^0 - e(\vec{p}))2e(\vec{p})} + \frac{i\sqrt{Z} a_{\text{in}}(-\vec{p})}{(p^0 + e(\vec{p}))2e(\vec{p})} + \dots \\ &= \frac{i\sqrt{Z}}{p^2 + m^2} (\theta(-p^0) a_{\text{in}}(-\vec{p}) - \theta(p^0) a_{\text{in}}^\dagger(\vec{p})) + \dots \end{aligned} \quad (10.34)$$

Here we only spelled out the singular terms at the mass shell $p^0 = \pm e(\vec{p})$ while discarding finite contributions or terms which are irrelevant when creating well-separated wave packets. The residues of these isolated poles can be identified as creation and annihilation operators for ingoing asymptotic particles.

An important subtlety is that the operators $a_{\text{in}}(\vec{p})$ belong to an asymptotic Fock space in the distant past. The S-matrix, however, is defined via operators $a(\vec{p})$ belonging to an asymptotic Fock space at the reference time $t = 0$. These Fock spaces are isomorphic but not identical, they are related by the appropriate evolution operator¹⁶

$$a_{\text{in}}(\vec{p}) = U_{\text{as}}(0, -\infty) a(\vec{p}) U_{\text{as}}(-\infty, 0). \quad (10.35)$$

We note:

¹⁵As usual, one could formally dampen this term by introducing some small imaginary part. This may be an approximation, but even in practice, one can *never* isolate a resonance perfectly.

¹⁶The subtlety is that the higher operatorial contributions to the field $\phi(x)$, which we never made explicit, will have a different form in both pictures. These operators can redistribute the energy and momentum with the particle excitations, and thus they contribute to the residues. In fact, they constitute the difference between a_{in} and a , and at the end of the day, they yield the S-matrix itself.

- The residues of the pole $1/(p^2 + m^2)$ isolate the creation and annihilation operators.
- The residues at positive and negative energies correspond to creation and annihilation operators, respectively.
- The residues do not depend on the final time t_2 .
- Bound state particles correspond to similar poles at different energies.

A similar expression with opposite sign is obtained for driving the field into the distant future

$$\begin{aligned} & \int_{t_1}^{\infty} dt \int d\vec{x}^3 e^{ip \cdot x} \phi(x) \\ &= -\frac{i\sqrt{Z}}{p^2 + m^2} (\theta(-p^0) a_{\text{out}}(-\vec{p}) - \theta(p^0) a_{\text{out}}^\dagger(\vec{p})) + \dots \end{aligned} \quad (10.36)$$

Here we have to identify

$$a_{\text{out}}(\vec{p}) = U_{\text{as}}(0, +\infty) a(\vec{p}) U_{\text{as}}(+\infty, 0). \quad (10.37)$$

LSZ Reduction. We want to express the elements of the S-matrix in terms of time-ordered correlation functions in momentum space. Let us start with the time-ordered expectation value¹⁷

$$\begin{aligned} F_{m,n}(p, q) &= \int \prod_{k=1}^m (dx_k^4 e^{ip_k \cdot x_k}) \prod_{k=1}^n (dy_k^4 e^{-iq_k \cdot y_k}) \\ &\quad \cdot \langle 0 | T[\phi(x_1) \dots \phi(x_m) \phi(y_1) \dots \phi(y_n)] | 0 \rangle. \end{aligned} \quad (10.38)$$

Consider just the integral of the quantum operator over one of the x_k

$$X := \int dx^4 e^{ip \cdot x} T[\phi(x) Y], \quad (10.39)$$

where Y stands for the product of remaining fields $\phi(x_j)$ and $\phi(y_j)$. Now split up the time integral into three regions at the times t_{\min} and t_{\max} representing the minimal and maximal times of fields within Y

$$\begin{aligned} X &= \int_{-\infty}^{t_{\min}} dt \int d\vec{x}^3 e^{ip \cdot x} T[Y] \phi(x) \\ &\quad + \int_{t_{\min}}^{t_{\max}} dt \int d\vec{x}^3 e^{ip \cdot x} T[\phi(x) Y] \\ &\quad + \int_{t_{\max}}^{+\infty} dt \int d\vec{x}^3 e^{ip \cdot x} \phi(x) T[Y]. \end{aligned} \quad (10.40)$$

According to the results of the above consideration of resonances, the two integrals extending to $t = \pm\infty$ produce a pole when the momentum is on shell, $p^2 = -m^2$.

¹⁷The two types of momenta p_k and q_k are equivalent up to their overall sign. In the following, the two will correspond to ingoing and outgoing momenta, respectively, and thus we assume that the energy components in all p_k and q_k are positive.

Conversely, the middle integral is finite and therefore does not produce a pole. We can express the residue of the pole using creation and annihilation operators of in and out particles. By assumption, the energy p^0 is positive, and hence only creation generators contribute

$$X \simeq \frac{-i\sqrt{Z}}{p^2 + m^2} \left(T[Y] a_{\text{in}}^\dagger(\vec{p}) - a_{\text{out}}^\dagger(\vec{p}) T[Y] \right), \quad (10.41)$$

where we discard finite contributions at $p^2 = -m^2$.

Performing this step recursively for all ingoing particles yields

$$\begin{aligned} F_{m,n} &\simeq \prod_{k=1}^m \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \int \prod_{k=1}^n (dy_k^4 e^{-iq_k \cdot y_k}) \\ &\quad \cdot \langle 0 | T[\phi(y_1) \dots \phi(y_n)] a_{\text{in}}^\dagger(\vec{p}_1) \dots a_{\text{in}}^\dagger(\vec{p}_m) | 0 \rangle \\ &= \prod_{k=1}^m \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \int \prod_{k=1}^n (dy_k^4 e^{-iq_k \cdot y_k}) \\ &\quad \cdot \langle 0 | T[\phi(y_1) \dots \phi(y_n)] U_{\text{as}}(0, -\infty) a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_m) | 0 \rangle. \end{aligned} \quad (10.42)$$

Note that all outgoing creation operators a_{out}^\dagger directly annihilate the vacuum $\langle 0 |$ and immediately drop from the relationship. Furthermore, the asymptotic evolution operator $U_{\text{as}}(-\infty, 0)$ governing the transformation between $a(\vec{p})$ to $a_{\text{in}}(\vec{p})$ acts trivially on the vacuum on the right-hand side by construction.

Now we perform equivalent steps for the outgoing particles. We first insert a trivial evolution operator $U_{\text{as}}(+\infty, 0)$ next to the vacuum $\langle 0 |$. We then use a similar relation as above dressed by factors of $U_{\text{as}}(+\infty, 0)$ and $U_{\text{as}}(0, -\infty)$

$$\begin{aligned} X &:= \int dy^4 e^{-iq \cdot y} U_{\text{as}}(+\infty, 0) T[\phi(y)Y] U_{\text{as}}(0, -\infty) \\ &\simeq \frac{i\sqrt{Z}}{q^2 + m^2} \left(U_{\text{as}}(+\infty, 0) T[Y] a_{\text{in}}(\vec{q}) U_{\text{as}}(0, -\infty) \right. \\ &\quad \left. - U_{\text{as}}(+\infty, 0) a_{\text{out}}(\vec{q}) T[Y] U_{\text{as}}(0, -\infty) \right) \\ &\simeq \frac{-i\sqrt{Z}}{q^2 + m^2} [a(\vec{q}), U_{\text{as}}(+\infty, 0) T[Y] U_{\text{as}}(0, -\infty)], \end{aligned} \quad (10.43)$$

where again Y represents the remaining product of fields $\phi(y_j)$. Here, we obtained annihilation operators as opposed to creation operators due to the assumption of positivity of the energy q^0 .¹⁸ For each particle this yields one commutator of the remaining fields $U_{\text{as}}(+\infty, 0) T[Y] U_{\text{as}}(0, -\infty)$ with an annihilation operator. After performing this step for all the outgoing particles, we are left with the S-matrix

$$U_{\text{as}}(+\infty, 0) T[1] U_{\text{as}}(0, -\infty) = U_{\text{as}}(+\infty, -\infty) = S. \quad (10.44)$$

¹⁸Note that the Fourier exponent for q is opposite to the one for p .

Altogether we find that the residue of $F_{m,n}$ is given by an element of the S-matrix

$$F_{m,n} \simeq \prod_{k=1}^m \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \prod_{k=1}^n \left(\frac{-i\sqrt{Z}}{q_k^2 + m^2} \right) \cdot \langle 0 | [a(\vec{q}_1), \dots [a(\vec{q}_n), S] \dots] a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_m) | 0 \rangle. \quad (10.45)$$

Here, the commutators make sure that all the operators $a(\vec{q}_k)$ and $a^\dagger(\vec{p}_m)$ connect only to the S-matrix: Suppose that S is expanded in terms of the kind $(a^\dagger)^{n'} a^{m'}$. Then, the commutators with the $a(\vec{q}_k)$ eliminate n factors a^\dagger from S . This only yields a non-trivial contribution if there are sufficiently many factors and if none remain to annihilate the vacuum $\langle 0 |$, so we may assume $n' = n$. Finally, each of the m' factors a need to connect to one of the m factors $a^\dagger(\vec{p}_k)$, hence also $m' = m$ for a non-trivial contribution. Thus, none of the operators $a(\vec{q}_k)$ will connect directly to the $a^\dagger(\vec{p}_l)$.^{19 20}

10.4 S-Matrix Reconstruction

We have seen that time-ordered correlation functions have poles when the external fields are on the mass shell of asymptotic particles. The residue of these poles describes the corresponding element of the scattering matrix. These relationships allow us to fully reconstruct the S-matrix from time-ordered correlation functions. We will now discuss the consequences and in particular spell out Feynman rules for S-matrix elements.

Two-Point Correlator. In the construction of the S-matrix, the two-point correlation function takes a special role. First, consider the above residue formula for two legs

$$F_{1,1} \simeq \frac{-i\sqrt{Z}}{p^2 + m^2} \frac{-i\sqrt{Z}}{q^2 + m^2} \langle 0 | a(\vec{q})(S - 1)a^\dagger(\vec{p}) | 0 \rangle. \quad (10.46)$$

Momentum conservation implies $p = q$, hence the residue of a double pole at $p^2 = -m^2$ is given by $\langle 0 | a(\vec{q})(S - 1)a^\dagger(\vec{p}) | 0 \rangle$. However, the S-matrix should act as the identity on single-particle states. We conclude that there is no double pole in $F_{1,1}$ at $p^2 = -m^2$. There is no reason to expect a double pole in the first place, therefore the above residue statement is empty for $m = n = 1$.

There is nevertheless a single pole at $p^2 = -m^2$ as can be shown using the spectral representation of the time-ordered two-point function

$$K(x - y) := i \langle \phi(x) \phi(y) \rangle = i \langle 0 | T[\phi(x) \phi(y)] | 0 \rangle. \quad (10.47)$$

¹⁹S-matrix elements are often denoted by $\langle \vec{q}_1, \dots, \vec{q}_n | S | \vec{p}_1, \dots, \vec{p}_m \rangle$, in the literature, but this expression (for $m > 2$) also allows direct connections between the external particles which are explicitly prohibited in the above formula.

²⁰In fact, the correlation function does contain pairwise contractions between external fields, but these have a single pole $(p_k^2 + m^2)^{-1}$ rather than a double pole $(p_k^2 + m^2)^{-1}(q_l^2 + m^2)^{-1}$. Therefore they do not appear in the above formula.

Using the spectral function $\rho(s)$ of the interacting field $\phi(x)$, it can be written in terms of the free Feynman propagator of mass \sqrt{s}

$$K(x-y) = \int_0^\infty \frac{ds}{2\pi} \rho(s) G_F(s; x-y). \quad (10.48)$$

Most importantly, its momentum space representation

$$K(p) = \int_0^\infty \frac{ds}{2\pi} \frac{\rho(s)}{p^2 + s - i\epsilon} = \frac{Z}{p^2 + m^2 - i\epsilon} + \dots \quad (10.49)$$

contains the parameters of the asymptotic particle: the function $K(p)$ has an isolated pole at the physical mass m , and its residue is the wave function renormalisation factor Z .

Now we can nicely expand K in terms of Feynman diagrams with two external legs and thus determine m and Z .

Amputation. The residue formula for the time-ordered correlation functions can be inverted to a complete expression for the S-matrix

$$S = 1 + \sum_{m,n=2}^\infty \int \prod_{k=1}^n \frac{d\vec{q}_k^3 a^\dagger(\vec{q}_k)}{(2\pi)^3 2e(\vec{q})} \prod_{k=1}^m \frac{d\vec{p}_k^3 a(\vec{p}_k)}{(2\pi)^3 2e(\vec{p})} \cdot \left(\frac{F_{m,n}(p,q)}{m! n!} \prod_{k=1}^m \frac{p_k^2 + m^2}{-i\sqrt{Z}} \prod_{k=1}^n \frac{q_k^2 + m^2}{-i\sqrt{Z}} \right). \quad (10.50)$$

Importantly, the poles and zeros of the latter term must be combined before the momenta are set on shell $p_k^2 = q_k^2 = -m^2$. The construction of this expression with $m, n \geq 2$ ensures that

- the vacuum does not scatter, $S|0\rangle = |0\rangle$,
- single-particle states do not scatter, $S|p\rangle = |p\rangle$,
- for two or more particles, the residue of $F_{m,n}$ is reproduced according to the above formula.

It is now convenient to replace each factor $(p_k^2 + m^2)$ by the inverse of the corresponding two-point function in the construction of the S-matrix

$$S = 1 + \sum_{m,n=2}^\infty \int \prod_{k=1}^n \frac{d\vec{q}_k^3 a^\dagger(\vec{q}_k)}{(2\pi)^3 2e(\vec{q})} \prod_{k=1}^m \frac{d\vec{p}_k^3 a(\vec{p}_k)}{(2\pi)^3 2e(\vec{p})} \cdot \left(\frac{F_{m,n}(p,q)}{m! n!} \prod_{k=1}^m \frac{\sqrt{Z}}{-iK(p_k)} \prod_{k=1}^n \frac{\sqrt{Z}}{-iK(q_k)} \right). \quad (10.51)$$

This formula has a useful interpretation in terms of Feynman graphs for $F_{m,n}$

$$F_n = \tilde{F}_n \text{ with } K \text{ on external legs} \quad (10.52)$$

In the second representation we have cut the graph into a smaller $(m+n)$ -function $\tilde{F}_{m,n}$ and $m+n$ 2-point functions K according to the rules:

- Each 2-point function connects an external leg to the $(m+n)$ -function at the core.
- Each 2-point function is maximal.
- The Feynman propagator that connects the 2-point function to the core is attributed to the 2-point function.

Essentially one chops each leg of the graph as much as possible. Such a graph is called *amputated*.

Now it is clear that each 2-point fragment of the graph is a Feynman graph for the two-point function K . Moreover all these graphs have natural relative weights. The sum of all Feynman graphs contributing to $F_{m,n}$ therefore contains the sum of all graphs contributing to K separately for each leg. What remains is a sum over all amputated Feynman graphs at the core. This expression separates cleanly into factors because all the weights are naturally defined

$$F_{m,n}(p, q) = (-i)^{m+n} \tilde{F}_{m,n}(p, q) \prod_{k=1}^m K(p_k) \prod_{k=1}^n K(q_k). \quad (10.53)$$

The function $\tilde{F}_{m,n}$ therefore is precisely what is needed for reconstruction of the S-matrix

$$S = 1 + \sum_{m,n=2}^{\infty} \int \prod_{k=1}^n \frac{d\vec{q}_k^3 a^\dagger(\vec{q}_k)}{(2\pi)^3 2e(\vec{q})} \prod_{k=1}^m \frac{d\vec{p}_k^3 a(\vec{p}_k)}{(2\pi)^3 2e(\vec{p})} \frac{\sqrt{Z}^{m+n}}{m! n!} \tilde{F}_{m,n}. \quad (10.54)$$

In other words, the elements of the S-matrix are determined precisely by the sum of amputated Feynman graphs multiplied by \sqrt{Z} for each external leg.

Note that this also solves one problem we encountered in the earlier naive computation of scattering matrix elements. At higher orders, one contribution was evaluated right on a pole of a Feynman propagator. However, this contribution is not amputated, and therefore does not actually contribute to the S-matrix.

General Picture. The general picture is as follows: Poles in the time-ordered two-point function $K(p)$ indicate stable asymptotic particle states.^{21 22}

- These may be deformations of the poles in the free theory.

²¹When a field has several components, the notion of pole is more subtle in the sense that the residue of a pole is typically a matrix of non-maximal rank, e.g. $\not{p} + m$ for spinor fields. In this case only the vectors which are not projected out correspond to asymptotic particles.

²²In practice one may not be able to distinguish an exact pole from a very narrow resonance. One might consider such resonances at the same level as stable external particles and allow them as legs of the S-matrix. Such an S-matrix would not rest on rigorous assumptions and therefore not all theorems apply in a strict sense. In this regard, one should remember that in quantum physics one has to make some separation of scales into the microscopic quantum regime and the regime of macroscopic classical objects. Alternatively, resonances can be viewed as asymptotic particles with a complex mass parameter.

- They may as well be poles corresponding to bound states.
- Also poles for correlators of composite fields are permissible.
- Unbound multi-particle states forming the continuum of the spectral function $\rho(s)$ do not (independently) contribute to the S-matrix: they are already taken into account in S by individual scattering of their constituent particles and moreover the continuum leads to branch cuts rather than pole singularities.

The location $p^2 = -m^2$ of the pole defines the mass m of the particle.

Time-ordered multi-point correlation functions have poles at these locations. Their overall residue yields the corresponding element of the S-matrix. Some comments:

- It is clear that all the external legs of the S-matrix must be exactly on shell.
- The S-matrix is completely determined in terms of time-ordered correlation functions. No reference is made to the original formulation of the QFT, e.g. the Lagrangian. This fact will be crucial when we go to higher perturbative orders where Feynman diagrams have internal loops.

Feynman Rules. Let us summarise the Feynman rules for elements of the S-matrix in ϕ^4 theory

$$\langle q_1, \dots, q_n | S | p_1, \dots, p_m \rangle. \quad (10.55)$$

The matrix element is given by the sum of all graphs with certain properties. The properties are similar to the properties of Feynman graphs for correlation functions in momentum space, but mainly the external legs are handled differently. Let us state the modified and additional rules:

- The graph has m ingoing and n outgoing external lines labelled by momenta p_k and q_k , respectively.

$$p_j \text{ } \xrightarrow{\quad} \quad \quad \quad \xrightarrow{\quad} q_k \quad (10.56)$$

- The external momenta must be on the mass shell, $p_k^2 = q_k^2 = -m^2$, and must have positive energy.
- The graph can have several connection components.
- Cutting the graph at any internal line must not split off a graph with only two external lines (amputated graph).



$$(10.57)$$

In particular, a connection component with two external lines must be a direct connection.

The Feynman rules for evaluating a graph are the same as for correlation functions in momentum space except:

- For each external line write a factor of \sqrt{Z} instead of a Feynman propagator $-i/(p_j^2 + m^2 - i\epsilon)$

$$p_j \text{ } \xrightarrow{\quad} \quad \quad \xrightarrow{\quad} q_k \longrightarrow \sqrt{Z}. \quad (10.58)$$

- Any external line which directly connects an ingoing to an outgoing particle contributes a factor of $\langle q_k | p_l \rangle = 2e(\vec{p}_l)(2\pi)^3 \delta^3(\vec{p}_l - \vec{q}_k)$. This line simply bypasses the S-matrix²³

$$p_j \text{---} \text{---} q_k \longrightarrow 2e(\vec{p}_j)(2\pi)^3 \delta^3(\vec{p}_j - \vec{q}_k). \quad (10.59)$$

- Each connection component contributes a factor of $(2\pi)^4 \delta^4(P_k)$ where P_k is the total momentum outflow of the component. One typically defines the matrix element M of a single connected component with the delta-function stripped off as

$$\langle q_1, \dots, q_n | S | p_1, \dots, p_m \rangle = (2\pi)^4 \delta^4(P - Q) iM. \quad (10.60)$$

The other Feynman rules then yield iM without the delta-function.

For quantum electrodynamics the Feynman rules for scattering matrix elements also have to be adjusted w.r.t. the Feynman rules in momentum space, namely:

- For each external spinor line, write a factor of $\sqrt{Z_\psi}$ along with $u_\alpha(\vec{q})$, $\bar{u}_\alpha(\vec{p})$, $v_\alpha(\vec{p})$ or $\bar{v}_\alpha(\vec{q})$ depending on whether the particle is ingoing (u , \bar{v}) or outgoing (v , \bar{u}) and whether it is an electron (v , \bar{v}) or a positron (u , \bar{u})

$$\begin{aligned} p_j \text{---} \text{---} \alpha_j &\longrightarrow \sqrt{Z_\psi} \bar{v}_{\alpha_j}(\vec{p}_j), & \text{---} \text{---} q_k \alpha_k &\longrightarrow \sqrt{Z_\psi} v_{\alpha_k}(\vec{q}_k), \\ p_j \text{---} \text{---} \alpha_j &\longrightarrow \sqrt{Z_\psi} u_{\alpha_j}(\vec{p}_j), & \text{---} \text{---} q_k \alpha_k &\longrightarrow \sqrt{Z_\psi} \bar{u}_{\alpha_k}(\vec{q}_k). \end{aligned} \quad (10.61)$$

- For external photon lines, write a factor of $\sqrt{Z_A}$ along with a normalised transverse polarisation vector $\varepsilon_\mu(\vec{p})$

$$p_j \text{---} \text{---} \alpha_j \longrightarrow \sqrt{Z_A} \varepsilon_{\alpha_j}(\vec{p}_j), \quad \text{---} \text{---} q_k \alpha_k \longrightarrow \sqrt{Z_A} \varepsilon_{\alpha_k}^*(\vec{q}_k). \quad (10.62)$$

10.5 Unitarity

The S-matrix is a unitary operator

$$S^\dagger = S^{-1}. \quad (10.63)$$

This is an essential feature of any physical QFT. However, when deriving the S-matrix from time-ordered correlators by means of the LSZ reduction, unitarity is not evident at all. Therefore, we can use the property to derive some non-trivial relations between elements of the S-matrix such as the famous optical theorem which relates the imaginary part of the S-matrix to total cross sections.

²³Such contributions do not directly correspond to the identity within S , i.e. they are present in $S - 1$, but only for at least 3 ingoing particles.

Optical Theorem. Commonly, an identity operator is removed from the S-matrix as

$$S = 1 + iT. \quad (10.64)$$

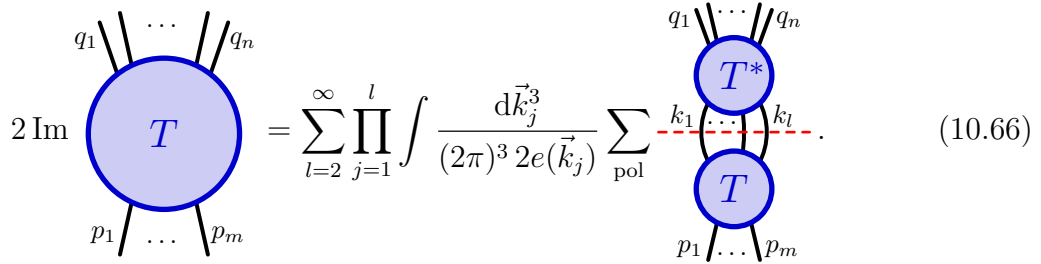
This split is useful because for small coupling T is small. Moreover, the identity in S is never seen in the LSZ reduction.

Unitarity $SS^\dagger = 1$ for the operator T is then written as the optical theorem

$$2 \operatorname{Im} T = -iT + iT^\dagger = TT^\dagger = T^\dagger T. \quad (10.65)$$

It relates the imaginary part of T to its absolute square. The latter is a quantity we have already encountered: In the form of matrix elements it appears in the scattering cross section. It allows to determine the total cross section of some process in terms of the imaginary part of a matrix element.²⁴ Alternatively, the imaginary part of T can be obtained as a total cross section.²⁵ The remaining real part of T can be reconstructed from arguments of complex analyticity.

A graphical representation of the optical theorem is as follows



$$2 \operatorname{Im} T = \sum_{l=2}^{\infty} \prod_{j=1}^l \int \frac{d\vec{k}_j^3}{(2\pi)^3 2e(\vec{k}_j)} \sum_{\text{pol}} \quad (10.66)$$

The optical theorem implies that one has to integrate and sum over all allowed degrees of freedom for the lines which connect T to T^\dagger . This is similar as for internal lines within T and T^\dagger with one important distinction: The cut lines originate from contracting two operators a and a^\dagger inside T and T^\dagger , respectively,

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p}) (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (10.67)$$

Therefore the momenta associated to these lines must be on shell, $p^2 = -m^2$, with *directed* flow of energy p^0 from T towards T^\dagger . Conversely, the internal lines are integrated over all off-shell momenta.

One-Loop Unitarity. A common application of the optical theorem is to express the imaginary part of a one-loop amplitude in terms of the total cross section at tree level. For example, the imaginary part of the one-loop scattering amplitude for two electrons is determined by the cross section we computed

²⁴In this matrix element one would choose the ingoing and outgoing momenta to be the same. Evidently, this requires to split off the momentum-conserving delta-function first.

²⁵In fact one needs a generalisation of the total cross section where the ingoing particles of T are chosen independently of the outgoing particles of T^\dagger .

earlier^{26 27}

$$2 \operatorname{Im} \left[\text{Diagram 1} \right] = \text{Diagram 2} . \quad (10.68)$$

This insight is helpful because the imaginary part of a complex analytic function determines to some extent the real part as well. In other words, unitarity can be used to construct higher-order perturbative terms from the lower orders.

Note that there are further contributions to the imaginary part from the annihilation of an electron with a positron

$$2 \operatorname{Im} \left[\text{Diagram 3} \right] = \text{Diagram 4} . \quad (10.69)$$

These actually happen in a different channel, i.e. in a different kinematical regime distinguished by the signs of the particle energies.²⁸

Another point to note is that the above unitarity relationship refers only to continuous contributions to the imaginary part. There are also discrete contributions which are harder to spot. These are present already at tree level where they can be identified easily, as we shall see below.

Tree Level. It is instructive to discuss the optical theorem at tree level. At first sight one might think that tree-level contributions to T are manifestly real because they are rational functions of the momenta and masses with real coefficients.²⁹ Although the $i\epsilon$ prescription for Feynman propagators appears negligible, it does have a considerable impact on the imaginary part

$$\frac{1}{p^2 + m^2 - i\epsilon} = \frac{1}{p^2 + m^2} + i\pi\delta(p^2 + m^2). \quad (10.70)$$

Now in the conjugate S-matrix T^\dagger all Feynman propagators are conjugated

$$G_F^*(p) = \frac{1}{p^2 + m^2 + i\epsilon} \neq G_F(p). \quad (10.71)$$

When computing the imaginary part of T one therefore frequently encounters the difference

$$\frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + m^2 + i\epsilon} = 2\pi i\delta(p^2 + m^2). \quad (10.72)$$

²⁶Recall that the total cross section for this process is divergent implying that the imaginary part of the loop amplitude is divergent, too. We can ignore this issue for the time being. We shall deal with divergent loop integrals in the next section (albeit only for UV divergences; here the divergence comes from the IR).

²⁷These equations are merely meant to sketch the unitarity relationship. Commonly, there are several similar diagrams contributing to either side of the equation.

²⁸In our case, the channels separate the various imaginary contributions. In more general situations at higher loops, typically many contributions overlap.

²⁹The various prefactors of i for propagators and interaction vertices conspire to cancel out in the appropriate definition of the imaginary part.

This identity replaces the Feynman propagator for an internal line by an on-shell correlator for a cut line connecting T and T^\dagger . The restriction to positive energies on the cut is a more subtle issue. It is resolved by the fact that in the sum over all possible cuts each line appears twice, once for every direction of energy flow.

With these remarks one can show that the optical theorem holds at tree level.³⁰ Here, we showed that at tree level T has an imaginary part concentrated at isolated momentum configurations. However, the optical theorem is most frequently applied at loop level where $\text{Im } T$ has continuous (non-distributional) contributions.

³⁰It is crucial to also take the disconnected contributions to T into account.

11 Loop Corrections

Finally, we will discuss some basic loop effects and how to deal with the divergent loop integrals we shall encounter. The following merely serves a sketch of the methods of regularisation and renormalisation, which will be treated more thoroughly in QFT II.

We use a scalar theory with a quartic potential

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{6}\kappa\mu\phi^3 - \frac{1}{24}\kappa^2\lambda\phi^4. \quad (11.1)$$

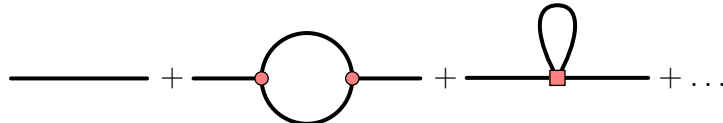
The mass is denoted by μ and there are two dimensionless constants κ and λ . We assume κ to be small and λ to be a freely tunable parameter.

11.1 Self Energy

First, we consider the time-ordered two-point correlator in order to isolate the pole associated to asymptotic particles. In momentum space we know by means of momentum conservation

$$i\langle\phi(p)\phi(q)\rangle = (2\pi)^4\delta^4(p+q)K(p). \quad (11.2)$$

Leading Orders. Let us evaluate the first few orders of the function K from Feynman diagrams.



$$\text{---} + \text{---} \bigcirc \text{---} + \text{---} \text{---} \text{---} + \dots \quad (11.3)$$

Obviously the leading contribution is the isolated Feynman propagator

$$K^{(0)}(p) = G_F(p) = \frac{1}{p^2 + \mu^2 - i\epsilon}. \quad (11.4)$$

The first correction involves one loop in the diagram. The tadpole diagram can be safely ignored.¹ The bubble diagram amounts to the following loop integral²

$$K^{(2)}(p) = \frac{i(-i\mu\kappa)^2(-i)^4}{(p^2 + \mu^2 - i\epsilon)^2} iI(-p^2),$$

$$I(-p^2) = \frac{1}{2} \int \frac{-i d\ell^4}{(2\pi)^4} \frac{1}{\ell^2 + \mu^2 - i\epsilon} \frac{1}{(p - \ell)^2 + \mu^2 - i\epsilon}. \quad (11.5)$$

¹We shall see later on how this can be achieved in practice.

²The resulting integral is Lorentz invariant and can therefore be written as a function of p^2 (or equivalently of $-p^2$).

The resulting quadruple integral is difficult to perform, in particular due to two different denominators. We will postpone the evaluation, and discuss the implications for a generic function $I(-p^2)$.

Mass Shift. Altogether we obtain for the two-point function

$$K(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} + \frac{\mu^2 \kappa^2 I(-p^2)}{(p^2 + \mu^2 - i\epsilon)^2} + \dots \quad (11.6)$$

Now this expression appears to have a double pole at $p^2 = -\mu^2$. From our earlier discussion we know that this should not happen; there should only be single poles in the two-point function.

Taking a peek at higher loop orders, we find, among others, a sequence of iterated loop integrals terms.

$$\text{---} + \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} \circ \text{---} + \dots \quad (11.7)$$

All of these we can express easily in the form of a geometric series

$$K(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} \sum_{k=0}^{\infty} \left(\frac{\mu^2 \kappa^2 I(-p^2)}{p^2 + \mu^2 - i\epsilon} \right)^k + \dots \quad (11.8)$$

Pushing convergence questions aside, we sum the series

$$K(p) = \frac{1}{p^2 + \mu^2 - \mu^2 \kappa^2 I(-p^2) - i\epsilon} + \dots \quad (11.9)$$

In other words we have moved the correction term to the denominator of the propagator.

Why should we include precisely these higher-order terms into our first order correction? There are several reasons:

- There is nothing wrong with it.³ We just have to make sure to eventually count every higher-order diagram with the correct weight.
- We avoid unwanted higher poles in the two-point function.⁴ The diagrams contributing to the denominator are one-particle irreducible. They cannot be cut into two parts by cutting a single internal line. Such diagrams are not expected to produce additional poles at the mass shell.
- The inverse of the two-point function at leading order is directly related to the action. It therefore appears somewhat more natural to expand the inverse $1/K(p)$ rather than the original function $K(p)$. Our correction terms are simply the first correction to $1/K(p)$.
- A mass counterterm (to be discussed below) yields the same result.

³Note that truncation of the perturbative series to a polynomial is just as good as any other choice that assumes some non-trivial contributions beyond the perturbative order of interest.

⁴The earlier general discussion of singularities in correlation functions and S-matrix elements revealed that the two-point function can only have simple poles.

- In QFT II we will introduce the effective action as a useful functional that produces precisely $1/K(p)$.

In this form we generically expect only single poles. Particularly, the pole which was originally at $p^2 + \mu^2 = 0$ may now have shifted to a new location $p^2 + m^2 = 0$. Assuming that this is the case, we can determine the new mass and also the residue at the pole. For the mass, we should solve the equation

$$-m^2 + \mu^2 - \mu^2 \kappa^2 I(m^2) + \dots = 0. \quad (11.10)$$

The assumption of the perturbative treatment is that κ is small and that m is approximated well by μ at this order with corrections contributing to yet higher orders. Hence we can replace $I(m^2)$ by $I(\mu^2)$ and therefore the new mass to leading order reads

$$m^2 = \mu^2 - \mu^2 \kappa^2 I(\mu^2) + \dots \quad (11.11)$$

How about the residue? The expansion of the denominator at $p^2 + m^2 = 0$ reads

$$(p^2 + m^2)(1 + \mu^2 \kappa^2 I'(m^2)) + \dots \quad (11.12)$$

Altogether the pole of the two-point function corresponding to the asymptotic particle takes the form

$$K(p) = \frac{Z}{p^2 + m^2 - i\epsilon} + \dots \quad (11.13)$$

with the field strength renormalisation

$$Z = 1 - \mu^2 \kappa^2 I'(\mu^2) + \dots \quad (11.14)$$

Spectral Function. We can now extract the spectral function $\rho(s)$ from $K(p)$. Recall their relationship

$$K(p) = \int_0^\infty \frac{ds}{2\pi} \frac{\rho(s)}{p^2 + s - i\epsilon}. \quad (11.15)$$

To that end, it is most convenient to consider the imaginary part originating from the $i\epsilon$ prescription

$$\frac{1}{x - i\epsilon} = \frac{1}{x} + i\pi\delta(x). \quad (11.16)$$

We thus find that the imaginary part of K is directly related to the spectral function up to a factor of 2

$$\rho(-p^2) = 2 \operatorname{Im} K(p). \quad (11.17)$$

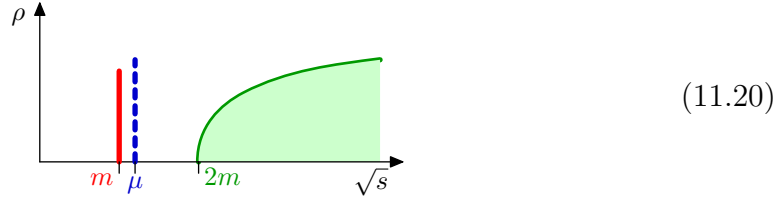
We can evaluate the imaginary part of our K starting from the naive perturbative expansion without making use of the above geometric series. Let us illustrate this option and check the consistency of both approaches. To extract the imaginary part, we use the above identity for the $i\epsilon$ prescription and its derivative

$$\frac{1}{(x - i\epsilon)^2} = \frac{1}{x^2} - i\pi\delta'(x). \quad (11.18)$$

We then find the spectral function^{5 6}

$$\begin{aligned}
\rho(s) &= 2\pi\delta(-s + \mu^2) - 2\pi\mu^2\kappa^2 \operatorname{Re} I(s) \delta'(-s + \mu^2) \\
&\quad + \frac{2\mu^2\kappa^2 \operatorname{Im} I(s)}{(-s + \mu^2)^2} + \dots \\
&= 2\pi\delta(s - \mu^2 + \mu^2\kappa^2 \operatorname{Re} I(\mu^2)) (1 - \mu^2\kappa^2 \operatorname{Re} I'(\mu^2)) \\
&\quad + \frac{2\mu^2\kappa^2 \operatorname{Im} I(s)}{(s - \mu^2)^2} + \dots \\
&= 2\pi Z\delta(s - m^2) + \frac{2\mu^2\kappa^2 \operatorname{Im} I(s)}{(s - \mu^2)^2} + \dots,
\end{aligned} \tag{11.19}$$

which matches precisely with our expectations of a shifted delta-peak and a continuum from $\operatorname{Im} I(s)$.



11.2 Loop Integral

We will now return to the loop integral $I(-p^2)$ and evaluate it.

Combining Denominators. A standard trick to proceed due to Feynman is to combine the denominators by virtue of a new integral. For two propagators we use the integral⁷

$$\frac{1}{AB} = \int_0^1 \frac{dz}{(zA + \bar{z}B)^2}, \quad \bar{z} := 1 - z. \tag{11.21}$$

An alternative is the trick used by Schwinger to convert each denominator to an exponent where they automatically sum up

$$\frac{1}{A} = \int_0^\infty dz e^{-Az}. \tag{11.22}$$

The loop integral then takes a form with a single squared denominator⁸

$$I(-p^2) = \int_0^1 dz \int \frac{-i d\ell^4}{32\pi^4} \frac{1}{(z\ell^2 + \bar{z}(p - \ell)^2 + \mu^2 - i\epsilon)^2}. \tag{11.23}$$

⁵Note that $f(s)\delta'(s - s_0) = f(s_0)\delta'(s - s_0) - f'(s_0)\delta(s - s_0)$.

⁶The combination of the delta-function and its derivative into a delta-function with shifted arguments is driven by the desire to avoid derivatives of delta-functions in the spectral function (in much the same way as the geometric series removes multiple poles). It is a non-trivial feature of Feynman diagrams that this absorption is perfect.

⁷There are similar formulas for more than two denominators and for higher powers.

⁸The advantage of this expression is that the new integrand is spherically symmetric for fixed value of z which simplifies integration drastically. The centre of the sphere, however, varies with z .

Now the denominator has quadratic (ℓ^2), linear ($\ell \cdot p$) and constant terms. We can remove the linear term by a shift $\ell \rightarrow \ell + \bar{z}p$ of the integration variable⁹ and thus simplify the integral

$$I(-p^2) = \int_0^1 dz \int \frac{d\ell^4}{32\pi^4} \frac{1}{(\ell^2 + z\bar{z}p^2 + \mu^2 - i\epsilon)^2}. \quad (11.24)$$

Momentum Integrals. Performing the momentum integrals is not very difficult. Let us start with the integral over the energy component.¹⁰ The integrand has double and single poles at

$$\ell_0 = \pm \sqrt{\vec{\ell}^2 + z\bar{z}p^2 + \mu^2 - i\epsilon}. \quad (11.25)$$

The integration contour is along the real axis and passes right between the two poles. It decays sufficiently fast at $|l_0| \rightarrow \infty$ so that we can close the contour by a large semicircle in the upper or lower half of the complex plane. Either of the single poles contributes the same residue

$$I(-p^2) = \int_0^1 dz \int \frac{d\vec{\ell}^3}{64\pi^3} \frac{1}{(\vec{\ell}^2 + z\bar{z}p^2 + \mu^2 - i\epsilon)^{3/2}}. \quad (11.26)$$

We notice that the integral merely reduces the exponent of the denominator by $1/2$ and multiplies by a suitable overall factor. The remaining three spatial momentum integrals yield a very similar result, each of them reducing the exponent by $1/2$. The last integral in fact is logarithmically divergent for large momenta. We have to cut it off at some bound $\|\vec{l}\| \simeq \Lambda_{\text{cut}}$, and we obtain

$$I(-p^2) = -\frac{1}{32\pi^2} \int_0^1 dz \log \frac{z\bar{z}p^2 + \mu^2 - i\epsilon}{\Lambda_{\text{cut}}^2}. \quad (11.27)$$

For a large UV cutoff Λ_{cut} the integral diverges logarithmically.

Wick Rotation. Another trick which is commonly used is to rotate the integration contour for the energy ℓ^0 from the real axis to the imaginary axis

$$\ell^0 = i\ell_E^4. \quad (11.28)$$

This Wick rotation is permissible since physical integrands are typically perfectly analytic in the first and third quadrants of the complex plane. For instance, poles

⁹For an infinite integration domain shifting the integration variable does not change the integral as long as it is convergent. For divergent integrals this point is subtle.

¹⁰The integrand is constant over surfaces of constant ℓ^2 whose area is infinite. Hence there is definitely a convergence issue in any loop integral. It is avoided by always starting with the integral over energy.

of Feynman propagators are located slightly below the positive real axis in quadrant four or slightly above the negative real axis in quadrant two.

(11.29)

By means of the residue theorem, the value of the integral does not change by the Wick rotation

$$\int d\ell^4 F(\ell^0, \vec{\ell}) = \int i d\ell_E^4 F(i\ell_E^4, \vec{\ell}). \quad (11.30)$$

Applying this rotation to our loop integral we obtain an integral over 4-dimensional Euclidean space

$$I(-p^2) = \int_0^1 dz \int \frac{(d\ell_E)^4}{32\pi^4} \frac{1}{(\ell_E^2 + z\bar{z}p^2 + \mu^2 - i\epsilon)^2}. \quad (11.31)$$

Now the integrand depends only on $\|\ell_E\|$, and we can use rotational symmetry to replace the integral over three spherical angles at fixed $\|\ell_E\|$ by the volume of a three-sphere $2\pi^2\|\ell_E\|^3$

$$\int d\ell_E^4 F(\|\ell_E\|) = 2\pi^2 \int_0^\infty \ell_E^3 d\ell_E F(\ell_E). \quad (11.32)$$

For our integral this implies

$$I(-p^2) = \int_0^1 dz \int_0^\infty \frac{d\ell_E}{16\pi^2} \frac{\ell_E^3}{(\ell_E^2 + z\bar{z}p^2 + \mu^2 - i\epsilon)^2}. \quad (11.33)$$

The integral over ℓ_E is divergent again and needs to be cut off at $\|\ell_E\| \simeq \Lambda_{\text{cut}}$. The result is compatible with the above expression up to some minor adjustment of the cutoff parameter Λ_{cut} .

Final Integral. Gladly, the remaining integral over the Feynman parameter z can be performed for our simple integral yielding

$$\begin{aligned} I(-p^2) = & -\frac{1}{16\pi^2} \sqrt{\frac{p^2 + 4\mu^2 - i\epsilon}{-p^2}} \arctan \sqrt{\frac{-p^2}{p^2 + 4\mu^2 - i\epsilon}} \\ & - \frac{1}{32\pi^2} \log \frac{\mu^2}{\Lambda_{\text{cut}}^2 e^2}. \end{aligned} \quad (11.34)$$

The integral is manifestly real for $0 < -p^2 < 4\mu^2$. It is also real for $-p^2 < 0$. However, for $-p^2 > 4\mu^2$ it develops an imaginary part

$$\text{Im } I(-p^2) = \frac{1}{32\pi} \sqrt{\frac{-p^2 - 4\mu^2}{-p^2}}. \quad (11.35)$$

It signals the opening of the two-particle creation channel at $-p^2 > 4\mu^2$.

Spectral Function. We can now write the spectral function for our model at next-to-leading order

$$\rho(s) = 2\pi Z \delta(s - m^2) + \frac{\mu^2 \kappa^2 \theta(s - 4\mu^2)}{16\pi(s - \mu^2)^2} \sqrt{\frac{s - 4\mu^2}{s}} + \dots \quad (11.36)$$

The two terms correspond to the asymptotic particle and the two-particle continuum. Let us consider its normalisation, we find

$$\int \frac{ds}{2\pi} \rho(s) = Z + \kappa^2 \frac{2\sqrt{3}\pi - 9}{288\pi^2} + \dots \quad (11.37)$$

Since $2\sqrt{3}\pi > 9$ we see that the correction term due to the two-particle continuum is indeed small and positive. Reassuringly the field strength renormalisation precisely compensates for the correction term

$$Z = 1 - \mu^2 \kappa^2 I'(\mu^2) + \dots, \quad I'(\mu^2) = \frac{2\sqrt{3}\pi - 9}{288\pi^2 \mu^2}. \quad (11.38)$$

11.3 Regularisation and Renormalisation

Above, we have encountered a divergent integral $I(-p^2)$ and in order to evaluate it anyway, we somewhat arbitrarily introduced a momentum cutoff Λ_{cut} . Fortunately, the cutoff has only a mild impact on the function $I(-p^2)$

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}}. \quad (11.39)$$

In particular, the dependence on Λ_{cut} does not mix at all with the dependence on the momenta and masses! This allowed us to extract some information from $I(-p^2)$ without caring too much about the cutoff.

In the following we will introduce the notion of regularisation and renormalisation as central aspects of QFT to be discussed further in QFT II, and we shall put our earlier results into this context.

Regularisation Schemes. In order to extract precise information from a QFT model at higher orders, one has to introduce a consistent regularisation scheme. Such a scheme should make all relevant quantities finite.

There are several schemes, e.g.:

- *Cutoff.* Our choice was to cut off momentum integrals at very large momenta (UV). Similarly one could cut off very small momenta (IR). Unfortunately, a cutoff is not easy to formulate consistently for all quantities. It is often used to quickly derive individual leading order results.

- *Pauli–Villars*. Replace Feynman propagators by a difference of two propagators

$$\frac{1}{p^2 + m^2 - i\epsilon} \rightarrow \frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + M^2 - i\epsilon}. \quad (11.40)$$

For large M^2 and small p^2 the second propagator is suppressed. For large p^2 , however, the two propagators almost cancel. Therefore this scheme suppresses UV divergences. It is similar to a UV cutoff, but it can be applied universally to Feynman diagrams.

- *Point Splitting*. In position space, the problem of UV divergences is related to putting several fields at the same point in spacetime. By separating the field insertion points in the action by a tiny amount, UV divergences can be avoided.
- *Lattice*. For the lattice regulator one approximates infinite spacetime by a finite lattice. For finitely many degrees of freedom there cannot be divergences, neither from the UV (finite spacing), nor from the IR (finite extent).
- *Dimensional Regularisation*. The types and degrees of divergences depend crucially on the number of spacetime dimensions D . In the dimensional regularisation scheme, one works in a spacetime of dimension D , where D is taken to be an unconstrained real (or even complex) number. Observables become functions of D , and divergences appear as poles in the D -dependence, e.g. $1/(D - 4)$. Although the definition of this scheme is somewhat abstract, it is one of the favourite ones because it works well in almost all circumstances.
- *Finite Observables*. Sometimes divergences can be avoided by considering physical observables only. In our example, one could try to argue that all observables can be deduced from $I'(s)$ which is perfectly finite. The constant term of $I(s)$ is an integration constant of $\int ds I'(s)$. The divergence happens to be located precisely in this undetermined coefficient.

In a regularised QFT, all observables are perfectly finite. However, they are not quite what we are interested in, we are interested in observables of the original QFT.

Renormalisation. The next step called renormalisation is to somehow absorb the divergences consistently. To that end the most important insights are the following:

- All physically relevant information and all observables for a QFT model are encoded into its quantum correlation functions.
Example. The spectrum of asymptotic particles is encoded into time-ordered two-point functions. Moreover, the scattering matrix can be derived from the poles of higher-point functions.
- The Lagrangian and the action are devices to derive suitable correlation functions. They are not fundamental objects, in particular their parameters such as masses and coupling constants are not directly observable.¹¹
Example. The mass terms (μ) in the Lagrangian do not exactly reflect the masses (m) of asymptotic particles.

¹¹Some traces of the complete Lagrangian or action may remain valid in the QFT, such as the exact equations of motion and normalisation of the interacting field, as well as Noether's theorem.

- Correlation functions depend on the so-called bare parameters of the Lagrangian. One should tune the parameter values such that the correlation functions behave as expected. At the end of the day, the numerical values of the parameters are not important.
Example. One would adjust μ and κ such that the physical mass m has the desired physical value.
- In terms of differential geometry: The parameters of a QFT form a manifold, where each point represents a particular physical setup. A Lagrangian description (with a particular regularisation, renormalisation and, where applicable, a particular gauge fixing scheme) represents a chart of the manifold. The parameter values correspond to coordinates on this particular chart. There is, however, no universal meaning to coordinates without reference to the specific chart.

In this picture, renormalisation is the step to adjust the Lagrangian parameters to the physical parameters. In the regularised and finite QFT this step is well-defined. We express the bare Lagrangian parameters in terms of the physical parameters, e.g.

$$\mu = \mu(m, \Lambda_{\text{cut}}) = m + \frac{1}{2}m\kappa^2 I(m^2, \Lambda_{\text{cut}}) + \dots \quad (11.41)$$

with

$$I(m^2, \Lambda_{\text{cut}}) = \frac{2 - \pi/\sqrt{3}}{32\pi^2} - \frac{1}{32\pi^2} \log \frac{m^2}{\Lambda_{\text{cut}}^2}. \quad (11.42)$$

Running Coupling. We can now remove the regulator by sending the regularisation parameters to some appropriately chosen limit. We shall keep the physical parameters fixed in the limit, but the resulting bare parameters may well be divergent

$$\mu(m, \Lambda_{\text{cut}}) \rightarrow \infty \quad \text{as } \Lambda_{\text{cut}} \rightarrow \infty. \quad (11.43)$$

This by itself is not a problem, since we attribute no meaning to μ . We just need to keep in mind that the definition of bare parameters such as μ depends on a scale such as Λ_{cut} . Changing the scale must be compensated by a change in the bare parameter. In our case this dependency is governed by the equation

$$\frac{d\mu}{d\Lambda_{\text{cut}}} = \frac{m\kappa^2}{16\pi^2\Lambda_{\text{cut}}} + \dots \quad (11.44)$$

Such an equation is often written in logarithmic form

$$\frac{d \log \mu}{d \log \Lambda_{\text{cut}}} = \frac{\kappa^2}{16\pi^2} + \dots, \quad (11.45)$$

which suggests the scaling behaviour

$$\mu \sim m(\Lambda_{\text{cut}}/m)^{\kappa^2/16\pi^2 + \dots}. \quad (11.46)$$

Note that the dependence of the bare mass μ on the physical mass m has become non-linear by quantum effects. In the quantum field theory the mass term has acquired a so-called anomalous dimension.

A similar effect can be observed for coupling constants governing the interactions of several particles. In quantum field theory one may find an anomalous dependence on the particle momenta, e.g. $\lambda \sim (p/\Lambda)^{*\kappa^2+\dots}$. It means that one will measure a different effective coupling strength depending on which length or energy scale the interaction is probed (e.g. the energy of the probe photon). Naturally, one would like to define a universally valid coupling strength to appear in the Lagrangian, e.g. by considering the limit of very high or very low energies. However, in this limit, the coupling strength often diverges. Therefore one needs to define the coupling constant at a particular energy scale Λ , and the value of this coupling constant depends on Λ . This effect is called a running coupling constant. An alternative interpretation is charge screening: The electromagnetic interactions depend on the product of the coupling constant and the charges. Hence a running of the coupling has the same effect as a scale-dependent effective charge of the particles. In electrodynamics one can view the bare charges of particles as partially screened by the surrounding quantum effects of the electromagnetic and matter fields. The effectively perceived charge then decreases slightly with the distance.

Renormalisability. The question is whether all physical quantities remain finite in the limit $\Lambda_{\text{cut}} \rightarrow \infty$, and whether they are independent of the chosen regularisation scheme. We can only adjust one bare parameter per physical parameter, are there sufficiently many bare parameters to absorb all the divergences?

- In the case of our model the answer is yes.
- QFT models where all divergences can be absorbed are called renormalisable.
- In principle, one can introduce further terms and couplings in the Lagrangian to compensate for more and more divergences. As long as only finitely many terms are needed to absorb the divergences at all perturbative orders, the model is called renormalisable.
- Some models, such as general relativity, appear to require infinitely many coupling constants to absorb all divergences. These models are called non-renormalisable. Here one would need infinitely many measurements to adjust infinitely many parameters, and effectively the model loses its predictive power.

Before discussing which models are renormalisable, let us consider some technical aspects of absorbing divergences into the coupling constants.

11.4 Counterterms

In principle, we know how to absorb divergences by writing the bare parameters of the Lagrangian as functions of the physical parameters. Let us discuss the origin of the divergence in detail towards absorbing the divergences into redefinitions of the Lagrangian parameters.

Localised Divergences. To that end we investigate the integral $I(-p^2)$ more closely. The overall dependence on p^2 is some inverse trigonometric function.¹² However, the divergent or cutoff-dependent contribution to $I(-p^2)$ is much simpler

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}} . \quad (11.47)$$

It is actually independent of p^2 !

This behaviour is in fact general; divergences typically couple to polynomials of the momenta only. This statement becomes more meaningful when translated to position space: Any polynomial of the momenta translates to a localised distribution such as $\delta^4(x)$ under a Fourier transformation.

In terms of the loop integral in position space we can localise the origin of the divergence. The loop integrand diverges as r^{-4} when the two vertices are nearby at a distance of r , otherwise it is perfectly finite.



(11.48)

This divergence cannot be compensated by the measure $dx^4 \sim r^3 dr$, and the region $r \approx 0$ contributes an infinite amount to the integral. The divergence is therefore localised in spacetime, and it can be absorbed by a suitable local term in the Lagrangian.



(11.49)

Asymptotic Lagrangian. Indeed we see that we can absorb the divergence by a suitable definition of the bare mass term $\mu = \mu(m, \Lambda_{\text{cut}})$ in the Lagrangian. The prescription may be somewhat confusing because the integral $I(-p^2)$ depends on the mass μ , so the definition of μ apparently is implicit. For our calculations we use the interaction picture where we decided to split the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{0,\text{int}}$ into a free and an interaction contribution

$$\mathcal{L}_0 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2, \quad \mathcal{L}_{0,\text{int}} = -\frac{1}{6}\kappa\mu\phi^3 - \frac{1}{24}\kappa^2\lambda\phi^4. \quad (11.50)$$

However, we are not forced to do this naive split, we are free to choose any free field as a reference. In this sense, it makes perfect sense to choose the asymptotic field with the physical mass m as a reference

$$\mathcal{L}_{\text{as}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2. \quad (11.51)$$

This will automatically position all the poles due to Feynman propagators at the desired physical location. Now we have to add a compensating mass term to the interaction terms¹³

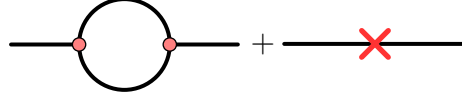
$$\mathcal{L}_{\text{as,int}} = -\frac{1}{2}\mu' m^2\phi^2 - \frac{1}{6}\kappa m\phi^3 - \frac{1}{24}\lambda\kappa^2\phi^4. \quad (11.52)$$

¹²We were extremely lucky to find a simple function. Typically one finds much more complicated special functions such as polylogarithms or hyper-geometric functions. And this only if one is lucky. Often the encountered integrals lead to functions which do not even have a name.

¹³The definition of μ' , κ and λ has changed, but this does not matter since the values of the bare parameters are not directly measurable.

The new term is called a counterterm. Its role is to compensate all potential mass shifts due to loop effects. The mass of the reference field will thus conveniently be the physical mass to all orders.¹⁴ In this picture, the loop integral $I(-p^2)$ is defined directly in terms of the physical mass m instead of μ .

When including the counterterm in our example, we obtain the following one-loop contributions¹⁵



$$(11.53)$$

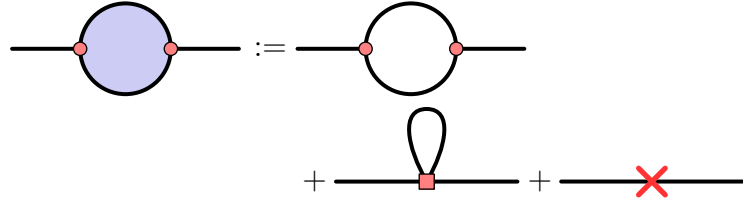
and the following corrected two-point function

$$K(p) = \frac{1}{p^2 + m^2 + \mu' m^2 - m^2 \kappa^2 I(-p^2) - i\epsilon} + \dots \quad (11.54)$$

We impose the consistency equation that the physical mass equals the asymptotical mass

$$m^2 = m^2 + \mu' m^2 - m^2 \kappa^2 I(m^2) + \dots, \quad (11.55)$$

which is solved by $\mu' = \kappa^2 I(m^2)$. Concerning the counterterm, the rule is that whenever a bubble with two legs appears in a Feynman graph, there is a compensating counterterm.¹⁶



$$(11.56)$$

It therefore makes sense to introduce a subtracted bubble integral

$$I_{\text{sub}}(-p^2) := I(-p^2) - I(m^2). \quad (11.57)$$

Obviously, this function is finite and satisfies

$$I_{\text{sub}}(m^2) = 0. \quad (11.58)$$

The corrected two-point function then takes the simplified form

$$K(p) = \frac{1}{p^2 + m^2 - m^2 \kappa^2 I_{\text{sub}}(-p^2) - i\epsilon} + \dots \quad (11.59)$$

¹⁴Only the mass is stabilised by the counterterm. The two Feynman diagrams do not cancel exactly because the loop integral has a complicated dependence on p^2 whereas the counterterm is a constant function.

¹⁵The loop order does not necessarily refer to the literal number of loops in a Feynman graph. It makes sense to also count counterterms to loop divergences as loops. The loop order commonly refers to the order in a small coupling constant, in our case κ^2 .

¹⁶We have reintroduced the contribution from the tadpole graph which is equivalent to a mass term with infinite mass and can therefore be absorbed entirely into a suitable redefinition of μ' .

We can even go one step further and decide to add a counterterm for the kinetic term

$$\mathcal{L}_{\text{as,int}} = -\frac{1}{2}\zeta(\partial\phi)^2 - \frac{1}{2}\mu'm^2\phi^2 - \frac{1}{6}\kappa m\phi^3 - \frac{1}{24}\lambda\kappa^2\phi^4. \quad (11.60)$$

This term changes the overall normalisation of the field $\phi(x)$ away from its canonical choice, but allows us to normalise the residue of the asymptotic particle pole in the two-point function to 1 at all orders. In this case, we can drop the field strength renormalisation factor Z by setting it to 1.

In conclusion, the asymptotic Hamiltonian describes the canonically normalised fields with appropriate physical masses. The interaction Hamiltonian contains all types of allowable interactions terms. Their parameters are tuned to stabilise the masses and normalisations and to match with physical interaction processes.

Power Counting. Can we understand under which circumstances a QFT model is renormalisable? We can use a crude argument in terms of the mass dimensions of interaction terms.

A Feynman diagram evaluates to an integral over a rational function of the momenta and masses of the particles

$$I \sim \int \frac{d\ell^D Q(\ell)}{P(\ell)}, \quad (11.61)$$

where the denominator $P(\ell)$ originates from Feynman propagators and the numerator $Q(\ell)$ from vertices and spinor traces.

- Earlier, we have discussed that UV divergences of the integral are polynomials of the momenta.
- Furthermore, we can argue that particle masses in the denominator P can be safely ignored for the purpose of UV divergences because they are always dominated by the momenta.¹⁷ Contributions from the masses to divergences therefore originate from Q alone (or equivalent parts of the integral), and will only appear as polynomials in the final answer.
- Evidently, the coupling constants will appear as overall factors.

Altogether this implies that the structure of UV divergences of a loop integral is given by polynomials in the momenta and the masses

$$\frac{dI}{d \log \Lambda} \in \text{Poly}(\alpha_k, p_k, m_k). \quad (11.62)$$

The polynomials in the momenta p_k determine the appropriate local counterterm.

This has an important consequence for a Lagrangian \mathcal{L} whose terms \mathcal{L}_k have a mass dimension bounded from above by the dimension D of spacetime

$$\mathcal{L} = \sum_k \alpha_k \mathcal{L}_k. \quad (11.63)$$

The class of potential divergences is restricted by the following consideration:

¹⁷A notable exception is the propagator of a massive vector field which has unaccompanied factors of mass in the denominator. Such factors upset the power counting scheme, and in order to preserve it, masses of vector field should be generated by the Higgs mechanism.

- The mass dimension of the integral is essentially determined by the external lines of the Feynman diagram. It is a fixed number usually bounded from above by the number of spacetime dimensions D , in our case $D = 4$.
- Since the mass dimension of the Lagrangian \mathcal{L} must equal D , all coupling constants α_k have a non-negative mass dimension.
- Then the overall polynomial divergence terms must have non-negative mass dimension. The remaining mass dimension must be carried by the momenta and masses.
- This implies that there is only a very restricted set of momentum polynomials which can carry divergences. In other words, only few counterterms are needed to compensate the divergences.
- These counterterms have mass a dimension bounded from above by D . The counterterm couplings in turn have non-negative mass dimension.
- Usually, there are finitely many such terms, and therefore such models are renormalisable.

Note that there is a crucial difference between coupling constants with positive mass dimension and dimensionless coupling constants. As the dimension of coupling constants accumulates, dimensionful coupling constants contribute divergences only at low perturbative orders. When there are also dimensionless coupling constants present, arbitrary loop orders will be divergent.

In our example, we have included all interaction terms of mass dimension bounded from above by 4. For each divergence which can possibly arise, there is a corresponding counterterm.

When considering Feynman diagrams which contain a divergent loop along with some other structures, it can be shown that the loop can be replaced by a universal counterterm corresponding to the loop to cancel the divergence. A subtle issue at higher loops are overlapping loops, where it may not be evident which counterterms to use. Gladly, it can be shown that this situation does not leave behind divergences which cannot be accounted for.

11.5 Vertex Renormalisation

Let us briefly discuss how to renormalise the remaining divergences at the one-loop level in our example.

Divergent Interactions. By power counting arguments we can derive that divergences can appear in our model only for two-sided loops.¹⁸ Loops with three or more sides are perfectly finite. There are only three potentially divergent terms in our model.


(11.64)

¹⁸One-sided tadpoles can be removed entirely by suitable counterterms.

One of them we have already discussed, the other two integrals can be made finite by adding appropriate counterterms of the form ϕ^3 and ϕ^4 to the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -\frac{1}{6}\kappa m\phi^3 - \frac{1}{24}\lambda\kappa^2\phi^4 + \mathcal{L}_{\text{ct}}, \\ \mathcal{L}_{\text{ct}} &= -\frac{1}{2}\mu'm^2\phi^2 - \frac{1}{6}\kappa'\kappa m\phi^3 - \frac{1}{24}\lambda'\kappa^2\phi^4 \\ &= \text{---}\text{X}\text{---} + \text{---}\text{X}\text{---} + \text{---}\text{X}\text{---}.\end{aligned}\quad (11.65)$$

Suitable counterterm coefficients to make all observables finite at one loop read

$$\mu' = \kappa^2 I(m^2), \quad \kappa' = 3\kappa^2 \lambda I(m^2), \quad \lambda' = 3\kappa^2 \lambda^2 I(m^2). \quad (11.66)$$

As discussed above, μ' is determined by a stable physical mass at m . There is no similar universal condition for the coefficients κ' and λ' ; any finite shift with respect to the above values is permissible, it merely leads to a reparametrisation of our model.

We have already seen that the ϕ^2 counterterm effectively replaces the loop integral by a finite subtracted loop integral

$$I(-p^2) \rightarrow I_{\text{sub}}(-p^2). \quad (11.67)$$

Exactly the same replacement is achieved by our above choice of counterterms for ϕ^3 and ϕ^4 .

$$\text{---}\text{---}\text{---} \quad \text{---}\text{---}\text{---} \quad \text{---}\text{---}\text{---} \quad (11.68)$$

Three-Point Function. Let us briefly consider the resulting one-loop contributions to the three-point function.

$$\begin{aligned}F_3 &= F_3^{(1)} + F_3^{(3)} + \dots, \\ F_3^{(1)} &= \text{---}\text{---}\text{---}, \\ F_3^{(3)} &= \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} \\ &\quad + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} \\ &= \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---}.\end{aligned}\quad (11.69)$$

The latter two terms involve bubbles only for which we know the integral already. The counterterms make both integrals finite.¹⁹ The first triangle integral $\sim \int d\ell^4 / \ell^6$ is UV finite by itself. In fact it is the most complicated contribution

$$\begin{aligned}
F_3^{(3)} = & -i\kappa^3 m^3 (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \\
& \cdot \int \frac{-i d\ell^4}{(2\pi)^4} \frac{1}{\ell^2 + m^2 - i\epsilon} \\
& \cdot \frac{1}{(\ell + p_1)^2 + m^2 - i\epsilon} \frac{1}{(\ell - p_3)^2 + m^2 - i\epsilon}
\end{aligned} \tag{11.70}$$

and the only one which allows all three momenta to interact non-trivially.

Four-Point Function. There are many diagrams contributing to a four-point process. Here we can merely plot all the graphs (up to permutations of the external legs).

$$\begin{aligned}
F_4 = & F_4^{(2)} + F_4^{(4)} + \dots, \\
F_4^{(2)} = & \text{[diagram 1]} + \text{[diagram 2]}, \\
F_4^{(4)} = & \text{[diagram 3]} + \text{[diagram 4]} + \text{[diagram 5]} \\
& + \text{[diagram 6]} + \text{[diagram 7]} + \text{[diagram 8]} \\
& + \text{[diagram 9]} + \text{[diagram 10]} + \text{[diagram 11]}.
\end{aligned} \tag{11.71}$$

¹⁹Note that because $I_{\text{sub}}(m^2) = 0$ the internal Feynman propagator which connects the bubble to the 3-vertex is precisely cancelled, and only one Feynman propagator with associated asymptotic particle pole remains for each leg.

Schedule of Lectures

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