Classical Mechanics

Problem Sets

ETH Zurich, 2020 HS

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The current version of this work can be found at:
`http://people.phys.ethz.ch/~nbeisert/lectures/`.

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1.1. Vector cross product

In this problem we consider some useful identities involving the cross product of two three-dimensional vectors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} \tag{1.1}$$

and the closely related totally anti-symmetric tensor ε^{ijk} with $i, j, k \in \{1, 2, 3\}$ which is defined via the following properties

$$\varepsilon^{ijk} = -\varepsilon^{jik} = -\varepsilon^{ikj}, \qquad \varepsilon^{123} := +1. \tag{1.2}$$

Hint: The parts may be considered in any convenient sequence.

a) Show the identity $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$.

b) Show the identities $(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}$.

c) Show the identity $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$.

d) Show that the cross product behaves as a vector under rotations $R \in \mathrm{SO}(3)$

$$(R\vec{a})\times(R\vec{b}) = R(\vec{a}\times\vec{b}). \tag{1.3}$$

How can this identity be generalised for $R \in \mathrm{O}(3)$?

e) How is ε^{ijk} related to ε^{jki}, ε^{kij} and ε^{kji}?

f) Determine all non-zero components of the totally anti-symmetric tensor ε^{ijk}.

g) Show that the cross product is related to the totally anti-symmetric tensor ε^{ijk} via the relationship

$$(\vec{a}\times\vec{b})^k = \sum_{i,j=1}^{3} \varepsilon^{ijk} a^i b^j. \tag{1.4}$$

h) Show the identity $\sum_{m=1}^{3} \varepsilon^{ijm}\varepsilon^{klm} = \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}$.

i) Show the identity $\sum_{i,j,k=1}^{3} \varepsilon^{ijk} a^i b^j c^k = (\vec{a}\times\vec{b})\cdot\vec{c}$.

j) Show the identity $\sum_{l,m,n=1}^{3} \varepsilon^{lmn} M^{li} M^{mj} M^{nk} = \varepsilon^{ijk} \det M$ for a 3×3 matrix M.

k) Determine a 3×3 matrix Ω for the vector $\vec{\omega}$ such that $\vec{\omega}\times\vec{a} = \Omega\vec{a}$ for all vectors $\vec{a}$. What property does the matrix Ω have?

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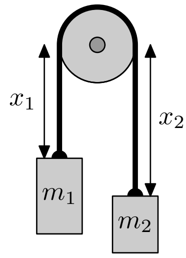
1.2. Atwood machine

The Atwood machine consists of two different masses m_1 and m_2 which are connected by a rope of length l and which are hanging from a pulley.



(1.5)

a) What two forces act on each of the masses? What forces act on the pulley?

b) Write the equation of motion for the position(s) of the masses.

c) Can you use this equation to determine the gravitational constant g? How can you make the measurement most accurate?

1.3. Free particle in spherical coordinates

Determine the equations of motion for a free particle in spherical coordinates

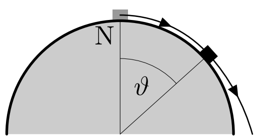
$$\vec{x} = (r \sin\vartheta \cos\varphi, r \sin\vartheta \sin\varphi, r \cos\vartheta). \tag{1.6}$$

In the case of an additional physical force $\vec{F}$ acting on the particle with mass m, show that the radial component of the acceleration is given by

$$\ddot{r} = r\dot{\vartheta}^2 + r\dot{\varphi}^2 \sin^2\vartheta + F_r/m. \tag{1.7}$$

1.4. Liftoff from a ball

A point mass is placed at the north pole of a ball within a homogeneous vertical gravitational field with acceleration g. The mass therefore resides in an unstable equilibrium from which it is removed by a negligibly small kick. It then glides without friction down the surface of the ball. At which angle ϑ does the mass lift off from the surface of the ball?



(1.8)

Hint: To obtain the velocity of the mass as a function of the angle, you can use conservation of energy. Alternatively, you can multiply the equation of motion for $\vartheta(t)$ by $\dot{\vartheta}$ and integrate over time.

2.1. Galilean group

Show that the Galilean transformations

$$(t, \vec{x}) \mapsto (t', \vec{x}') = (\lambda t + a, R\vec{x} + \vec{v}t + \vec{b}), \tag{2.1}$$

where $R \in \mathrm{O}(3)$, $\vec{v}, \vec{b} \in \mathbb{R}^3$, $a \in \mathbb{R}$ and $\lambda = \pm 1$, form a group.

2.2. Galilean invariance

We consider a mechanical system of n particles in $\mathbb{R}^3$ which is described by the following Galilean invariant law of forces:

$$m_k \ddot{\vec{x}}_k = -\frac{\partial}{\partial \vec{x}_k} V(\vec{x}_1, \ldots, \vec{x}_n). \tag{2.2}$$

a) Consider the case of two particles ($n = 2$) which are initially at rest. Show that the movement of the two particles takes place on the line which connects their starting points.

b) Formulate and prove an analogous statement for the case of three particles ($n = 3$) which are initially at rest.

c) Considering two particles which are not necessarily initially at rest, show that there is an inertial system in which the movement takes place in a plane.

2.3. Conserved quantities in a central potential

We consider a particle in a central potential, i.e. the potential V only depends on the distance $r = \|\vec{x}\|$ to the origin. The equation of motion thus reads

$$m \ddot{\vec{x}} = -\frac{\partial}{\partial \vec{x}} V(\vec{x}) = -\frac{\vec{x}}{r} \frac{\mathrm{d}V(r)}{\mathrm{d}r}\,. \tag{2.3}$$

a) Show that the angular momentum $\vec{L} = \vec{x} \times \vec{p}$ is conserved in this case.

b) We demand that the Laplace–Runge–Lenz vector $\vec{A} = \dot{\vec{x}} \times \vec{L} + V(r)\,\vec{x}$ be conserved as well. What additional constraint does that put on the r-dependence of the potential?

Hint: The differential equation can be treated by using the method of separation of variables. Alternatively, you may consider an ansatz of the form $V(r) \sim \alpha\, \mathrm{e}^{\lambda r}\, r^{\gamma}$.

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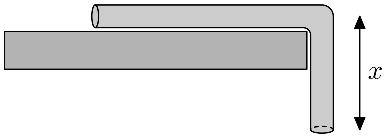
2.4. Sliding rope

We consider a rope of mass m and length l which hangs over the edge of a table. The gravitational field of the earth is acting in the x-direction and we neglect the friction between the rope and the surface.



(2.4)

a) Find and solve the equation of motion for the end of the rope under the following initial conditions: at time $t = 0$ the rope is at rest and a part of length x_0 is hanging over the edge.

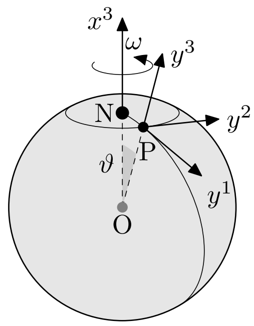
b) What is the velocity of the rope at the moment when it leaves the surface of the table?

Classical Mechanics **Problem Set 3**

ETH Zurich, 2020 HS — Prof. N. Beisert, Dr. M. Schiavina

3.1. The Foucault pendulum

Consider a pendulum with a wire of length l at a location P on earth with latitude $\frac{1}{2}\pi - \vartheta$ as illustrated in the figure. We use a co-rotating coordinate system with coordinates y^1, y^2, y^3 and origin at the point P, which is also the equilibrium position of the pendulum. The earth rotates around the axis x^3 with angular velocity ω.



(3.1)

a) Show that the equations of motion for the pendulum bob for small oscillations with $\vec{y} = (y^1, y^2)$ in the limit $\omega \ll \sqrt{g/l}$ can be written as

$$\ddot{\vec{y}} = -\frac{g}{l}\,\vec{y} + 2\omega\cos\vartheta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{\vec{y}}. \tag{3.2}$$

b) Determine the period T, where $T/2$ is defined as the time between two zero-crossings of $\dot{\vec{y}}$. Further determine the precession period τ, i.e. the time it takes until the plane, in which the pendulum oscillates, has completed a full rotation.

Hint: In order to solve the differential equation, it is convenient to consider the complex function $z(t) = y^1(t) + \mathring{\imath} y^2(t)$.

3.2. Rocket science

Rockets are propelled through the momentum of the ejected gas which leaves the rocket with velocity v_{g}. Since these gases are produced from the rocket fuel, the mass of the rocket is not constant, but is reduced due to the consumption of fuel.

a) We now consider a rocket which is shot up vertically in a homogeneous gravitational field, neglecting the friction with air. Find an equation for the acceleration $a(t) = \dot{v}(t)$ in order to describe the motion of the rocket.

b) The payload of the rocket has mass m_0 and initially the rocket carries fuel of additional mass m_1. We assume that the fuel is burnt at a constant rate and is used up after a time τ. Find the velocity $v(t)$ for the acceleration phase $0 < t < \tau$.

c) Formulate a criterion for the lift-off of the rocket. At what height is the fuel used up? What height does it reach in total?

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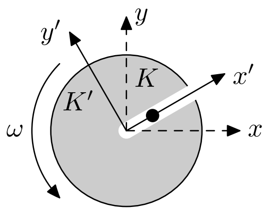
3.3. Rotating disc with a ball in a pipe

We consider a disc of radius R that rotates with constant angular velocity ω in an inertial system K. At $t = 0$, we shoot a ball of mass m from the centre of the disc through a pipe which is mounted on the disc. We assume that the friction between the ball and the pipe is such that the velocity of the ball in the co-rotating system K' is constant. This requires a special set-up. We are aiming to understand which forces have to act in order for this to work out. We ignore gravitational effects.



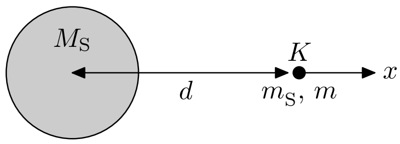
(3.3)

a) List the real and fictitious forces acting on the ball in the system K' and solve the equations of motion. What are the forces in K?

b) Derive the trajectory of the ball in the K system.

3.4. Tidal forces and the equivalence principle

Consider a freely falling particle with inertial mass m and (a priori different) gravitational mass m_{S} in the gravitational field of a planet of gravitational mass M_{S}. We assume that $M_{\mathrm{S}} \gg m_{\mathrm{S}}$ and thus we consider the approximation of a particle moving in the static potential of the planet. At time $t = 0$ the particle is at a distance d from the centre of the planet. We first consider the system K, which is related to the stationary system of the planet by a constant shift. The shift is chosen such that at $t = 0$ the particle is at the position $x = 0$.



(3.4)

a) Write down Newton's second law for the particle in K. Show by expanding the gravitational force around $x = 0$ that one obtains the differential equation

$$\ddot{x} = -\frac{m_{\mathrm{S}}}{m} G M_{\mathrm{S}} \left(\frac{1}{d^2} - \frac{2x}{d^3} + \dots \right). \tag{3.5}$$

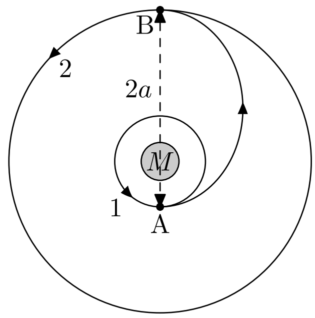
When is it a good approximation to only keep the first two terms? How do you interpret the first two terms in the expansion? How does the problem depend on the physical properties of the particle if the weak equivalence principle $m_{\mathrm{S}} = m$ holds?

b) In the following we keep only the first two terms in (3.5) and *assume* that $m_{\mathrm{S}} = m$. Show that the first term in (3.5) can be cancelled by transforming to a uniformly accelerated frame K' relative to K. Interpret yet again the terms in Newton's second law in K'. When can the arising time-dependent part be neglected?

c) Neglect the time-dependent part of the solution from part b). Let us introduce two additional free-falling particles in K whose initial positions are $x(0) = \pm\delta$ and are at rest for $t = 0$. Calculate the distance between the two particles as a function of time.

4.1. Hohmann transfer orbits

We consider the Hohmann transfer (named after Walter Hohmann) of a satellite of mass m_2 between two circular orbits of radii r_1 and r_2 around a massive object of mass m_1. In order to go from orbit 1 to 2, we use the rocket engines at the points A and B (see figure) to give two kicks, which we describe as instantaneous changes of the momentum. Between these points, we follow an elliptical orbit with semi-major axis a.



(4.1)

Hint: For this exercise consider the center of mass frame.

a) Give the initial and final speeds of the satellite. What are the speeds at A, B in the elliptical orbit and how much do you need to speed up/slow down to exit/enter the orbits 1 and 2, respectively?

b) How much time is required to complete the transfer?

c) Use the Hohmann transfer to approximate travel from Earth to Mars. Calculate the time needed for the transfer and compare to the NASA missions Viking 1 (1976) and Curiosity (2012). Is there a discrepancy? Why (not)?

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4.2. Disintegration of a binary star

Two stars of masses m_1 and m_2 move along circular orbits around their mutual centre of gravity. The second star explodes in a spherically symmetric way and leaves a mass of m_2' behind. The ejected mass $\Delta m_2 = m_2 - m_2'$ leaves the binary star immediately and without colliding with the other star.

a) What is the total energy $E = T_{\text{kin}} + V$ of the binary star before the explosion? Show that $V = -2T_{\text{kin}}$.

b) Write the kinetic energy T_{kin} before the explosion in the centre of mass frame as a function of the difference of velocities $\dot{\vec{x}}_1 - \dot{\vec{x}}_2$.

c) Under which general condition is such a system bound? Show that in this case the system remains bound after the explosion if

$$\Delta m_2 < \tfrac{1}{2}(m_1 + m_2). \tag{4.2}$$

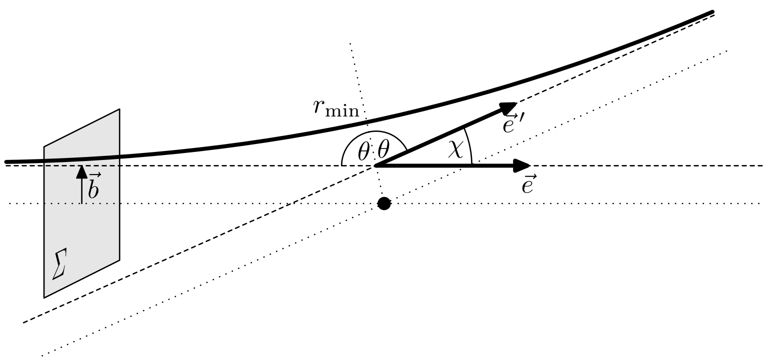
Give an example of masses m_1, m_2, and Δm_2 such that the system is not bound after the explosion.

4.3. Scattering cross section for a repulsive central force

Consider the scattering of a particle with energy $E > 0$ in a repulsive central force field

$$\vec{F}(\vec{x}) = \frac{C}{\|\vec{x}\|^4}\,\vec{x}, \tag{4.3}$$

where C is a constant.



(4.4)

a) Calculate the scattering angle $\chi = \pi - 2\theta$ as a function of the impact parameter b.
Hint: Use a substitution to put the integral into the following form:

$$\int_0^\infty \frac{\mathrm{d}u}{\cosh u} = \frac{\pi}{2}\,. \tag{4.5}$$

You can try to compute this integral yourself. Is it possible to get the result of the above integral without calculation?

b) Use the previous result to derive the differential cross section:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}^2\Omega} = \frac{\pi^2 C}{2E}\,\frac{\pi - \chi}{\chi^2(2\pi - \chi)^2 \sin\chi}\,. \tag{4.6}$$

Hint: $x(2 - x) = 1 - (1 - x)^2$.

5.1. Stability of a circular orbit

Consider a mass m moving on a circular orbit under the influence of a central force of potential $V(r)$. Study small oscillations around the circular orbit, i.e. the motion of the particle when it is slightly deflected out of the equilibrium trajectory. Consider oscillations occurring in the same plane of the circular orbit and with the same angular momentum.

a) We perturb the path to linear order around the circular orbit,

$$r(t) = r_0 + \epsilon\,\rho(t) + \mathcal{O}(\epsilon^2), \qquad \varphi(t) = \omega t + \epsilon\,\theta(t) + \mathcal{O}(\epsilon^2), \tag{5.1}$$

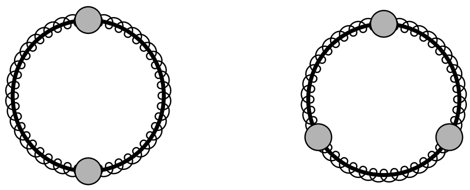
where r_0 is the radius of the circular orbit, ω is the equilibrium frequency and $\epsilon \ll 1$. Determine the equations of motion for $\rho(t)$ and $\theta(t)$ at leading order in ϵ.

b) Let the potential be of the form $V(r) = -k\,r^{-\alpha+1}$. Show that stable oscillations occur if $\alpha < 3$.

Hint: Determine a relation between ω^2 and $V'(r_0)$ and use it.

5.2. Ring oscillator

We consider n identical masses which are connected by n identical springs. Both the masses and the springs are constrained to move along a circle, see figure. Find the normal modes of the system for the cases $n = 2$ and $n = 3$.



(5.2)

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5.3. Perihelion precession

The solution of the two-body problem in Newtonian Mechanics shows that the planets move along ellipses around the sun. The observation of Mercury shows, however, that its point of closest approach to the sun, the perihelion, is turning slowly around the sun. The recorded value of this turning is 574.10 ± 0.65 arc-seconds per 100 years. It is mainly due to the influence of the other planets, based on which one expects a perihelion precession of 531.63 ± 0.69 arc-seconds per 100 years. The discrepancy to the recorded value was explained by Albert Einstein in 1915 using general relativity theory. This was the first empirical test of the theory, and we are aiming to reproduce it within this exercise.

General relativity implies a correction of the gravitational potential,

$$V(r) = -\frac{g\mu}{r} + \frac{\alpha}{r^3}\,, \tag{5.3}$$

where we consider $\alpha = -g\mu l^2 c^{-2}$ to be a small perturbation parameter, in which we will expand in the following.

a) In this exercise, we are working with the path $r(\varphi)$. It will prove to be helpful to introduce the parameter $u = r^{-1}$. In order to do this, first rewrite the equation

$$\frac{\mu}{2}\left(\dot{r}^2 + r^2\,\dot{\varphi}^2\right) = E - V(r) \tag{5.4}$$

using the relation $\dot{\varphi} = lr^{-2}$ as an equation for $u(\varphi)$ and $u'(\varphi)$. Take a φ-derivative in order to reach the equation

$$\mu\, l^2(u'' + u) = -\tilde{V}'(u), \qquad \tilde{V}(u) := V(u^{-1}). \tag{5.5}$$

b) In the lecture, we have learned that the above equation for $\alpha = 0$ is solved by the ellipse

$$u_0(\varphi) = gl^{-2}(1 + \varepsilon\cos\varphi). \tag{5.6}$$

We now consider the expansion $u = u_0 + \alpha v + \mathcal{O}(\alpha^2)$ around the ellipse. Employ equation (5.5) to derive the following equation for $v(\varphi)$

$$v''(\varphi) + v(\varphi) = -3g^2\mu^{-1}l^{-6}(1 + 2\varepsilon\cos\varphi + \varepsilon^2\cos^2\varphi). \tag{5.7}$$

Let the first perihelion be located at $\varphi = 0$. We use the initial conditions $u(0) = u_0(0)$ and $u'(0) = 0$. Derive initial conditions for $v(\varphi)$ from these and solve the above equation using these conditions.

Hint: Show first that for an arbitrary inhomogeneity $b(\varphi)$, the function

$$v(\varphi) = \int_0^{\varphi} \mathrm{d}\psi\, b(\psi)\sin(\varphi - \psi) \tag{5.8}$$

is a solution to the ODE $v''(\varphi) + v(\varphi) = b(\varphi)$.

c) We now determine the perihelion shift $\Delta\varphi$ after one revolution from the condition $u'(2\pi + \Delta\varphi) = 0$. Here, we assume that $\Delta\varphi$ is of order α. Use the solution $v(\varphi)$ you found above to derive the relation

$$\Delta\varphi = 6\pi g^2 l^{-2} c^{-2}. \tag{5.9}$$

Classical Mechanics — Problem Set 6

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6.1. Coupled pendula

We consider a system of two identical pendula of length $l_1 = l_2 = l$ and masses $m_1 = m_2 = m$ in a homogeneous gravitational field with acceleration g. The two pendula are moving in the same plane and we denote the (small) deflection angles by θ_1 and θ_2. Moreover, the pendula are connected by a massless spring, whose length equals the distance of the points to which the pendula are attached. We define $\omega_{\mathrm{g}}^2 = g/l$ and $\omega_{\mathrm{s}}^2 = k/m$.

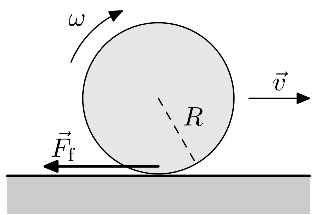
a) Find the equations of motion for θ_1, θ_2 and the normal modes of the system.

b) At time $t = 0$, the two pendula are at rest. Then, we push one of them such that it has the initial velocity $l\,\dot{\theta}_1 = v$. Show that the first pendulum is almost at rest after a time T, which you should determine. Assume that $\omega_{\mathrm{s}} \ll \omega_{\mathrm{g}}$.

Hint: $\sin a + \sin b = 2\cos\frac{1}{2}(a-b)\sin\frac{1}{2}(a+b)$.

6.2. Bowling

Consider a disc with moment of inertia $I = \beta\, m\, R^2$ on a horizontal plane with friction. Initially, the disc slides without rolling, i.e. the velocity of its centre of mass is v_0 and its angular velocity is zero. As it slides, the friction with the ground makes it spin up such that it eventually rolls without slipping.



(6.1)

a) Find the velocity of the centre of mass of the disc when it rolls without slipping.

Hint: You do not have to make any assumption on the nature of the friction force.

b) Consider the contact point P between the disc and the ground. What can you say about the torque of the disc around P? Use your result to rederive the results of part a).

c) Find the kinetic energy dissipated while sliding.

d) Now assume that the modulus of the friction force is $|F_{\mathrm{f}}| = \mu\, m\, g$. Find the time t and distance d at which the slipping stops.

e) Verify that the work done by friction equals the energy dissipation calculated in part c).

Hint: Naively multiplying the friction force for the distance computed in part d) does not agree with the result of part c). Why? What distance should you instead consider?

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6.3. Forced pendulum

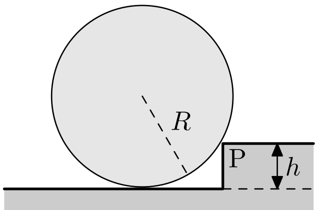
Consider a child sitting in a swing of length l. She starts swinging from an angle θ_0 with the vertical. Her father pushes her along the arc of the circle $0 \leq \theta \leq \theta_0$, with a constant force $F = \gamma mg$. We will model the child as a pointlike body of mass m and the swing as a massless rope of length l.

a) What is the relative error due to the small angle approximation $\sin\theta \approx \theta$ up to an angle of $\theta = 30° = \pi/6$?

b) How long does it take for the child to reach the configuration $\theta = 0$? What is the velocity of the child at $\theta = 0$?

6.4. Rolling over a bump *Optional:*

A ball with radius R and moment of inertia $I = \frac{2}{5} mR^2$ rolls without slipping on the ground. The initial speed of the centre of mass is v_0. The ball then hits a step of height $h < R$.



(6.2)

We will assume that the collision of the ball with the bump is inelastic and instantaneous. Moreover, we will assume that right after the collision, the point P of the ball which hit the step is at rest and the remainder of the ball merely rotates about this point.

a) Find the angular velocity ω' with which the ball rotates around P.

Hint: Energy is not conserved during the collision. What quantity is instead conserved? Why? Note that the instantaneous momentum transfer during the collision at P is directed away from P.

b) Can the ball roll up over the bump after the collision? Suppose that the ball sticks to the corner of the step and its rotation around P results into an upward motion. What is the minimal value of v_0 for the ball to roll up over the bump?

Hint: Figure out a conserved quantity suitable for this situation.

c) Under which condition does the ball lift off at the moment of collision?

Hint: The radius of curvature of a planar curve $\vec{x}(t) = (x_1(t), x_2(t))$ is given by

$$r = \left| \frac{(\dot{x}_1^2 + \dot{x}_2^2)^{3/2}}{\dot{x}_1\,\ddot{x}_2 - \dot{x}_2\,\ddot{x}_1} \right|. \tag{6.3}$$

7.1. Atwood machine with moment of inertia

We revisit problem 1.2, but this time take the pulley's inertia into account. As a reminder, the Atwood machine consists of two different masses m_1 and m_2 which are connected by a rope of length l and which are hanging from a pulley. The pulley is modelled by a homogeneous cylinder of radius r, thickness d and mass m_P.

a) Calculate the relevant moment of inertia I of the pulley. Using your result, verify that the angular momentum of the pulley as a function of the velocity v of the masses is given by $L = \frac{1}{2}\, m_\mathrm{P} R v$.

b) Write down the equation of motion for the position(s) of the masses.

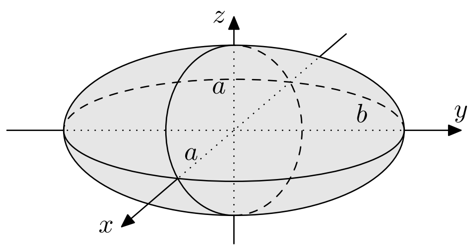
c) Solve the equations of motion and show that

$$x_1(t) = \frac{(m_1 - m_2) g t^2}{2(m_1 + m_2 + \frac{1}{2} m_\mathrm{P})}\,. \tag{7.1}$$

d) Determine the potential energy V, the kinetic energy T_lin of the masses and the rotational energy T_rot of the pulley as functions of time t. Verify that the total energy is conserved. How large is the fraction $T_\mathrm{rot}/T_\mathrm{lin}$?

7.2. Freely rotating ellipsoid

Consider an ellipsoid with semi-axes a, b, a. The ellipsoid has a uniform mass density ρ and total mass m. It is freely rotating in an inertial frame.



(7.2)

a) Calculate the moments of inertia tensor and show that it is given by

$$I = \tfrac{1}{5} m \operatorname{diag}(a^2 + b^2, 2a^2, a^2 + b^2). \tag{7.3}$$

b) What are the conserved quantities in the inertial frame? What do they imply for the allowed vectors of angular momentum $\vec{\omega}$?

c) Write down the Euler equations and solve with general initial conditions on $\vec{\omega}$. What is the frequency of precession?

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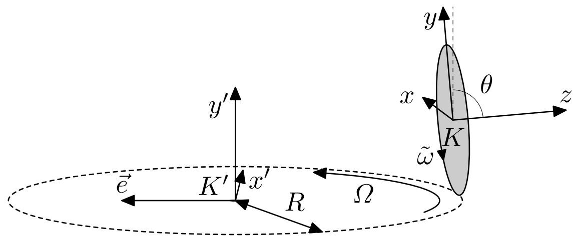
7.3. Rolling coin

Consider a thin coin of radius r with uniform mass density and mass m rolling without slipping in a circle of radius R. The coin rolls around the circle with a constant angular velocity Ω such that it makes a constant angle θ with the floor as indicated in the figure.



(7.4)

We define an accelerated frame K fixed to the centre of the coin with the z-axis aligned to its symmetry axis and the y-axis pointing in a largely upward direction. The inertial frame K' is fixed to the centre of the circle with the y'-axis aligned to the axis of the circle.

a) What forces act on the coin in the inertial frame K' and what is the relation between Ω and the angular velocity $\tilde{\omega}$ of the coin around its symmetry axis? Express the overall angular velocity vector $\vec{\omega}$ of the coin in terms of R, r, θ, Ω in the accelerated frame K.

b) Determine the principal moments of inertia of the coin. Write down the angular momentum vector $\vec{S}$ in the accelerated frame K and the component $\vec{e}\cdot\vec{L}$ pointing horizontally outward in the inertial frame K' where $\vec{e}$ is the unit vector pointing away from the support of the coin (see picture).

c) Determine the torque $\vec{M}$ acting on the coin and its norm in the inertial frame K'. Find an expression for Ω in terms of g, R, r, θ using the precession condition $\|\dot{\vec{L}}\| = \Omega\, \vec{e}\cdot\vec{L}$. Show that we must have $R > {}^5\!/_6\, r\cos\theta$ for the motion to be possible.

8.1. Lorentz transformations

Consider the one-parameter family of matrices $L(\theta)$, $\theta \in \mathbb{R}$, given by

$$L(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{8.1}$$

a) Show that these satisfy

$$L(\theta_1)\, L(\theta_2) = L(\theta_1 + \theta_2), \tag{8.2}$$

and deduce that they form a group.

Hint: The following addition identities will be useful:

$$\begin{aligned} \sinh(x+y) &= \sinh x \cosh y + \sinh y \cosh x, \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y. \end{aligned} \tag{8.3}$$

b) Show that $L(\theta)$ leaves the Minkowski metric $\eta := \operatorname{diag}(-1, +1)$ invariant

$$L(\theta)^{\mathsf{T}}\, \eta\, L(\theta) = \eta. \tag{8.4}$$

8.2. Lorentz boosts and communication in spacetime

In this exercise, we consider transformations between inertial systems.

a) An inertial system moves with velocity $\vec{v} = (v\cos\varphi, v\sin\varphi, 0)$ with respect to another one. Show that the associated Lorentz transformation $L_\varphi(v)$ is given by the following matrix (with $\beta := v/c$ and $\gamma := 1/\sqrt{1-\beta^2}$)

$$L_\varphi(v) = \begin{pmatrix} \gamma & \gamma\beta\cos\varphi & \gamma\beta\sin\varphi & 0 \\ \gamma\beta\cos\varphi & \gamma\cos^2\varphi + \sin^2\varphi & (\gamma-1)\sin\varphi\cos\varphi & 0 \\ \gamma\beta\sin\varphi & (\gamma-1)\sin\varphi\cos\varphi & \gamma\sin^2\varphi + \cos^2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.5}$$

What do you get for a boost along a diagonal of the x,y-plane with $\varphi = 135°$?

Two spaceships A and B start at the same time from a space station with velocities $\vec{v}_A$ and $\vec{v}_B$ relative to the space station. The angle between the velocities as measured in the rest frame of the space station is φ.

b) Spaceship B starts a clock when departing from the space station. After the spaceship gets hit by an asteroid, the clock breaks, showing the time t_B. What amount of time t_A passes in the rest frame of spaceship A from the departure from the space station until the breaking of the clock at spaceship B?

c) The calculation in part b) is unphysical. Why?

d) A more meaningful comparison of times arises if spaceship B sends its time t_B via a light signal to A, and times are compared on spaceship A. Assume that B sends the signal at time t_B to the space station first. The space station instantaneously re-sends it to A. At what time $t_{A,\text{signal}}$ does the signal arrive on the spaceship A?

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8.3. Roundtrip in spacetime

The effects of special relativity are sometimes counter-intuitive. In this problem we consider a composition of Lorentz transformations with a curious outcome.

Consider a spaceship which goes on a roundtrip in space(time) at relativistic velocities. The spaceship accelerates to the velocity v, and it then makes a turn at a right angle (without decelerating). This maneouvre is repeated four times. We disregard all displacements along the journey and consider only the orientation and the motion of the spaceship.

Hint: It suffices to assume a spacetime of $2+1$ dimensions, i.e. you may drop the z-coordinate for convenience.

a) We will need two transformations of spacetime: a Lorentz boost in the x-direction by velocity v and a rotation in the x,y-plane by 90°. Find the matrices R and L that describe the rotation and boost, respectively.

b) Show that $R^4 = \text{id}$ and interpret the finding.

c) The roundtrip (up to displacements) is described by the Lorentz transformation $L_4 := (RL)^4$. Compute L_4 and show that you get the matrix ($\beta := v/c$ and $\gamma := 1/\sqrt{1-\beta^2}$)

$$L_4 = \begin{pmatrix} \gamma^4 - 2\gamma^3\beta^2 & \gamma^4\beta - \gamma^3\beta - \gamma^3\beta^3 & \gamma^3\beta - \gamma^2\beta \\ -\gamma^3\beta + \gamma^2\beta & -\gamma^3\beta^2 + \gamma^2 & -\gamma^2\beta^2 \\ -\gamma^4\beta + \gamma^3\beta + \gamma^3\beta^3 & -\gamma^4\beta^2 + 2\gamma^3\beta^2 & -\gamma^3\beta^2 + \gamma^2 \end{pmatrix}. \tag{8.6}$$

Note that some signs may differ depending on the choice of conventions.

d) Expand L_4 for small β to third order. Show that the transformation describes a rotation by the angle β^2 and a Lorentz boost of velocity $\frac{1}{2}\sqrt{2}\,\beta^3 c$ along the diagonal in the x,y-plane, with $\varphi = 135°$.

e) Suppose the spaceship performs the maneouvre a fifth time. Determine $L_5 := (RL)^5$. What do you get for $\beta = (\frac{1}{2}\sqrt{5} - \frac{1}{2})^{1/2} \approx 0.786$? How do you interpret the result?

9.1. Relativistic Doppler effect

In this exercise we will derive the relativistic Doppler effect for electromagnetic waves and light. Consider a light source which moves with constant velocity v towards an observer at rest in system K. The source emits light with a frequency f_0 as measured in the co-moving frame K'. The observer measures a wave train consisting of n periods of the electromagnetic wave during some time Δt. Our goal is to find the relation between the frequency f measured by the observer and the frequency f_0 as emitted by the source.

a) What is the total distance between the front and the rear of the wave train? Deduce from this the wavelength λ and frequency f as observed by the observer.

b) Now consider the situation from the point of view of the source. Give an expression for n in terms of f_0.

c) Find an expression for the time interval $\Delta t'$ in the K' in terms of the time interval Δt in the K frame using

$$t' = \frac{t + xv/c^2}{\sqrt{1 - v^2/c^2}}, \qquad x' = \frac{x + vt}{\sqrt{1 - v^2/c^2}}. \tag{9.1}$$

Use this result, which is called *time dilation*, to put everything together to obtain an expression for f as a function of f_0 and v.

d) Repeat the analysis for a source moving away from an observer with velocity v and show that

$$f = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\, f_0. \tag{9.2}$$

e) Recall that for the regular, non-relativistic, Doppler effect of sound we have that

$$f = \frac{\tilde{c} + v}{\tilde{c} - v_0}\, f_0, \tag{9.3}$$

where $\tilde{c}$ denotes the speed of sound, v denotes the speed of the observer, and v_0 denotes the speed of the source. What qualitative difference from the relativistic Doppler effect do you notice? Explain why this is not a violation of the principle of relativity.

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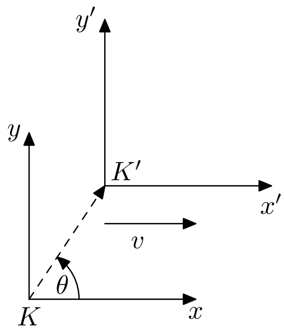
9.2. Relativistic Doppler effect from energy considerations

Consider a light source which passes an observer at some finite distance and at constant relative velocity v. The frequency of the emitted light is f_0 in the rest frame K of the source. Let θ denote the emission angle in K of the light which is currently being observed by the observer.



(9.4)

a) Recall that the energy of a photon is $E = hf$, where h is the Planck constant and f denotes the frequency. Write down the four-momentum of a photon in K.

b) Apply the appropriate Poincaré transformation to obtain the four-momentum of the photon in the frame K' of the observer. Use this to express the frequency f' as observed by the observer in K' in terms of f_0, v and the angle θ.

Hint: What effect does the spacetime translation that relates the origins of the frames K and K' have on the four-momentum?

c) At what angle θ' will the observer in K' see the light source at this instant.

9.3. Relativistic kinematics

The production of Higgs particles at the LHC was achieved through the collision of two collinear beams of protons. We consider the collisions in the rest frame K of the LHC. We denote by $M = 125\,\mathrm{GeV}/c^2$ the mass of the Higgs particle, and by $m = 938\,\mathrm{MeV}/c^2$ the mass of a proton.

a) Suppose two protons move along the x-axis with equal but opposite velocities to produce a Higgs particle at rest upon collision. Find the required energy and velocity of the protons.

b) Suppose a moving proton hits another proton at rest in the frame K. What is the energy and velocity such that a Higgs particle can be produced? What do you conclude from this?

Next, we consider the decay of the Higgs particle: Among many other options, it may decay into a pair of photons. This decay process is very rare but interesting since the resulting two highly energetic photons are easily recognised.

c) Suppose we had measured two photons with energies E_1 and E_2 at an angle θ between the two rays. Assuming the photons result from the decay of a single particle, what is the mass of the decaying particle?

d) Determine the speed and the direction of the decaying particle in the frame K.

10.1. Atwood machine in Lagrangian formalism

Consider now again the Atwood machine of problem 1.2 and problem 7.1. Two different masses m_1 and m_2 are connected by a rope of length l and are hanging from a pulley of radius r in a homogeneous gravitational field. This time we want to analyse it using the Lagrangian formalism.

At first, we will ignore the inertia of the pulley.

a) What are suitable generalised coordinate(s) to use to ensure that the length of the rope is constant? Write down the potential and kinetic energies as well as the Lagrangian function.

b) Write down the Euler–Lagrange equation for the problem. How does it relate to the acceleration from problem 1.2? Why does the string tension not show up in the problem?

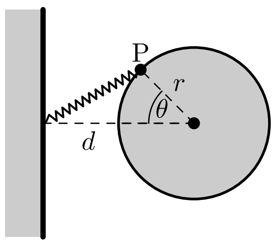
Next, let us additionally take the inertia of the pulley into account which is modelled by a homogeneous cylinder of thickness d and mass m_{P}.

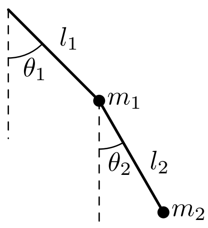
c) What generalised coordinate is suitable to describe the motion of the pulley? Eliminate that generalised coordinate by introducing a suitable constraint.

d) State the kinetic energy of the pulley and the full Lagrangian. Write down the Euler–Lagrange equation for the system. Verify that the answer to problem 7.1 solves the equation of motion. Why does the torque not show up here?

10.2. Lagrangians and equations of motion

Find the Lagrangians and the equations of motion for the systems illustrated below. You do not need to solve them.





(10.1)

a) A disc of radius r and mass m is fixed at its centre, around which it can rotate freely. Additionally, a spring of constant k and rest length l_0 connects a point P on the rim of the disc to a nearby wall. The spring is attached to the wall at the same height as the centre of the disc and its distance to the centre is d.

b) A double pendulum with fixed lengths l_i in a constant gravitational field.

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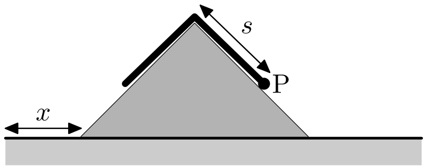
10.3. Sliding chain

A uniform chain of length l and mass m lies on top of a triangle block of mass $2m$ with two equal sides of length $\geq l$ and the top corner forming a right angle. The block is free to slide on a frictionless horizontal plane. At the right end P of the chain, a mass m is attached. The system is subjected to the gravity force and we ignore friction. At time $t = 0$ it is at rest with a length $s(0) = l/2$ of chain lying from the right side of the block.



(10.2)

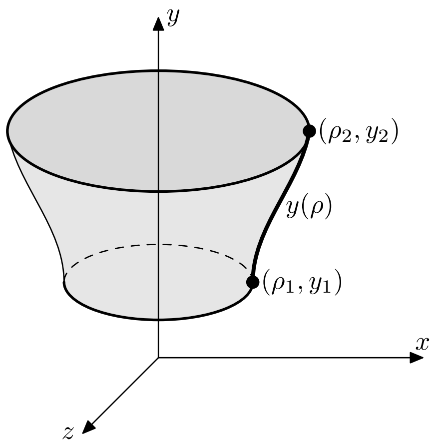
a) Find the Lagrangian of the system.

b) Solve the equations of motion for the given initial conditions as long as $s(t) \leq l$.

c) Compute the time t_f at which the left end of the chain will pass over the block's top, and the length x_f by which the block will have moved until then.

10.4. Minimal surface of revolution

Consider a surface originating from the revolution of a curve $y(\rho)$ around the y-axis. The curve connects the points (ρ_1, y_1) and (ρ_2, y_2), and we assume that $y_1 < y_2$ and $\rho_1 \leq \rho_2$. We may alternatively describe the curve by a function $\rho(y)$ and we will consider both descriptions in this exercise. In either case, we would like to determine $y(\rho)$ or $\rho(y)$ in such a way that the surface has minimal area using the calculus of variations.



(10.3)

a) Do you see an issue with either of the two descriptions?

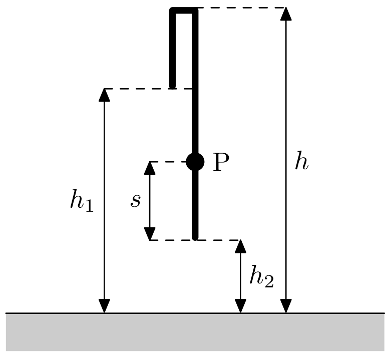
Hint: The figure does not display the issue. Think about the boundary conditions for the case $\rho_1 = \rho_2$.

b) Write the surface area A as a functional $A[y(\rho)]$ and as a functional $A[\rho(y)]$.

c) Write down the Euler–Lagrange equations for the problem using both descriptions. Select the description you find simpler and solve the Euler–Lagrange equations. Using your solution, find a solution for the other description.

11.1. Falling rope

A rope of length l and uniform linear mass density ρ is thrown vertically into the air and released, such that it forms a shape as shown in the figure below. We assume that the rope is infinitely flexible, so the bend is negligibly thin and the rope simply forms two vertical segments of length $h - h_1$ and $h - h_2$ (which add up to its total length l).



(11.1)

a) Use h_1 and h_2 as generalised coordinates, and show that the Lagrangian is given by

$$\begin{aligned} L = {} & \tfrac{1}{2}(h - h_1)\rho\dot{h}_1^2 + \tfrac{1}{2}(h - h_2)\rho\dot{h}_2^2 \\ & - \tfrac{1}{2}\rho g(h - h_1)(h + h_1) - \tfrac{1}{2}\rho g(h - h_2)(h + h_2). \end{aligned} \tag{11.2}$$

Find the corresponding Euler–Lagrange equations.

b) Substitute $x = h_1 - h_2$ and $y = h_1 + h_2$, and show that the coordinate x has the equation of motion

$$(l^2 - x^2)\ddot{x} = x\dot{x}^2. \tag{11.3}$$

c) Using the equation of motion found in part b), find $\dot{x}$ as a function of l, x and an integration constant in the regime where $\dot{x}(t) \neq 0$. Show that $\dot{x}$ goes to infinity when the rope changes its shape and the bend reaches the end of the rope (i.e. it causes a "whip cracking" effect). How do you interpret the other (trivial) solution $\dot{x}(t) = 0$ for all t?

Hint: To determine an expression for $\dot{x}$, write out expressions for $(\mathrm{d}/\mathrm{d}t)\log(\dot{x})$ and $(\mathrm{d}/\mathrm{d}t)\log(l^2 - x^2)$ and compare them with one another using (11.3).

d) The tension in the rope at any given point P at distance s from the lower end of the rope (see figure) is $Z = \rho s(\ddot{h}_2 + g)$. Show that for $\dot{x} \neq 0$, it holds that

$$Z = \frac{\rho s c^2}{2(l + x)(l^2 - x^2)}, \tag{11.4}$$

where c is a constant determined by initial conditions. How does Z behave as $x \to l$?

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11.2. Particle on a cylinder and paraboloid

In this problem, two forces act on a particle of mass m: gravitation (in negative z-direction) and another force $\vec{F} = -k\vec{x}$ pointing to the coordinate origin.

First, the movement of the particle is restricted to the surface of a cylinder described by the equation $x^2 + y^2 = R^2$.

a) Choosing appropriate generalised coordinates, and determine the Lagrangian of the system.

b) Write down the Euler–Lagrange equations of motion, and deduce that angular momentum is conserved.

Now, the point particle is constrained to move on a parabolic surface described by the equation $z = c^2(x^2 + y^2)$ under the influence of the same two forces.

c) Using a Lagrange multiplier λ to express the constraint, write the Lagrangian for this system in cylindrical coordinates (r, φ, z). Deduce the Euler–Lagrange equations of the system.

d) Show that for any value of angular momentum $L_z = mr^2\dot{\varphi} \neq 0$, the circular path $r(t) = r_0$ is a solution of the equations of motion if r_0 verifies an equation of the form

$$Ar_0^6 + Br_0^4 - L_z^2 = 0. \tag{11.5}$$

Find expressions for A and B.

11.3. Separable systems

Consider a system where the kinetic and potential energies in terms of the generalised coordinates q_i take the form

$$T = \sum_i f_i(q_i)\dot{q_i}^2, \qquad V = \sum_i V_i(q_i). \tag{11.6}$$

Note that this is not the most general form that T and V could take for an arbitrary system; here we are considering a special kind of system and/or choice of coordinates.

a) Show that the Euler–Lagrange equations for this system are separable, i.e. that each equation does not involve more than one of the generalised coordinates and its derivatives.

b) Using the equations of motion, show that each of the quantities $E_i = f_i(q_i)\dot{q_i}^2 + V_i(q_i)$ is a constant of motion, i.e. there is an effective energy associated to each generalised coordinate that is conserved as the system evolves.

c) Using the fact that E_i is a constant of motion, find an integral that expresses the time t as a function of the coordinate q_i, i.e. $t = g_i(q_i)$ for some function g_i. We shall assume there is some interval on which this function g_i is bijective, in which case its inverse would give the time evolution of the coordinate, $q_i = g_i^{-1}(t)$.

12.1. Galilean invariance and Lagrangian mechanics

In this exercise we want to derive the Lagrangian for a particle from the requirement of Galilean invariance.

a) Consider the undetermined Lagrangian of a particle $L(\vec{x}, \dot{\vec{x}}, t)$. If you require that your theory is invariant under rotations and translations, which constraints can you place on the functional dependence of L? Can you derive Newton's first law from it?

b) Consider two generic Lagrangians related by the total time derivative of another function

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} F(q, t). \tag{12.1}$$

How do the equations of motion generated by L and L' differ?

c) Consider the restricted Lagrangian from part a). How does it change under a Galilean boost $\dot{\vec{x}}' = \dot{\vec{x}} + \vec{\kappa}$? Use this expression to write down the explicit form of L that satisfies Galilean invariance. Interpret the result.

12.2. Symmetry transformations

Consider a solution $x = x(t)$ for a harmonic oscillator with mass m and spring constant k. Show that the transformation $x' = \phi_\lambda(x, t) := x + \lambda \cos(\omega t)$ with $\omega = \sqrt{k/m}$ is a symmetry transformation of the harmonic oscillator. Derive the associated conserved quantity and show by explicit calculation that the conserved quantity you found is indeed conserved.

Hint: Show that the Lagrangian is invariant up to a total time derivative of a function $K_\lambda(x, t)$.

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12.3. Symmetry and Laplace–Runge–Lenz vector

In this exercise we consider the symmetry variation associated to the Laplace–Runge–Lenz vector by Noether's theorem. In the lecture, we found that it is generated by the vector field

$$\vec{v} = \frac{\delta_{\vec{\kappa}}\vec{x}}{\delta\lambda} = \mu\big(2(\vec{\kappa}\cdot\vec{x})\dot{\vec{x}} - (\vec{\kappa}\cdot\dot{\vec{x}})\vec{x} - (\vec{x}\cdot\dot{\vec{x}})\vec{\kappa}\big), \tag{12.2}$$

which describes the deformation of a solution $\vec{x}$ of the equations of motion into another solution

$$\vec{x}\,' = \vec{x} + \delta\lambda\,\vec{v} + \mathcal{O}(\delta\lambda^2). \tag{12.3}$$

As a starting point, we consider the following solution for a circular orbit

$$\vec{x}(t) = (R\cos\omega t, R\sin\omega t, 0), \qquad \omega = \sqrt{g}\,R^{-3/2}. \tag{12.4}$$

We apply the above transformation, and we shall show that it deforms the circular orbit into an elliptic orbit.

a) First, show that the transformation generates the following deformation for the choice $\vec{\kappa} = (\kappa\cos\vartheta, \kappa\sin\vartheta, \rho)$

$$r(t) = R + \lambda\mu\kappa R^2\omega\sin(\omega t - \vartheta) + \mathcal{O}(\lambda^2). \tag{12.5}$$

b) In a second step, find the eccentricity, direction and semi-major axis of the deformation at linear order in the deformation parameter λ, by comparing (12.5) with the expansion of the relation

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\varphi-\varphi_0)}\,, \tag{12.6}$$

describing an ellipse which has one of its focal point in the origin. Here, a denotes the length of the semi-major axis, ε the eccentricity and φ_0 determines the orientation of the ellipse within the plane.

c) Consider the variations of the conserved quantities E, $\vec{L}$ and $\vec{A}$ around a generic solution of the equations of motion and show that they are given by

$$\frac{\delta_{\vec{\kappa}}E}{\delta\lambda} = 0, \qquad \frac{\delta_{\vec{\kappa}}\vec{L}}{\delta\lambda} = \vec{\kappa}\times\vec{A}, \qquad \frac{\delta_{\vec{\kappa}}\vec{A}}{\delta\lambda} = -2\mu E\,\vec{\kappa}\times\vec{L}. \tag{12.7}$$

Hint: Use the equation of motion $\ddot{\vec{x}} = -g\vec{x}/r$ to show that

$$\frac{\delta_{\vec{\kappa}}\dot{\vec{x}}}{\delta\lambda} = \mu\Big[(\vec{\kappa}\cdot\dot{\vec{x}})\dot{\vec{x}} - \frac{g}{r^3}\,(\vec{\kappa}\cdot\vec{x})\vec{x} - \Big(\dot{\vec{x}}^2 - \frac{g}{r}\Big)\vec{\kappa}\Big]. \tag{12.8}$$

d) *Optional:* Verify that the results of part c) are consistent with the deformations of the path parameters found in part b).

e) *Advanced:* Express the relations (12.7) as a Lie algebra of rotations and $\vec{\kappa}$-transformations (or equivalently as a Poisson algebra of conserved charges $\vec{L}$ and $\vec{A}$). Can you interpret the algebra?

13.1. Constrained particle in Hamiltonian framework

In problem 11.2 we considered the motion of a particle of mass m along a given surface (cylinder or paraboloid) while subject to gravitation (in negative z-direction) and another force $\vec{F} = -k\vec{x}$ pointing to the coordinate origin.

When the particle is constrained to move on the surface of a cylinder of radius R, the Lagrangian is given in cylindrical coordinates by

$$L = \tfrac{1}{2}m(R^2\dot{\varphi}^2 + \dot{z}^2) - \tfrac{1}{2}k(z^2 + R^2) - mgz. \tag{13.1}$$

a) Obtain the corresponding Hamiltonian. Derive and solve the Hamilton equations of motion.

When the particle is constrained to move on a parabolic surface described by the equation $z = c^2(x^2 + y^2)$ the Lagrangian in cylindrical coordinates reads after eliminating the z-coordinate by means of the constraint

$$L = \tfrac{1}{2}m\big((1 + 4c^4r^2)\dot{r}^2 + r^2\dot{\varphi}^2\big) - \tfrac{1}{2}k(c^4r^2 + 1)r^2 - mgc^2r^2. \tag{13.2}$$

b) Obtain the Hamiltonian and the corresponding Hamilton equations of motion.

13.2. Poisson algebra of angular momentum

In the lecture, we introduced the Hamiltonian vector field $\mathcal{D}[F]$ associated to a function $F(x, p)$ on the phase space as the differential operator

$$\mathcal{D}[F] := \sum_{i=1}^{3}\left(\frac{\partial F}{\partial x_i}\frac{\partial}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i}\right). \tag{13.3}$$

Here, we restrict to the case of one particle moving in three-dimensional Euclidean space and consider $\{\partial/\partial x_i, \partial/\partial p_i\}$ to be the basis of the tangent space of $\mathbb{R}^6$ at the point (x, p). We moreover discussed different ways to obtain the Poisson brackets of F from its Hamiltonian vector field in the lecture and we will apply them below.

a) Determine the Hamiltonian vector fields associated to the phase space functions position x_i, momentum p_i and angular momentum $L_i = \sum_j \sum_k \varepsilon_{ijk}x_jp_k$.

b) Employ the relation $\mathcal{D}[F]G = \{F, G\}$ to derive the Poisson brackets

$$\{L_i, x_j\} = \sum_{k=1}^{3}\varepsilon_{ijk}x_k, \qquad \{L_i, p_j\} = \sum_{k=1}^{3}\varepsilon_{ijk}p_k, \qquad \{L_i, L_j\} = \sum_{k=1}^{3}\varepsilon_{ijk}L_k. \tag{13.4}$$

c) Now, use the symplectic form

$$\omega = \sum_{i=1}^{3}\mathrm{d}x_i \wedge \mathrm{d}p_i \tag{13.5}$$

to derive the above Poisson brackets from the relation $\omega(\mathcal{D}[F], \mathcal{D}[G]) = \{F, G\}$.

$\longrightarrow$

13.3. Canonical transformations

Prove that the following transformation is canonical:

$$\begin{aligned} Q_1 &= q_1, & P_1 &= +p_1 - 2p_2, \\ Q_2 &= p_2, & P_2 &= -2q_1 - q_2. \end{aligned} \tag{13.6}$$

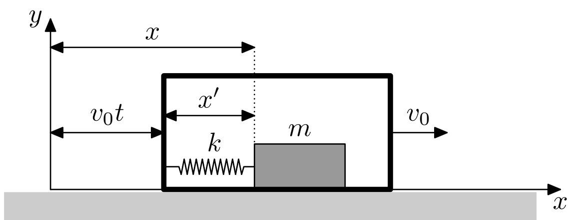
a) Show that the canonical Poisson brackets $\{Q_i, Q_j\}$, $\{Q_i, P_j\}$, $\{P_i, P_j\}$ of the new coordinates have the expected form.

b) Show that the criterion $A^\mathsf{T} \varepsilon A = \varepsilon$ holds where A is the Jacobian of the transformation and ε the symplectic structure.

c) Show that the symplectic form $\sum_i \mathrm{d}Q_i \wedge \mathrm{d}P_i$ is invariant.

13.4. Harmonic oscillator in a moving box

A box moves frictionless along the x-axis with constant velocity v_0. Inside the box – on its bottom and in x-direction as well – a mass m is oscillating. It is attached to the rear wall of the box via a spring of strength k and rest length d.



(13.7)

a) Derive the Hamiltonian H in the laboratory system. Is H a conserved quantity? Is H equal to the total energy E? In addition to that calculate Hamilton's equations.

b) Derive the Hamiltonian H in the co-moving system of the box. Consider the same tasks as in the former subproblem. What does change?

c) Let p' denote the momentum conjugate to the coordinate x' in the co-moving frame. Show that the following transformation with $\omega = \sqrt{k/m}$ is canonical

$$x' = \sqrt{\frac{2P}{m\omega}} \sin Q + d, \qquad p' = \sqrt{2Pm\omega} \cos Q. \tag{13.8}$$

d) Write the Hamiltonian in the co-moving system of the box in terms of P and Q and solve the Hamiltonian equations of motion in order to find $Q(t)$ and $P(t)$.

e) Write the solution in the coordinates $x'(t)$ and $p'(t)$ and plot its trajectory in phase space.

14.1. Hamiltonian with dissipative force

A particle of mass m moves in one dimension in a potential $V(q)$ and is subject to a damping force $F = -2m\gamma\dot{q}$ proportional to its velocity.

a) Show that the equation of motion can be obtained from the Lagrangian

$$L(q, \dot{q}, t) = \mathrm{e}^{2\gamma t}\left(\tfrac{1}{2}m\dot{q}^2 - V(q)\right). \tag{14.1}$$

b) Compute the canonical momentum p conjugate to q and the Hamiltonian $H(q, p, t)$.

c) Consider the canonical transformation defined by the following generating function

$$F_2(q, \bar{p}, t) = q\,\bar{p}\,\mathrm{e}^{\gamma t}\,. \tag{14.2}$$

Find the transformed Hamiltonian $\bar{H}(\bar{q}, \bar{p}, t)$.

Now, consider the harmonic oscillator potential $V(q) = \frac{1}{2}m\omega^2 q^2$.

d) Which of the Hamiltonians H and $\bar{H}$ is a constant of motion? Why?

e) In the underdamped case $\gamma < \omega$, obtain the solution $q(t)$ and express the integration constant related to the oscillation amplitude in terms of the conserved quantity.

14.2. Time-dependent Hamilton–Jacobi equation

We consider a particle of mass m in two dimensions (x, z) subject to the gravitational force $F_z = -mg$.

a) Set up the Hamiltonian for this system. Argue why in this case $H = T + V$ holds.

b) Formulate and solve the time-dependent Hamilton–Jacobi equation for this problem. How does $S(q, t)$ depend on t?

Hint: Use a separation ansatz $S(q, t) = \tilde{S}(q) + S_t(t)$.

c) Use the solution for $S(q, t)$ to determine $x(t)$, $z(t)$ and $z(x)$. Check that the solutions match your intuition.

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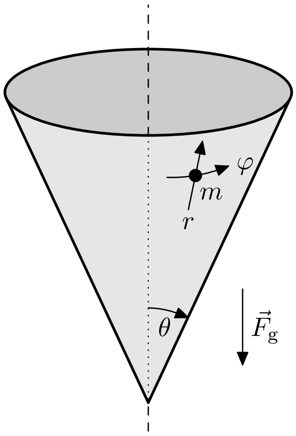
14.3. Time-independent Hamilton–Jacobi equation

Consider a particle of mass m moving on the surface of a cone with aperture 2θ. It is subject to a homogeneous gravitational field which acts downward along the axis.



(14.3)

a) Derive the Hamiltonian in spherical coordinates, determine the equations of motion and name two constants of motion. Do not forget to make use of the constraints.

b) Now, set up the time-independent Hamilton–Jacobi equation for this problem and solve it.

Hint: Use a separation ansatz in the (φ, r) coordinates of the form $S(\varphi, r) = S_1(\varphi) + S_2(r)$ in order to solve the Hamilton–Jacobi equation.

c) How can the solution to the Hamilton–Jacobi equation be used to derive an (implicit) solution to the above found equations of motion? You do not need to calculate the integrals explicitly. How can one interpret the constants showing up in the solution?

14.4. Canonical flows and generating functions

We have seen in the lecture that we can associate a generating function F to a canonical flow by the relation

$$\frac{\partial F}{\partial z}\left(\phi_\lambda(z)\right) = \varepsilon\,\frac{\partial \phi_\lambda}{\partial \lambda}(z). \tag{14.4}$$

Here, $z = (z^1, \ldots, z^{2N}) = (q^1, \ldots, q^N, p_1, \ldots, p_N)$ denotes the coordinates and

$$\varepsilon = \begin{pmatrix} 0 & +\,\mathrm{id}_N \\ -\,\mathrm{id}_N & 0 \end{pmatrix} \tag{14.5}$$

the coefficients of the symplectic form. We now consider a two-dimensional harmonic oscillator with Hamiltonian

$$H = T + V = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{k}{2}\left(x^2 + y^2\right). \tag{14.6}$$

a) Write the canonical flow which describes rotations in the x,y-plane and find the associated generating function. Is it conserved? Interpret your result.

b) Write the canonical flow which describes translations in the x-direction and find the associated generating function. Is it conserved? Interpret your result.